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Dilucidationes super methodo elegantissima, qua illustris de la Grange usus est in integranda aequatione differentiali $dx/\sqrt{X} = dy/\sqrt{Y}$

Leonhard Euler

DILVCIDATIONES SVPER METHODO ELEGANTISSIMA, QVA ILLVSTRIS *DE LA GRANGE* VSVS EST IN INTEGRANDA AEQVATIONE DIFFERENTIALI

$$\frac{dx}{\sqrt{X}} = \frac{du}{\sqrt{Y}}.$$

Auctore

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§. 1.

Postquam diu et multum in perſcrutanda aequatione differentiali $\frac{dx}{\sqrt{X}} = \frac{dy}{\sqrt{Y}}$ deſudaſſem, atque imprimis in methodum *directam*, quae via facili ac plana ad eius integrale perduceret, nequicquam inquiſiuiſſem; penitus obſtupui, cum mihi nunciaretur, in volumine quarto *Miſcellaneorum Taurinenſium* ab Illuſtri *de la Grange* talem methodum eſſe expoſitam, cuius ope pro caſu, quo

$$X = A + Bx + Cxx + Dx^3 + Ex^4 \text{ et}$$
$$Y = A + By + Cyy + Dy^3 + Ey^4$$

propoſitae aequationis differentialis hoc integrale algebraicum atque adeo completum feliciſſimo ſucceſſu elicuit.

$$\frac{\sqrt{X} + \sqrt{Y}}{x - y} = \sqrt{(\Delta + D(x+y) + E(x+y)^2)}$$

vbi Δ denotat quantitatem conſtantem arbitrariam per integrationem ingreſſam.

§. 2.

§. 2. Iſtud autem egregium inuentum eo magis ſum admiratus, quod equidem ſemper putaueram, talem methodum in inueſtigando idoneo factore, quo aequatio propoſita integrabilis redderetur, quaeri oportere, cum vulgo omnis methodus integrandi vel in ſeparatione variabilium, vel in idoneo multiplicatore contineri videatur, etiamſi certis caſibus quoque ipſa differentiatio ad integrale perducere queat, quemadmodum tam a me ipſo quam ab aliis per plurima exempla eſt oſtenſum. Ad hanc autem tertiam viam illa ipſa methodus *Grangiana* rite referri poſſe videtur.

§. 3. Quanquam autem facile eſt inuentis aliquid addere, tamen in re tam ardua plurimum intererit, hanc methodum ab Illuſtri *la Grange* adhibitam accuratius perpendiſſe atque ad vſum analyticum magis accommodaſſe; ſiquidem totum negotium multo facilius ac ſimplicius expediri poſſe videtur; quamobram, quae de hoc argumento, quod merito maximi momenti eſt cenſendum, ſum meditatus, hic data opera fuſius ſum expoſiturus.

§. 4. Quoniam autem hoc integrale ab Illuſtri *la Grange* inuentum, ab iis formis quas ipſe olim dederam, plurimum diſcrepat, ac ſimplicitate non mediocriter antecellit, ante omnia viſum eſt ſcitari, quomodo aequationi differentiali ſatisfaciat. Hunc in finem pono breu. gr. $\sqrt{X} + \sqrt{Y} = V$, vt habeam

$$\frac{V}{x-y} = \sqrt{(\Delta + D(x+y) + E(x+y)^2)},$$

quam aequationem ita differentiare oportet, vt conſtans arbitraria Δ ex differentiali excedat. Sumtis igitur quadratis erit

$\frac{V^2}{(x-y)^2} = \Delta + D(x+y) + E(x+y)^2$, quae differentiata dat

$$\frac{2 V dV}{(x-y)^2} - \frac{2 VV (dx-dy)}{(x-y)^3} -$$
$$D(dx+dy) - 2E(x+y)(dx+dy) = 0.$$

§. 5. Quo nunc calculus planior reddatur, seorsim partes vel per dx vel per dy affectas inuestigemus. Pro elemento igitur dx, si y vt constans spectetur, erit

$$dV = \frac{X^1 dx}{2\sqrt{X}},$$

vnde singulae partes ita se habebunt:

$$dx\left(\frac{V X^1}{(x-y)^2 \sqrt{X}} - \frac{2VV}{(x-y)^3} - D - 2E(x+y)\right)$$

vbi notetur esse $V = \sqrt{X} + \sqrt{Y}$, hincque

$$VV\sqrt{X} = (X+Y)\sqrt{X} + 2X\sqrt{Y}$$

vnde hic duplicis generis termini occurrunt, dum vel per $\sqrt{X}$ vel per $\sqrt{Y}$ sunt affecti. Duo autem termini adsunt $\sqrt{Y}$ affecti, qui sunt

$$-\frac{4X\sqrt{Y}}{(x-y)^3} + \frac{X^1\sqrt{Y}}{(x-y^2)},$$

qui ergo iunctim sumti dabunt

$$\frac{\sqrt{Y}}{(x-y)^3}(X^1(x-y) - 4X),$$

quae forma ob

$$X = A + Bx + Cxx + Dx^3 + Ex^4, \text{ hincque}$$
$$X^1 = B + 2Cx + 3Dxx + 4Ex^3, \text{ dabit}$$
$$X^1(x-y) - 4X = -4A - B(3x+y)$$
$$-2C(xx + xy) - D(x^3 + 3xxy) - 4Ex^3y$$

Termini autem per $\sqrt{x}$ affecti sunt

$$\frac{\sqrt{X}}{(x-y)^3}(X^1(x-y) - 2(X+Y) - D(x-y)^3 - 2E(x+y)(x-y)^3).$$

Cum

Cum igitur ſit

$$X + Y = 2A + B(x+y) + C(x^2+y^2) + D(x^3+y^3) + E(x^4+y^4)$$

facta ſubſtitutione iſte poſtremus factor erit

$$-4A - B(x+3y) - 2C(xy+yy) - D(3xyy+y^3) - 4Exy^3$$

quae forma a praecedente hoc tantum diſcrepat, quod litterae x et y ſunt permutatae.

§. 6. Quod ſi ergo breu. gr. ponamus

$$M = 4A + B(3x+y) + 2C(xx+xy) + D(x^3+3xxy) + 4Ex^3y$$

$$N = 4A + B(x+3y) + 2C(yy+xy) + D(y^3+3xyy) + 4Exy^3$$

hinc pars elemento dx affecta ita erit expreſſa:

$$-\frac{dx}{(x-y)^3\sqrt{X}}(M\sqrt{Y} + N\sqrt{X}).$$

§. 7. Simili modo

ob $dV = \frac{Y'dy}{2\sqrt{Y}}$,

partes elemento dy affectae erunt

$$\frac{dy}{\sqrt{Y}}\left(\frac{VY'}{(x-y)^2} + \frac{2VV\sqrt{Y}}{(x-y)^3} - D\sqrt{Y} - 2E(x+y)\sqrt{Y}\right).$$

Haec iam forma ob

$$V = \sqrt{X} + \sqrt{Y} \text{ et } VV\sqrt{Y} = (X+Y)\sqrt{Y} + 2Y\sqrt{X}$$

continebit ſequentes terminos per $\sqrt{X}$ affectos,

$$\frac{\sqrt{X}}{(x-y)^3}(Y'(x-y) + 4Y)$$

quae forma ex priore praecedentis calculi oritur, ſi litterae x et y permutentur, ſimulque ſigna; vnde patet hanc expreſſipreſſionem praebere valorem $+ N$. Reliqui autem termini per $\surd Y$ effecti erunt

$$\frac{\surd Y}{(x-y)^3}\left(Y'(x-y) + 2(X+Y) - D(x-y)^3 - 2E(x+y)(x-y)^3\right).$$

Haec forma iterum ex permutatione litterarum et ſignorum ex forma praecedentis calculi oritur, quae ergo cum eſſet $-N$, haec erit $+M$. Hoc igitur modo partes elementum dy continentes erunt

$$\frac{+dy}{(x-y)^3\surd Y}(N\surd X + M\surd Y)$$

§. 8. Coniungendis igitur his membris aequatio differentialis ex forma *Grangiana* orta erit

$$\left(\frac{dy}{\surd Y} - \frac{dx}{\surd X}\right)\left(\frac{N\surd X + M\surd Y}{(x-y)^3}\right) = 0,$$

quae per factorem comunem diuiſa praebet ipſam aequationem differentialem propoſitam $\frac{dx}{\surd X} = \frac{dy}{\surd Y}$; vnde ſimul patet aequationem integralem exhibitam recte ſe habere, atque adeo valorem litterae Δ arbitrio noſtro penitus relinqui.

§. 9. Antequam autem methodum *Grangianum* ad ipſam aequationem differentialem $\frac{dx}{\surd X} = \frac{dy}{\surd Y}$ in omni extenſione acceptam applicemus, a caſu ſimpliciore inchoëmus, quo aequatio adeo rationalis proponitur haec:

$$\frac{dx}{a+2bx+cxx} = \frac{dy}{a+2by+cyy}.$$

Analyſis.

