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De corporibus regularibus per doctrinam sphaericam determinatis; ubi simul nova methodus globos sive coelestes sive terrestres charta obducendi traditur

Leonhard Euler

DE

CORPORIBVS REGVLARIBVS
PER DOCTRINAM SPHAERICAM DETERMINATIS; VBI SIMVL NOVA METHODVS, GLOBOS SIVE COELESTES SIVE TERRESTRES CHARTA OBDVCENDI, TRADITVR.

Auctore
L. EVLERO.

§. 1.

Corpus regulare vocatur Polyedrum circumquaque hedris planis regularibus et inter ſe aequalibus incluſum, et cuius omnes anguli ſolidi a totidem angulis planis formantur. Hedrae ergo erunt vel triangula aequilatera, vel quadrata, vel pentagona regularia, vel etiam hexagona; ad angulos autem ſolidos conſtituendos vel ternae hedrae, vel quatuor, vel quinque, vel etiam ſex concurrunt.

§. 2. In ſingulis hedris praeter numerum laterum cuiusque, qui ſit $= n$, in computum venire debet quantitas ſingulorum laterum, quae ſit $= x$, ita vt perimeter cuiusque hedrae ſit $= n x$. Deinde quia quaelibet hedra eſt polygonum regulare, ponatur radius circuli circumſcripti $= y$, hincque reperitur area cuiusque hedrae

$$= \tfrac{1}{2} n x \surd (y y - \tfrac{1}{4} x x).$$

Denique vocetur radius ſphaerae ipſi polyedro circumſcriptae $= r$, eritque $\surd (r r - y y)$ perpendiculum ex centro ſphaerae in quamlibet hedram demiſſum, cuius pars tertia in aream hedrae ducta et per numerum omnium hedrarum multiplicata dabit ſoliditatem totius polyedri ſeu corporis regularis. Ita ſi numerus omnium hedrarum fuerit $= N$, erit ſuperficies polyedri $= \tfrac{1}{2} N n x \surd (y y - \tfrac{1}{4} x x)$, tota autem ſoliditas $= \tfrac{1}{6} N n x \surd (y y - \tfrac{1}{4} x x)(r r - y y)$. Hinc ſi fuerit ſuperficies $= S$, erit haec ſoliditas

$$= \tfrac{1}{3} S \surd (r r - y y).$$

§. 3. Concipiatur igitur corpori regulari ſphaera circumſcripta, cuius radium vocauimus $= r$, et omnes anguli ſolidi reperientur in ſuperficie ſphaerae, qui ſi arcubus circulorum maximorum iungantur, cuilibet hedrae in ſuperficie ſphaerae reſpondebit polygonum ſphaericum regulare totidem laterum n; hocque modo tota ſuperficies ſphaerae, quae eſt $= 4 \pi r r$, diuidetur in tot huiusmodi polygona regularia ſphaerica quot habentur hedrae, quarum numerus cum ſit $= N$, area cuiusque horum Polygonorum ſphaericorum erit $= \frac{4 \pi r r}{N}$.

§. 4. Quando ergo ternae hedrae planae ad angulos ſolidos conſtituendos concurrunt, tum in ſuperficie ſphaerica etiam ternae hedrae ſphaericae in ſingulis angulis ſolidis conuenient; vnde patet in his polygonis ſphaericis

ricis ſingulos angulos fore $= 120$ gr. Sin autem quaternae hedrae planae in angulis ſolidis concurrant, in hedris ſphaericis omnes anguli debent eſſe recti ſeu 90 graduum. At ſi quinae hedrae planae concurrant, in hedris ſphaericis ſinguli anguli erunt 72 gr. Denique ſi adeo ſex hedrae planae conueniant, in polygonis ſphaericis ſinguli anguli erunt 60 gr. Ulterius enim progredi ipſa rei natura prohibet.

§. 5. His praemiſſis omnes caſus, qui quidem occurrere poſſunt, ſeorſim euoluamus. Ac primo quidem ſint omnes hedrae triangulares, quarum vel ternae, vel quaternae, vel quinae, vel ſenae in ſingulis angulis ſolidis concurrere poſſunt: ſecundo pro hedris quadrungularibus vel ternae, vel etiam quaternae concurrere poterunt: pro pentagonis autem plures quam tres occurrere non poſſe manifeſtum eſt, quod multo magis pro hexagonis valet.

Caſus Primus.

Pro hedris triangularibus, quarum ternae in angulis ſolidis occurrunt.

