1778

De Theoria Lunae ad maiorem perfectionis gradum evehenda

Leonhard Euler

DE

THEORIA LUNAE

AD

MAIOREM PERFECTIONIS GRADUM EVEHENDA.

Auctore

L. EVLERO.

Cum motum Lunae ad centrum Terrae, tanquam fixum spectatum, referre conueniat, ante omnia locum Lunae per ternas coordinatas determinare opportet, quarum directiones sint fixae & inter se normales. Tum vero omnes vires sollicitantes secundum easdem directiones sunt resoluendae, vt ex principiis motus ternae aequationes differentiales secundi gradus obtineri queant. Ita si planam tabulae eclipticam referat, in quo punctum T sit centrum Terrae, recta autem T ♈ ad punctum coeli fixum dirigatur, locus Lunae, qui sit in Z, determinetur per has ternas coordinatas: $TX = X$, $XY = Y$ et $YZ = Z$; tum vires sollicitantes, posito elemento temporis $= dt$, huiusmodi ternas aequationes suppeditant: Tab. XIII. Fig. 1.

$$\frac{ddX}{dt^2} = L, \quad \frac{ddY}{dt^2} = M \text{ et } \frac{ddZ}{dt^2} = N,$$

vbi litterae M et N denotant certas functiones tam ipsarum coordinatarum X, Y, Z, quam quantitatum locum Solis definientium. Hic igitur, quoniam Luna nunquam vltra

vltra certos terminos ab ecliptica diuagatur, quantitas Z perpetuo intra limites fatis arctos continebitur; binae vero reliquae coordinatae X et Y per totam orbitam lunarem variari poterunt; quam ob rem eas ita ad alium axem reduci conueniet, vt earum variatio certos limites transgredi nequeat. Hanc ob rem ducatur recta T M, Lunae longitudinem mediam exhibens, ad quam ex Y agatur normalis Y x, ita vt nunc locus Lunae per iftas coordinatas T x, x Y, Y Z definiatur, quarum variatio vtique intra limites fatis arctos coërcebitur. Nam fi longitudo Lunae media, feu angulus ♈ T M vocetur $= \theta$, qui ergo tempori erit proportionalis, tum erit $T x = X \cos. \theta + Y \sin. \theta$ et $x Y = Y \cos. \theta - X \sin. \theta$; tertia autem coordinata manebit Z. Quod fi iam a denotet diftantiam Lunae mediam a Terra, pofita diftantia Terrae media a Sole $= 1$, ita vt a fit fractio valde parua, fcilicet $a = \frac{1}{390}$, ac ponatur recta $T x = a(1 + x)$, $x Y = a y$ et $Y Z = a z$, euidens eft has nouas quantitates x, y, z femper fore fatis exiguas, ita vt termini, in quibus ea ad plures dimenfiones affurgunt, mox pro nihilo haberi queant. His igitur in calculum introductis, fiquidem per eas erit

$$X = a(1 + x) \cos. \theta - a y \sin. \theta \text{ et}$$
$$Y = a(1 + x) \sin. \theta + a y \cos. \theta \text{ et } Z = a z,$$

totum negotium perducitur ad inueftigandos valores quantitatum exiguarum x, y, z, et facta euolutione omnium formularum, quae in calculum ingrediuntur, ternae aequationes fundamentales ad feries terminorum maxime convergentes reuocabuntur. Deinde, hoc calculo expedito, praeter quantitatem conftantem a infuper introducantur: 1°. excentricitas orbitae lunaris $= K$; 2°. excentricitas orbitae folaris $= \kappa$; 3°. fractio i ab inclinatione orbitae lunaris

naris pendens; tum vero ternae nostrae quantitates incognitae ita referantur, vt sit

$$\begin{aligned} x = {} & \mathfrak{O} + K\mathfrak{P} + K^2\mathfrak{Q} + K^3\mathfrak{R} + a\mathfrak{S} + aK\mathfrak{T} + \varkappa\mathfrak{U} \\ & + \varkappa K\mathfrak{V} + \varkappa K^2\mathfrak{W} + a\varkappa\mathfrak{w} \\ & + ii\mathfrak{X} + iiK\mathfrak{Y} + iiK^2\mathfrak{Z} \end{aligned}$$

$$\begin{aligned} y = {} & O + KP + K^2Q + K^3R + aS + aKT + \varkappa U \\ & + \varkappa KV + \varkappa K^2W + a\varkappa w \\ & + i^2X + i^2KY + i^2K^2Z \end{aligned}$$

$$z = i\mathfrak{p} + iK\mathfrak{q} + iK^2\mathfrak{r} + i\varkappa\mathfrak{s} + i^3\mathfrak{t}$$

et, facta substitutione, ternae aequationes differentiales secundi gradus in totidem partes discerpantur, quot hic occurrunt diuersi coefficientes K, K^2, K^3, $\varkappa$ etc. dum scilicet termini iisdem affecti coefficientibus seorsim inter se aequantur. Hoc modo loco ternarum aequationum fundamentalium totidem ordines aequationum particularium obtinebuntur, quas singulas haud difficulter integrare licebit.

Hac scilicet methodo vsus, omnes inaequalitates motus Lunae olim sum persecutus, in tractatu: *Theoria motuum Lunae noua methodo pertractata*, edito Petropoli 1772, vnde simul tabulas construxi, quarum ope locus Lunae ad quoduis tempus multo commodius et accuratius determinari posse videbatur, quam per alias tabulas vsu receptas, si modo singula elementa, quibus innituntur, ex obseruationibus exquisitissimis rite determinentur. Interim tamen nonnullas inaequalites non satis accurate definire licuit, cuius autem defectus caussa non ipsi Theoriae est imputanda, sed potius difficultati calculi numerici, quem non vltra sextam figuram in fractionibus decimalibus sum prosecutus, cum eum vsque ad octauam figuram extendere oportuisset.

 Post-

Poſtquam autem deinceps ſingula momenta, quibus haec methodus erat ſuperſtructa, accuratius perpendiſſem, primo quidem ſtatim perſpexi, hanc methodum adhiberi non potuiſſe, ſi inclinatio orbitae lunaris ad eclipticam multo maior fuiſſet quam reuera deprehenditur; haec autem circumſtantia parui aeſtimanda eſt momenti, cum in vero motu Lunae locum non habeat. Verum ſi quis eandem methodum ad motum ſatellitum Jovis vel etiam Saturni accommodare voluerit, tum vtique maior inclinatio, quae forte in orbitis horum ſatellitum occurreret, omnem ſucceſſum impedire poſſet. Praeterea vero, quia hic ſtatim Lunae motum ad planum eclipticae reduxi, hoc ipſo nonnullae inaequalitates ſe immiſcere debuerunt, quas fere penitus euitaſſem, ſi motum Lunae quouis tempore ſaltem ad planum orbitae eius mediae reuocaſſem; hoc autem modo non ſolum calculus prolixior euaderet, ſed etiam ipſa determinatio Lunae, ex valoribus x, y, z petenda, maiorem calculi apparatum requireret. At vero ſi adeo omnes inaequalitates ſimpliciores fierent et ad minorem numerum redigerentur, hoc modo totus labor largiter compenſaretur; quam ob rem vſu non cariturum fore arbitror, ſi hanc ideam motum Lunae repraeſentandi accuratius expoſuero.

SECTIO PRIMA.

De reductione coordinatarum principalium ad planum orbitae Lunaris mediae.

Tab. XIII. Fig. 2. §. 1. Conſtitutis vt ante ternis coordinatis principalibus $TX = X$, $XY = Y$, $YZ = Z$, ad quas principia motus ſunt accommodanda, ducamus primo rectam

T ☊

T ☊, quae ad tempus propositum referat lineam nodorum, at vocetur longitudo media nodi ascendentis, seu angulus $\Upsilon T ☊ = ☊$; tum vero ducta ex Y normali $Y X'$, vocentur coordinatae ad istum axem relatae $T X' = X'$, $X' Y = Y'$ manente $Y Z = Z$, eritque $X' = X \cos. ☊ + Y \sin. ☊$ et $Y' = Y \cos. ☊ - X \sin. ☊$. Iam concipiatur planum secundum rectam T ☊ ad eclipticam inclinatum sub angulo $= \iota$, (qui aequetur inclinationi mediae orbitae lunaris, aestimata circiter $= 5°. 9'$) quod planum secet rectam $Y Z$ in O; tum ad rectam $X' O$, productam ex Z, demittatur perpendiculum $Z Y''$, et nunc in hoc plano vocemus coordinatas $T X' = X'' = X'$; $X' Y'' = Y''$ et $Y'' Z = Z''$. Hinc erit $Y'' = Y' \cos. \iota + Z \sin. \iota$ et $Z'' = Z \cos. \iota - Y' \sin. \iota$; sicque habebimus $X'' = X \cos. ☊ + Y \sin. ☊$; $Y'' = Y \cos. ☊ \cos. \iota - X \sin. ☊ \cos. \iota + Z \sin. \iota$; $Z'' = Z \cos. \iota - Y \cos. ☊ \sin. \iota + X \sin. ☊ \sin. \iota$.

§. 2. Referat nunc tabula planum orbitae lunaris, in quo habentur ternae coordinatae modo inuentae $T X' = X''$, $X' Y'' = Y''$ et $Y'' Z = Z''$; tum vero in eodem plano ducatur recta T M, motum Lunae medium referens, quam pro axe assumamus, in quem ex Y'' ducatur perpendiculum $Y'' x$; et quia iam istae coordinatae $T x$, $x Y''$, $Y'' Z$ satis exiguis variationibus sunt obnoxiae, statuamus distantiam Lunae mediam a Terra $= 1$, atque $T x = 1 + x$, $x Y'' = y$ et $Y'' Z = z$. Nunc igitur angulus ☊ T M designat argumentum latitudinis Lunae medium, quod vocemus r, ita vt sit ☊ $T M = r$, iamque habebimus $1 + x = X'' \cos. r + Y'' \sin. r$; $y = Y'' \cos. r - X'' \sin. r$ et $z = Z''$. Quare, si valores ante inuenti substituantur, habebimus sequentes formulas: Tab. XIII. Fig. 3.

$1 + x$

$$1 + x = X(\cos.\Omega \cos. r - \cos.\iota \sin.\Omega \sin. r)$$
$$+ Y(\sin.\Omega \cos. r + \cos.\iota \cos.\Omega \sin. r) + Z \sin.\iota \sin. r$$
$$y = - X(\cos.\iota \sin.\Omega \cos. r + \cos.\Omega \sin. r)$$
$$+ Y(\cos.\iota \cos.\Omega \cos. r - \sin.\Omega \sin. r) + Z \sin.\iota \cos. r$$
$$z = X \sin.\iota \sin.\Omega - Y \sin.\iota \cos.\Omega + Z \cos.\iota.$$

§. 3. Praeterea vero etiam necesse est vt coordinatae X, Y, Z etiam per nostras quantitates exiguas x, y, z exprimantur, quem in finem primo habebimus

$$X'' = (1 + x) \cos. r - y \sin. r,$$
$$Y'' = (1 + x) \sin. r + y \cos. r \quad \text{et} \quad Z'' = z.$$

Deinde ex paragr. 1°. colligimus

$$Y'' \cos.\iota - Z'' \sin.\iota = - X \sin.\Omega + Y \cos.\Omega$$

Cum igitur sit $X'' = X \cos.\Omega + Y \sin.\Omega$ erit

$$X'' \cos.\Omega - Y'' \cos.\iota \sin.\Omega + Z'' \sin.\iota \cos.\Omega = X$$

Porro

$$X'' \sin.\Omega + Y'' \cos.\iota \cos.\Omega - Z'' \sin.\iota \cos.\Omega = Y$$

Denique $Y'' \sin.\iota + Z'' \cos.\iota = Z$. Quare si hic valores ante inuenti substituantur, reperietur:

$$X = (1 + x)(\cos. r \cos.\Omega - \cos.\iota \sin. r \sin.\Omega)$$
$$- y(\sin. r \cos.\Omega + \cos.\iota \cos. r \sin.\Omega) + z \sin.\iota \sin.\Omega$$
$$Y = (1 + x)(\cos. r \sin.\Omega + \cos.\iota \sin. r \cos.\Omega)$$
$$- y(\sin. r \sin.\Omega - \cos.\iota \cos. r \cos.\Omega) - z \sin.\iota \cos.\Omega$$
$$Z = (1 + x) \sin.\iota \sin. r + y \sin.\iota \cos. r + z \cos.\iota.$$

§. 4. Quo has formulas simpliciores reddamus ponamus breuitatis gratia:

$$\cos. r \cos.\Omega - \cos.\iota \sin. r \sin.\Omega = m$$
$$\cos. r \sin.\Omega + \cos.\iota \sin. r \cos.\Omega = n$$
$$\sin. r \sin.\Omega - \cos.\iota \cos. r \cos.\Omega = \mu$$
$$\sin. r \cos.\Omega + \cos.\iota \cos. r \sin.\Omega = \nu$$

qui-

quibus valoribus introductis habebimus:

$$1 + x = m\,X + n\,Y + Z\,\text{fin.}\,\iota\,\text{fin.}\,r$$
$$y = -\nu\,X - \mu\,Y + Z\,\text{fin.}\,\iota\,\text{cof.}\,r$$
$$z = X\,\text{fin.}\,\iota\,\text{fin.}\,\Omega - Y\,\text{fin.}\,\iota\,\text{cof.}\,\Omega + Z\,\text{cof.}\,\iota$$

deinde

$$X = m\,(1 + x) - \nu\,y + z\,\text{fin.}\,\iota\,\text{fin.}\,\Omega$$
$$Y = n\,(1 + x) - \mu\,y - z\,\text{fin.}\,\iota\,\text{cof.}\,\Omega$$
$$Z = (1 + x)\,\text{fin.}\,\iota\,\text{fin.}\,r + y\,\text{fin.}\,\iota\,\text{cof.}\,r + z\,\text{cof.}\,\iota$$

vbi notasse iuuabit sequentes relationes

I. $m\,m + \nu\,\nu + \text{fin.}\,\iota^2\,\text{fin.}\,\Omega^2 = 1$

II. $n\,n + \mu\,\mu + \text{fin.}\,\iota^2\,\text{cof.}\,\Omega^2 = 1$

III. $m\,m + n\,n + \text{fin.}\,\iota^2\,\text{fin.}\,r^2 = 1$

IV. $\nu\,\nu + \mu\,\mu + \text{fin.}\,\iota^2\,\text{cof.}\,r^2 = 1$

tum vero etiam

V. $m\,n + \mu\,\nu - \text{fin.}\,\iota^2\,\text{fin.}\,\Omega\,\text{cof.}\,\Omega = 0$

VI. $-m\,\nu - n\,\mu + \text{fin.}\,\iota^2\,\text{fin.}\,r\,\text{cof.}\,r = 0$

VII. $m\,\text{fin.}\,\iota\,\text{fin.}\,r - \nu\,\text{fin.}\,\iota\,\text{cof.}\,r + \text{fin.}\,\iota\,\text{cof.}\,\iota\,\text{fin.}\,\Omega = 0$

VIII. $n\,\text{fin.}\,\iota\,\text{fin.}\,r - \mu\,\text{fin.}\,\iota\,\text{cof.}\,r - \text{fin.}\,\iota\,\text{cof.}\,\iota\,\text{cof.}\,\Omega = 0$

IX. $m\,\text{fin.}\,\iota\,\text{fin.}\,\Omega - n\,\text{fin.}\,\iota\,\text{cof.}\,\Omega + \text{fin.}\,\iota\,\text{cof.}\,\iota\,\text{fin.}\,r = 0$

X. $-\nu\,\text{fin.}\,\iota\,\text{fin.}\,\Omega + \mu\,\text{fin.}\,\iota\,\text{cof.}\,\Omega + \text{fin.}\,\iota\,\text{cof.}\,\iota\,\text{cof.}\,r = 0$

Ratio harum comparationum in eo est sita, quod duplici modo sit quadratum distantiae Lunae a Terra

$$TZ^2 = X^2 + Y^2 + Z^2 = (1 + x)^2 + y^2 + z^2.$$

SECTIO II.