Pro integratione aequationis differentialis

$$\frac{dx}{a+2bx+cxx} = \frac{dy}{a+2by+cyy}.$$

§. 10. Ponamus br. gr. $a + 2bx + cxx = X$ et $a + 2by + cyy = Y$, vt fieri debeat $\frac{dx}{X} = \frac{dy}{Y}$, quae formu-

formulae cum inter ſe debeant eſſe aequales, vtraque per idem elementum dt deſignetur, ita vt nanciſcamur has duas formulas: $\frac{dx}{dt} = X$ et $\frac{dy}{dt} = Y$.
Quod ſi ergo iam ſtatuamus

$$x - y = q, \text{ erit } \frac{dq}{dt} = X - Y = 2bq + cq(x + y)$$

vnde per q diuidendo erit $\frac{dq}{qdt} = 2b + c(x + y)$.

§. 11. Nunc primas formulas differentiemus, ſumto elemento dt conſtante, et facto

$$dX = X' dx \text{ et } dY = Y' dy$$

orientur hae duae aequationes:

$$\frac{ddx}{dxdt} X' \text{ et } \frac{ddy}{dydt} = Y',$$

quae inuicem additae praebent

$$\frac{ddx}{dxdt} + \frac{ddy}{dydt} = X' + Y'.$$

Quare cum ſit

$$X' = 2b + 2cx \text{ et } Y' = 2b + 2cy \text{ erit}$$

$$\frac{1}{dt}\left(\frac{ddx}{dx} + \frac{ddy}{dy}\right) = 4b + 2c(x + y).$$

§. 12. Quoniam igitur hic poſtremus valor duplo maior eſt praecedente $\frac{dq}{qdt}$, hoc modo deducti ſumus ad hanc aequationem:

$$\frac{ddx}{dx} + \frac{ddy}{dy} = \frac{2dq}{q},$$

quae integrata dat $l\,dx + l\,dy = 2\,l\,q +$ conſt, hincque in numeris erit

$$dx\,dy = Cqq\,dt^2, \text{ ita vt ſit } C = \frac{dxdy}{qqdt^2}.$$

Quare cum ſit

$$\frac{dx}{dt} = X \text{ et } \frac{dy}{dt} Y, \text{ aequatio integralis erit}$$

$$\frac{XY}{(x-y)^2} = C, \text{ quae ergo non ſolum eſt algebraica,}$$

ſed etiam completa.

§. 13. Si igitur propofita fuerit haec aequatio differentialis:

$$\frac{dx}{a+2bx+cxx}=\frac{dy}{a+2by+cyy},$$

eius integrale completum ita erit expreffum:

$$\frac{(a+2bx+cxx)(a+2by+cyy)}{(x-y)^2}=C$$

quae, vtrinque addendo $bb-ac$, induet hanc formam:

$$\frac{aa+2ab(x+y)+2acxy+bb(x+y)^2+2bcxy(x+y)+ccxxyy}{(x-y)^2}=\Delta\Delta,$$

ficque, extracta radice, integrale hanc formam habebit:

$$\frac{a+b(x+y)+cxy}{x-y}=\Delta,$$

quae fine dubio eft fimpliciffima, quandoquidem tam y per x quam x per y facillime exprimi poteft, cum fit

$$y=\frac{(\Delta-b)x-a}{\Delta+b+cx} \text{ et } x=\frac{a+(\Delta+b)y}{\Delta-b-cy}.$$

§. 14. Calculum, quo hic vfi fumus, perpendenti facile patebit, in his formis X et Y, non vltra quadrata progredi licere. Si enim ipfi X info̧per tribuamus terminum dx^3 et ipfi Y terminum dy^3, pro priore forma prodit

$$\frac{X-Y}{x-y}=2b+c(x+y)+d(xx+xy+yy)=\frac{dq}{qdt};$$

pro altera autem forma eft

$$X'+Y'=4b+2c(x+y)+3d(xx+yy)=\frac{ddx}{dxdt}+\frac{ddy}{dydt}.$$

Quare fi hinc duplum praecedentis aufferamus, colligitur

$$\frac{ddx}{dxdt}+\frac{ddy}{dydt}-\frac{2dq}{qdt}=d(x-y)^2,$$

quam aequationem non amplius integrare licet.

§. 15. Facile autem oftendi poteft, talem aequationem differentialem, in qua vltra quadratum proceditur, nullo amplius modo algebraice integrari poffe. Si enim tan-

tantum hic casus proponeretur: $\frac{dx}{1+x^2} = \frac{dy}{1+y^2}$, notum est, vtrinque integrale partim logarithmos partim arcus circulares inuoluere, ideoque quantitates transcendentes diuersos, quae nullo modo inter se comparari possunt. Huiusmodi scilicet comparationes iis tantum casibus locum habere possunt, quando vtrinque vnius generis tantum quantitates transcendentes occurrunt.

Analysis.

Pro integratione aequationis

$$\frac{dx}{a+2bx+cxx} + \frac{dy}{a+2by+cyy} = 0.$$

§. 16. Quod si hic vt ante ponamus $\frac{dx}{a+2bx+cxx} = dt$, statui debebit $\frac{dy}{a+2by+cyy} = -dt$: at vero si calculum simili modo quo ante instituere velimus, nihil plane proficimus. Postquam autem omnes difficultates probe perpendissem, tandem in artificium incidi, quo hunc casum expedire licuit, ita vt hinc non contemnendum incrementum methodo *Grangianae* attulisse mihi videar.

§. 17. Quoniam igitur has duas habeo aequationes: $\frac{dx}{dt} = X$ et $\frac{dy}{dt} = -Y$, hinc formo istam nouam aquationem:

$$\frac{ydx+xdy}{dt} = yX - xY.$$

Iam facio $xy = u$, vt habeam

$$\frac{du}{dt} = a(y-x) + cxy(x-y),$$

vnde posito

$$x - y = q \text{ erit } \frac{du}{dt} = q(cu-a),$$

quae

quae aequatio per $cu-a$ diuisa ductaque in c praebet $\frac{c\,du}{dt(cu-a)}=cq$, hocque modo nacti sumus differentiale logarithmicum.

§. 18. Dein vero aequationes principales vt ante differentiemus, et obtinebimus

$$\frac{ddx}{dt\,dx}=X' \text{ et } \frac{ddy}{dt\,dy}=-Y',$$

quae inuicem additae dant

$$\frac{1}{dt}\left(\frac{ddx}{dx}+\frac{ddy}{dy}\right)=X'-Y'=2cq;$$

quare si hinc duplum praecedentis aequationis subtrahamus, remanebit

$$\frac{1}{dt}\left(\frac{ddx}{dx}+\frac{ddy}{dy}-\frac{2c\,du}{cu-a}\right)=0,$$

vnde per dt multiplicando et integrando nanciscimur $l\,dx+l\,dy-2\,l(cu-a)=l\,C$, ideoque $\frac{dx\,dy}{(cu-a)^2}=C\,dt^2$. Cum igitur sit $dx=X\,dt$ et $dy=-Y\,dt$, aequatio integralis nostra erit $-\frac{XY}{(cu-a)^2}=C$.

§. 19. Per hanc ergo analysin deducti sumus ad hanc aequationem integralem aequationis propositae:

$$\frac{(a+2bx+cxx)(a+2by+cyy)}{(a-cxy)^2}=C.$$

quae aequatio, si vtrinque vnitas subtrahatur, reducitur ad hanc formam:

$$\frac{2ab(x+y)+ac(x+y)^2+4bbxy+2bcxy(x+y)}{(a-cxy)^2}=C.$$

§. 20. Illustremus hanc integrationem exemplo, ponendo $a=1$, $b=0$ et $c=1$, ita vt proposita sit haec aequatio differentialis: $\frac{dx}{1+xx}+\frac{dy}{1+yy}=0$, cuius integrale nouimus esse $A\,\text{tang.}\,x+A\,\text{tang.}\,y=A\,\text{tang.}\,\frac{x+y}{1-xy}=C$,

sicque

sicque nouimus esse $\frac{x+y}{1-xy}=C$. At vero nostra postrema formula dat pro hoc casu

$$\frac{(x+y)^2}{(1-xy)^2}=C \text{ ideoque } \frac{x+y}{1-xy}=C$$

quod egregie conuenit.

§. 21. Consideremus etiam casum, quo $a=1$, $b=\frac{1}{2}$ et $c=1$, ita vt proponatur haec aequatio:

$$\frac{dx}{1+x+xx}+\frac{dy}{1+y+yy}=0,$$

cuius integrale est

$$\frac{2}{\sqrt{3}}\text{ A tang. }\frac{x\sqrt{3}}{2+x}+\frac{2}{\sqrt{3}}\text{ A tang. }\frac{y\sqrt{3}}{2+y}=C,$$

vnde sequitur fore

$$\text{A tang. }\frac{2(x+y+xy)\sqrt{3}}{4+2(x+y)-2xy}=C,$$

ideoque etiam $\frac{x+y+xy}{2+x+y-xy}=C$. At vero forma integralis inuenta pro hoc casu dabit

$$\frac{x+y+(x+y)^2+xy+xy(x+y)}{(1-xy)^2}=C$$

quae in factores resoluta dat

$$\frac{(1+x+y)(x+y+xy)}{(1-xy)^2}=C.$$

Prior vero aequatio

$$\frac{x+y+xy}{2+x+y-xy}=C \text{ inuersa praebet } \frac{2+x+y-xy}{x+y+xy}=C,$$

et vnitate subtracta $\frac{1-xy}{x+y+xy}=C$, atque haec in praecedentem ducta dat $\frac{1+x+y}{1-xy}=C$.