§. 6. Hic igitur eſt $n = 3$, ſitque in ſuperficie ſphaerica triangulum ſphaericum ABC cuique hedrae planae reſpondens, et quia eius ſinguli anguli A, B, C debent eſſe $= 120°$ eorum ſumma fit $360° = 2\pi$, vnde eius area colligitur $= \pi r r$, quae cum etiam ſit $= \frac{4\pi r r}{N}$, erit $N = 4$. Hinc patet quatuor tantum hedras requiri ad hoc corpus regulare formandum, vnde etiam iſtud corpus regulare Tetraedron appellatur. Quia ergo omnium angulorum planorum numerus eſt $= 12$, terni autem in ſingulis angulis ſolidis concurrant, angulorum ſolidorum numerus quoque erit $= 4$. Tab. I. Fig. 1.

§. 7. Iam quia in triangulo sphaerico A B C dantur omnes anguli $A = B = C = 120$ gr. inde etiam latera definiri poterunt per regulas trigonometriae sphaericae. Si enim terni anguli fuerint α, β et γ, et latera ipsis opposita a, b et c, erit

$$\text{cos. } a = \frac{\text{cos. } \alpha + \text{cos. } \beta . \text{cos. } \gamma}{\text{sin. } \beta . \text{sin. } \gamma} .$$

et quia omnes anguli sunt inter se aequales, erit

$$\text{cos. } a = \frac{\text{cos. } \alpha + \text{cos. } \alpha^2}{\text{sin. } \alpha^2} = \frac{\text{cos. } \alpha}{1 - \text{cos. } \alpha} .$$

Quoniam igitur nostro casu est $\alpha = 120$ gr. erit cos. $\alpha = -\frac{1}{2}$, ideoque cos. $a = -\frac{1}{3}$; vnde intelligitur, singula latera $AB = AC = BC$ esse quadrante maiora, ita vt excessus cuiusque supra 90 gr. sinus sit $= \frac{1}{3}$, vnde iam vnum latus erit $= 109°. 28'$; quare cum latus hedrae planae x sit subtensa arcus A B erit $\frac{x}{2r} = \text{sin. } \frac{1}{2} a$. Est vero

$$\text{sin. } \tfrac{1}{2} a = \sqrt{\frac{(1 - \text{cos. } a)}{2}} = \sqrt{\tfrac{2}{3}}$$

hinc ergo colligimus latus cuiusque hedrae $x = 2r\sqrt{\frac{2}{3}}$.

§. 8. Vt autem simul etiam radium circuli hedrae planae circumscripti y inuestigemus, consideremus centrum hedrae nostrae sphaericae, quod sit in O, ex quo, ductis arcubus O A et O B, in latus A B demittamus perpendiculum O P, latus B A in P bisecans, eritque

$$\frac{x}{2r} = \text{sin. A P et } \frac{y}{r} = \text{sin. O A}.$$

Quod si ergo in genere ponamus angulum $PAO = \alpha$ et angulum $AOP = \beta$, ob angulum A P O rectum, erit

$$\text{cos. A P} = \frac{\text{cos. } \beta}{\text{sin. } \alpha} \text{ et cos. O A} = \text{cot. } \alpha . \text{ cot. } \beta .$$

Nostro autem casu est $\alpha = 60$ gr. et $\beta = 60$ gr. hinc

$$\text{sin. } \alpha = \text{sin. } \beta = \frac{\sqrt{3}}{2} ; \text{ cos. } \alpha = \text{cos. } \beta = \frac{1}{2} \text{ et cot. } \alpha = \frac{1}{\sqrt{3}} ;$$

vnde

vnde colligitur cof. A P $= \frac{1}{\sqrt{3}}$ et cof. O A $= \frac{1}{3}$, hincque fin. A P $= \sqrt{\frac{2}{3}} = \frac{x}{2r}$ et fin. O A $= \frac{2\sqrt{2}}{3} = \frac{y}{r}$. Pro noftro igitur cafu obtinetur latus hedrae $x = 2r\sqrt{\frac{2}{3}}$ et radius circuli circumfcripti $y = \frac{2r\sqrt{2}}{3}$.

§. 9. Ex his pro x et y inuentis valoribus fequitur fore, $\sqrt{(yy - \frac{1}{4}xx)} = \frac{r\sqrt{2}}{3}$ et $\sqrt{(rr - yy)} = \frac{r}{3}$; vnde pro tetraëdro, fphaerae, cuius radius $= r$, infcripto, fequentes nancifcimur determinationes, quas fimul in fractionibus decimalibus adiungamus:

I. Latus hedrae	$= 2r\sqrt{\frac{2}{3}}$	$= 1{,}632993.\,r$
II. Radius circuli hedrae circumfcripti	$= \frac{2r\sqrt{2}}{3}$	$= 0{,}942809.\,r$
III. Area cuiusque hedrae	$= \frac{2rr}{\sqrt{3}}$	$= 1{,}154701.\,rr$
IV. Superficies tetraëdri	$= \frac{8rr}{\sqrt{3}}$	$= 4{,}618804.\,rr$
V. Soliditas tetraëdri	$= \frac{8r^3}{9\sqrt{3}}$	$= 0{,}513200.\,r^3$.

Cafus Secundus.