De differentiatione nouarum coordinatarum x, y, z.

§. 5. Cum principia mechanica huiusmodi ternas formulas suppeditent:

$$d\,d\,X = L\,d\,t^2;\quad d\,d\,Y = M\,d\,t^2;\quad d\,d\,Z = N\,d\,t^2$$

vbi

vbi litterae L, M, N sunt certae functiones ipsarum X, Y, Z, hinc valores differentio-differentialium ddx, ddy, ddz elici opportet. Hic igitur tenendum est angulos ☊ et r esse variabiles, inclinationem vero ι esse constantem; vnde formularum m, n, μ, ν differentialia sunt quaerenda, scilicet cum sit:

$$m = \text{cos.}\, r \,\text{cos.}\, ☊ - \text{cos.}\, ι \,\text{sin.}\, r \,\text{sin.}\, ☊$$
$$n = \text{cos.}\, r \,\text{sin.}\, ☊ + \text{cos.}\, ι \,\text{sin.}\, r \,\text{cos.}\, ☊$$
$$\mu = \text{sin.}\, r \,\text{sin.}\, ☊ - \text{cos.}\, ι \,\text{cos.}\, r \,\text{cos.}\, ☊$$
$$\nu = \text{sin.}\, r \,\text{cos.}\, ☊ + \text{cos.}\, ι \,\text{cos.}\, r \,\text{sin.}\, ☊$$

differentiando reperietur

$$dm = -n\,d☊ - \nu\,dr;\quad dn = m\,d☊ - \mu\,dr$$
$$d\mu = \nu\,d☊ + n\,dr;\quad d\nu = -\mu\,d☊ + m\,dr.$$

§. 6. His differentialibus inuentis sumamus primam nostram aequationem:

$$1 + x = mX + nY + Z \,\text{sin.}\, ι \,\text{sin.}\, r$$

quae differentiata dat

$$dx = mdX + ndY + dZ \,\text{sin.}\, ι \,\text{sin.}\, r - dr(\nu X + \mu Y - Z \,\text{sin.}\, ι \,\text{cos.}\, r)$$
$$- d☊(nX - mY)$$

Est vero

$$\nu X + \mu y - Z \,\text{sin.}\, ι \,\text{cos.}\, r = -y$$

et ex posterioribus formulis

$$nX - mY = y(m\mu - n\nu) + z \,\text{sin.}\, ι (m \,\text{cos.}\, ☊ + n \,\text{sin.}\, ☊)$$

vbi cum sit

$$m\mu - n\nu = -\text{cos.}\, ι \quad\text{et}\quad m \,\text{cos.}\, ☊ + n \,\text{sin.}\, ☊ = \text{cos.}\, r$$

habebimus

$$dx = m\,dX + n\,dY + dZ \,\text{sin.}\, ι \,\text{sin.}\, r + y\,dr$$
$$+ d☊(y \,\text{cos.}\, ι - z \,\text{sin.}\, ι \,\text{cos.}\, r)$$

vnde viciffim colligitur

$$m\,dX + n\,dY + dZ \,\text{sin.}\, ι \,\text{sin.}\, r = dx - y(dr + d☊ \,\text{cos.}\, ι)$$
$$+ z\,d☊ \,\text{sin.}\, ι \,\text{cos.}\, r.$$

§. 7.

§. 7. Eodem modo tractetur secunda aequatio

$$y = -\nu X - \mu Y + Z \text{ fin. } \iota \text{ cof. } r,$$

quae differentiata praebet

$$dy = -\nu dX - \mu dY + dZ \text{ fin. } \iota \text{ cof. } r - dr (mX + nY + Z \text{ fin. } \iota \text{ fin. } r) + d\Omega (\mu X - \nu Y).$$

Est vero

$$m X + n Y + Z \text{ fin. } \iota \text{ fin. } r = 1 + x$$

et ex posterioribus formulis

$$\mu X - \nu Y = (m\mu - n\nu)(1 + x) + z \text{ fin. } \iota (\mu \text{ fin. } \Omega + \nu \text{ cof. } \Omega)$$

siue ob

$$m\mu - n\nu = -\text{cof. } \iota \text{ et } \mu \text{ fin. } \Omega + \nu \text{ cof. } \Omega = \text{fin. } r$$

erit

$$\mu X - \nu Y = -(1 + x) \text{ cof. } \iota + z \text{ fin. } \iota \text{ fin. } r$$

vnde prodit

$$dy = -\nu dX - \mu dY + dZ \text{ fin. } \iota \text{ cof. } r - dr (1 + x) - d\Omega ((1 + x) \text{ cof. } \iota - z \text{ fin. } \iota \text{ fin. } r)$$

hincque vicissim colligitur

$$\nu dX + \mu dY - dZ \text{ fin. } \iota \text{ cof. } r = -dy - (1 + x)(dr + d\Omega \text{ cof. } \iota) + z d\Omega \text{ fin. } \iota \text{ fin. } r.$$

§. 8. Tertia vero aequatio:

$$z = X \text{ fin. } \iota \text{ fin. } \Omega - Y \text{ fin. } \iota \text{ cof. } \Omega + Z \text{ cof. } \iota$$

differentiata dat

$$dz = dX \text{ fin. } \iota \text{ fin. } \Omega - dY \text{ fin. } \iota \text{ cof. } \Omega + dZ \text{ cof. } \iota + d\Omega (X \text{ fin. } \iota \text{ cof. } \Omega + Y \text{ fin. } \iota \text{ fin. } \Omega)$$

est vero

$$X \text{ fin. } \iota \text{ cof. } \Omega + Y \text{ fin. } \iota \text{ fin. } \Omega = (1 + x)(m \text{ fin. } \iota \text{ cof. } \Omega + n \text{ fin. } \iota \text{ fin. } \Omega) - y (\mu \text{ fin. } \iota \text{ fin. } \Omega + \nu \text{ fin. } \iota \text{ cof. } \Omega).$$

Cum igitur sit

$$m \text{ fin. } \iota \text{ cof. } \Omega + n \text{ fin. } \iota \text{ fin. } \Omega = \text{fin. } \iota \text{ cof. } r \text{ et}$$

$$\mu \text{ fin. } \iota \text{ fin. } \Omega + \nu \text{ fin. } \iota \text{ cof. } \Omega = \text{fin. } \iota \text{ fin. } r, \text{ erit}$$

$$X \text{ fin. } \iota \text{ cof. } \Omega + Y \text{ fin. } \iota \text{ fin. } \Omega = (1 + x) \text{ fin. } \iota \text{ cof. } r - y \text{ fin. } \iota \text{ fin. } r$$

 qui-

quibus substitutis prodibit

$$dz = dX \sin. \iota \sin. \Omega - dY \sin. \iota \cos. \Omega + dZ \cos. \iota$$
$$+ d\Omega ((1+x) \sin. \iota \cos. r - y \sin. \iota \sin. r)$$

vnde vicissim colligitur

$$dX \sin. \iota \sin. \Omega - dY \sin. \iota \cos. \Omega + dZ \cos. \iota = dz$$
$$-(1+x) d\Omega \sin. \iota \cos. r + y d\Omega \sin. \iota \sin. r.$$

§. 9. Ante autem quam ad differentialia secunda descendamus conducet differentialia dX, dY, dZ per literas minusculas exprimere. Cum igitur sit

$$X = m(1+x) - \nu y + z \sin. \iota \sin. \Omega$$

erit differentiando

$$dX = m\,dx - \nu\,dy + dz \sin. \iota \sin. \Omega$$
$$+ d\Omega (z \sin. \iota \cos. \Omega + \mu y - n(1+x)) - dr(my + \nu(1+x)).$$

Eodem modo secunda aequatio

$$Y = n(1+x) - \mu y - z \sin. \iota \cos. \Omega$$

differentiata praebet

$$dY = n\,dx - \mu\,dy - dz \sin. \iota \cos. \Omega$$
$$+ d\Omega (z \sin. \iota \sin. \Omega - \nu y + m(1+x)) - dr(ny + \mu(1+x)).$$

Denique tertia aequatio

$$Z = (1+x) \sin. \iota \sin. r + y \sin. \iota \cos. r + z \cos. \iota$$

differentiata dat

$$dZ = dx \sin. \iota \sin. r + dy \sin. \iota \cos. r + dz \cos. \iota$$
$$+ dr \sin. \iota ((1+x) \cos. r - y \sin. r).$$

§. 10. His praeparatis progrediamur ad differentio-differentialia; vbi primo notandum est: quia anguli r et Ω sunt tempori proportionales, eorum differentialia dr et $d\Omega$ pariter esse constantia, perinde atque elementum temporis dt. Hinc primam euoluamus aequationem §. 6. datam, quae erat:

$dx -$

$$dx - y(dr + d\Omega\,\text{cof.}\,\iota) + z\,d\Omega\,\text{fin.}\,\iota\,\text{cof.}\,r$$
$$= m\,dX + n\,dY + dZ\,\text{fin.}\,\iota\,\text{fin.}\,r$$

quae differentiata dat

$$ddx - dy(dr + d\Omega\,\text{cof.}\,\iota) + dz\,d\Omega\,\text{fin.}\,\iota\,\text{cof.}\,r - z\,dr\,d\Omega\,\text{fin.}\,\iota\,\text{fin.}\,r$$
$$= m\,ddX + n\,ddY + ddZ\,\text{fin.}\,\iota\,\text{fin.}\,r$$
$$- dr(\nu\,dX + \mu\,dY - dZ\,\text{fin.}\,\iota\,\text{cof.}\,r)$$
$$- d\Omega(n\,dX - m\,dY)$$

est vero

$$\nu\,dX + \mu\,dY - dZ\,\text{fin.}\,\iota\,\text{cof.}\,r = -dy$$
$$-(1+x)(dr + d\Omega\,\text{cof.}\,\iota) + z\,d\Omega\,\text{fin.}\,\iota\,\text{fin.}\,r \quad \text{et}$$
$$n\,dX - m\,dY = +dy(m\mu - n\nu) + dz\,\text{fin.}\,\iota(n\,\text{fin.}\,\Omega + m\,\text{cof.}\,\Omega)$$
$$+(1+x)(dr(m\mu - n\nu) - d\Omega(mm+nn))$$
$$+y\,d\Omega(\mu n + \nu m) + z\,d\Omega\,\text{fin.}\,\iota(n\,\text{cof.}\,\Omega - m\,\text{fin.}\,\Omega)$$

pro cuius formulae vlteriori reductione reperitur fore:

I. $m\mu - n\nu = -\text{cof.}\,\iota$

II. $n\,\text{fin.}\,\Omega + m\,\text{cof.}\,\Omega = \text{cof.}\,r$

III. $mm + nn = 1 - \text{fin.}\,\iota^2\,\text{fin.}\,r^2$

IV. $\mu n + \nu m = \text{fin.}\,\iota^2\,\text{fin.}\,r\,\text{cof.}\,r$

V. $n\,\text{cof.}\,\Omega - m\,\text{fin.}\,\Omega = \text{cof.}\,\iota\,\text{fin.}\,r$

ita vt fit

$$n\,dX - m\,dY = -dy\,\text{cof.}\,\iota + dz\,\text{fin.}\,\iota\,\text{cof.}\,r$$
$$-(1+x)(dr\,\text{cof.}\,\iota + d\Omega(1 - \text{fin.}\,\iota^2\,\text{fin.}\,r^2))$$
$$+y\,d\Omega\,\text{fin.}\,\iota^2\,\text{fin.}\,r\,\text{cof.}\,r + z\,d\Omega\,\text{fin.}\,\iota\,\text{cof.}\,\iota\,\text{fin.}\,r$$

quibus valoribus substitutis aequatio nostra erit

$$ddx - dy\,dr - dy\,d\Omega\,\text{cof.}\,\iota + dz\,d\Omega\,\text{fin.}\,\iota\,\text{cof.}\,r$$
$$-z\,dr\,d\Omega\,\text{fin.}\,\iota\,\text{fin.}\,r = m\,ddX + n\,ddY$$
$$+ddZ\,\text{fin.}\,\iota\,\text{fin.}\,r + dy\,dr + dr^2(1+x)$$
$$-z\,d\Omega\,dr\,\text{fin.}\,\iota\,\text{fin.}\,r + d\Omega\,dy\,\text{cof.}\,\iota$$
$$-d\Omega\,dz\,\text{fin.}\,\iota\,\text{cof.}\,r + d\Omega^2(1+x)$$
$$-d\Omega^2(1+x)\,\text{fin.}\,\iota^2\,\text{fin.}\,r^2 - y\,d\Omega^2\,\text{fin.}\,\iota^2\,\text{fin.}\,r\,\text{cof.}\,r$$
$$-z\,d\Omega^2\,\text{fin.}\,\iota\,\text{cof.}\,\iota\,\text{fin.}\,r + 2(1+x)\,dr\,d\Omega\,\text{cof.}\,\iota$$

quae

quae porro ob

$$d\,d\,X = L\,dt^2,\quad d\,d\,Y = M\,dt^2,\quad d\,d\,Z = N\,dt^2,$$

reliquis terminis a dextra ad finiftram transpofitis, induet hanc formam:

$$\begin{aligned} ddx - 2dr\,dy \quad & -2(1+x)dr\,d\Omega\,\text{cof.}\,\iota \\ -2d\Omega\,dy\,\text{cof.}\,\iota \quad & +2d\Omega\,dz\,\text{fin.}\,\iota\,\text{cof.}\,r - d\Omega^2(1+x) \\ & +d\Omega^2(1+x)\,\text{fin.}\,\iota^2\,\text{fin.}\,r^2 \\ & -dr^2(1+x) \\ & +y\,d\Omega^2\,\text{fin.}\,\iota^2\,\text{fin.}\,r\,\text{cof.}\,r + z\,d\Omega^2\,\text{fin.}\,\iota\,\text{cof.}\,\iota\,\text{fin.}\,r \\ & = dt^2(mL + nM + N\,\text{fin.}\,\iota\,\text{fin.}\,r). \end{aligned}$$