§. 22. Videamus igitur, vtrum haec posteriores aequationes inter se conueniant, et quia constantes vtrinque inter se discrepare possunt, ambas aequationes ita referamus:

$$\frac{1-xy}{x+y+xy}=\alpha \text{ et } \frac{1+x+y}{1-xy}=\beta;$$

vnde

vnde cum sit $\frac{1}{\alpha} = \frac{x+y+xy}{1-xy}$, euidens est fore $\beta - \frac{1}{\alpha} = 1$, ex quo pulcherrimus consensus inter ambas formulas elucet. Ex his exemplis intelligitur aequationem generalem supra inuentam hoc modo per factores repraesentari posse:

$$\frac{(2b+c(x+y))(a(x+y)+2bxy)}{(a-cxy)^2}.$$

Ceterum consideratio harum formularum haud iniucundas speculationes suppeditare poterit.

§. 23. Sequenti autem modo forma illa integralis inuenta:

$$\frac{(2b+c(x+y))(a(x+y)+2bxy)}{(a-cxy)^2} = C$$

statim ad formam simplicissimam reduci potest; si enim eius factores statuamus

$$\frac{2b+c(x+y)}{a-cxy} = P \text{ et } \frac{a(x+y)+2bxy}{a-cxy} = Q$$

vt esse debeat $PQ = c$, erit

$$aP - cQ = \frac{2ab - 2bcxy}{a-cxy} = 2b \text{ vnde fit } Q = \frac{aP-2b}{c},$$

sicque quantitati constanti aequari debet haec forma: $\frac{aPP-2bP}{c}$; ex quo patet, etiam ipsam quantitatem P constanti aequari debere, ita vt iam aequatio nostra integralis sit

$$\frac{2b+c(x+y)}{a-cxy} = C, \text{ vel etiam } \frac{a(x+y)+2bxy}{a-cxy} = C.$$

Alia solutio facillima eiusdem aequationis

$$\frac{dx}{a+2bx+cxx} + \frac{dy}{a+2by+cyy} = 0.$$

§. 24. Postrema reductione probe perpensa, comperui, statim ab initio ad formam integralis simplicissimam perueniri posse, atque adeo non necesse esse ad differentialia secunda ascendere. Si enim vt ante ponamus $x+y$ $= p$

$= p$; $x - y = q$ et $xy = u$, ex formulis

$$\frac{dx}{dt} = X \text{ et } \frac{dy}{dt} = -Y$$

ſtatim deducimus

$$\frac{dp}{dt} = X - Y = 2bq + cpq, \text{ vnde fit } \frac{dp}{2b+cp} = q\,dt.$$

§. 25. Porro vero erit

$$\frac{y\,dx + x\,dy}{dt} = \frac{du}{dt} = yX - xY = -aq + cqu,$$

vnde fit $\frac{du}{cu-a} = q\,dt$, quam ob rem hinc ſtatim colligimus hanc aequationem: $\frac{dp}{2b+cp} = \frac{du}{cu-a}$, cuius integratio praebet $l(2b + cp) = l(cu - a) + lC$; vnde deducitur haec aequatio algebraica: $\frac{2b+cp}{cu-a} = C$, quae, reſtitutis literis x et y, dat $\frac{2b+c(x+y)}{cxy-a} = C$, quae eſt forma ſimpliciſſima aequationis integralis deſideratae. Hic imprimis notatu dignum occurrit, quod caſum primum hac ratione reſolvere non licet.

§. 26. Ex forma autem integrali inuenta facile aliae deriuantur, veluti ſi addamus $\frac{2b}{a}$, orietur haec forma $\frac{a(x+y)+2bxy}{cxy-a} = C$, quae per praecedentem diuiſa denuo nouam formam ſuppeditat, ſcilicet: $\frac{2b+c(x+y)}{a(x+y)+2bxy} = C$, quae formae quomodo ſatisfaciant operae pretium erit oſtendiſſe. Et quidem poſtrema forma, differentiata, erit

$$\frac{-2ab(dx+dy) - 4bb(y\,dx+x\,dy) - 2bc(yy\,dx+xx\,dy)}{(a(x+y)+2bxy)^2}$$

quae in ordinem redacta praebet

$$dx(2ab + 4bby + 2bcyy) + dy(2ab + 4bbx + 2bcxx) = 0.$$

Haec per $2b$ diuiſa et ſeparata dat

$$\frac{dx}{a+2bx+cxx} + \frac{dy}{a+2by+cyy} = 0$$

quae eſt ipſa propoſita.

Ana-

Analysis.

Pro integratione aequationis

$$\frac{dx}{\sqrt{(A+Bx+Cxx)}} = \frac{dy}{\sqrt{(A+By+Cyy)}}.$$

§. 27. Introducto nouo elemento dt, deinceps pro constanti habendo, oriuntur hae duae aequationes:

$$\frac{dx}{dt} = \sqrt{X} \text{ et } \frac{dy}{dt} = \sqrt{Y},$$

vbi literis X et Y valores initio assignatos tribuamus. Videbimus autem, pro methodo, qua hic vtemur, terminos litteris D et E affectos omitti debere. Sumtis ergo quadratis erit

$$\frac{dx^2}{dt^2} = X \text{ et } \frac{dy^2}{dt^2} = Y.$$

§. 28. Nunc istas formulas differentiemus, positoque, vt fieri solet, $dX = X' dx$ et $dY = Y' dy$ nanciscemur has aequationes:

$$\frac{2ddx}{dt^2} = X' \text{ et } \frac{2ddy}{dt^2} = Y',$$

ac posito $x + y = p$ fiet $\frac{2ddp}{dt^2} = X' + Y'$. Cum iam sit

$$X' = B + 2Cx + 3Dxx + 4Ex^3 \text{ et}$$
$$Y' = B + 2Cy + 3Dyy + 4Ey^3 \text{ erit}$$
$$X' + Y' = 2B + 2Cp + 3D(xx+yy)$$
$$+ 4E(x^3 + y^3) = \frac{2ddp}{dt^2},$$

quae aequatio manifesto integrationem admittet, si fuerit et $D = 0$ et $E = 0$, quemadmodum assumsimus. Multiplicando igitur per dp et integrando nanciscimur

$$\frac{dp^2}{dt^2} = \Delta + 2Bp + Cpp$$

et radicem extrahendo

$$\frac{dp}{dt} = \sqrt{(\Delta + 2Bp + Cpp)}.$$

Cum

Cum igitur ſit $\frac{dp}{dt} = \sqrt{X} + \sqrt{Y}$, aequatio integralis, quam ſumus adepti, erit

$$\sqrt{X} + \sqrt{Y} = \sqrt{(\Delta + 2B(x+y) + C(x+y)^2)},$$

quae adeo eſt algebraica; vbi notetur eſſe

$$X = A + Bx + Cxx \text{ et } Y = A + By + Cyy.$$

§. 29. Sumamus igitur quadrata, et noſtra aequatio integralis erit

$$2A + B(x+y) + C(x^2+y^2) + 2\sqrt{XY}$$
$$= \Delta + 2B(x+y) + C(x+y)^2, \text{ ſiue}$$
$$2A - B(x+y) - 2Cxy + 2\sqrt{XY} = \Delta,$$

quae penitus ab irrationalitate liberata, poſito $\Delta - 2A = \Gamma$ praebebit

$$4XY = 4AA + 4AB(x+y) + 4AC(xx+yy)$$
$$+ 4BBxy + 4BCxy(x+y) + 4CCxxyy$$
$$= \Gamma^2 + 2\Gamma B(x+y) + 4\Gamma Cxy + BB(x+y)^2$$
$$+ 4BCxy(x+y) + 4CCxxyy$$

ſiue

$$(4AA - \Gamma^2) + 2B(2A - \Gamma)(x+y) + 4(BB - \Gamma C)xy$$
$$+ 4AC(xx+yy) - B^2(x+y)^2 = 0.$$

§. 30. Quod ſi iam hanc aequationem rationalem cum formula *canonica*, qua olim ſum vſus ad huiusmodi integrationes expediendas, comparemus, quae erat

$$\alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy = 0,$$

dum ſcilicet loco $(x+y)^2$ ſcribamus $(xx+yy) + 2xy$, reperiemus fore

$$\alpha = 4AA - \Gamma^2;\ \beta = B(2A - \Gamma);\ \gamma = 4AC - B^2;$$
$$\delta = BB - 2\Gamma C.$$

§. 31. Alio vero insuper modo eandem aequationem differentialem propositam integrare poterimus, introducendo literam $q = x - y$; tum enim habebimus

$$\frac{2\,ddq}{dt} = X' - Y'. \text{ At vero erit}$$

$$X' - X' = 2\,C\,q + 3\,D\,q\,(x + y)$$

vbi iterum patet statui debere tam $D = 0$ quam $E = 0$, vt integratio, multiplicando per dq, succedat. Hoc autem notato erit integrale $\frac{dq^2}{dt^2} = \text{Const} + C\,q\,q$, ideoque

$$\frac{dq}{dt} = \sqrt{(\Delta + C\,q\,q)}.$$

§. 32. Cum igitur sit $\frac{dq}{dt} = \sqrt{X} - \sqrt{Y}$, hoc integrale ita erit expressum:

$$\sqrt{X} - \sqrt{Y} = \sqrt{(\Delta + C\,q\,q)}$$

quae aequatio sumtis quadratis abit in hanc:

$$2\,A + B\,(x + y) + C\,(x\,x + y\,y) - 2\sqrt{X\,Y}$$
$$= \Delta + C\,(x - y)^2 \text{ siue}$$

$$2\,A + B\,(x + y) + 2\,C\,x\,y - 2\sqrt{X\,Y} = \Delta$$

vnde fit

$$2\sqrt{X\,Y} = 2\,A - \Delta + B\,(x + y) + 2\,C\,x\,y,$$

vbi si ponatur $2\,A - \Delta = -F$ aequatio ab ante inventa prorsus non discrepat.