Pro hedris triangularibus, quarum quaternae in angulis folidis concurrunt.

§. 10. Hic ergo eft iterum $n = 3$, fitque in fuperficie fphaerica triangulum fphaericum A B C cuique hebrae refpondens, et quia quatuor coniunguntur, quilibet angulus erit quarta pars totius peripheriae, ideoque $= \frac{\pi}{2}$, vnde fumma trium angulorum erit 270 gr. $= \frac{3\pi}{2}$. Huius igitur trianguli fphaerici area erit $= \frac{\pi rr}{2}$, quae in tota fuperficie fphaerae octies continetur, ita ut hoc Polyedrum conftet octo haedris triangularibus, ideoque fiat $N = 8$; vnde de tiam Octaedron vocari ſolet. Quia ergo numerus angulorum planorum eſt 24, quorum quaterni occurrunt in ſingulis angulis ſolidis, numerus angulorum ſolidorum erit $= 6$.

§. 11. Sit nunc iterum O centrum trianguli ſphaerici, ex quo in latus A B demittatur perpendiculum O P, vt obtineatur $\frac{x}{2r} =$ ſin. A P & $\frac{y}{r} =$ ſin. O A. Quia nunc in triangulo A O P eſt $\alpha = 45$ gr. & $\beta = 60$ gr. erit ſin. $\alpha =$ coſ. $\alpha = \frac{1}{\sqrt{2}} =$ et cot. $\alpha = 1$; tum vero erit vt ante ſin. $\beta = \frac{\sqrt{3}}{2}$; coſ. $\beta = \frac{1}{2}$ & cot. $\beta = \frac{1}{\sqrt{3}}$; vnde ex formulis ſuperioribus colligitur coſ. A P $= \frac{\text{coſ.}\ \beta}{\text{ſin.}\ \alpha} = \frac{1}{\sqrt{2}}$, ideoque ſin. A P $= \frac{1}{\sqrt{2}} = \frac{x}{2r}$; porro coſ. O A $=$ cot. α cot. $\beta = \frac{1}{\sqrt{3}}$, hincque ſin. O A $= \sqrt{\frac{2}{3}} = \frac{y}{r}$, ideoque $x = r\sqrt{2}$ & $y = r\sqrt{\frac{2}{3}}$; ex quibus valoribus colligimus $\sqrt{(yy - \frac{1}{4}xx)} = \frac{r}{\sqrt{6}}$ et $\sqrt{(rr - yy)} = \frac{r}{\sqrt{3}}$.

§. 12. Ex his autem valoribus pro octaedro ſequentes naſcuntur determinationes:

I. Latus hedrae $= r\sqrt{2} = 1{,}414214.\ r$

II. Radius circuli $= r\sqrt{\frac{2}{3}} = 0{,}816496.\ r$

III. Area hedrae $= \frac{rr\sqrt{3}}{2} = 0{,}866025.\ rr$

IV. Superficies } octaedri $= 4rr\sqrt{3} = 6{,}928203.\ rr$

V. Soliditas } octaedri $= \frac{4r^3}{3} = 1{,}333333.\ r^3$

Caſus Tertius.

Pro hedris triangularibus, quarum quinae in angulis ſolidis concurrunt.

§. 13. Hic ergo iterum eſt $n = 3$, ac in triangulo ſphaerico A B C ſinguli anguli erunt $= \frac{360}{5} = 72$ gr. vnde

vnde ſumma angulorum $= 216 = \frac{6\pi}{5}$. Hinc area trianguli ſphaerici erit $= \frac{\pi r r}{5}$, quae cum ſit vicies minor quam tota ſuperficies ſphaerae, hoc polyedron conſtabit ex viginti hedris, ita vt ſit $N = 20$, vnde hoc corpus Icoſaedron appellari ſolet. Quia porro omnino 60 anguli plani adſunt, quorum quini in angulum ſolidum coëunt, numerus angulorum ſolidorum erit $= 12$.

§. 14. Iam in triangulo ſphaerae rectangulo A O P erit angulus $\alpha = 36$ gr. manente $\beta = 60$ gr. Cum igitur conſtet eſſe

$$\text{coſ. } \alpha = \text{ſin } 54 \text{ gr.} = \frac{\sqrt{5}+1}{4} \text{ erit}$$

$$\text{ſin. } \alpha = \frac{\sqrt{10-2\sqrt{5}}}{4}, \text{ hinque}$$

$$\text{cot. } \alpha = \frac{\sqrt{5}+1}{\sqrt{(10-2\sqrt{5})}} = \frac{4}{\sqrt{(80-32\sqrt{5})}} = \frac{1}{\sqrt{(5-2\sqrt{5})}}$$

$$= \frac{\sqrt{(5+2\sqrt{5})}}{\sqrt{5}} = \sqrt{\left(1 + \frac{2}{\sqrt{5}}\right)}.$$

tum vero vt ante erit

$$\text{coſ. } \beta = \frac{1}{2} \ \& \ \text{cot. } \beta = \frac{1}{\sqrt{3}}.$$