§. 11. Secunda aequatio §. 7. inuenta ita fe habet:

$$dy + (1+x)(dr + d\Omega\,\text{cof.}\,\iota) - z\,d\Omega\,\text{fin.}\,\iota\,\text{fin.}\,r = -\nu\,dX - \mu\,dY + dZ\,\text{fin.}\,\iota\,\text{cof.}\,r$$

quae differentiata praebet

$$\begin{aligned} & d\,dy + dx(dr + d\Omega\,\text{cof.}\,\iota) - dz\,d\Omega\,\text{fin.}\,\iota\,\text{fin.}\,r \\ & - z\,dr\,d\Omega\,\text{fin.}\,\iota\,\text{cof.}\,r = -\nu\,d\,dX - \mu\,d\,dY \\ & + d\,dZ\,\text{fin.}\,\iota\,\text{cof.}\,r - dr(m\,dX + n\,dY + dZ\,\text{fin.}\,\iota\,\text{fin.}\,r) \\ & + d\Omega(\mu\,dX - \nu\,dY). \end{aligned}$$

Ex praecedentibus autem patet effe

$$m\,dX + n\,dY + dZ\,\text{fin.}\,\iota\,\text{fin.}\,r = dx - y(dr + d\Omega\,\text{cof.}\,\iota) + z\,d\Omega\,\text{fin.}\,\iota\,\text{cof.}\,r$$

tum vero reperitur

$$\begin{aligned} \mu\,dX - \nu\,dY = {} & dx(m\mu - n\nu) + dz\,\text{fin.}\,\iota(\mu\,\text{fin.}\,\Omega + \nu\,\text{cof.}\,\Omega) \\ & - (1+x)d\Omega(n\mu + m\nu) + y(d\Omega(\mu\mu + \nu\nu) - dr(m\mu - n\nu)) \\ & + z\,d\Omega\,\text{fin.}\,\iota(\mu\,\text{cof.}\,\Omega - \nu\,\text{fin.}\,\Omega) \end{aligned}$$

pro cuius formulae reductione vlteriori notetur effe

I. $m\mu + n\nu = -\text{cof.}\,\iota$

II. $\mu\,\text{fin.}\,\Omega + \nu\,\text{cof.}\,\Omega = \text{fin.}\,r$

III. $n\mu + m\nu = \text{fin.}\,\iota^2\,\text{fin.}\,r\,\text{cof.}\,r$

IV.

$$\text{IV.}\quad \mu\mu + \nu\nu = 1 - \sin.\iota^2 \cos.r^2$$

$$\text{V.}\quad \mu \cos.\Omega - \nu \sin.\Omega = -\cos.\iota \cos.r$$

vnde fiet

$$\mu\, dX \pm \nu\, dY = - dx \cos.\iota + dz \sin.\iota \sin.r$$
$$- (1 + x)\, d\Omega \sin.\iota^2 \sin.r \cos.r$$
$$+ y\,(d\Omega\,(1 - \sin.\iota^2 \cos.r^2) + dr \cos.\iota)$$
$$+ z\, d\Omega \sin.\iota \cos.\iota \cos.r$$

et substitutis his valoribus aequatio nostra fiet

$$d\,dY + dx\, dr + (dx\, d\Omega \cos.\iota - dz\, d\Omega \sin.\iota \sin.r$$
$$- z\, dr\, d\Omega \sin.\iota \cos.\iota = -\nu\, ddX - \mu\, ddY$$
$$+ ddZ \sin.\iota \cos.r - dx\, dr + y\, dr^2$$
$$+ y\, d\Omega\, dr \cos.\iota - z\, d\Omega\, dr \sin.\iota \cos.r$$
$$- dx\, d\Omega \cos.\iota + dz\, d\Omega \sin.\iota \sin.r$$
$$- (1 + x)\, d\Omega^2 \sin.\iota^2 \sin.r \cos.r + y\, d\Omega^2$$
$$- y\, d\Omega^2 \sin.\iota^2 \cos.r^2 + y\, dr\, d\Omega \cos.\iota$$
$$- z\, d\Omega^2 \sin.\iota \cos.\iota \cos.r.$$

Quod si iam loco ddX, ddY, ddZ valores dati substituantur et reliqui membri a dextra ad sinistram transferantur, prodibit ista aequatio:

$$ddy + 2\,dx\,dr \quad - 2\,dz\,d\Omega \sin.\iota \sin.r + (1 + x)\,d\Omega^2 \sin.\iota^2 \sin.r \cos.r - y\,dr^2$$
$$+ 2\,dx\,d\Omega \cos.\iota \text{ [illegible] } - y\,d\Omega^2$$
$$\text{[illegible] } - 2y\,d\Omega\,dr \cos.\iota$$
$$+ y\,d\Omega^2 \sin.\iota^2 \cos.r^2$$
$$+ z\,d\Omega^2 \sin.\iota \cos.\iota \cos.r$$
$$= - dt^2\,(\nu L + \mu M - N \sin.\iota \cos.r).$$

§. 12. Tertia denique aequatio differentianda, ex §. 8. sumta est:

$$dz - (1 + x)\, d\Omega \sin.\iota \cos.r + y\, d\Omega \sin.\iota \sin.r$$
$$= dX \sin.\iota \sin.\Omega - dY \sin.\iota \cos.\Omega + dZ \cos.\iota$$

 vnde

vnde colligitur

$$d\,d\,z - 2\,d\,x\,d\,\Omega\,\text{fin.}\,\iota\,\text{cof.}\,r + 2\,d\,y\,d\,\Omega\,\text{fin.}\,\iota\,\text{fin.}\,r$$
$$+ (1+x)\,d\,r\,d\,\Omega\,\text{fin.}\,\iota\,\text{fin.}\,r + y\,d\,r\,d\,\Omega\,\text{fin.}\,\iota\,\text{cof.}\,r$$
$$= d\,d\,X\,\text{fin.}\,\iota\,\text{fin.}\,\Omega - d\,d\,Y\,\text{fin.}\,\iota\,\text{cof.}\,\Omega + d\,d\,Z\,\text{cof.}\,\iota$$
$$+ d\,\Omega\,(d\,X\,\text{fin.}\,\iota\,\text{cof.}\,\Omega + d\,Y\,\text{fin.}\,\iota\,\text{fin.}\,\Omega)$$

vbi est

$$d\,X\,\text{cof.}\,\Omega + d\,Y\,\text{fin.}\,\Omega = d\,x\,(m\,\text{cof.}\,\Omega + n\,\text{fin.}\,\Omega)$$
$$- d\,y\,(\nu\,\text{cof.}\,\Omega + \mu\,\text{fin.}\,\Omega) + (1+x)\,d\,\Omega\,(m\,\text{fin.}\,\Omega - n\,\text{cof.}\,\Omega)$$
$$- (1+x)\,d\,r\,(\nu\,\text{cof.}\,\Omega + \mu\,\text{fin.}\,\Omega) + y\,d\,\Omega\,(\mu\,\text{cof.}\,\Omega - \nu\,\text{fin.}\,\Omega)$$
$$- y\,d\,r\,(m\,\text{cof.}\,\Omega + n\,\text{fin.}\,\Omega) + z\,d\,\Omega\,\text{fin.}\,\iota$$

Supra autem iam obseruauimus esse:

$$1^\circ.\ m\,\text{cof.}\,\Omega + n\,\text{fin.}\,\Omega = \text{cof.}\,r$$
$$2^\circ.\ \nu\,\text{cof.}\,\Omega + \mu\,\text{fin.}\,\Omega = \text{fin.}\,r$$
$$3^\circ.\ m\,\text{fin.}\,\Omega - n\,\text{cof.}\,\Omega = -\text{cof.}\,\iota\,\text{fin.}\,r$$
$$4^\circ.\ \mu\,\text{cof.}\,\Omega - \nu\,\text{fin.}\,\Omega = -\text{cof.}\,\iota\,\text{cof.}\,r$$

sicque erit

$$d\,X\,\text{cof.}\,\Omega + d\,Y\,\text{fin.}\,\Omega = d\,x\,\text{cof.}\,r - d\,y\,\text{fin.}\,r$$
$$- (1+x)\,d\,\Omega\,\text{cof.}\,\iota\,\text{fin.}\,r - (1+x)\,d\,r\,\text{fin.}\,r$$
$$- y\,d\,\Omega\,\text{cof.}\,\iota\,\text{cof.}\,r - y\,d\,r\,\text{cof.}\,r + z\,d\,\Omega\,\text{fin.}\,\iota$$

vnde tandem obtinetur sequens aequatio:

$$ddz - 2dxd\Omega\,\text{fin.}\iota\,\text{cof.}r + 2(1+x)drd\Omega\,\text{fin.}\iota\,\text{fin.}r + 2ydrd\Omega\,\text{fin.}\iota\,\text{cof.}r$$
$$+ 2dyd\Omega\,\text{fin.}\iota\,\text{fin.}r + (1+x)d\Omega^2\,\text{fin.}\iota\,\text{cof.}\iota\,\text{fin.}r + yd\Omega^2\,\text{fin.}\iota\,\text{cof.}\iota\,\text{cof.}r$$
$$- z\,d\,\Omega^2\,\text{fin.}\,\iota^2$$
$$= d\,t^2\,(L\,\text{fin.}\,\iota\,\text{fin.}\,\Omega - M\,\text{fin.}\,\iota\,\text{cof.}\,\Omega + N\,\text{cof.}\,\iota).$$

§. 13. Quoniam hae euolutiones summam attentionem postulant, quo de harum formularum veritate magis conuincamur alia easdem methodo inuestigemus, vtramque scilicet differentiationem simul peragamus. Cum igitur in genere sit

$$d\,d.\,p\,q = p\,d\,d\,q + 2\,d\,p\,d\,q + q\,d\,d\,p$$

supra

supra autem inuenerimus

$$dm = -n\,d\Omega - \nu\,dr;\quad dn = m\,d\Omega - \mu\,dr;\quad d\mu = \nu\,d\Omega + n\,dr;$$
$$d\nu = -\mu\,d\Omega + m\,dr$$

erit denuo differentiando:

$$ddm = -m\,dr^2 + 2\mu\,dr\,d\Omega - m\,d\Omega^2$$
$$ddn = -n\,dr^2 + 2\nu\,dr\,d\Omega - n\,d\Omega^2$$
$$dd\mu = -\mu\,dr^2 + 2m\,dr\,d\Omega - \mu\,d\Omega^2$$
$$dd\nu = -\nu\,dr^2 + 2n\,dr\,d\Omega - \nu\,d\Omega^2$$

§. 14. Cum igitur prima nostrarum formularum sit

$$1 + x = mX + nY + Z\,\text{fin.}\,\iota\,\text{fin.}\,r$$

erit statim bis differentiando

$$ddx = m\,ddX + n\,ddY + ddZ\,\text{fin.}\,\iota\,\text{fin.}\,r \;\text{- - - (I)}$$
$$+ 2dX\,dm + 2dY\,dn + 2dZ\,dr\,\text{fin.}\,\iota\,\text{cof.}\,r \;\text{- - (II)}$$
$$+ X\,ddm + Y\,ddn - Z\,dr^2\,\text{fin.}\,\iota\,\text{fin.}\,r \;\text{- - - - (III)}$$

quas ternas partes seorsim euoluamus, ac primo quidem erit, vt supra vidimus

$$\text{(I) - - } dt^2(mL + nM + N\,\text{fin.}\,\iota\,\text{fin.}\,r)$$

(II) in haec duo membra resoluitur:

$$\text{(II) } + 2dr(-\nu\,dX - \mu\,dY + dZ\,\text{fin.}\,\iota\,\text{cof.}\,r)$$
$$+ 2d\Omega(-n\,dX + m\,dY)$$

at vero tertia pars in haec tria membra discerpitur

$$\text{(III) - - } + dr^2(-mX - nY - Z\,\text{fin.}\,\iota\,\text{fin.}\,r)$$
$$+ 2dr\,d\Omega(\mu X - \nu Y)$$
$$+ d\Omega^2(-mX - nY)$$

supra autem vidimus esse:

$$-\nu\,dX - \mu\,dY + dZ\,\text{fin.}\,\iota\,\text{cof.}\,r = dy + (1+x)(dr + d\Omega\,\text{cof.}\,\iota)$$
$$- z\,d\Omega\,\text{fin.}\,\iota\,\text{fin.}\,r$$

$-n\,d$

III

$$-n\,dX+m\,dY=dy\,\text{cof.}\,\iota-dz\,\text{fin.}\,\iota\,\text{cof.}\,r$$
$$+(1+x)\,(dr\,\text{cof.}\,\iota+d\Omega\,(1-\text{fin.}\,\iota^2\,\text{fin.}\,r^2)$$
$$-y\,d\Omega\,\text{fin.}\,\iota^2\,\text{fin.}\,r\,\text{cof.}\,r-z\,d\Omega\,\text{fin.}\,\iota\,\text{cof.}\,\iota\,\text{fin.}\,r$$

pro tertia autem parte pariter iam supra inuenimus:

$$-mX-nY-Z\,\text{fin.}\,\iota\,\text{fin.}\,r=-(1+x)$$
$$\mu X-\nu Y=-(1+x)\,\text{cof.}\,\iota+z\,\text{fin.}\,\iota\,\text{fin.}\,r$$
$$-mX-nY=+(1+x)+Z\,\text{fin.}\,\iota\,\text{fin.}\,r=-(1+x)(1-\text{fin.}\,\iota^2\,\text{fin.}\,r^2)$$
$$+y\,\text{fin.}\,\iota^2\,\text{fin.}\,r\,\text{cof.}\,r+z\,\text{fin.}\,\iota\,\text{cof.}\,\iota\,\text{fin.}\,r$$

His igitur colligendis erit

$$(II)+(III)=\begin{array}{l}+2\,dr\,dy\\+2\,d\Omega\,dy\,\text{cof.}\,\iota\end{array}\quad\begin{array}{l}-2\,d\Omega\,dz\,\text{fin.}\,\iota\,\text{cof.}\,r+(1+x)\,dr^2\\+2(1+x)\,dr\,d\Omega\,\text{cf.}\,\iota\\+(1+x)\,d\Omega^2\\-(1+x)\,d\Omega^2\,\text{fi.}\,\iota^2\,\text{fi.}\,r^2\end{array}$$

$$-y\,d\Omega^2\,\text{fin.}\,\iota^2\,\text{fin.}\,r\,\text{cof.}\,r-z\,d\Omega^2\,\text{fin.}\,\iota\,\text{cof.}\,\iota\,\text{cof.}\,r$$

his igitur terminis ad finiftram translatis aequatio prodit:

$$ddx-2\,dr\,dy+2\,d\Omega\,dz\,\text{fin.}\,\iota\,\text{cof.}\,r-(1+x)\,dr^2$$
$$-2\,d\Omega\,dy\,\text{cof.}\,\iota\qquad-2(1+x)\,dr\,d\Omega\,\text{cof.}\,\iota$$
$$-(1+x)\,d\Omega^2$$
$$+(1+x)\,d\Omega^2\,\text{fin.}\,\iota^2\,\text{fin.}\,r^2+y\,d\Omega^2\,\text{fin.}\,\iota^2\,\text{fin.}\,r\,\text{cof.}\,r$$
$$+z\,d\Omega^2\,\text{fin.}\,\iota\,\text{cof.}\,\iota\,\text{fin.}\,r=(mL+nM+N\,\text{fin.}\,\iota\,\text{fin.}\,r)\,dt^2$$

quae cum praecedente §. 10. exhibita prorfus congruit.