§. 33. Quod si autem proposita fuisset aequatio

$$\frac{dx}{\sqrt{(A + B\,x\,x + C\,x\,x)}} + \frac{dy}{\sqrt{(A + B\,y + C\,y\,y)}} = 0,$$

integralia ante inuenta ad hunc casum referentur, si modo loco $\sqrt{Y}$ scribatur $-\sqrt{Y}$; vnde patet pro hoc casu haberi hanc aequationem:

$$\sqrt{X} - \sqrt{Y} = \sqrt{(\Delta + 2\,B\,(x + y) + C\,(x + y)^2)}$$

vel

vel etiam

$$\sqrt{X} + \sqrt{Y} = \sqrt{(\Delta + C(x-y)^2)}$$

§. 34. Hic singularis casus occurrit, quando formulae $A + Bx + Cxx$ sunt quadrata. Sit enim

$$X = (a + bx)^2 \text{ et } Y = (a + by)^2 \text{ ideoque}$$

$$A = a^2,\ B = 2ab,\ C = bb;$$

tum enim prior forma integralis erit

$$b(x-y) = \sqrt{(\Delta + 4ab(x+y) + bb(x+y)^2)}$$

sumtisque quadratis

$$-4bbxy = \Delta + 4ab(x+y), \text{ ideoque}$$

$$\Delta = a(x+y) + bxy$$

cuius aequationis differentiale est

$$a(dx + dy) + b(x\,dy + y\,dx) = 0 \text{ ideoque}$$

$$dx(a + by) + dy(a + bx) = 0.$$

Sin autem altera formula vtatur, erit

$$2a + b(x+y) = \sqrt{(\Delta + bb(x-y)^2)}$$

vnde quadratis sumtis, positoque $\Delta - 4aa = \Gamma$ prodit vt ante $\Gamma = a(x+y) + bxy$.

Analysis

Pro integranda aequatione

$$\frac{dx}{\sqrt{X}} = \frac{dy}{\sqrt{Y}}$$

existente $X = A + Bx + Cxx + Dx^3 + Ex^4$

et $Y = A + By + Cyy + Dy^3 + Ey^4$

§. 35. Introducto iterum elemento dt, vt fit

$$\frac{dx}{dt} = \sqrt{X} \text{ et } \frac{dy}{dt} = \sqrt{Y},$$

ideoque fumtis quadratis

$$\frac{dx^2}{dt^2} = X \text{ et } \frac{dy^2}{dt^2} = Y,$$

ftatuamus $x + y = p$ et $x - y = q$, et quia hinc prodit

$$dx^2 - dy^2 = dp\,dq, \text{ erit}$$

$$\frac{dp\,dq}{dt^2} = X - Y = B(x - y) + C(x^2 - y^2) + D(x^3 - y^3) + E(x^4 - y^4)$$

§. 36. Quoniam igitur eft

$$x = \frac{p+q}{2} \text{ et } y = \frac{p-q}{2}$$

his valoribus introductis reperietur

$$X - Y = Bq + Cpq + \tfrac{1}{4}Dq(3pp + qq) + \tfrac{1}{2}Epq(pp + qq)$$

vnde per q diuidendo oritur

$$\frac{dp\,dq}{q\,dt^2} = B + Cp + \tfrac{1}{4}D(3pp + qq) + \tfrac{1}{2}Epq(pp + qq).$$

§. 37. Nunc etiam formulas quadratas primo exhibitas differentiemus, et ftatuendo vt ante

$$dX = X'dx \text{ et } dX = Y'dy \text{ habebimus}$$

$$\frac{2\,ddx}{dt^2} = X' \text{ et } \frac{2\,ddy}{dt^2} = Y', \text{ hincque addendo}$$

$$\frac{2\,ddp}{dt^2} = X' + Y'. \text{ Cum vero fit}$$

$$X' = B + 2Cx + 3Dxx + 4Ex^3 \text{ et}$$

$$Y' = B + 2Cy + 3Dyy + 4Ey^3$$

erit $X' + Y' = 2B + 2Cp + \frac{3}{2}D(pp + qq) + Ep(pp + 3qq)$,

ita

ita vt substituto hoc valore fiat

$$\frac{dd p}{d t^2} = B + C p + \tfrac{3}{2} D (pp + qq) + \tfrac{1}{2} E p (pp + 3 qq)$$

a qua aequatione si priorem $\frac{dp\, dq}{q\, dt^2}$ subtrahamus, remanebit sequens:

$$\frac{dd p}{dt^2} - \frac{dp\, dq}{q\, dt^2} = \tfrac{1}{2} D qq + E p qq.$$

§. 38. Haec iam aequatio per qq diuisa producit istam:

$$\frac{1}{dt^2}\left(\frac{dd p}{qq} - \frac{dp\, dq}{q^3}\right) = \tfrac{1}{2} D + E p,$$

quae ducta in $2\, dp$ manifesto fit integrabilis: prodit enim

$$\frac{dp^2}{qq\, dt^2} = \Delta + D p + E pp$$

ex qua radice extracta colligitur:

$$\frac{dp}{q\, dt} = \sqrt{(\Delta + D p + E pp)}.$$

Cum igitur posuerimus

$$p = x + y \text{ et } q = x - y, \text{ erit } \frac{dp}{dt} = \sqrt{X} + \sqrt{Y},$$

vnde resultat haec aequatio integralis algebraica:

$$\frac{\sqrt{X} + \sqrt{Y}}{x - y} = \sqrt{(\Delta + D (x + y) + E (x + y)^2)}$$

quae est ipsa forma ab Illustri *la Grange* inuenta.

§. 39. Euoluamus vlterius hanc formam, ac sumtis quadratis erit

$$\frac{X + Y + 2\sqrt{XY}}{(x - y)^2} = \Delta + D (x + y) + E (x + y)^2.$$

Est vero

$$X + Y = 2 A + B (x + y) + C (xx + yy) + D (x^3 + y^3) + E (x^4 + y^4)$$

vnde si auferamus

$$(D (x + y) + E (x + y)^2)\, (x - y)^2$$

remanebit

$$2 A + B (x + y) + C (x^2 + y^2) + D xy (x + y) + 2 E xxyy,$$

quo

quo substituto aequatio integralis erit

$$\frac{2A+B(x+y)+C(x^2+y^2)+Dxy(x+y)+2Ex^2y^2+2\sqrt{XY}}{(x-y)^2}=\Delta$$

§. 40. Haec aequatio aliquanto concinnior reddi potest subtrahendo vtrinque C et statuendo $\Delta - C = \Gamma$: habebitur enim hoc facto

$$\frac{2A+B(x+y)+2Cxy+Dxy(x+y)+2Exxyy+2\sqrt{XY}}{(x-y)^2}=\Gamma$$

vnde deducimus

$$2\sqrt{XY}=\Gamma(x-y)^2-2A-B(x+y)-2Cxy$$
$$-Dxy(x+y)-2Exxyy$$

siue ponendo

$$2A+B(x+y)+2Cxy+Dxy(x+y)+2Exxyy=V$$

aequatio nostra erit

$$2\sqrt{XY}=\Gamma(x-y)^2-V,$$ quae sumtis quadratis abit in hanc:

$$4XY=\Gamma^2(x-y)^4-2\Gamma V(x-y)^2+VV$$ siue

$$4XY-VV=\Gamma^2(x-y)^4-2\Gamma V(x-y)^2$$

§. 41. Facta nunc substitutione erit

$$4XY=4A^2+4AB(x+y)+4AC(xx+yy)$$
$$+4AD(x^3+y^3)+4AE(x^4+y^4)+4BBxy$$
$$+4BCxy(x+y)+4BDxy(xx+yy)$$
$$+4BExy(x^3+y^3)+4CCxxyy$$
$$+4CDxxyy(x+y)+4CExxyy(xx+yy)$$
$$+4DDx^3y^3+4DEx^3y^3(x+y)$$
$$+4EEx^4y^4.$$

At vero porro colligitur fore

$$VV=4AA+4AB(x+y)+8ACxy$$
$$+4ADxy(x+y)+8AExxyy+BB(x+y)^2$$
$$+4B$$

$$+4BCxy(x+y)+2BDxy(x+y)^2$$
$$+4BE(x+y)\,xxyy+4CCxxyy$$
$$+4CD(x+y)\,xxyy+8CEx^3y^3$$
$$+DDxxyy(x+y)^2+4DEx^3y^3(x+y)$$
$$+4EEx^4y^4$$