Hinc igitur fiet

$$\text{coſ. } A P = \frac{2}{\sqrt{(10-2\sqrt{5})}}, \text{ vnde colligitur}$$

$$\text{ſin. } A P^2 = \frac{(6-2\sqrt{5})}{10-2\sqrt{5}} = \frac{40-8\sqrt{5}}{80} = \frac{5-\sqrt{5}}{10}, \text{ ideoque}$$

$$\text{ſin. } A P = \frac{\sqrt{(5-\sqrt{5})}}{\sqrt{10}} = \frac{x}{2r}, \text{ vnde fit}$$

$$x = \frac{2r\sqrt{5-\sqrt{5}}}{\sqrt{10}} = \frac{r\sqrt{10-2\sqrt{5}}}{\sqrt{5}}. \text{ Deinde vero cum ſit}$$

$$\text{coſ. } O A = \frac{\sqrt{(5+2\sqrt{5})}}{\sqrt{15}} \text{ erit}$$

$$\text{ſinus } O A = \frac{\sqrt{(10-2\sqrt{5})}}{\sqrt{15}} = \frac{y}{r}$$

$$\text{hinc } y = \frac{r\sqrt{(10-2\sqrt{5})}}{\sqrt{15}}$$

§. 15. Cum igitur ſit

$$\tfrac{1}{2}\, x = \frac{r\sqrt{(10 - 2\sqrt{5})}}{2\sqrt{5}}, \text{ erit } \tfrac{1}{4}\, x\, x = \frac{r\, r\, (5 - \sqrt{5})}{10},$$

quod ſubtractum ab

$$y\, y = \frac{r\, r\, (10 - 2\sqrt{5})}{15} \text{ relinquit}$$

$$\frac{(5 - \sqrt{5})}{30}\, r\, r; \text{ ſicque erit } \sqrt{(y\, y - \tfrac{1}{4}\, x\, x)} = \frac{r\sqrt{(5 - \sqrt{5})}}{\sqrt{30}}.$$

Deinde vero erit

$$r\, r - y\, y = \frac{(5 + 2\sqrt{5})}{15}\, r\, r \text{ ideoque}$$

$$\sqrt{(r\, r - y\, y)} = \frac{r\sqrt{(5 + 2\sqrt{5})}}{\sqrt{15}}$$

hinc igitur fiet

$$x\sqrt{(y\, y - \tfrac{1}{4}\, x\, x)} = \frac{r\, r\sqrt{(60 - 20\sqrt{5})}}{\sqrt{5}.\sqrt{30}} = \frac{r\, r\sqrt{(6 - 2\sqrt{5})}}{\sqrt{15}}$$

$$= \frac{r\, r\sqrt{(5 - 1)}}{\sqrt{15}}.$$

§. 16. Hinc igitur pro Icoſaëdro ſequentes nanciſcimur determinationes:

I. Latus hedrae $= \frac{r\sqrt{10 - 2\sqrt{5}}}{\sqrt{5}} = 1{,}051462.\ r$

II. Radius circuli $= \frac{r\sqrt{10 - 2\sqrt{5}}}{\sqrt{15}} = 0{,}607062.\ r$

III. Area hedrae $= \frac{3\, r\, r\, (5 - \sqrt{5})}{10\sqrt{3}} = 0{,}478727.\ r\, r$

IV. Superficies $= \frac{6\, r\, r.\, (5 - \sqrt{5})}{\sqrt{3}} = 9{,}574542.\ r\, r$

V. Soliditas $= \frac{2\, r^3\sqrt{(10 + 2\sqrt{5})}}{3} = 2{,}536150\ r^3$

Pro harum formularum euolutione numerica notaſſe iuuabit, eſſe $\sqrt{10 - 2\sqrt{5}} = 4$ ſin. 36 gr., $\sqrt{10 + 2\sqrt{5}} = 4$ coſ. 18 gr. $= 4$ ſin. 72 gr. et $5 - \sqrt{5} = 4\sqrt{5}$ ſin. 18 gr. praeterea pro hedris triangularibus ſemper eſſe $y = \frac{x}{\sqrt{3}}$.

Caſus

Casus Quartus.

Pro hedris triangularibus, quarum sex in angulis solidis concurrunt.

§. 17. Quia hic sex anguli plani concurrunt, in hedris sphaericis singuli anguli erunt $\frac{360\ gr.}{6} = 60$ gr., quorum summa in quolibet triangulo cum sit 180 gr. $= \pi$, area horum triangulorum euanescit et numerus hedrarum fiet infinitus. Scilicet tota superficies sphaerica in infinita triangula diuisa concipi potest, ita vt ipsa sphaera hoc corpus regulare exhibeat, cuius superficies est

$$= 4\pi rr = 12,56637060 . rr,$$

soliditas vero $= \frac{4}{3}\pi r^3 = 4,18879020 . r^3$; vnde intelligitur, sphaeram merito inter corpora regularia numerari. Manifestum autem est simili modo superficiem sphaerae etiam in innumera quadrata, vel etiam hexogona regularia, diuisam concipi posse.