§. 15. Deinde cum altera noftra aequatio fuerit

$$y=-\nu X-\mu Y+Z\,\text{fin.}\,\iota\,\text{cof.}\,r$$

erit pariter bis differentiando

$$ddy=-\nu\,ddX-\mu\,ddY+ddZ\,\text{fin.}\,\iota\,\text{cof.}\,r\;---\;(I)$$
$$-2\,dX\,d\nu-2\,dY\,d\mu-2\,dZ\,dr\,\text{fin.}\,\iota\,\text{fin.}\,r\;(II)$$
$$-X\,dd\nu-Y\,dd\mu-Z\,dr^2\,\text{fin.}\,\iota\,\text{cof.}\,r\;---\;(III)$$

quae fingulae partes euolutae praebent

$$(I)\quad dt^2(-\nu L-\mu M+N\,\text{fin.}\,\iota\,\text{cof.}\,r$$
$$(II)\quad 2\,dr(-m\,dX-n\,dY-dZ\,\text{fin.}\,\iota\,\text{fin.}\,r)$$
$$+2\,d\Omega(+\mu\,dX-\nu\,dY)$$

(III)

$$\begin{aligned}(\mathrm{III})\; & dr^2(+\nu X + \mu Y - Z\,\mathrm{fin.}\,\iota\,\mathrm{cof.}\,r) \\ & + 2\,dr\,d\Omega(nX - mY) \\ & + d\Omega^2(\nu X + \mu Y)\end{aligned}$$

noti autem funt fequentes valores:

$$-m\,dX - n\,dY - dZ\,\mathrm{fin.}\,\iota\,\mathrm{fin.}\,r = -dx + y(dr + d\Omega\,\mathrm{cof.}\,\iota) - z\,d\Omega\,\mathrm{fin.}\,\iota\,\mathrm{cof.}\,r$$

$$\begin{aligned}\mu\,dX - \nu\,dY = & -dx\,\mathrm{cof.}\,\iota + dz\,\mathrm{fin.}\,\iota\,\mathrm{fin.}\,r - (1+x)\,d\Omega\,\mathrm{fin.}\,\iota^2\,\mathrm{fin.}\,r\,\mathrm{cof.}\,r \\ & + y(d\Omega(1 - \mathrm{fin.}\,\iota^2\,\mathrm{cof.}\,r^2) + dr\,\mathrm{cof.}\,\iota) - z\,d\Omega\,\mathrm{fin.}\,\iota\,\mathrm{cof.}\,\iota\,\mathrm{cof.}\,r\end{aligned}$$

$$\nu X + \mu Y - Z\,\mathrm{fin.}\,\iota\,\mathrm{cof.}\,r = -y$$

$$nX - mY = -y\,\mathrm{cof.}\,\iota + z\,\mathrm{fin.}\,\iota\,\mathrm{cof.}\,r$$

$$\begin{aligned}\nu X + \mu Y = -y + Z\,\mathrm{fin.}\,\mathrm{cof.}\,r = & (1+x)\,\mathrm{fin.}\,\iota^2\,\mathrm{fin.}\,r\,\mathrm{cof.}\,r \\ & - y(1 - \mathrm{fin.}\,\iota^2\,\mathrm{cof.}\,r^2) + z\,\mathrm{fin.}\,\iota\,\mathrm{cof.}\,\iota\,\mathrm{cof.}\,r)\end{aligned}$$

quibus collectis erit

$$\begin{aligned}(\mathrm{II})+(\mathrm{III}) = & -2\,dr\,dx + 2\,d\Omega\,dz\,\mathrm{fin.}\,\iota\,\mathrm{fin.}\,r - (1+x)\,d\Omega^2\,\mathrm{fin.}\,\iota^2\,\mathrm{fin.}\,r\,\mathrm{cof.}\,r + y\,dr^2 \\ & - 2\,d\Omega\,dx\,\mathrm{cof.}\,\iota + 2y\,dr\,d\Omega\,\mathrm{cof.}\,\iota \\ & + y\,d\Omega^2 \\ & - y\,d\Omega^2\,\mathrm{fi.}\,\iota^2\,\mathrm{cf.}\,r^2 \\ & - z\,d\Omega^2\,\mathrm{fin.}\,\iota\,\mathrm{cof.}\,\iota\,\mathrm{cof.}\,r\end{aligned}$$

quibus ad finiftram translatis prodit:

$$\begin{aligned}& ddy + 2\,dr\,dx - 2\,d\Omega\,dz\,\mathrm{fi.}\,\iota\,\mathrm{fi.}\,r + (1+x)\,d\Omega^2\,\mathrm{fi.}\,\iota^2\,\mathrm{fi.}\,r\,\mathrm{cf.}\,r - y\,dr^2 \\ & + 2\,d\Omega\,dx\,\mathrm{cof.}\,\iota - 2y\,dr\,d\Omega\,\mathrm{cof.}\,\iota \\ & - y\,d\Omega^2 \\ & + y\,d\Omega^2\,\mathrm{fi.}\,\iota^2\,\mathrm{cf.}\,r^2 \\ & + z\,d\Omega^2\,\mathrm{fin.}\,\iota\,\mathrm{cof.}\,\iota\,\mathrm{cof.}\,r \\ & = -dt^2(\nu L + \mu M - N\,\mathrm{fin.}\,\iota\,\mathrm{cof.}\,r)\end{aligned}$$

quae cum praecedente §. 11. exhibita pariter congruit.

§. 16. Tertia aequatio erat

$$z = X\,\mathrm{fin.}\,\iota\,\mathrm{fin.}\,\Omega - Y\,\mathrm{fin.}\,\iota\,\mathrm{cof.}\,\Omega + Z\,\mathrm{cof.}\,\iota.$$

quae bis differentiata dat

 ddz

$$ddz = ddX \sin\iota \sin\Omega - ddY \sin\iota \cos\Omega + ddZ \cos\iota \quad - \; (\mathrm{I})$$
$$+ 2dX\, d\Omega \sin\iota \cos\Omega + 2dY\, d\Omega \sin\iota \sin\Omega \quad - - \; (\mathrm{II})$$
$$- X\, d\Omega^2 \sin\iota \sin\Omega + Y\, d\Omega^2 \sin\iota \cos\Omega \quad - - \; (\mathrm{III})$$

quae partes facile ita reducuntur:

$$(\mathrm{I}) \; - - \; dt^2 (L \sin\iota \sin\Omega - M \sin\iota \cos\Omega + N \cos\iota$$
$$(\mathrm{II}) \; - - \; 2\, d\Omega \sin\iota (dX \cos\Omega + dY \sin\Omega)$$
$$(\mathrm{III}) \; - - \; d\Omega^2 \sin\iota (-X \sin\Omega + Y \cos\Omega).$$

Est vero

$$dX \cos\Omega + dY \sin\Omega = dx \cos r - dy \sin r - (1+x) d\Omega \cos\iota \sin r$$
$$- (1+x) dr \sin r - y\, d\Omega \cos\iota \cos r - y\, dr \cos r + z\, d\Omega \sin\iota$$
$$- X \sin\Omega + Y \cos\Omega = (1+x) \cos\iota \sin r + \cos\iota \cos r - z \sin\iota$$

quibus substitutis colligimus

$$(\mathrm{II}) + (\mathrm{III}) = 2\, d\Omega\, dx \sin\iota \cos r - 2\, d\Omega\, dy \sin\iota \sin r$$
$$- (1+x) d\Omega^2 \sin\iota \cos\iota \sin r - y\, d\Omega^2 \sin\iota \cos\iota \cos r$$
$$+ z\, d\Omega^2 \sin\iota^2 - 2(1+x)\, dr\, d\Omega \sin\iota \sin r$$
$$- 2y\, dr\, d\Omega \sin\iota \cos r$$

quibus ad sinistram translatis aequatio ita se habebit:

$$ddz - 2d\Omega\, dx \sin\iota \cos r + 2d\Omega\, dy \sin\iota \sin r + (1+x) d\Omega^2 \sin\iota \cos\iota \sin r$$
$$+ y\, d\Omega^2 \sin\iota \cos\iota \cos r - z\, d\Omega^2 \sin\iota^2 =$$
$$= dt^2 (L \sin\iota \sin\Omega - M \sin\iota \cos\Omega + N \cos\iota)$$
$$+ 2(1+x)\, dr\, d\Omega \sin\iota \sin r + 2y\, dr\, d\Omega \sin\iota \cos r.$$

quae etiam congruit cum illa §. 12. exhibita.

§. 17. Ternae ergo aequationes differentio-differentiales, ex quibus motum Lunae determinari oportet, sequenti modo aspectui exponantur:

$$\mathrm{I}^{\circ}\; ddx - 2dy(dr + d\Omega \cos\iota) + 2d\Omega\, dz \sin\iota \cos r$$
$$- (1+x)(dr^2 + 2dr\, d\Omega \cos\iota) + d\Omega^2 (1 - \sin\iota^2 \sin r^2)$$
$$+ y\, d\Omega^2 \sin\iota^2 \sin r \cos r + z\, d\Omega^2 \sin\iota \cos\iota \sin r$$
$$= dt^2 (mL + nM + N \sin\iota \sin r$$

II

$$\text{II. } ddy + 2dx(dr + d☊ \cos\iota) - 2☊ dz \sin\iota \sin r - y(dr^2 + 2dr\, d☊ \cos\iota + d☊^2 (1 - \sin\iota^2 \cos r^2) + (1+x) d☊^2 \sin\iota^2 \sin r \cos r + z d☊^2 \sin\iota \cos\iota \cos r = -dt^2(\nu L + \mu M - N \sin\iota \cos r)$$

$$\text{III } ddz - 2d☊ dx \sin\iota \cos r + 2d☊ dy \sin\iota \sin r - z d☊^2 \sin\iota^2 + (1+x)(2dr\, d☊ \sin\iota \sin r + d☊^2 \sin\iota \cos\iota \sin r) + y(2dr\, d☊ \sin\iota \cos r + d☊^2 \sin\iota \cos\iota \cos r) = dt^2(L \sin\iota \sin ☊ - M \sin\iota \cos ☊ + N \cos\iota)$$

vbi meminisse iuuabit breuitatis gratia nos posuisse

$$m = \cos r \cos ☊ - \cos\iota \sin r \sin ☊$$
$$n = \cos r \sin ☊ + \cos\iota \sin r \cos ☊$$
$$\mu = \sin r \sin ☊ - \cos\iota \cos r \cos ☊$$
$$\nu = \sin r \cos ☊ + \cos\iota \cos r \sin ☊$$

Praeterea vero formulae principales, vnde has aequationes deduximus, erant:

$$1+x = mX + nY + Z \sin\iota \sin r \quad\Big|\quad X = m(1+x) - \nu y + z \sin\iota \sin ☊$$
$$y = -\nu X - \mu Y + Z \sin\iota \cos r \quad\Big|\quad Y = n(1+x) - \mu y - z \sin\iota \cos ☊$$
$$z = X \sin\iota \sin ☊ - Y \sin\iota \cos ☊ + Z \cos\iota \quad\Big|\quad Z = (1+x)\sin\iota \sin r + y \sin\iota \cos r + z \cos\iota$$

SECTIO III.

De valoribus litterarum L, M, N, ex principiis motus deducendis.

Tab. XIII Fig. 4.

§. 18. Denotent signa ♁, ☉, ☾ massas Terrae, Solis et Lunae, quas, posita massa Terrae ♁ = 1, ex nouissimis obseruationibus concludimus fore ☉ = 360000; at vero massam Lunae ☾ = $\frac{1}{70}$ propemodum. Parum autem interest quanta sit massa Lunae, quandoquidem Newtonus eam statuit = $\frac{1}{40}$, Illustris *Daniel Bernoulli* eam reduxit ad $\frac{1}{70}$ tantum. His positis sit, vti initio, Z locus Lunae

Lunae per ternas coordinatas $TX = X$, $XY = Y$, $YZ = Z$ determinatus, exiſtente T centro Terrae et recta $T\Upsilon$ directione fixa ad aequinoctium vernum tendente; tum vero, vocetur diſtantia Lunae a Terra $TZ = v$, ita vt ſit $vv = X^2 + Y^2 + Z^2$. Nouimus autem per nouas coordinatas $1 + x$, y, z eſſe etiam

$$v^2 = (1 + x)^2 + yy + zz.$$

Praeterea vero ſit centrum Solis in S, pro quo binae coordinatae ſint $TU = \mathfrak{x}$ et $US = \mathfrak{y}$, ipſa autem Solis a Terra diſtantia vocetur $TS = u$, ita vt ſit $uu = \mathfrak{x}\mathfrak{x} + \mathfrak{y}\mathfrak{y}$; vnde ſi ponatur longitudo Solis vera, ſeu angulus $\Upsilon TS = \Phi$ erit $\mathfrak{x} = u \cos\Phi$ et $\mathfrak{y} = u \sin\Phi$. Porro vero vocetur diſtantia Solis a Luna, ſeu recta $SZ = w$ eritque

$$ww = (\mathfrak{x} - X)^2 + (\mathfrak{y} - Y)^2 + ZZ = uu - 2\mathfrak{x}X - 2\mathfrak{y}Y + vv$$

ſiue

$$w^2 = uu - 2uX\cos\Phi - 2uY\sin\Phi + vv.$$

§. 19. Quod ſi iam $d\tau$ denotet elementum temporis indefinitum, cui coefficientum Δ adiungamus, huius valor ſtatim determinabitur ac menſuram temporis idoneam ſtabiliuerimus. His poſitis ex principiis motus facile deducuntur ſequentes tres aequationes ſecundi gradus:

$$\text{I.} \quad \frac{ddX}{\Delta d\tau^2} = -\frac{(\text{♁} + ☾)X}{v^3} + \frac{\odot(\mathfrak{x} - X)}{w^3} - \frac{\odot\mathfrak{x}}{u^3}$$

$$\text{II.} \quad \frac{ddY}{\Delta d\tau^2} = -\frac{(\text{♁} + ☾)Y}{v^3} + \frac{\odot(\mathfrak{y} - Y)}{w^3} - \frac{\odot\mathfrak{y}}{u^3}$$

$$\text{III.} \quad \frac{ddZ}{\Delta d\tau^2} = -\frac{(\text{♁} + ☾)Z}{v^3} - \frac{\odot Z}{w^3}.$$

Ante autem quam has aequationes ad noſtrum inſtitutum accommodare queamus, certam temporis menſuram ſtabilire oportet, id quod ſequenti modo praeſtabimus.

§. 20.