§. 42. Quod si iam posteriorem formulam a priore subtrahamus et singulos terminos ordine analogos disponamus, reperiemus

$$4XY-VV=4AC(x-y)^2+4AD(x+y)(x-y)^2$$
$$+4AE(x+y)^2(x-y)^2-B^2(x-y)^2$$
$$+2BDxy(x-y)^2+4BExy(x+y)(x-y)^2$$
$$+4CExxyy(x-y)^2-DDxxyy(x-y)^2$$

quae expressio factorem habet communem $(x-y)^2$, per quem ergo si diuidamus perueniemus ad hanc aequationem concinniorem:

$$4AC+4AD(x+y)+4AE(x+y)^2-BB$$
$$+2BDxy+4BExy(x+y)+(4CE-DD)xxyy$$
$$=\Gamma\Gamma(x-y)^2-4\Gamma A-2\Gamma B(x+y)-4\Gamma Cxy$$
$$-2\Gamma Dxy(x+y)-4\Gamma Exxyy.$$

§. 43. Transferamus nunc omnes terminos ad partem sinistram et loco $(x+y)^2$ scribamus $(xx+yy)+2xy$, tum vero $(xx+yy)-2xy$ loco $(x-y)^2$, quo facto talis oritur aequatio meae canonicae respondens:

$$0=\begin{cases}4AC+4AD(x+y)+4AE(x^2+y^2)+2BDxy+4BExy(x+y)+4CExxyy\\-BB+2\Gamma C(x+y)-\Gamma\Gamma(x^2+y^2)+8AExy+2\Gamma Dxy(x+y)-DDxxyy\\+4\Gamma A\qquad\qquad\qquad\qquad\qquad +2\Gamma^2xy\qquad\qquad\qquad +4\Gamma Exxyy\\\qquad\qquad\qquad\qquad\qquad\qquad +4\Gamma Cxy\end{cases}$$

§. 44. Hinc ergo pro aequatione canonica literae graecae α, β, γ, δ, etc. per latinas A, B, C, D, E, vna cum constante Γ sequenti modo determinantur:

$$\alpha = 4\,AC + 4\,\Gamma A - BB$$
$$\beta = 2\,AD + \Gamma B$$
$$\gamma = 4\,AE - \Gamma\Gamma$$
$$\delta = BD + 4\,AE + \Gamma^2 + 2\,\Gamma C$$
$$\varepsilon = 2\,BE + \Gamma D$$
$$\zeta = 4\,CE + 4\,\Gamma E - DD$$

ita vt aequatio canonica, qua olim sum vsus, sit

$$\alpha + 2\,\beta(x+y) + \gamma(xx+yy) + 2\,\delta xy + 2\,\varepsilon xy(x+y) + \zeta xxyy = 0.$$

§. 45. Haec autem aequatio integralis ad rationalitatem perducta latius patet quam aequatio proposita differentialis $\frac{dx}{\sqrt{X}} - \frac{dy}{\sqrt{Y}} = 0$: simul enim complectitur integrale huius: $\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$. Scilicet haec aequatio complectitur duos factores, quorum alteruter alterutri satisfacit. Ex genesi autem patet hanc aequationem esse productum ex his factoribus: $\Delta\,(x-y)^2 - V + 2\sqrt{XY}$, et $\Delta\,(x-y)^2 - V - 2\sqrt{XY}$.

§. 46. Supra iam obseruauimus, eiusdem aequationis differentialis integrale hoc quoque modo exhiberi posse: $\frac{M\sqrt{Y} + N\sqrt{X}}{(x-y)^3} = C$ (vide §. 8. et praec.) existente

$$M = 4\,A + B(3\,x+y) + 2\,C(xx+xy) + D\,xx(x+3y) + 4\,E\,x^3y$$
$$N = 4\,A + B(3\,y+x) + 2\,C\,y(x+y) + D\,yy(y+3\,x) + 4\,E\,xy^3$$

vbi

vbi notasse iuuabit esse

$$M + N = 8A + 4B(x+y) + 2C(x+y)^2$$
$$+ D(x+y)^3 + 4Exy(xx+yy)$$
$$M - N = 2B(x-y) + 2C(x+y)(x-y)$$
$$+ D(x-y)(x^2+4xy+y^2)$$
$$+ 4Exy(x+y(x-y)$$

Interim tamen haud facile intelligitur, quomodo haec forma cum ante inuenta consentiat, dum tamen de consensu certi esse possumus.

§. 47. Ex iis, quae hactenus sunt allata, satis liquet, eandem aequationem integralem innumeris modis exhiberi posse, prout constans arbitraria alio atque alio modo repraesentatur; vnde plurimum intererit certam legem stabilire, secundum quam quouis casu constantem illam arbitrariam exprimere velimus. Hunc in finem ista regula obseruetur: vt perpetuo integralia ita capiantur, vt posito $y = 0$ fiat $x = k$, hincque secundum legem compositionis $X = K$, existente

$$K = + A + Bk + Ckk + Dk^3 + Ek^4.$$

Hac enim lege obseruata omnia integralia, vtcunque diuersa videantur, ad perfectum consensum perduci poterunt. Hoc igitur modo quae hactenus inuenimus sequentibus Theorematibus complectamur.

Theorema I.

§. 48. Si haec aequatio differentialis

$$\frac{dx}{a+bx+cxx} - \frac{dy}{a+by+cyy} = 0,$$

ita integretur, vt posito $y = 0$ fiat $x = k$, integrale ita se habebit:

$$\frac{2a+b(x+y)+2cxy}{x-y} = \frac{2a+bk}{k}.$$

Theorema II.

§. 49. Si haec aequatio differentialis:

$$\frac{dx}{a+bx+cxx} + \frac{dy}{a+by+cyy} = 0$$

ita integretur, vt posito $y=0$ fiat $x=k$, integrale supra triplici modo est inuentum; erit enim:

I. $\frac{b+c(x+y)}{cxy-a} = -\frac{b+ck}{a}$;

II. $\frac{a(x+y)+bxy}{cxy-a} = -k$

III. $\frac{b+c(x+y)}{a(x+y)+bxy} = \frac{b+ck}{ak}$.

Theorema III.

§. 50. Si haec aequatio differentialis:

$$\frac{dx}{\sqrt{(A+Bx+Cxx)}} - \frac{dy}{\sqrt{(A+By+Cyy)}} = 0$$

ita integretur, vt posito $y=0$ fiat $x=k$, integrale erit

$$-B(x+y) - 2Cxy + 2\sqrt{(A+Bx+Cxx)}\sqrt{(A+By+Cyy)} =$$
$$-Bk + 2\sqrt{A(A+Bk+Ckk)}, \text{ siue}$$
$$B(k-x-y) - 2Cxy = 2\sqrt{A(A+Bk+Ckk)} - 2\sqrt{(A+Bx+Cxx)(A+By+Cyy)}$$

Corollarium.

§. 51. Hinc ergo patet, si aequatio differentialis proposita, fuerit ista:

$$\frac{dx}{\sqrt{(A+Bx+Cxx)}} = \frac{dy}{\sqrt{(A+By+Cyy)}} = 0,$$

eaque integretur ita, vt posito $y=0$ fiat $x=k$, integrale fore

$$B(k-x-y) - 2Cxy = 2\sqrt{(A+Bx+Cxx)}.\sqrt{(A+By+Cyy)} - 2\sqrt{A(A+Bk+Ckk)}.$$

Theo-

Theorema IV.

§. 52. Si posito br. gr.

$$X = A + Bx + Cxx + Dx^3 + Ex^4$$
$$Y = A + By + Cyy + Dy^3 + Ey^4$$
$$K = A + Bk + Ckk + Dk^3 + Ek^4$$

haec proponetur aequatio differentialis: $\frac{dx}{\sqrt{X}} - \frac{dy}{\sqrt{Y}} = 0$, quae ita integrari debeat, vt posito $y = 0$ fiat $x = k$, eius integrale ita erit expressum:

$$\frac{2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy + 2\sqrt{XY}}{(x+y)^2} = \frac{2A + Bk + 2\sqrt{AK}}{kk}.$$

Sin autem aequatio proposita fuerit

$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$, eius integrale erit

$$\frac{2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy - 2\sqrt{XY}}{(x-y)^2} = \frac{2A + Bk - 2\sqrt{Ak}}{kk}.$$

Corollarium I.

§. 53. Quod si hic ponamus $D = 0$ et $E = 0$, casus oritur Theorematis tertii, pro aequatione

$$\frac{dx}{\sqrt{A + Bx + Cxx}} - \frac{dy}{\sqrt{A + By + Cyy}} = 0,$$

cuius ergo integrale hinc erit

$$\frac{2A + B(x+y) + 2Cxy + 2\sqrt{(A + Bx + Cxx)(A + By + Cyy)}}{(x-y)^2} = \frac{2A + Bk + 2\sqrt{A(A + Bk + Ckk)}}{kk},$$

quae forma si cum superiori comparetur, formulae irrationales eliminari poterunt. Quoniam enim ex priore est

$$2\sqrt{XY} = 2\sqrt{A(A + Bk + Ckk)} - B(k - x - y) + 2Cxy$$

erit

erit hoc integrale postrenum

$$\frac{2A + B(2x + 2y - k) + 4Cxy + 2\sqrt{A(A + Bk + Ckk)}}{(x-y)^2} =$$

$$\frac{2A + BK + 2\sqrt{A(A + Bk + Ckk)}}{kk}$$

vnde statim deduci potest aequatio canonica

$$\alpha + 2\beta(x+y) + \gamma(xx + yy) + 2\delta xy = 0.$$

Corollarium II.