Casus Quintus.

Pro hedris quadratis, quarum ternae in angulis solidis concurrunt.

§. 18. Hic igitur est $n = 4$ et cuilibet hedrae quadratae planae in superficie sphaerica respondet quadrilineum sphaericum A B C D, cuius singuli anguli erunt $= 120$ gr. $= \frac{2\pi}{3}$, quoniam tres tales anguli totam peripheriam complere debent. Hinc summa quatuor angulorum erit $= \frac{8\pi}{3}$, vnde, ablatis 2π, remanent $\frac{2\pi}{3}$, ita vt area futura sit $= \frac{2\pi}{3}rr$, quae in tota superficie sphaerae sexies continetur, ita vt hoc polyëdrum ex sex hedris planis quadratis formetur, vnde etiam Hexaëdron vocatur, quae denominatio cum cubo congruit. Quia igitur in iis dantur 24 anguli plani, eo- Tab. I. Fig. 2.

rum-

rumque terni in ſingulis angulis ſolidis concurrunt, numerus angulorum ſolidorum erit $= 8$.

§. 19. Sit nunc punctum O centrum cuiusque hedrae ſphaericae A B C D, vnde ad vnum angulum A ducto arcu O A et demiſſo in latus A B perpendiculo O P, in triangulo O A P debet eſſe

$$\text{ſin. } AP = \frac{x}{2r} \text{ et ſin. } AO = \frac{y}{r}.$$

Quia nunc angulus

$$\alpha = 60 \text{ gr. et } \beta = 45, \text{ fit ſin. } \alpha = \frac{\sqrt{3}}{2},$$

$$\text{coſ. } \alpha = \tfrac{1}{2}, \text{ cot. } \alpha = \frac{1}{\sqrt{3}};$$

$$\text{ſin. } \beta = \text{coſ. } \beta = \frac{1}{\sqrt{2}} \text{ et cot. } \beta = 1.$$

Hinc igitur ex regulis ſupra datis colligimus

$$\text{coſ. } AP = \frac{\text{coſ. } \beta}{\text{ſin. } \alpha} = \sqrt{\tfrac{2}{3}} \text{ et}$$

$$\text{coſ. } AO = \text{cot. } \alpha. \text{ cot. } \beta = \frac{1}{\sqrt{3}}.$$

Hinc ergo porro fit

$$\text{ſin. } AP = \frac{1}{\sqrt{3}} = \frac{x}{2r} \text{ et ſin. } AO = \sqrt{\tfrac{2}{3}} = \frac{y}{r},$$

vnde habebimus

$$x = \frac{2r}{\sqrt{3}} \text{ et } y = \frac{r\sqrt{2}}{\sqrt{3}}$$

ſicque colligetur

$$\sqrt{(yy - \tfrac{1}{4}xx)} = \frac{r}{\sqrt{3}} \text{ et } \sqrt{(rr - yy)} = \frac{r}{\sqrt{3}}$$

§. 20. Ex his ergo valoribus pro Hexaëdro ſeu cubo ſequentes adipiſcimur valores:

I. Latus hedrae $= \frac{2r}{\sqrt{3}} = 1{,}154700.\, r$

II. Radius circuli $= \frac{r\sqrt{2}}{\sqrt{3}} = 0{,}816496.\, r$

III. Area

III. Area hedrae $= \frac{4rr}{3} = 1,333333 \cdot rr$
IV. Superficies $= 8\,rr = 8,000000.\,rr$
V. Soliditas $= \frac{8r^3}{3\sqrt{3}} = 1,539600\,r^3$

Casus Sextus.

Pro hedris pentagonis, quarum ternae in angulis ſolidis concurrunt.

§. 21. Hic igitur eſt $n = 5$, et ſinguli anguli hedrae ſphaericae pentagonae iterum erunt 120 gr. $= \frac{2\pi}{3}$, et talium quinque angulorum ſumma erit $\frac{10\pi}{3}$, vnde, ablatis 3π, remanet $\frac{\pi}{3}$. Hinc area erit $\frac{\pi rr}{3}$, quae in tota ſuperficie ſpaerae duodecies continetur, ſicque hoc corpus formabitur ex duodecim hedris pentagonis, vnde etiam Dodecaedron appellari ſolet, et quia omnino dantur $5 \cdot 12 = 60$ anguli plani, eorumque terni in angulis ſolidis coëunt, numerus angulorum ſolidorum erit $= 20$.