§. 20. Consideremus solum Solem circa Terram quasi fixam in circulo reuolui, cuius radius sit distantia Solis media a Terra $= a$, ac formulae eius motum determinantes erunt

$$\frac{dd\mathfrak{x}}{\Delta d\tau^2} = -\frac{(\odot + \female)\mathfrak{x}}{u^3} \text{ et } \frac{dd\mathfrak{y}}{\Delta d\tau^2} = -\frac{(\odot + \female)\mathfrak{y}}{u^3}.$$

Ponamus nunc tempore τ a Sole describi anomaliam mediam $= t$, et cum sit $u = a$, ponere licebit

$$\mathfrak{x} = a \operatorname{cof.} t \text{ et } \mathfrak{y} = a \operatorname{fin.} t$$

vnde fit

$$dd\mathfrak{x} = -a\,dt^2 \operatorname{cof.} t \text{ et } dd\mathfrak{y} = -a\,dt^2 \operatorname{fin.} t$$

quibus valoribus substitutis prior aequatio euadit

$$\frac{a\,dt^2 \operatorname{cof.} t}{\Delta d\tau^2} = +\frac{(\odot + \female) \operatorname{cof.} t}{aa} \text{ vnde fit}$$

$$\Delta d\tau^2 = +\frac{a^3\,dt^2}{\odot + \female} \text{ hincque } \frac{1}{\Delta d\tau^2} = \frac{\odot + \female}{a^3\,dt^2}$$

altera vero aequatio sponte idem praebet. Nunc autem motum Lunae ita inuestigemus, quasi a sola Terra attracta in circulo reuolueretur ad distantiam mediam $= 1$, ac tempore τ angulum circa Terram describeret $= \theta$, qui scilicet longitudinem Lunae mediam designabit. Pro hoc ergo motu, quem in ecliptica fieri concipiamus, habebimus has aequationes:

$$\frac{dd\mathrm{X}}{\Delta d\tau^2} = -\frac{(\female + ☾)\mathrm{X}}{v^3} \text{ et } \frac{dd\mathrm{Y}}{\Delta d\tau^2} = -\frac{(\female + ☾)\mathrm{Y}}{v^3}$$

quare, cum in hac hypothesi sit $v = 1$, $\mathrm{X} = \operatorname{cof.} \theta$ et $\mathrm{Y} = \operatorname{fin.} \theta$ erit

$$dd\mathrm{X} = -d\theta^2 \operatorname{cof.} \theta \text{ et } dd\mathrm{Y} = -d\theta^2 \operatorname{fin.} \theta$$

quibus valoribus substitutis vtraque aequatio praebet

$$\frac{d\theta^2}{\Delta d\tau^2} = \female + ☾,$$

vnde si loco $\Delta d\tau^2$ valor modo inuentus substituatur, prodibit

dibit $\frac{d\theta^2(\odot + \text{♁})}{a^3\, dt^2} = \text{♁} + \text{☾}$. Cum igitur ratio inter motum medium Lunae ac ſolis vt cognita ſpectari poſſit, ſtatuamus $\frac{d\theta}{dt} = \lambda$, fietque $\frac{\lambda\lambda(\odot + \text{♁})}{a^3} = \text{♁} + \text{☾}$, vnde concludimus $a^3 = \frac{\lambda\lambda(\odot + \text{♁})}{\text{♁} + \text{☾}}$ hincque $\frac{1}{\Delta\, d\tau^2} = \frac{\text{♁} + \text{☾}}{\lambda\lambda\, dt^2}$.

§. 21. Hos igitur valores in noſtras aequationes principales introducamus ac reperiemus

$$\text{I.}\quad \frac{(\text{♁} + \text{☾})\, ddX}{\lambda\lambda\, dt^2} = -\frac{(\text{♁} + \text{☾})X}{v^3} + \frac{\odot(\mathfrak{x} - X)}{w^3} - \frac{\odot\,\mathfrak{x}}{u^3}$$

$$\text{II.}\quad \frac{(\text{♁} + \text{☾})\, ddY}{\lambda\lambda\, dt^2} = -\frac{(\text{♁} + \text{☾})Y}{v^3} + \frac{\odot(\mathfrak{y} - Y)}{w^3} - \frac{\odot\,\mathfrak{y}}{u^3}$$

$$\text{III.}\quad \frac{(\text{♁} + \text{☾})\, ddZ}{\lambda\lambda\, dt^2} = -\frac{(\text{♁} + \text{☾})Z}{v^3} - \frac{\odot Z}{w^3}.$$

Cum igitur ſumſerimus

$\frac{ddX}{dt^2} = L$, $\frac{ddY}{dt^2} = M$, $\frac{ddZ}{dt^2} = N$

hinc reperiemus has litteras L, M, N ſequenti modo

$$1^\circ.\quad L = -\frac{\lambda\lambda X}{v^3} + \frac{\lambda\lambda\odot(\mathfrak{x} - X)}{(\text{♁} + \text{☾})w^3} - \frac{\lambda\lambda\odot\,\mathfrak{x}}{(\text{♁} + \text{☾})u^3}$$

$$2^\circ.\quad M = -\frac{\lambda\lambda Y}{v^3} + \frac{\lambda\lambda\odot(\mathfrak{y} - Y)}{(\text{♁} + \text{☾})w^3} - \frac{\lambda\lambda\odot\,\mathfrak{y}}{(\text{♁} + \text{☾})u^3}$$

$$3^\circ.\quad N = -\frac{\lambda\lambda Z}{v^3} - \frac{\lambda\lambda\odot Z}{(\text{♁} + \text{☾})w^3}.$$

In

In posterioribus autem terminis signo ⊙ affectis pro λ scribamus eius valorem $\lambda\lambda = \frac{a^3(♁ + ☾)}{⊙ + ♁}$, vnde fit $\frac{\lambda\lambda ⊙}{♁ + ☾} = \frac{a^3 ⊙}{⊙ + ♁} = a^3$, quandoquidem Terrae massa est quasi infinite parua respectu massae Solis hinc igitur habebimus

$$L = -\frac{\lambda\lambda X}{v^3} + \frac{a^3(\mathfrak{x} - X)}{w^3} - \frac{a^3 \mathfrak{x}}{u^3}$$

$$M = -\frac{\lambda\lambda Y}{v^3} + \frac{a^3(\mathfrak{y} - Y)}{w^3} - \frac{a^3 \mathfrak{y}}{u^3}$$

$$N = -\frac{\lambda\lambda Z}{v^3} - \frac{a^3 Z}{w^3}$$

§. 22. Hos valores secundum litteras $\mathfrak{x}$, $\mathfrak{y}$ et X, Y, Z partiamur, et obtinebimus sequentes valores:

$$L = -X\left(\frac{\lambda\lambda}{v^3} + \frac{a^3}{w^3}\right) + \mathfrak{x}\left(\frac{a^3}{w^3} - \frac{a^3}{u^3}\right)$$

$$M = -Y\left(\frac{\lambda\lambda}{v^3} + \frac{a^3}{w^3}\right) + \mathfrak{y}\left(\frac{a^3}{w^3} - \frac{a^3}{u^3}\right)$$

$$N = -Z\left(\frac{\lambda\lambda}{v^3} + \frac{a^3}{w^3}\right).$$

Hic igitur breuitatatis gratia statuamus $\frac{\lambda\lambda}{v^3} + \frac{a^3}{w^3} = F$ et $\frac{a^3}{w^3} - \frac{a^3}{u^3} = G$, ita vt habeamus

$$L = -FX + G\mathfrak{x};\ M = -FY + G\mathfrak{y};\ N = -FZ.$$

§. 23. Hinc igitur facile formulas in principalibus nostris, aequationibus in fine sectionis praecedentis, exhibitis, quae ad dextram partem erant dispositae, euoluemus: reperiemus enim

$$mL + nM + N\,\text{sin.}\,i\,\text{sin.}\,r = -F(\mathfrak{x} + x) + G(m\mathfrak{x} + n\mathfrak{y})$$

$$\nu L + \mu M - N\,\text{sin.}\,i\,\text{cos.}\,r = +Fy + G(\nu\mathfrak{x} + \mu\mathfrak{y})$$

$$L\,\text{sin.}\,i\,\text{sin.}\,☊ - M\,\text{sin.}\,i\,\text{cos.}\,☊ + N\,\text{cos.} = -Fz + G(\mathfrak{x}\,\text{sin.}\,i\,\text{sin.}\,☊ - \mathfrak{y}\,\text{sin.}\,i\,\text{cos.}\,☊)$$

§. 24. Operae igitur pretium erit, hos valores in aequationibus nostris fundamentalibus substituere, vt in posterum nulla amplius mentio literarum L, M, N occurrat:

$$\left.\begin{aligned} & ddx - 2dy(dr + d☊\cos.\iota) + 2d☊dz \sin.\iota\cos.r \\ & \quad - (1+x)(dr^2 + 2drd☊\cos.\iota + d☊^2(1 - \sin.\iota^2\sin.r^2)) \\ & \quad + yd☊^2\sin.\iota^2\sin.r\cos.r + zd☊^2\sin.\iota\cos.\iota\sin.r \end{aligned}\right\}$$

$$= -F(1+x)dt^2 + G(m\mathfrak{x} + n\mathfrak{y})dt^2$$

$$\left.\begin{aligned} & ddy + 2dx(dr + d☊\cos.\iota) - 2d☊dz\sin.\iota\sin.r \\ & \quad - y(dr^2 + 2drd☊\cos.\iota + d☊^2(1 - \sin.\iota^2\cos.r^2)) \\ & \quad + (1+x)d☊^2\sin.\iota^2\sin.r\cos.r + zd☊^2\sin.\iota\cos.\iota\cos.r \end{aligned}\right\}$$

$$= -Fydt^2 - G(\nu\mathfrak{x} + \mu\mathfrak{y})dt^2$$

$$\left.\begin{aligned} & ddz - 2d☊dx\sin.\iota\cos.r + 2d☊dy\sin.\iota\sin.r - zd☊^2\sin.\iota^2 \\ & \quad + (1+x)(2drd☊\sin.\iota\sin.r + d☊^2\sin.\iota\cos.\iota\sin.r) \\ & \quad + y(2drd☊\sin.\iota\cos.r + d☊^2\sin.\iota\cos.\iota\cos.r) \end{aligned}\right\}$$

$$= -Fzdt^2 + G(\mathfrak{x}\sin.\iota\sin.☊ - \mathfrak{y}\sin.\iota\cos.☊)dt^2.$$

SECTIO IV.

De exstirpatione literarum $\mathfrak{x}$, $\mathfrak{y}$, et X, Y.

§. 25. Introducendo longitudinem Solis veram ♈TS $= \Phi$, cum eius distantia a Terra TS $= u$, vidimus esse $\mathfrak{x} = u\cos.\Phi$ et $\mathfrak{y} = u\sin.\Phi$, vnde sequentes formulae euoluendae occurrunt:

I. $m\mathfrak{x} + n\mathfrak{y} = u(m\cos.\Phi + n\sin.\Phi)$

cum igitur sit

$$m = \cos.r\cos.☊ - \cos.\iota\sin.r\sin.☊ \text{ et}$$

$$n = \cos.r\sin.☊ + \cos.\iota\sin.r\cos.☊ \text{ erit}$$

$$m\cos.\Phi + n\sin.\Phi = \cos.r\cos.(\Phi - ☊) + \cos.\iota\sin.r\sin.(\Phi - ☊)$$

hinc igitur erit

$$m\mathfrak{x} + n\mathfrak{y} = u(\cos.r\cos.(\Phi - ☊) + \cos.\iota\sin.r\sin.(\Phi - ☊)).$$

§. 26.

§. 26. Deinde in secunda aequatione occurrit ista formula: $\nu \mathfrak{x} + \mu \mathfrak{y}$; quia igitur est

$$\nu \mathfrak{x} + \mu \mathfrak{y} = u (\nu \operatorname{cos.} \Phi + \mu \operatorname{sin.} \Phi), \text{ ob}$$
$$\mu = \operatorname{sin.} r \operatorname{sin.} \Omega - \operatorname{cos.} \iota \operatorname{cos.} r \operatorname{cos.} \Omega \text{ et}$$
$$\nu = \operatorname{sin.} r \operatorname{cos.} \Omega + \operatorname{cos.} \iota \operatorname{cos.} r \operatorname{sin.} \Omega \text{ erit}$$
$$\nu \operatorname{cos.} \Phi + \mu \operatorname{sin.} \Phi = \operatorname{sin.} r \operatorname{cos.} (\Phi - \Omega) - \operatorname{cos.} \iota \operatorname{cos.} r \operatorname{sin.} (\Phi - \Omega)$$

vnde nostra formula reducenda erit

$$\nu \mathfrak{x} + \mu \mathfrak{y} = u (\operatorname{sin.} r \operatorname{cos.} (\Phi - \Omega) - \operatorname{cos.} \iota \operatorname{cos.} r \operatorname{sin.} (\Phi - \Omega)).$$

§. 27. In tertia tandem aequatione occurrit:

$$\mathfrak{x} \operatorname{sin.} \Omega - \mathfrak{y} \operatorname{cos.} \Omega = u (\operatorname{cos.} \Phi \operatorname{sin.} \Omega - \operatorname{sin.} \Phi \operatorname{cos.} \Omega)$$

quae manifesto abit in $- u \operatorname{sin.} (\Phi - \Omega)$, ita vt sit

$$\mathfrak{x} \operatorname{sin.} \Omega - \mathfrak{y} \operatorname{cos.} \Omega = - u \operatorname{sin.} (\Phi - \Omega).$$

§. 28. Quodsi ergo breuitatis gratia ponamus $\Phi - \Omega = \psi$, ternae formulae hic reductae erunt:

$$m \mathfrak{x} + n \mathfrak{y} = u (\operatorname{cos.} r \operatorname{cos.} \psi + \operatorname{cos.} \iota \operatorname{sin.} r \operatorname{sin.} \psi)$$
$$\nu \mathfrak{x} + \mu \mathfrak{y} = u (\operatorname{sin.} r \operatorname{cos.} \psi - \operatorname{cos.} \iota \operatorname{cos.} r \operatorname{sin.} \psi)$$
$$\mathfrak{x} \operatorname{sin.} \Omega - \mathfrak{y} \operatorname{cos.} \Omega = - u \operatorname{sin.} \psi.$$

Cum igitur porro sit

$$\operatorname{cos.} r \operatorname{cos.} \psi = \tfrac{1}{2} \operatorname{cos.} (r - \psi) + \tfrac{1}{2} \operatorname{cos.} (r + \psi) \text{ et}$$
$$\operatorname{sin.} r \operatorname{sin.} \psi = \tfrac{1}{2} \operatorname{cos.} (r - \psi) - \tfrac{1}{2} \operatorname{cos.} (r + \psi),$$

tum vero

$$\operatorname{sin.} r \operatorname{cos.} \psi = \tfrac{1}{2} \operatorname{sin.} (r - \psi) + \tfrac{1}{2} \operatorname{sin.} (r + \psi), \text{ et}$$
$$\operatorname{cos.} r \operatorname{sin.} \psi = - \tfrac{1}{2} \operatorname{sin.} (r - \psi) + \tfrac{1}{2} \operatorname{sin.} (r + \psi)$$

his valoribus substitutis binae formulae priores fient

$$m \mathfrak{x} + n \mathfrak{y} = \tfrac{1}{2} u (1 + \operatorname{cos.} \iota) \operatorname{cos.} (r - \psi) + \tfrac{1}{2} u (1 - \operatorname{cos.} \iota) \operatorname{cos.} (r + \psi)$$
$$\nu \mathfrak{x} + \mu \mathfrak{y} = \tfrac{1}{2} u (1 + \operatorname{cos.} \iota) \operatorname{sin.} (r - \psi) + \tfrac{1}{2} u (1 - \operatorname{cos.} \iota) \operatorname{sin.} (r + \psi).$$