§. 54. Ponamus nunc esse $A = 0$ et $B = 0$, vt sit

$$X = xx(C + Dx + Exx) \text{ et } Y = yy(C + Dy + Eyy)$$
$$\text{et } K = kk(C + Dk + Ekk)$$

aequatio differentialis integranda fiet

$$\frac{dx}{x\sqrt{(C + Dx + Exx)}} - \frac{dy}{y\sqrt{(C + Dy + Eyy)}} = 0,$$

cuius ergo integrale erit

$$\frac{xy(2C + D(x+y) + 2Exy) + 2xy\sqrt{(C + Dx + Exx)(C + Dy + Eyy)}}{(x-y)^2} = \Delta$$

atque hic constantem Δ per k definire non licebit: positio enim $y = 0$ incongruum iam inuoluit. Interim tamen et haec integratio maxime est memoratu digna.

Corollarium III.

§. 55. Quod si autem in hac postrema integratione loco x et y scribamus $\frac{1}{x}$ et $\frac{1}{y}$ primo aequatio differentialis erit

$$\frac{dy}{\sqrt{(Cyy + Dy + E)}} - \frac{dx}{\sqrt{(Cxx + Dx + E)}} = 0;$$

tum vero integrale sequentem induet formam:

$$\frac{2Cxy + D(x+y) + 2E + 2\sqrt{(Cxx + Dx + E)(Cyy + Dy + E)}}{(y-x)^2} = \Delta$$

$$= \frac{Dk + 2E + 2\sqrt{E(Chk + Dk + E)}}{kk}.$$

Si

Si igitur hic loco literarum E, D, C, scribamus A, B, C, prodibit aequatio differentialis supra tractata

$$\frac{dx}{\sqrt{(A+Bx+Cxx)}} - \frac{dx}{\sqrt{(A+By+Cyy)}} =$$

cuius ergo integrale erit

$$\frac{2A+B(x+y)+2Cxy+2\sqrt{(A+Bx+Cxx)(A+By+Cyy)}}{(x-y)^2} = \frac{Bk+2A+2\sqrt{A}(A+Bk+Ckk)}{kk},$$

quae egregie conuenit cum ea in Coroll. I. data.

Corollarium IV.

§. 56. Contemplemur nunc etiam casum, quo formula $A+Bx+Cxx+Dx^3+Ex^4$ fit quadratum, quod sit $(a+bx+cxx)^2$, ita vt iam habeamus

$$A=aa,\ B=2ab,\ C=bb+2ac,\ D=2bc,\ E=cc,$$

tum vero

$$\sqrt{X}=a+bx+cxx,\ \sqrt{Y}=a+by+cyy,$$
$$\sqrt{K}=a+bk+ckk$$

atque aequatio differentialis pro priore casu erit

$$\frac{dx}{a+bx+cxx} - \frac{dy}{a+by+cyy} = 0,$$

cuius ergo integrale erit

$$(2aa+2ab(x+y)+2(bb+2ac)xy+2bcxy(x+y)$$
$$+2ccxxyy+2(a+bx+cxx)(a+by+cyy)):$$
$$(x-y)^2=\Delta,$$

quae reducitur ad

$$\frac{aa+ab(x+y)+(bb+ac)xy+bcxy(x+y)+ccxxyy}{(x-y)^2} =$$

$\frac{aa+abk}{kk}$. Quod si iam vtrinque addamus $\frac{1}{4}bb$, prodibit

$$\frac{(a+\frac{1}{2}b(x+y)+cxy)^2}{(x-y)^2} = \frac{(a+\frac{1}{2}bk)^2}{k^2}$$

vnde

vnde extracta radice obtinetur forma integralis in theoremate primo assignata.

§. 57. Sin autem hoc modo alterum casum aequationis

$$\frac{dx}{a+bx+cxx}+\frac{dy}{a+by+cyy}=0$$

euoluere velimus, peruenimus ad hanc aequationem:

$$\frac{2aa+2ab(x+y)+2(bb+2ac)xy+2bcxy(x+y)+2ccxxyy}{(x-y)^2}$$
$$-\frac{2(a+bx+cxx)(a+by+cyy)}{(x-y)^2}=\Delta,$$

quae euoluta praebet $\Delta=-2ac$, haecque aequatio manifesto est absurda, et nihil circa integrale quaesitum declarat, cuius rationem maximi momenti erit perscrutari.

Insigne Paradoxon.

§. 58. Cum huius aequationis differentialis

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$$

integrale in genere inuentum sit

$$\frac{2A+B(x+y)+2Cxy+Dxy(x+y)+2Exxyy-2\sqrt{XY}}{(x-y)^2}=\Delta$$

casu autem, quo statuitur

$$\sqrt{X}=a+bx+cxx \text{ et } \sqrt{Y}=a+by+cyy$$

aequatio absurda inde oriatur, quaeritur enodatio huius insignis difficultatis ac praecipue modus, hinc verum integralis valorem inuestigandi.

Enodatio Paradoxi.

§. 59. Quemadmodum scilicet in Analysi eiusmodi formulae occurrere solent, quae certis casibus indeterminatae atque adeo nihil plane significare videntur: ita hic

hic ſimile quid vſu venit, ſed longe alio modo, cum neque ad fractionem, cuius numerator et denominator ſimul euaneſcunt, neque ad differentiam inter duo infinita perveniatur, quod exemplum eo magis eſt notatu dignum, quod non memini, ſimilem caſum mihi vnquam ſe obtuliſſe. Iſtud ſingulare phaenomenon ſe nimirum exerit, quando ambae formulae X et Y euadunt quadrata, ad quod ergo reſoluendum ad ſimile artificium recurri oportet, quo formulae X et Y non ipſis quadratis aequales ſed ab iis infinite parum diſcrepare aſſumuntur.

§. 60. Statuamus igitur

$$X=(a+bx+cxx)^2+\alpha \text{ et } Y=(a+by+cyy)^2+\alpha,$$

ita vt pro litteris maiuſculis A, B, C, D, E, fiat $A=aa+\alpha$, $B=2ab$, $C=2ac+bb$, $D=2bc$, $E=cc$, vbi α denotat quantitatem infinite paruam, deinceps nihilo aequalem ponendam. Hinc ergo ſi br. gr. ponamus

$$a+bx+cxx=R \text{ et } a+by+cyy=S \text{ erit}$$

$$\sqrt{X}=R+\tfrac{\alpha}{2R} \text{ et } \sqrt{Y}=S+\tfrac{\alpha}{2S}.$$

§. 61. Nunc igitur conſideremus formam integralis primo inuentam, quae erat

$$\frac{\sqrt{X}-\sqrt{Y}}{x-y}=\sqrt{(\Delta+D(x+y)+E(x+y)^2)}$$

pro qua igitur habebimus

$$\sqrt{X}-\sqrt{Y}=R-S-\frac{\alpha(R-S)}{2RS}.$$

Quia vero hic erit

$$R-S=b(x-y)+c(xx-yy) \text{ fiet } \frac{R-S}{x-y}=b+c(x+y).$$

At poſito br. gr. $x+y=p$ erit $\frac{R-S}{x-y}=b+cp$, vnde

aequa-

aequatio noſtra erit

$$b + cp - \frac{\alpha(b+cp)}{2RS} = \sqrt{\Delta + 2bcp + ccpp}.$$

§. 62. Sumantur nunc vtrinque quadrata et aequatio noſtra ſequentem induet formam: $bb - \frac{\alpha}{RS}(b+cp)^2 = \Delta$. Alteriores ſcilicet poteſtates ipſius α hic vbique praetermittuntur. Hic ergo ratio noſtri Paradoxi manifeſto in oculos incidit, quia poſito $\alpha = 0$ oritur $bb = \Delta$; vnde, vt Δ maneat conſtans arbitraria, euidens eſt, differentiam inter bb et Δ etiam infinite paruam ſtatui debere; quam obrem ponamus $\Delta = bb - \alpha\Gamma$, ac obtinebitur iſta aequatio penitus determinata $\frac{(b+cp)^2}{RS} = \Gamma$, ſiue

$$(b + c(x+y))^2 = \Gamma(a + bx + cxx)(a + by + cyy)$$

quae forma non multum diſcrepat a formula ſupra inuenta.