§. 22. Si hic vt hactenus conſtituatur triangulum ſphaericum rectangulum O A P, erit angulus $\alpha = 60$ gr. et $\beta = 36$ gr. vnde habetur

$$\text{ſin.}\ \alpha = \frac{\sqrt{3}}{2},\ \text{cot.}\ \alpha = \frac{1}{\sqrt{3}},$$
$$\text{ſin.}\ \beta = \frac{\sqrt{10 - 2\sqrt{5}}}{4},\ \text{coſ.}\ \beta = \frac{\sqrt{5}+1}{4}\ \text{et}$$
$$\text{cot.}\ \beta = \frac{\sqrt{5 + 2\sqrt{5}}}{\sqrt{5}}$$

hincque colligimus

$$\text{coſ. A P} = \frac{\text{coſ.}\ \beta}{\text{ſin.}\ \alpha} = \frac{\sqrt{5}+1}{2\sqrt{3}}\ \text{et}$$
$$\text{coſ. O A} = \frac{\sqrt{5 + 2\sqrt{5}}}{\sqrt{15}},$$

vnde fit

ſin.

$$\text{fin. } AP = \frac{\sqrt{3-\sqrt{5}}}{\sqrt{6}} = \frac{x}{2r}, \text{ ideoque}$$

$$x = \frac{2r\sqrt{3-\sqrt{5}}}{\sqrt{6}} = \frac{r\sqrt{6-2\sqrt{5}}}{\sqrt{3}} = \frac{r(\sqrt{5}-1)}{\sqrt{3}},$$

vbi notetur esse $\sqrt{5} - 1 = 4$ fin. 18 gr.
tum vero erit

$$\text{fin. } OA = \frac{\sqrt{10-2\sqrt{5}}}{\sqrt{15}} = \frac{y}{r} \text{ ideoque}$$

$$y = \frac{r\sqrt{10-2\sqrt{5}}}{\sqrt{15}},$$

vbi notetur esse $\sqrt{(10 - 2\sqrt{5})} = 4$ fin. 36 gr. Ex his autem colligitur

$$\sqrt{(yy - \tfrac{1}{4}xx)} = \frac{r\sqrt{10+2\sqrt{5}}}{2\sqrt{15}}$$

(vbi notasse iuuabit esse $\sqrt{(10+2\sqrt{5})} = 4$ cof. 18 gr.) et

$$\sqrt{(rr - yy)} = \frac{r\sqrt{5+2\sqrt{5}}}{\sqrt{15}},$$

tum autem erit

$$x\sqrt{(yy - \tfrac{1}{4}xx)} = \frac{\sqrt{10-2\sqrt{5}}}{3\sqrt{5}}.$$

§. 23. Hinc igitur pro dodecaedro, Sphaerae, cuius radius $= r$, infcripto fequentes inuenimus determinationes:

I. Latus hedrae	$= \frac{r\sqrt{5}-1}{\sqrt{3}}$	$= 0{,}713644\,r$
II. Radius circuli	$= \frac{r\sqrt{10-2\sqrt{5}}}{\sqrt{15}}$	$= 0{,}607062\,r$
III. Area hedrae	$= rr\,\frac{\sqrt{5}}{6}\sqrt{10-2\sqrt{5}}$	$= 0{,}876218\,rr$
IV. Superficies	$= 2\,rr\sqrt{5}.\sqrt{10-2\sqrt{5}}$	$= 10{,}514616\,rr$
V. Soliditas	$= \frac{2\,r^3(5+\sqrt{5})}{3\sqrt{3}}$	$= 2{,}785164\,r^3$

pro vltimo valore notetur esse $5 + \sqrt{5} = 4\sqrt{5}$ fin. 54. gr.

§. 24. Sic igitur nacti fumus quinque illa corpora regularia, quorum inueftigatio vulgo laborem non parum

rum molestum, atque imprimis figuras maxime intricatas, requirere solet, quorum naturam hic tam facile et propemodum sine figuris ex theoria sphaerica deduximus, vnde simul patet ipsi sphaerae sextum locum merito concedi. Quo autem ea facilius inter se comparare liceat omnes eorum determinationes hic iunctim conspectui exponamus:

Corpus regulare.	Latus cuiusque hedrae.	Radius circl. circumscripti	Area vnius hedrae.	Superficies tota.	Soliditas corporis.
Tetraëdron.	1,63299r	0.94281r	1,15470rr	4,61880rr	0,51320r^3
Octaëdron.	1,41421r	0,81650r	0,86602rr	6,92820rr	1,33333r^3
Icosaëdron.	1,05146r	0,60706r	0,47873rr	9,57454rr	2,53615r^3
Hexaëdron.	1,15470r	0,81650r	1,33333rr	8,00000rr	1,53960r^3
Dodecaëdron.	0,71364r	0,60706r	0,87622rr	10,51462rr	2,78516r^3
Sphaera.	0,00000r	1,00000r	0,00000rr	12,56637rr	4,18870r^3