§. 29. Quia denique est

$$\frac{1 + \operatorname{cos.} \iota}{2} = \operatorname{cos.} \tfrac{1}{2} \iota^2 \text{ et } \frac{1 - \operatorname{cos.} \iota}{2} = \operatorname{sin.} \tfrac{1}{2} \iota^2$$

 tres

tres nostrae formulae reductae ita succincte exprimentur:

$$\text{I. } m\mathfrak{x}+n\mathfrak{y}=u\,\text{cof.}\tfrac{\iota^2}{2}\,\text{cof.}(r-\psi)+u\,\text{fin.}\tfrac{\iota^2}{2}\,\text{cof.}(r+\psi)$$

$$\text{II. } \nu\mathfrak{x}+\mu\mathfrak{y}=u\,\text{cof.}\tfrac{\iota^2}{2}\,\text{fin.}(r-\psi)+u\,\text{fin.}\tfrac{\iota^2}{2}\,\text{fin.}(r+\psi)$$

$$\text{III. } \mathfrak{x}\,\text{fin.}\,\text{☊}-\mathfrak{y}\,\text{cof.}\,\text{☊}=-u\,\text{fin.}\,\psi.$$

§. 30. Praeterea vero etiam literae X et Y insunt in valore $w^2=uu-2u\,\text{X}\,\text{cof.}\,\Phi-2u\,\text{Y}\,\text{fin.}\,\Phi+vv$; quare, si earum loco valores supra dati substituantur, prodibit:

$$\begin{aligned}\text{X}\,\text{cof.}\,\Phi+\text{Y}\,\text{fin.}\,\Phi=&\,m(1+x)\,\text{cof.}\,\Phi-\nu y\,\text{cof.}\,\Phi+z\,\text{fin.}\,\iota\,\text{fin.}\,\text{☊}\,\text{cof.}\,\Phi\\&+n(1+x)\,\text{fin.}\,\Phi-\mu y\,\text{fin.}\,\Phi-z\,\text{fin.}\,\iota\,\text{cof.}\,\text{☊}\,\text{fin.}\,\Phi\end{aligned}$$

at vero erit ex praecedentibus reductionibus

$$\begin{aligned}m\,\text{cof.}\,\Phi+n\,\text{fin.}\,\Phi=&\,\text{cof.}\,r\,\text{cof.}\,\psi+\text{cof.}\,\iota\,\text{fin.}\,r\,\text{fin.}\,\psi=\\&\text{cof.}\tfrac{\iota^2}{2}\,\text{cof.}(r-\psi)+\text{fin.}\tfrac{\iota^2}{2}\,\text{cof.}(r+\psi).\end{aligned}$$

Eodem modo pro terminis y et z habebimus

$$\nu\,\text{cof.}\,\Phi+\mu\,\text{fin.}\,\Phi=\text{cof.}\tfrac{\iota^2}{2}\,\text{fin.}(r-\psi)+\text{fin.}\tfrac{\iota^2}{2}\,\text{fin.}(r+\psi) \text{ et}$$

$$\text{fin.}\,\text{☊}\,\text{cof.}\,\Phi-\text{cof.}\,\text{☊}\,\text{fin.}\,\Phi=-\text{fin.}(\Phi-\text{☊})=-\text{fin.}\,\psi$$

quibus valoribus substitutis erit

$$\begin{aligned}\text{X}\,\text{cof.}\,\Phi+\text{Y}\,\text{fin.}\,\Phi=&\,(1+x)\big(\text{cof.}\tfrac{\iota^2}{2}\,\text{cof.}(r-\psi)+\text{fin.}\tfrac{\iota^2}{2}\,\text{cof.}(r+\psi)\big)\\&-y\big(\text{cof.}\tfrac{\iota^2}{2}\,\text{fin.}(r-\psi)+\text{fin.}\tfrac{\iota^2}{2}\,\text{fin.}(r+\psi)\big)\\&-z\,\text{fin.}\,\iota\,\text{fin.}\,\psi\end{aligned}$$

§. 31. Quo has formulas ad calculum adhuc commodiores reddamus, ponamus breuitatis gratia

$$\text{A}=\text{cof.}\tfrac{\iota^2}{2}\,\text{cof.}(r-\psi)+\text{fin.}\tfrac{\iota^2}{2}\,\text{cof.}(r+\psi) \text{ et}$$

$$\text{B}=\text{cof.}\tfrac{\iota^2}{2}\,\text{fin.}(r-\psi)+\text{fin.}\tfrac{\iota^2}{2}\,\text{fin.}(r+\psi)$$

eruntque omnes nostrae formulae reductae

$$m\mathfrak{x}+n\mathfrak{y}=\text{A}u,\quad \nu\mathfrak{x}+\mu\mathfrak{y}=\text{B}u,\quad \mathfrak{x}\,\text{fin.}\,\text{☊}-\mathfrak{y}\,\text{cof.}\,\text{☊}=-u\,\text{fin.}\,\psi$$

$$\text{X}\,\text{cof.}\,\Phi+\text{Y}\,\text{fin.}\,\Phi=\text{A}(1+x)-\text{B}y-z\,\text{fin.}\,\iota\,\text{fin.}\,\psi$$

sicque tandem habebimus

$$w^2=uu-2u(\text{A}(1+x)-\text{B}y-z\,\text{fin.}\,\iota\,\text{fin.}\,\psi)+vv.$$

SECTIO

SECTIO V.

De euolutione formulae $\frac{1}{v^3}$.

§. 32. Cum fit $vv = (1+x)^2 + yy + zz$, erit

$$\frac{1}{v^3} = ((1+x)^2 + yy + zz)^{-\frac{3}{2}}$$

vnde, quia quantitates yy et zz funt tam paruae refpectu primae partis $(1+x)^2$, vt earum quadrata et altiores poteftates fine errore negligi queant, erit pro noftro inftituto fatis exacte:

$$\frac{1}{v^3} = \frac{1}{(1+x)^3} - \frac{3(yy+zz)}{2(1+x)^4}.$$

§. 33. Cum vero etiam quantitas x fit valde parua prae vnitate, erit

$$\frac{1}{(1+x)^3} = 1 - 3x + 6xx - 10x^3 \text{ atque } \frac{1}{(1+x)^4} = 1 - 4x;$$

hi ergo valores fubftituantur et fecundum dimenfiones quantitatum exiguarum x, y, z disponantur, hocque facto obtinebitur fequens expreffio:

$$\frac{1}{v^3} = 1 - 3x - \tfrac{3}{2}yy - \tfrac{3}{2}zz + 6xx - 10x^3 + 6xyy + 6xzz$$

vbi vltimum membrum tres continens dimenfiones iam tam exiguum deprehendetur, vt tuto negligi queat.

SECTIO VI.

De euolutione formulae $\frac{1}{w^3}$.

§. 34. Cum, vti inuenimus, fit

$$ww = uu - 2u(A(1+x) - By - z \sin. \iota \sin. \psi) + vv$$

ante omnia obferuandum eft, literam u, quae diftantiam Solis a Terra defignat, prae vnitate, qua diftantia media Lunae a Terra exprimitur, effe valde magnam, fiquidem propemodum erit $u = 400$; tum vero etiam nouimus literas x, y, z denotare fractiones prae vnitate valde paruas, vnde postremum membrum erit $vv = 1 + 2x + xx + yy + zz$.

§. 35. His perpensis si breuitatis gratia statuatur

$$\Pi = A(1 + x) - By - z \sin. \iota \sin. \psi$$

vt habeamus

$$ww = uu - 2\Pi u + vv$$

vnde fit

$$\frac{1}{w^3} = (uu - 2\Pi u + vv)^{-\frac{3}{2}}$$

hinc euoluendo prodit:

$$\frac{1}{w^3} = \frac{1}{u^3} + \frac{3\Pi}{u^4} - \frac{3vv}{2u^5} + \frac{15\Pi\Pi}{2u^5}$$

neque vero opus erit hanc seriem vlterius continuare.

§. 36. Disponamus nunc membra huius expressionis secundum dimensiones quantitatum exiguarum x, y, z, eritque $\Pi = A + Ax - By - z \sin. \iota \sin. \psi$, pro membro secundo; pro membro autem tertio, quia diuisum est per u^5, termini x, y, z inuoluentes tuto omitti poterunt, vnde erit $vv = 1$ et $\Pi\Pi = AA$. His igitur substitutis reductio ita se habebit:

$$\frac{1}{w^3} = \frac{1}{u^3} + \frac{3A}{u^4} + \frac{3(Ax - By - z \sin. \iota \sin. \psi)}{u^4} - \frac{3 + 15AA}{2u^5}.$$

§. 37. Hinc igitur iam poterimus valores litterarum F et G, sublata omni irrationalitate, sequenti modo exhibere:

I°. $F = \lambda\lambda - 3\lambda\lambda x + 6\lambda\lambda xx - \frac{3}{2}\lambda\lambda yy - \frac{3}{2}\lambda\lambda zz$
$- 10\lambda\lambda x^3 + 6\lambda\lambda xyy + 6\lambda\lambda xzz$
$+ \frac{a^3}{u^3} + \frac{3Aa^3}{u^4} + \frac{3a^3(Ax - By - z \sin. \iota \sin. \psi)}{u^4} - \frac{(3 - 15AA)a^3}{2u^5}.$

II. G

II. $G = \frac{3a^3 A}{u^4} + \frac{3a^3(Ax - By - z\,\mathrm{fin.}\,i\,\mathrm{fin.}\,\psi)}{u^4} - \frac{a^3(3 - 15AA)}{2u^5}$

vbi meminiſſe oportet eſſe

$$A = \mathrm{cof.}\,\tfrac{i^2}{2}\,\mathrm{cof.}\,(r - \psi) + \mathrm{fin.}\,\tfrac{i^2}{2}\,\mathrm{cof.}\,(r + \psi) \text{ et}$$

$$B = \mathrm{cof.}\,\tfrac{i^2}{2}\,\mathrm{fin.}\,(r - \psi) + \mathrm{fin.}\,\tfrac{i^2}{2}\,\mathrm{fin.}\,(r + \psi).$$

SECTIO VII.

De eliminatione literarum u et ϕ.

§. 38. Iam diximus, literam x denotare diſtantiam Solis a Terra, et angulum t anomaliam Solis mediam, vnde ſi excentricitas orbitae ſolaris vocetur $= \varepsilon$, cuius valor circiter eſt $\frac{1}{59}$, ex theoria Planetarum notum eſt fore diſtantiam Solis a Terra veram $u = a(1 + \varepsilon\,\mathrm{cof.}\,t)$. Tum vero ſi ζ denotet longitudinem Solis mediam, erit eius longitudo vera ſatis exacte $\phi = \zeta - 2\varepsilon\,\mathrm{fin}\,t$. His notatis literam u eliminabimus ope harum formularum:

$$\frac{1}{uu} = \frac{1 - 2\varepsilon\,\mathrm{cof.}\,t}{aa}, \quad \frac{1}{u^3} = \frac{1 - 3\varepsilon\,\mathrm{cof.}\,t}{a^3},$$

$$\frac{1}{u^4} = \frac{1 - 4\varepsilon\,\mathrm{cof.}\,t}{a^4}, \quad \frac{1}{u^5} = \frac{1 - 5\varepsilon\,\mathrm{cof.}\,t}{a^5}.$$

§. 39. Primo autem litera u occurrebat in valore $F = \frac{\lambda\lambda}{v^3} + \frac{a^3}{w^3}$ cuius pars poſterior $\frac{a^3}{w^3}$ ita reducetur, vt ſit:

$$\frac{a^3}{w^3} = 1 + \frac{3A}{a} + \frac{3Ax}{a} - \frac{3By}{a} - \frac{3Z\,\mathrm{fin.}\,i\,\mathrm{fin.}\,\psi}{a} - \frac{3}{2aa} + \frac{15AA}{2aa}$$

$$- 3\varepsilon\,\mathrm{cof.}\,t - \frac{12A\varepsilon\,\mathrm{cof.}\,t}{a} - \frac{12A\varepsilon x\,\mathrm{cof.}\,t}{a} + \frac{12B\varepsilon y\,\mathrm{cof.}\,t}{a}$$

$$+ \frac{12\varepsilon z\,\mathrm{fin.}\,i\,\mathrm{fin.}\,\psi\,\mathrm{cof.}\,t}{a} + \frac{15\varepsilon\,\mathrm{cof.}\,t}{2aa} - \frac{75AA\varepsilon\,\mathrm{cof.}\,t}{2aa}.$$

Hic autem pro noſtro inſtituto non ſolum terminos per aa diuiſos, ſed etiam eos qui continent $\frac{\varepsilon}{a}$, tuto ob paruitatem negligere licet, ita vt ſit:

F =

$$F = \lambda\lambda - 3\lambda\lambda x + 6\lambda\lambda xx - \tfrac{3}{2}\lambda\lambda yy - \tfrac{3}{2}\lambda\lambda zz$$
$$- 10\lambda\lambda x^3 + 6\lambda\lambda xyy + 6\lambda\lambda xzz$$
$$+ 1 + \frac{3A}{a} + \frac{3Ax}{a} - \frac{3By}{a} - \frac{3z\,\text{fin.}\,\iota\,\text{fin.}\,\psi}{a} - 3\varepsilon\,\text{cof.}\,t.$$

Vbi partes posteriores per a diuisas, pariter ac terminum litera ε affectum, probe a reliquis distingui conueniet.