§. 63. Haec quidem forma magis eſt complicata quam ſolutiones §§ 24 et ſeqq. inuentae: Sequenti autem artificio ad formam ſimpliciſſimam redigi poterit. Cum haec fractio $\frac{RS}{(b-cp)^2}$ debeat eſſe quantitas conſtans, ſit ea $= F$, vt eſſe debeat $F(cp+b)^2 = RS$, et quemadmodum hic poſuimus $x + y = p$, ponamus porro $xy = u$, fietque:

$$RS = aa + abp + ac(pp - 2u) + bbu + bcpu + ccuu$$

atque aequatio iam ſecundum poteſtates ipſius p diſpoſita erit

$$F(cp+b)^2 = \begin{array}{l} acpp + abp + aa \\ \qquad\quad + bcpu + bbu \\ \qquad\qquad\qquad - 2acu \\ \qquad\qquad\qquad + ccuu \end{array}$$

vb

vbi primo vtrinque diuidamus, quatenus fieri poteſt, per $cp + b$, ac reperietur

$$F(cp + b) = ap + bu + \frac{(a - cu)^2}{cp + b}.$$

Diuidamus nunc porro per $cp + b$, quatenus fieri poteſt, ac fiet

$$F = \frac{a}{c} - \frac{b}{c}\frac{(a - cu)}{(cp + b)} + \frac{(a - cu)^2}{(cp + b)^2}.$$

§. 64. Hac forma inuenta, ſi ſtatuamus

$\frac{a - cu}{cp + b} = V$, erit $F = \frac{a}{c} - \frac{b}{c}.V + VV$.

Cum igitur iſta expreſſio aequari debeat quantitati conſtanti, euidens eſt ipſam quantitatem V conſtantem eſſe debere, ita vt iam noſtrum integrale reductum ſit ad hanc formam:

$$\frac{a - cu}{cp + b} = \frac{a - cxy}{c(x + y) + b} = \text{Conſt.}$$

Subtrahamus vtrinque $\frac{a}{b}$

fietque $\frac{cxy + a(x + y)}{b + c(x + y)} = \text{Conſt.}$

quae forma per priorem diuiſa producit hanc:

$$\frac{a(x + y) + cxy}{cxy - a} = \text{Conſt.}$$

quae formae conueniunt cum ſupra exhibitis.

Theorema V.

§. 65. Si in genere haec ratio deſignandi adhibeatur: vt ſit $Z = A + Bz + Czz + Dz^3 + Ez^4$, atque valor huius formulae integralis $\int \frac{dz}{\sqrt{Z}}$, ita ſumtus vt euaneſcat poſito $z = 0$, deſignetur hoc charactere $\Pi : z$; tum, vt fiat $\Pi : k = \Pi : x + \Pi : y$, neceſſe eſt vt inter quantitates k, x, y iſta relatio ſubſiſtat:

$$\frac{2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy \mp 2\sqrt{XY}}{(x-y)^2} = \frac{2A + Bk \mp 2\sqrt{AK}}{kk},$$

cuius ratio ex superioribus est manifesta. Cum enim k denotet quantitatem constantem erit

$$d.\Pi : x + d.\Pi : y = 0 \text{ siue } \frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0,$$

cuius integrale modo ante vidimus ita exprimi:

$$\frac{2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy \mp 2\sqrt{XY}}{(x-y)^2} = \Delta.$$

Quare cum esse debeat $\Pi : x + \Pi : y = \Pi : k$, manifestum est posito $y = 0$ fieri debere $\Pi : x = \Pi : k$ ideoque $x = k$ vnde constans indefinita Δ eodem prorsus modo definitur, vti est exhibita.

Corollarium I.

§. 66. Hinc si formulae $\Pi : z$ exprimat arcum cuiuspiam lineae curuae abscissae siue applicatae Z respondentem, in hac curua omnes arcus eodem modo inter se comparare licebit, quo arcus circulares inter se comparantur, quandoquidem, propositis duobus arcubus $\Pi : x$ et $\Pi : y$, tertius arcus $\Pi : k$ semper exhiberi poterit vel summae vel differentiae eorum arcuum aequalis.

Corollarium II.

§. 67. Ita si in hac forma $\Pi : k = \Pi : x + \Pi : y$ statuatur $y = x$, prodibit $\Pi\, k = 2\,\Pi : x$; sicque arcus reperitur duplo alterius aequalis. At vero si in nostra forma faciamus $y = x$, tam numerator quam denominator in nihilum abeunt. Vt autem eius verum valorem eruamus, vtamur aequatione primum (§. 38.) inuenta:

$$\frac{\sqrt{X} - \sqrt{Y}}{x - y} = \sqrt{(\Delta + D(x+y) + E(x+y)^2)},$$

et iam in membro finiftro fpectetur y vt conftans; ipfi x vero valorem tribuamus infinite parum difcrepantem, fiue, quod eodem redit, loco numeratoris et denominatoris eorum differentialia fubftituantur, fumta fola x variabili, hocque modo pro cafu $y = x$ membrum finiftrum euadit $\frac{X'}{2\sqrt{X}}$, vbi eft $X' = B + 2Cx + 3Dxx + 4Ex^3$. Nunc ergo fumtis quadratis habebitur:

$$\frac{X'X'}{4X} = \Delta + 2Dx + 4Exx,$$

exiftente Δ vt ante $= \frac{2A + Bk - 2\sqrt{AK}}{kk}$.

Corollarium III.

§. 68. Verum fine his ambagibus duplicatio arcus ex altera forma $\Pi : k = \Pi : x - \Pi y$ deduci poteft, ponendo $y = k$, fiquidem hinc fit $\Pi : x = 2\Pi : k$, pro quo ergo cafu relatio inter x et k hac aequatione exprimetur:

$$\frac{2A + B(k+x) + 2Ckx + Dkx(k+x) + 2Ekkxx + 2\sqrt{KX}}{(x-k)^2}$$
$$= \frac{2A + Bk + 2\sqrt{AK}}{kk}.$$

Facile autem patet quomodo hinc ad triplicationem, quadruplicationem et quamlibet multiplicationem arcuum progredi debeat, quod argumentum olim fufius fum tractatus.

Theorema VI.

§. 69. Si in formis fupra inuentis ponatur tam $B = 0$ quam $D = 0$, vt fit $X = A + Cxx + Ex^4$ et $Y = A + Cyy + Ey^4$ et $K = A + Ckk + Ek^4$; tum fi ifta aequatio $\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$ ita integretur, vt pofito $y = 0$ fiat $x = k$, tum aequatio integralis erit:

$$\frac{A + Cxy + Exxyy \mp \sqrt{XY}}{(x-y)^2} = \frac{A \mp \sqrt{AK}}{kk}.$$

Corollarium I.

§. 70. Hic notari meretur, istum casum adhuc alio modo ex forma generali deduci posse, si scilicet sumatur $A = 0$ et $E = 0$, tum enim prodit ista aequatio differentialis:

$$\frac{dx}{\sqrt{(Bx + Cxx + Dx^3)}} \pm \frac{dy}{\sqrt{(By + Cyy + Dy^3)}} = 0$$

cuius ergo integrale erit

$$\frac{2B(x+y) + 2Cxy + Dxy(x+y) \mp 2\sqrt{(Bx + Cxx + Dx^3)(By + Cyy + Dy^3)}}{(x-y)^2}$$

$= \frac{Bk}{kk} = \frac{B}{k}$, vbi valor constantis admodum simplex euasit. Nunc in his formilis loco x et y scribamus xx et yy, at vero loco literarum B et D scribamus A et E, fietque aequatio differentialis

$$\frac{dx}{\sqrt{(A + Cxx + Dx^4)}} \pm \frac{dy}{\sqrt{(A + Cyy + Ey^4)}} = 0$$

cuius ergo integrale etiam hoc modo exprimetur

$$\frac{A(xx+yy) + 2Cxxyy + Exxyy(xx+yy) \mp 2xy\sqrt{XY}}{(xx-yy)^2} = \frac{A}{kk}$$

Corollarium II.

§. 71. Ecce ergo hac ratione peruenimus ad aliam integralis formam non minus notabilem priore, atque adeo nunc ex earum combinatione formula radicalis $\sqrt{XY}$ eliminari poterit, quandoquidem ex posteriore fit

$$\mp 2\sqrt{XY} = \frac{A(xx-yy)^2}{kkxy} - \frac{A(xx+yy)}{xy} - 2Cxy$$
$$- Exy\ (xx+yy)$$

qui valor in priore substitutus conducit ad hanc aequationem rationalem:

$$2A + 2Cxy + 2Exxyy$$
$$+ \frac{A(xx-yy)^2}{kkxy} - \frac{A(xx+yy)}{xy} - 2Cxy - Exy(xx+yy)$$
$$= \frac{2A(x-y)^2}{kk} \mp \frac{2(x-y)^2\sqrt{AK}}{kk}$$

quae

quae porro reducta et per $(x-y)^2$ diuisa reuocatur ad hanc formam:

$$\frac{2A \mp 2\sqrt{AK}}{kk} = \frac{A(x+y)^2}{kkxy} - Exy - \frac{A}{xy}$$

siue ad hanc:

$$\frac{A}{kk}(xx+yy-kk) - Exxyy \pm \frac{2xy\sqrt{AK}}{kk} = 0$$

quae egregie conuenit cum aequatione canonica, qua olim sum vsus: scil.

$$0 = \alpha + \gamma(xx+yy) + 2\delta xy + \zeta xxyy$$

si quidem est

$$\alpha = -\frac{A}{kk};\ \gamma = +\frac{A}{kk};\ 2\delta = \pm\frac{2\sqrt{AK}}{kk};\ \zeta = -E$$

Corollarium III.