§. 25. Hinc igitur, patet Dodecaëdron prae reliquis tam maximam superficiem quam soliditatem habere, vnde merito suspicari licet, totam sphaerae superficiem commodissime duodecim pentagonis planis obduci posse. Etsi enim superficies sphaerica nullo modo figuris planis, nisi sint quam minimae, exacte repraesentari potest: tamen nouimus, sphaeras per lacinias chartaceas, quae in polis coeant, satis accurate obduci solere, dum scilicet charta aliquantillum in medio se expandi patitur, ita vt quampiam gibbositatem recipiat convexitati sphaerae conuenientem. Talem igitur effectum multo magis in pentagonis chartaceis exspectare licebit, hicque modus vulgari longe anteferendus videtur, ideo quod hic nusquam plures tribus anguli plani sint iungendi, cum more solito ad

minimum

minimum duodecim laciniae in polis concurrere debeant, id quod plerumque ſine inſigni vitio vix praeſtare licet.

§. 26. Operae igitur pretium erit inueſtigare, quam exacte duodecim illa pentagona, in quae ſuperficies globi ab inſcripto Dodecaëdro diuiditur, figuris planis obduci queant; Hunc in finem primo omnes dimenſiones vnius pentagoni ſphaerici accurate determinemus, vbi quidem ſufficiet vni-cum ſectorem A O B condideraſſe, exiſtente ſcilicet cen-tro talis pentagoni in puncto O, et radio ſphaerae ma-nente perpetuo $= r$. Hic igitur demiſſo perpendiculo O P erunt anguli $\alpha = 60°$ et $\beta = 36°$, area vero talis pen-tagoni ſupra eſt inuenta $\frac{\pi rr}{3} = 1{,}0471975 . rr$: tantam igitur etiam aream figura plana inducenda recipere de-bebit.

§. 27. Nunc more ſolito triangulum ſphaericum A O P euoluamus, ope formularum coſ. A P $= \frac{cof.\ \beta}{\sin.\ \alpha}$ et coſ. O A $=$ cot. α. cot. β

a *l.* coſ. $\beta = 9{,}9079576$	ad. *l.* cot. $\beta = 10{,}1387390$
ſubtr. *l.* ſin. $\alpha = 9{,}9375306$	adde *l.* cot. $\alpha = 9{,}7614394$
l. coſ. A P $= 9{,}9704270$	*l.* coſ. O A $= 9{,}9001784$
ergo A P $= 20°. 54'. 19''$	hinc O A $= 37°. 22'. 38'$
ſive A P $= 75259''$	ſiue O A $= 134558''$
ad. *l.* A P $= 4{,}8765584$	ad. *l.* O A $= 5{,}1289095$
adde $= 4{,}6855749$	adde - $4{,}6855749$
$9{,}5621333$	$9{,}8144844$
Arcus A P $= 0{,}3648660 . r$	Arcus O A $= 0{,}6523556 . r$

§. 28.

§. 28. Eodem modo etiam computemus quantitatem arcus O P ex formula

$$\text{cof. } O P = \frac{\text{cof. } \alpha}{\text{fin. } \beta}, \text{ vt fequitur}$$

a l. cof. α	$= 9{,}6989700$	ad l. O P	$= 5{,}0576014$
fubtr. l. fin. β	$= 9{,}7692187$	adde	$4{,}6855749$
l. cof. O P	$= 9{,}9297513$		$9{,}7431763$
ergo O P	$= 31^{\circ}.\ 43'.\ 3''$	Arcus O P	$= 0{,}5535749.r$
fiue O P	$= 114183$ fec.		

§. 29. Confideremus nunc pari modo pentagonum regulare planum, feu potius tantum eius partem quintam, vni lateri refpondentem $a\,o\,b$, cuius latus fit ab, centrum circuli circumfcripti o, eiusque radius $o\,a$, qui, quia tanquam incognitus fpectari debet, ponatur $o\,a = z$, ac demiffo ex o in latus $a\,b$ perpendiculo $o\,p$, ob angulum $a\,o\,p = 36$ gr. erit $a\,p = z$ fin. 36 gr. et $o\,p = z$ cof. 36 gr. ideoque $a\,p = 0{,}5877853.z$ et $o\,p = 0{,}8090170.z$; area vero trianguli $a\,o\,p$ erit $= 0{,}2377671.zz$. Tab. I. Fig. 2.