§. 40. Cum deinde sit $G = \frac{a^3}{w^3} - \frac{a^3}{u^3}$, factis iisdem substitutionibus et praetermissis terminis tam per $\frac{1}{aa}$ quam per $\frac{\varepsilon}{a}$ affectis quia litera G vbique multiplicatur per u, tum vero pro prima aequatione fit:

$$G(m\mathfrak{x} + n\mathfrak{y}) = AGu$$

pro secunda vero

$$G(\nu\mathfrak{x} + \mu\mathfrak{y}) = BGu$$

et pro tertia

$$G\,\text{fin.}\,\iota\,(\mathfrak{x}\,\text{fin.}\,\Omega - \mathfrak{y}\,\text{cof.}\,\Omega) = -Gu\,\text{fin.}\,\iota\,\text{fin.}\,\psi\,)$$

habebitur:

$$Gu = 3A + 3Ax - 3By - 3z\,\text{fin.}\,\iota\,\text{fin.}\,\psi - \frac{3}{2a} + \frac{15AA}{2a}$$
$$- 9A\varepsilon x\,\text{cof.}\,t + 9B\varepsilon y\,\text{cof.}\,t + 9\varepsilon z\,\text{fin.}\,\iota\,\text{fin.}\,\psi\,\text{cof.}\,t.$$

§. 41. Vt autem etiam angulum ϕ ex formulis nostris elidamus, cum sit

$\phi = \zeta - 2\varepsilon\,\text{fin.}\,t$ erit $\psi = \zeta - \Omega - 2\varepsilon\,\text{fin.}\,t$;

breuitatis gratia autem ponamus $\zeta - \Omega = \eta$, ita vt η denotet distantiam loci medii Solis a nodo ascendente, sicque angulus hic η etiam tempori sit proportionalis. Quare cum sit $\psi = \eta - 2\varepsilon\,\text{fin.}\,t$, vbi pars posterior tanquam angulus vehementer paruus spectari potest; cuius sinus ipsi $2\varepsilon\,\text{fin.}\,t$, cosinus vero vnitati aequalis censeri queat, hoc obseruato erit

$$\text{fin.}\,\psi = \text{fin.}\,\eta - 2\varepsilon\,\text{fin.}\,t\,\text{cof.}\,\eta \text{ et}$$
$$\text{cof.}\,\psi = \text{cof.}\,\eta + 2\varepsilon\,\text{fin.}\,t\,\text{fin.}\,\eta$$

pro-

pro angulis autem $r - \psi$ et $r + \psi$ habebimus:

$$\text{fin.}\,(r - \psi) = \text{fin.}\,(r - \eta) + 2\,\varepsilon\,\text{fin.}\,t\,\text{cof.}\,(r - \eta)$$
$$\text{cof.}\,(r - \psi) = \text{cof.}\,(r - \eta) - 2\,\varepsilon\,\text{fin.}\,t\,\text{fin.}\,(r - \eta)$$
$$\text{fin.}\,(r + \psi) = \text{fin.}\,(r + \eta) - 2\,\varepsilon\,\text{fin.}\,t\,\text{cof.}\,(r + \eta)$$
$$\text{cof.}\,(r + \psi) = \text{cof.}\,(r + \eta) + 2\,\varepsilon\,\text{fin.}\,t\,\text{fin.}\,(r + \eta).$$

§. 42. Hinc igitur ambas literas A et B ad institutum nostrum magis accommodatas exprimere poterimus, quibus per analogiam adiungamus pro tertia aequatione literam $C = \text{fin.}\,\iota\,\text{fin.}\,\psi$. Nanciscemur igitur hos valores:

$$A = \text{cof.}\,\tfrac{\iota^2}{2}\,\text{cof.}\,(r - \eta) + \text{fin.}\,\tfrac{\iota^2}{2}\,\text{cof.}\,(r + \eta)$$
$$- 2\,\varepsilon\,\text{cof.}\,\tfrac{\iota^2}{2}\,\text{fin.}\,t\,\text{fin.}\,(r - \eta) + 2\,\varepsilon\,\text{fin.}\,\tfrac{\iota^2}{2}\,\text{fin.}\,t\,\text{fin.}\,(r + \eta)$$

$$B = \text{cof.}\,\tfrac{\iota^2}{2}\,\text{fin.}\,(r - \eta) + \text{fin.}\,\tfrac{\iota^2}{2}\,\text{fin.}\,(r + \eta)$$
$$+ 2\,\varepsilon\,\text{cof.}\,\tfrac{\iota^2}{2}\,\text{fin.}\,t\,\text{cof.}\,(r - \eta) - 2\,\varepsilon\,\text{fin.}\,\tfrac{\iota^2}{2}\,\text{fin.}\,t\,\text{cof.}\,(r + \eta)$$

$$C = \text{fin.}\,\iota\,\text{fin.}\,\eta - 2\,\varepsilon\,\text{fin.}\,\iota\,\text{fin.}\,t\,\text{cof.}\,\eta.$$

§. 43. In his formulis potissimum occurrit angulus $r - \eta$, qui reperitur, si ab argumento latitudinis Lunae r subtrahatur distantia Solis a Nodo media $\eta = \zeta - ☊$. Cum igitur angulus r reperiatur, si a loco Lunae medio in orbita θ subtrahatur locus nodi ☊, vt sit $r = \theta - ☊$, fiet iste angulus $r - \eta = \theta - \zeta$, qui ergo habebitur, si a loco lunae medio θ subtrahatur longitudo Solis media ζ. Ponamus igitur breuitatis gr. hunc angulum $r - \eta = \theta - \zeta = p$, eritque $\eta = r - p$ et hinc $r + \eta = 2\,r - p$. Porro sit etiam breuitatis ergo $\text{cof.}\,\tfrac{\iota^2}{2} = \mu$ et $\text{fin.}\,\tfrac{\iota^2}{2} = \nu$, ita vt fit $\mu + \nu = 1$ et $\mu - \nu = \text{cof.}\,\iota$; vbi notasse iuuabit, ob angulum ι satis exiguum, fore $\mu = 1$ et ν fractionem valde paruam. His igitur denominationibus introductis habebimus:

A =

$$A = \mu \text{ cof. } p + \nu \text{ cof. } (2r - p) - 2\mu\varepsilon \text{ fin. } t \text{ fin. } p$$
$$+ 2\nu\varepsilon \text{ fin. } t \text{ fin. } (2r - p)$$

$$B = \mu \text{ fin. } p + \nu \text{ fin. } (2r - p) + 2\mu\varepsilon \text{ fin. } t \text{ cof. } p$$
$$- 2\nu\varepsilon \text{ fin. } t \text{ cof. } (2r - p)$$

$$C = \text{fin. } \iota \text{ fin. } (r - p) - 2\varepsilon \text{ fin. } \iota \text{ fin. } t \text{ cof. } (r - p).$$

§. 44. Quia in formulis F et Gu adhuc occurrit fin. ψ, eius loco fubftituatur valor fupra inuentus

$$\text{fin. } \psi = \text{fin. } (r - p) - 2\varepsilon \text{ fin. } t \text{ cof. } (r - p)$$

quo valore fubftituto obtinebimus fcribendo literam C loco fin. ι fin. ψ:

$$F = \lambda\lambda - 3\lambda\lambda x + 6\lambda\lambda xx - \tfrac{3}{2}\lambda\lambda yy - \tfrac{3}{2}\lambda\lambda zz$$
$$- 10\lambda\lambda x^3 + 6\lambda\lambda xyy + 6\lambda\lambda xzz$$
$$+ 1 + \frac{3A}{a} + \frac{3Ax}{a} - \frac{3By}{a} - \frac{3Cz}{a}$$
$$- 3\varepsilon \text{ cof. } t$$

$$Gu = 3A + 3Ax - 3By - 3Cz - \frac{3}{2a} + \frac{15AA}{2a}$$
$$- 9A\varepsilon x \text{ cof. } t + 9B\varepsilon y \text{ cof. } t + 9C\varepsilon z \text{ cof. } t$$

hocque modo omnes quantitates peregrinas ex calculo expulimus, ita vt praeter ternas noftras incognitas x, y, z aliae quantitates variabiles cognitae non occurrant praeter ternos angulos p, r et t.

§. 45. His igitur valoribus ita definitis tres noftrae aequationes principales pro motu Lunae fequenti modo referentur:

$$\text{I.}\quad \left.\begin{array}{l} ddx - 2dy(dr + d\Omega \text{ cof. } \iota) + 2dz\,d\Omega \text{ fin. } \iota \text{ cof. } r \\ -(1+x)(dr^2 + 2dr\,d\Omega \text{ cof. } \iota + d\Omega^2(1 - \text{fin. } \iota^2 \text{ fin. } r^2)) \\ + y\,d\Omega^2 \text{ fin. } \iota^2 \text{ fin. } r \text{ cof. } r + z\,d\Omega^2 \text{ fin. } \iota \text{ cof. } \iota \text{ fin. } r \end{array}\right\}$$
$$= -F(1+x)dt^2 + A\,Gu\,dt^2$$

II.

$$\text{II.}\quad \left.\begin{aligned} & ddy + 2dx(dr + d\Omega \,\text{cof.}\, \iota) - 2 dz d\Omega \,\text{fin.}\, \iota \,\text{fin.}\, r \\ & - y(dr^2 + 2 dr d\Omega \,\text{cof.}\, \iota + d\Omega^2 (1 - \text{fin.}\, \iota^2 \,\text{cof.}\, r^2)) \\ & + (1+x) d\Omega^2 \,\text{fin.}\, \iota^2 \,\text{fin.}\, r \,\text{cof.}\, r + z\, d\Omega^2 \,\text{fin.}\, \iota \,\text{cof.}\, \iota \,\text{cof.}\, r \\ & \quad = - F y dt^2 - B G u dt^2 \end{aligned}\right\}$$

$$\text{III.}\quad \left.\begin{aligned} & ddz - 2dx . d\Omega \,\text{fin.}\, \iota \,\text{cof.}\, r + 2 dy . d\Omega \,\text{fin.}\, \iota \,\text{fin.}\, r - z d\Omega^2 \,\text{fin.}\, \iota^2 \\ & + (1+x)(2 dr d\Omega \,\text{fin.}\, \iota \,\text{fin.}\, r + d\Omega^2 \,\text{fin.}\, \iota \,\text{cof.}\, \iota \,\text{fin.}\, r) \\ & + y(2 dr d\Omega \,\text{fin.}\, \iota \,\text{cof.}\, r + d\Omega^2 \,\text{fin.}\, \iota \,\text{cof.}\, \iota \,\text{cof.}\, r) \\ & \quad = - F z dt^2 - C G u dt^2. \end{aligned}\right\}$$

SECTIO VIII.

De reductione differentialium ad elementum anomaliae mediae Solis dt.

§. 46. Cum omnes anguli, qui in noſtras aequationes ingrediuntur, ſint tempori proportionales, eorum differentialia ad elementum temporis dt, ex anomalia media Solis deſumtum, datas tenebunt rationes, quas ex tabulis mediorum motuum Lunae et Solis depromere licet. Quanquam enim motus lineae nodorum ex ipſa Theoria definiri poteſt: tamen iam ſatis eſt euictum, motum nodorum ex obſeruationibus concluſum perfecte cum Theoria conſentire; quam ob rem ſtatuamus vt ſequitur: $dr = l\,dt$, $dp = m\,dt$, ita vt hi valores l et m ex tabulis colligi queant; vbi quidem, quia nulli logarithmi vsquam occurrent, litera l ſine ambiguitate vtor. Deinde quia linea nodorum in antecedentia promouetur, valor rationis $\frac{d\Omega}{dt}$ erit negatiuus, quam ob rem ſtatuamus

$$d\Omega \,\text{cof.}\, \iota = - \alpha\, dt \quad \text{et} \quad d\Omega \,\text{fin.}\, \iota = - \beta\, dt$$

vbi manifeſtum eſt literam β multo minorem eſſe quam α.

§. 47. His igitur rationibus mere numericis constitutis erit pro nostris binis prioribus aequationibus

$$dr + d\Omega \cos. \iota = (l - \alpha)\, dt,$$

deinde vero quia

$$dr^2 + 2\, dr\, d\Omega \cos. \iota + d\Omega^2 = (dr + d\Omega \cos. \iota)^2 + d\Omega^2 \sin. \iota^2$$

erit haec formula

$$dr^2 + 2\, dr\, d\Omega \cos. \iota + d\Omega^2 = (l - \alpha)^2\, dt^2 + \beta\beta\, dt^2.$$

Quare si nostras aequationes per dt^2 diuidamus, eae sequentes induent formas:

$$\text{I.}\ \left.\begin{array}{l} \frac{ddx}{dt^2} - \frac{2dy}{dt}(l-\alpha) - 2\beta\frac{dz}{dt}\cos. r \\ -(1+x)((l-\alpha)^2 + \beta\beta \cos. r^2) \\ + \beta\beta y \sin. r \cos. r + \alpha\beta z \sin. r \end{array}\right\} = -F(1+x) + AGu$$

$$\text{II.}\ \left.\begin{array}{l} \frac{ddy}{dt^2} + \frac{2dx}{dt}(l-\alpha) + 2\beta\frac{dz}{dt}\sin. r \\ -y((l-\alpha)^2 + \beta\beta \sin. r^2) \\ + \beta\beta(1+x)\sin. r \cos. r + \alpha\beta z \cos. r \end{array}\right\} = -Fy - BGu$$

$$\text{III.}\ \left.\begin{array}{l} \frac{ddz}{dt^2} + 2\beta\frac{dx}{dt}\cos. r - 2\beta\frac{dy}{dt}\sin. r - \beta\beta z \\ -(1+x)(2\beta l \sin. r - \alpha\beta \sin. r) \\ -y(2\beta l \cos. r - \alpha\beta \cos. r) \end{array}\right\} = -Fz - CGu.$$

§. 48. Totum ergo negotium iam huc est reductum, vt tres istae aequationes differentio-differentiales per duplicem integrationem rite resoluantur, vnde pro quouis tempore proposito valores trium nostrarum incognitarum x, y, z assignari queant. Ante autem quam hunc laborem suscipiamus, operae pretium erit ostendere, quemadmodum ex inuentis quantitatibus x, y, z verum Lunae locum in coelo determinare oporteat.

Tab. XIII. Fig. 5. §. 49. Referat igitur tabula planum orbitae lunaris mediae pro tempore proposito, in quo sit T centrum Terrae

Terrae et recta T☊ linea nodorum, cuius longitudo in ecliptica ſit $=$ ☊; tum vero in eodem plano ducatur recta TM ad locum Lunae medium in ſua orbita, ita vt ſit angulus ☊TM $= r$, locus autem Lunae verus ſit ſupra hoc planum in z; vnde demiſſo perpendiculo zy, et yx ad TM normali, erit vti ſtatuimus: $Tx = 1 + x$, $xy = y$ et $yz = z$ quos ergo valores vt cognitos ſpectabimus. Hinc ductis rectis Ty et Tz vocentur anguli $xTy = \rho$ et $yTz = \omega$, eritque tang. $\rho = \frac{y}{1+x}$, hincque diſtantia $Ty = (1+x)$ ſec. ρ; vnde porro reperitur tang. $\omega = \frac{z}{1+x}$ ſec. ρ; ac denique vera diſtantia Lunae a Terra $Tz = Ty$ ſec. ω, cui parallaxis Lunae horizontalis reciproce eſt proportionalis. His angulis definitis erit argumentum Lunae verum, ſeu angulus ☊$Ty = r + \rho$ et declinatio Lunae ab hoc plano $= \omega$.

§. 50. Transferamus nunc has determinationes in coelum, in quo ſit circulus maximus ♈☊p, in eoque punctum ♈ aequinoxium vernum et ſignum ☊ nodus Lunae aſcendens; vnde ducatur arcus ☊y cum ecliptica faciens angulum p☊$y = \iota$, inclinationi conſtanti aequalem, et capiatur arcus ☊$y = r + \rho$; tum vero fiat arculus $yz = \omega$, eritque z verus Lunae locus in coelo ex centro Terrae viſus. Vnde ſi ad eclipticam normaliter ducatur arcus zq hic exhibebit latitudinem Lunae veram; longitudo vero eius vera erit arcus ♈☊q = ☊ + ☊q. Tab. XIII. Fig. 6.