§. 72. Methodo posteriore, qua hic vsi sumus ad hanc aequationem integrandam, aequatio multo generalior tractari poterit, vbi in formulis radicalibus potestates vsque ad sextam dimensionem assurgunt. Namque si tantum statuamus $A = 0$, vt sit aequatio

$$\frac{dx}{\sqrt{x(B+Cx+Dxx+Ex^3)}} \pm \frac{dy}{\sqrt{y(B+Cy+Dyy+Ey^3)}} = 0$$

eius integrale est

$$\frac{B(x+y)+2Cxy+Dxy(x+y)+2Exxyy}{(x-y)^2} \mp \frac{2\sqrt{xy(B+Cx+Dxx+Ex^3)(B+Cy+Dyy+Ay^3)}}{(x-y)^2} = \frac{B}{k}.$$

Quod si iam hic loco x et y scribamus xx et yy, aequatio differentialis fiet

$$\frac{dx}{\sqrt{(B+Cxx+Dx^4+Ex^6)}} \pm \frac{dy}{\sqrt{(B+Cyy+Dy^4+Ey^6)}} = 0,$$

cuius ergo integrale erit

$$\frac{B(xx+yy)+2Cxxyy+Dxxyy(xx+yy)+2Ex^4y^4}{(xx-yy)^2} \mp \frac{2xy\sqrt{(B+Cxx+Dx^4+Ex^6)(B+Cyy+Dy^4+Ey^6)}}{(xx-yy)^2} = \frac{B}{kk}.$$

Nunc autem ostendamus, quomodo ope methodi Illustris *de la Grange* idem integrale impetrari queat.

Analy-

Analyſis.

Pro integratione aequationis differentialis
$\frac{dx}{\sqrt{X}} \pm \frac{dy}{\sqrt{Y}} = 0$, exiſtente $X = B + Cxx + Dx^4 + Ex^6$
$Y = B + Cyy + Dy^4 + Ey^6$

§. 73. Poſito igitur
$\frac{dx}{\sqrt{X}} = dt$ erit $\frac{dy}{\sqrt{Y}} = \mp dt$
hincque ſumtis quadratis
$$\frac{dx^2}{dt^2} = X \text{ et } \frac{dy^2}{dt^2} = Y.$$
Hinc formentur hae aequationes:
$$\frac{xxdx^2}{dt^2} = xxX \text{ et } \frac{yydy^2}{dt^2} = yyY.$$
Iam introducantur duae nouae variabiles p et q vt ſit $xx+yy=2p$ et $xx-yy=2q$, ex quo fit $xdx+ydy=dp$, hincque $xxdx^2 - yydy^2 = dpdq$; quam obrem habebimus
$$\frac{dpdq}{dt^2} = xxX - yyY,$$
quae aequatio diuidatur per $xx-yy=2q$, prodibitque
$$\frac{dpdq}{2qdt^2} = \frac{xxX-yyY}{xx-yy}$$
quae forma, valoribus pro X et Y ſubſtitutis, dabit
$$\frac{dpdq}{2qdt^2} = B+2Cp+D(3pp+qq)+4E(p^3+pqq)$$

§. 64. Nunc porro aequationes $\frac{dx^2}{dt^2}$ et $\frac{dy^2}{dt^2}$ differentiatae dabunt
$$\frac{2ddx}{dt^2} = X' \text{ et } \frac{2ddy}{dt^2} = Y'.$$
Ex priore fit $\frac{2xddx}{dt} = xX'$, cui addatur $\frac{2dx^2}{dt^2} = 2X$, vt prodeat $\frac{2(xddx+dx^2)}{dt^2} = \frac{2d.xdx}{dt^2} = xX' + 2X$.

Simili

Simili modo erit $\frac{2d.ydy}{dt^2} = yY' + 2Y$, quae duae aequationes inuicem additae dabunt

$$\frac{2d.dp}{dt^2} = \frac{2ddp}{dt^2} = xX' + yY' + 2(X+Y).$$

Substitutis autem valoribus et facta substitutione respectu literarum p et q reperitur

$$2X+2Y = 4B+4Cp+4D(pp+qq)+4Ep(pp+3qq).$$

Deinde ob

$$X'x = 2Cxx + 4Dx^4 + 6Ex^6 \text{ et}$$
$$yY' = 2Cyy + 4Dy^4 + 6Ey^6 \text{ erit}$$
$$xX'+yY' = 4Cp + 8D(pp+qq) - 12Ep(pp+3qq)$$

ex quibus coniunctis fit

$$\frac{2ddp}{dt^2} = 4B + 8Cp + 12D(pp+qq) + 16Ep(pp+3qq).$$

§. 75. Ab hac formula subtrahatur supra inuenta $\frac{dpdq}{2qdt^2}$ quater sumta, ac remanebit

$$\frac{2ddp}{dt^2} - \frac{2dpdq}{qdt^2} = 8Dqq + 32Epqq.$$

Nunc vtrinque multiplicetur per $\frac{dp}{qq}$ et prodibit

$$\frac{1}{dt^2}\left(\frac{2dpddp}{qq} - \frac{2dp^2dq}{q^3}\right) = 8Ddp + 32Epdp$$

cuius integrale sponte se offert ita expressum

$$\frac{dp^2}{qqdt^2} = 4\Delta + 8Dp + 16Epp$$

ideoque extracta radice

$$\frac{dp}{qdt} = 2\sqrt{\Delta + 2Dp + 4Epp}.$$

§. 76. Cum nunc sit

$$\frac{dp}{dt} = x\sqrt{X} \mp y\sqrt{Y} \text{ et } 2q = xx - yy$$

facta substitutione orietur haec aequatio:

$$\frac{x\sqrt{X} \mp y\sqrt{Y}}{xx-yy} = \sqrt{(\Delta + D(xx+yy) + E(xx+yy)^2)}$$

quae

quae sumtis quadratis reducetur ad istam formam:

$$\frac{xxX + yyY \mp 2xy\sqrt{XY}}{(xx - yy)^2} = \Delta + D(xx + yy)$$
$$+ E(xx + yy)^2.$$

Est vero

$$xxX + yyY = B(xx + yy) + C(x^4 + y^4)$$
$$+ D(x^6 + y^6) + E(x^8 + y^8)$$

hincque peruenietur ad hanc aequationem

$$\frac{B(xx + yy) + C(x^4 + y^4) + Dxxyy(xx + yy) + 2Ex^4y^4 \mp 2xy\sqrt{XY}}{(xx - yy)^2} = \Delta.$$

§. 77. Sumamus nunc vt supra constantem Δ ita vt posito

$y = o$ fiat $x = k$, et $X = K = B + Ckk + Dk^4 Ek^6$

et aequatio integralis induet hanc formam:

$$\frac{B(xx + yy) + C(x^4 + y^4) + Dxxyy(xx + yy) + 2Ex^4y^4 \mp 2xy\sqrt{XY}}{(xx - yy)^2} = \frac{B + Ckk}{kk},$$

quae aliquanto simplicior euadit si vtrinque subtrahamus C: erit enim

$$\frac{B(xx + yy) + 2Cxxyy + Dxxyy(xx + yy) + 2Ex^4y^4 \mp 2xy\sqrt{XY}}{(x^2 - y^2)^2} = \frac{B}{kk}$$

quae egregie conuenit cum integrali supra §. 72. exhibito.

§. 78. Hic casus notatu dignus se offert, dum $B = o$, tum autem aequatio differentialis ita se habebit:

$$\frac{dx}{x\sqrt{(C + Dxx + Ex^4)}} \pm \frac{dy}{y\sqrt{(C + Dyy + Ey^4)}} = o$$

cuius ergo integrale per constantem Δ expressum erit

$$\frac{C(x^4 + y^4) + Dxxyy(xx + yy) + 2Ex^4y^4 \mp 2xy\sqrt{XY}}{(xx - yy)^2} = \Delta.$$

Hoc autem casu integratio non ita determinari potest, vt posito $y = o$ fiat $x = k$; quia integrale posterioris membri hoc casu manifesto abit in infinitum; quam obrem alio

alio modo integrationem determinari conueniet veluti vt posito $y = b$ fiat $x = a$, tum autem erit ista constans

$$\Delta = \frac{C(a^4 + b^4) + D a^2 b^2 (aa + bb) + 2 E a^4 b^4 \mp 2 ab \sqrt{AB}}{(aa - bb)^2}$$

existente

$$A = C + D aa + E a^4 \text{ et } B = C + D bb + E b^4.$$

Conclusio.

§. 79. Qui processum Analyseos hic vsitatae comparare voluerit cum methodo, qua Illustris D. *de la Grange* vsus est in *Miscellan. Taur. Tom. IV.* facile perspiciet, eam multo facilius ad scopum desideratum perducere atque multo commodius ad quosuis casus applicari posse. Introduxerat autem Vir. Ill. in calculum formulam $\frac{dt}{T}$, cuius loco hic simplici elemento dt sumus vsi, ac deinceps quantitatem T tanquam functionem literarum p et q spectauit, quae positio satis difficiles calculos postulauit, dum nostra methodo longe concinnius easdem integrationes inuestigare licuit. Quanquam autem nullum est dubium, quin ista Analyseos species insigne incrementum polliceatur, tamen nondum patet, quemadmodum ad alias integrationes ea accommodari queat, praeter hos ipsos casus, quos hic tractauimus et quos olim ex aequatione canonica deriuaueram.