§. 30. Quia autem in hac figura angulus $o\,a\,p$ eft tantum 54 gr. dum in fphaera erat 60 gr., eo angulus fphaericus neutiquam obtegi poterit, fed nimis paruus eft 6 gradibus. Ad hunc ergo defectum fupplendum lateri $a\,b$ adiungatur fegmentum circulare $a\,\pi\,b$, cuius arcus cum chorda faciat angulum $p\,a\,\pi = 6$ gr. vt fiat angulus $o\,a\,\pi = 60$ gr. angulo fcilicet O A P obtegendo aptus. Huius arcus centrum fit in v, ductaque recta $v\,a = v\,\pi$, ob angulum $v\,a\,\pi = 90$, erit angulus $a\,v\,\pi = 6$ gr.; quare fi ponamus $v\,a = v$, erit $\frac{a\,p}{v} =$ fin. 6 gr. ideoque

$$v = \frac{a\,p}{\text{fin. } 6\text{ gr.}} = \frac{z \text{ fin. } 36\text{ gr.}}{\text{fin. } 6\text{ gr.}}$$

vnde

vnde ſit $v = 5{,}6232078.z$ et $l.\frac{v}{z} = 0{,}7499841.$
Cum igitur arcus $a\pi$ contineat 6 gr. $= \frac{\pi}{30}$, eius quantitas
erit $a\pi = \frac{\pi}{30}\,v = 0{,}5888643.z$, qui ſi ducatur in $\frac{1}{2}v$,
prodibit area ſectoris $av\pi = 1{,}655652.z^2$. Hinc auferatur
area trianguli $vpa = \frac{1}{2}ap.vp = \frac{1}{2}v$ ſin. 6 gr. v coſ. 6 gr. $= \frac{1}{4}vv$
ſin. 12 gr. $= 1{,}643566.z^2$, et remanebit area ſemi-ſeg-
menti $\pi ap = 0{,}012086\,z^2$, quae addita ad aream trian-
guli $opa = 0{,}237767.zz$ producet aream trilinei

$$o\,a\,\pi = 0{,}249853.zz$$

quae decies ſumta dabit aream totius figurae planae quam
quaerimus $= 2{,}49863\,zz$. Denique cum ſit $v\pi = v$
haec linea a
$vp = v$ coſ. 6 gr. $= 5{,}592403$,
ſubducta relinquet ſagittam $p\pi = 0{,}030804$, vnde ſit re-
cta $o\pi = 0{,}839821.z$

§. 31. Omnes igitur has determinationes, quas tam pro figura ſphaerica quam pro figura plana inuenimus, ita repraeſentemus, vt vno obtutu percipi queant

Pro Pentagono ſphaerico	Pro Pentagono plano.
$OA = 0{,}6523556.r$	$oa = z$
$AP = 0{,}3648660.r$	$a\pi = 0{,}5888643.z$
$OP = 0{,}5535749.r$	$o\pi = 0{,}8398210.z$
tota area $= 1{,}0471975.rr$	tota area $= 2{,}49863.zz$

§. 32. Vt nunc hae duae figurae ſatis exacte inter ſe congruere queant, dum ſcilicet charta circa medium o aliquantillum expanditur, quod humectatione facillime obtinetur, manifeſtum eſt, extremitates, vbi talis expanſio locum non habet, exacte inter ſe conuenire debere, ita vt fiat $a\pi = AP$ ſiue $0{,}5888643\,z = 0{,}3648660.r$, vnde colligitur

$$z = \frac{0{,}3648660\,r}{0{,}5888643} = 0{,}6196096.r \text{ et } \frac{l\,z}{r} = 9{,}7921125$$

hoc

hoc igitur valore fubftituto praecedentes determinationes ad fequentes valores reducentur:

Pro Pentagono fphaerico	Pro Pentagono plano.
$AO = 0{,}6523556.r$	$oa = 0{,}6196097.r$
$AP = 0{,}3648612.r$	$a\pi = 0{,}3648612.r$
$OP = 0{,}5535749.r$	$o\pi = 0{,}5175824.r$
tota area $= 1{,}0471975.rr$	tota area $= 0{,}9591895.rr$

Vnde apparet per expanfionem chartae rectam oa incrementum capere debere $= 0{,}0327459.r$, quod eft fere vicefima pars totius longitudinis; at vero recta $o\pi$ capere debet incrementum $0{,}0359925.r$, quod valet circiter partem decimam quartam; denique totius fuperficiei incrementum erit $= 0{,}0880080$, quae eft pars fere vndecima: tantam autem expanfionem chartam recipere poffe experientia fatis declarat, hinc itaque fequens problema commode refoluere poterimus.

Problema.

Superficiem dati globi per duodecim pentagona plana quam fieri poteft exactiffime obducere.

Solutio.

Sit radius globi dati $= r$, et in plano defcribatur circulus radio $z = 0{,}6196097.r$, cui infcribatur pentagonum regulare, fuper cuius fingulis lateribus adiungantur fegmenta circularia $a\pi b$ radio

$$= v = 5{,}6232078.z = 3{,}484114.r$$

tum huiusmodi duodecim pentagona accurate ex charta exfcindantur, iisque ternis ad angulos coniungendis fuperficies globi fatis perfecte obtegi poterit.