§. 51. Quo autem iſte calculus facilior reddatur, primo ex y ad eclipticam demittatur arcus yp; vnde, ſi more ſolito pro angulo inclinationis ι conſtruatur tabula, pro ſingulis argumentis latitudinis exhibens tam reductionem ad eclipticam quam latitudinem, ex ea reperietur arcus ☊p, ac deinde arcus py, quibus inuentis, quia arculus $yz = \omega$ ſemper eſt quam minimus, propterea quod Luna ab orbita

 media

media nunquam notabiliter deflectere potest, ex hoc arculo $zy = \omega$ facile colligentur correctionas in longitudine oriundae. Hunc in finem ducatur ex y ad zq perpendiculum yr et quia in triangulo zyr habetur latus $yz = \omega$ et angulus $zyr =$ angulo ☊yp, qui angulus si vocetur $= \sigma$, erit $yr = \omega \operatorname{cos}. \sigma$ et $zr = \omega \operatorname{sin}. \sigma$, quae posterior particula zr manifesto dabit correctionem arcus yp, quippe quae ad yp addita dabit arcum zq seu latitudinem Lunae veram. Deinde vero pro longitudine notetur esse $pq : yr = 1 : \operatorname{cos}. yp$, vnde fit $pq = \frac{\omega \operatorname{cos}. \sigma}{\operatorname{cos}. yp}$, sicque particula pq ab arcu ☊p subtracta dabit arcum ☊q, vnde prodibit longitudo vera ☊ + ☊q.

§. 52. Quia angulum ☊yp vocauimus $= \sigma$, ex trigometricis constat fore:

$$\operatorname{tang}. \iota \operatorname{tang}. \sigma = \operatorname{sec}. ☊ y = \frac{1}{\operatorname{cos}. (r + \varrho)}, \text{ ideoque}$$

$$\operatorname{tang}. \sigma = \frac{\operatorname{cos}. \iota}{\operatorname{sin}. \iota \operatorname{cos}. (r + \varrho)}. \text{ Deinde autem est}$$

$$\operatorname{sin}. yp = \operatorname{sin}. (r + \varrho) \operatorname{sin}. \iota, \text{ vnde fit}$$

$$\operatorname{cos}. yp = \sqrt{1 - \operatorname{sin}. \iota^2 \operatorname{sin}. (r + \varrho)^2}$$

Ex priori vero formula est

$$\operatorname{cos}. \sigma = \frac{\operatorname{sin}. \iota^2 \operatorname{cos}. (r + \varrho)}{\sqrt{\operatorname{cos}. \iota^2 + (\operatorname{sin}. \iota \operatorname{cos}. (r + \varrho)^2}}$$

vbi manifesto est

$$1 - \operatorname{sin}. \iota^2 \operatorname{sin}. (r + \varrho)^2 = \operatorname{cos}. \iota^2 + \operatorname{sin}. \iota^2 \operatorname{cos}. (r + \varrho)^2$$

hinc igitur fit

$$\frac{\operatorname{cos}. \sigma}{\operatorname{cos}. yp} = \frac{\operatorname{sin}. \iota \operatorname{cos}. (r + \varrho)}{1 - \operatorname{sin}. \iota^2 \operatorname{sin}. (r + \varrho)^2}$$

Praestat autem prioribus formulis vti, postquam angulus σ est exploratus.

§. 53. Cum autem haec correctio calculum non exiguum requirat, hoc negotium multo facilius expediri posse videtur, si, missa omni approximatione, calculum accurate insti-

instituamus. Scilicet ex triangulo sphaerico $☊yz$ ex datis lateribus $☊y = r + \varsigma$ et $yz = \omega$ quaeratur

$$\text{cos.}\,☊z = \text{cos.}\,☊y\,\text{cos.}\,yz \text{ et } \text{tang.}\,y☊z = \frac{\text{tang.}\,yz}{\text{sin.}\,☊y},$$

tum iste angulus $y☊z$ addatur inclinationi $q☊y = \iota$, vt habeatur angulus $q☊z$; et ex triangulo Sphaerico $☊qz$ computetur

$$\text{sin.}\,zq = \text{sin.}\,☊z\,\text{sin.}\,q☊z \text{ et } \text{tang.}\,☊q = \text{tang.}\,☊z\,\text{cos.}\,q☊z$$

quo facto statim habebitur longitudo Lunae $♈q = ♈☊ + ☊q$ et latitudo Lunae zq. Hoc igitur modo tabula illa memorata reductionum et latitudinum carere poterimus.

SEXTIO IX.

De prima appropinquatione ad motum Lunae.

§. 54. Quoniam valores nostrarum incognitarum x, y, z aliter nisi per approximationes definiri non licet, initium harum appropinquationum ita faciamus, vt primo in nostris aequationibus remoueamus terminos per a diuisos, quippe qui eas inaequalitates lunares inuoluunt, quae a Parallaxi Solis pendent et *Parallacticae* vocari solent, quippe quae sunt quam minimae. Deinde etiam excludamus omnes terminos excentricitatem Solis ε continentis, vnde nascuntur inaequalitates *Solares* dictae, quae etiam sunt vehementer paruae. Tertio vero etiam in ipso limine excentricitatem orbitae lunaris excludamus, quippe quae singularem inuestigationem postulat; ac denique etiam reiiciamus terminos, in quibus incognitae x, y, z duas pluresue dimensiones occupant, quippe qui prae reliquis sunt valde parui et in hoc negotio tanquam euanescentes spectari poterunt.

§. 55. His igitur obseruatis pro literis A, B, C hos habebimus valores:

$A = \mu \cos. p + \nu \cos. (2r - p)$; $B = \mu \sin. p + \nu \sin. (2r - p)$;
$C = \sin. \iota \sin. (r - p)$

tum vero pro formulis F & Gu his vtemur valoribus:

$F = \lambda\lambda - 3\lambda\lambda x + 1$ et $Gu = 3A + 3Ax - 3By - 3Cz$

siue

$$Gu = 3\mu(1 + x)\cos. p + 3\nu(1 + x)\cos.(2r - p) - 3\mu y \sin. p - 3\nu y \sin.(2r - p) - 3z \sin. \iota \sin.(r - p).$$

§. 56. Pro prima igitur nostra aequatione membrum ad dextram positum euoluatur, omissis terminis, vbi x ad duas dimensiones assurgeret, ac reperietur

$$F(1 + x) = 1 + \lambda\lambda + x - 2\lambda\lambda x \text{ et}$$
$$AGu = 3AA + 3AAx - 3ABy - 3ACz$$

vbi producta AA, AB, AC ad simplices cosinus reducantur. Reperietur igitur

$$AA = \mu\mu \cos. p^2 + 2\mu\nu \cos. p \cos.(2r - p) + \nu\nu \cos.(2r - p)^2 = \tfrac{1}{2}(\mu\mu + \nu\nu) + \tfrac{1}{2}\mu\mu \cos. 2p + \mu\nu \cos. 2r + \mu\nu \cos.(2r - 2p) + \tfrac{1}{2}\nu\nu \cos.(4r - 2p)$$

$$AB = \mu\mu \sin. p \cos. p + \mu\nu \sin. p \cos.(2r - p) + \mu\nu \cos. p \sin.(2r - p) + \nu\nu \sin.(2r - p)\cos.(2r - p) =$$
$$\tfrac{1}{2}\mu\mu \sin. 2p + \mu\nu \sin. 2r + \tfrac{1}{2}\nu\nu \sin.(4r - 2p)$$

$$AC = \mu \sin. \iota \cos. p \sin.(r - p) + \nu \sin. \iota \sin.(r - p)\cos.(2r - p) = \tfrac{1}{2}\mu \sin. \iota \sin. r - \tfrac{1}{2}\nu \sin. \iota \sin. r + \tfrac{1}{2}\mu \sin. \iota \sin.(r - 2p) + \tfrac{1}{2}\nu \sin. \iota \sin.(3r - 2p).$$

Hinc igitur pro prima nostra aequatione colligitur.

Pars dextra.

$$-1 - \lambda\lambda + \tfrac{3}{2}(\mu\mu + \nu\nu) + \tfrac{3}{2}\mu\mu \cos. 2p + 3\mu\nu \cos. 2r + 3\mu\nu \cos.(2r - 2p) + \tfrac{3}{2}\nu\nu \cos.(4r - 2p) - x$$

$$-x(1-2\lambda\lambda)+\tfrac{1}{2}(\mu\mu+\nu\nu)x+\tfrac{3}{2}\mu\mu x \cos. 2p+3\mu\nu x \cos. 2r$$
$$+3\mu\nu x \cos.(2r-2p)+\tfrac{3}{2}\nu\nu x \cos.(4r-2p)$$
$$-\tfrac{3}{2}\mu\mu y \sin. 2p-3\mu\nu y \sin. 2r-\nu\nu y \sin.(4r-2p)$$
$$-\tfrac{3}{2}\mu z \sin. i \sin. r+\tfrac{3}{2}\nu z \sin. i \sin. r-\tfrac{3}{2}\mu z \sin. i \sin.(r-2p)$$
$$-\tfrac{3}{2}\nu z \sin. i \sin.(3r-2p)$$

§. 57. Pro parte autem finiſtra eiusdem aequationis vtamur pariter iisdem reductionibus, ſcilicet $\sin. r^2=\tfrac{1}{2}-\tfrac{1}{2}\cos. 2r$, $\cos. r^2=\tfrac{1}{2}+\tfrac{1}{2}\cos. 2r$ et $\sin. r \cos. r=\tfrac{1}{2}\sin. 2r$; tum vero terminos abſolutos, qui nullas incognitas inuoluunt, a reliquis ſeparemus, ſicque pars ſiniſtra hoc modo repraeſentabitur:

$$-(l-\alpha)^2-\tfrac{1}{2}\beta\beta-\tfrac{1}{2}\beta\beta \cos. 2r$$
$$+\frac{ddx}{dt^2}-\frac{2dy}{dt}(l-\alpha)-2\beta\frac{dz}{dt}\cos. r$$
$$-(l-\alpha)^2 x-\tfrac{1}{2}\beta\beta x-\tfrac{1}{2}\beta\beta x \cos. 2r+\tfrac{1}{2}\beta\beta y \sin. 2r+\alpha\beta z \sin. r.$$

Hic vero ante omnia eſt obſeruandum, membra conſtantia, quae in vtraque parte reperiuntur, ſe mutuo ſeorſim deſtruere debere, quia alioquin diſtantia media non amplius foret $=1$, neque motus medius rite definitus; vnde ex hac conditione ſtatim colligimus hanc aequalitatem:

$$-(l-\alpha)^2-\tfrac{1}{2}\beta\beta=-1-\lambda\lambda+\tfrac{1}{2}(\mu\mu+\nu\nu), \text{ vnde fit}$$
$$\lambda\lambda=(l-\alpha)^2+\tfrac{1}{2}\beta\beta+\tfrac{1}{2}(\mu\mu+\nu\nu)-1.$$

Cum enim λ denotet celeritatem angularem, qua Luna circa Terram ad diſtantiam mediam $=1$ in circulo reuolueretur, remota perturbatione a Sole oriunda, eius valor vtique per ſe nondum eſt cognitus, ideoque eum ex hac conditione determinare neceſſe erat, quandoquidem valores literarum λ, α, β, μ et ν ex Phaenomenis ſunt cogniti. Hoc igitur valore ſubſtituto pro prima aequatione omnes terminos incognitos ad partem ſiniſtram, cognitos vero

ad

ad dextram exhibeamus, quo facto aequatio prima sequentem induet formam:

$$\frac{ddx}{dt^2} - \frac{2dy}{dt}(l-\alpha) - 2\beta\frac{dz}{dt}\,\mathrm{cof.}\,r - 3\lambda\lambda x$$
$$-\tfrac{1}{2}\beta\beta x\,\mathrm{cof.}\,2r - \tfrac{3}{2}\mu\mu x\,\mathrm{cof.}\,2p - 3\mu\nu x\,\mathrm{cof.}\,2r - 3\mu\nu x\,\mathrm{cof.}(2r-2p)$$
$$-\tfrac{3}{2}\nu\nu x\,\mathrm{cof.}(4r-2p)$$
$$+\tfrac{1}{2}\beta\beta y\,\mathrm{fin.}\,2r + \tfrac{3}{2}\mu\mu y\,\mathrm{fin.}\,2p + 3\mu\nu y\,\mathrm{fin.}\,2r$$
$$+\tfrac{3}{2}\nu\nu y\,\mathrm{fin.}(4r-2p)$$
$$+\alpha\beta z\,\mathrm{fin.}\,r + \tfrac{3}{2}\mu z\,\mathrm{fin.}\,\iota\,\mathrm{fin.}\,r - \tfrac{3}{2}\nu z\,\mathrm{fin.}\,\iota\,\mathrm{fin.}\,r$$
$$+\tfrac{3}{2}\mu z\,\mathrm{fin.}\,\iota\,\mathrm{fin.}(r-2p) + \tfrac{3}{2}\nu z\,\mathrm{fin.}\,\iota\,\mathrm{fin.}(3r-2p)$$
$$= \tfrac{1}{2}\beta\beta\,\mathrm{cof.}\,2r + 3\mu\nu\,\mathrm{cof.}\,2r + \tfrac{3}{2}\mu\mu\,\mathrm{cof.}\,2p + 3\mu\nu\,\mathrm{cof.}(2r-2p)$$
$$+\tfrac{3}{2}\nu\nu\,\mathrm{cof.}(4r-2p).$$

§. 58. Eodem modo tractemus secundam aequationem principalem, pro cuius parte dextra habebimus $Fy = y + \lambda\lambda y$ et $BGu = 3AB + 3ABx - 3BBy - 3BCz$ pro qua formula iam obseruauimus esse

$$AB = \tfrac{1}{2}\mu\mu\,\mathrm{fin.}\,2p + \mu\nu\,\mathrm{fin.}\,2r + \tfrac{1}{2}\nu\nu\,\mathrm{fin.}(4r-2p)$$

tum vero erit

$$BB = \tfrac{1}{2}(\mu\mu+\nu\nu) - \tfrac{1}{2}\mu\mu\,\mathrm{fin.}\,2p - \tfrac{1}{2}\nu\nu\,\mathrm{fin.}(4r-2p)$$
$$+\mu\nu\,\mathrm{cof.}(2r-2p) - \mu\nu\,\mathrm{cof.}\,2r$$
$$BC = \tfrac{1}{2}\mu\,\mathrm{fin.}\,\iota\,\mathrm{cof.}\,r + \tfrac{1}{2}\nu\,\mathrm{fin.}\,\iota\,\mathrm{cof.}\,r + \tfrac{1}{2}\mu\,\mathrm{fin.}\,\iota\,\mathrm{cof.}(r-2p)$$
$$-\tfrac{1}{2}\nu\,\mathrm{fin.}\,\iota\,\mathrm{cof.}(3r-2p)$$

His igitur valoribus substitutis erit

Pars dextra.

$$= \tfrac{3}{2}\mu\mu\,\mathrm{fin.}\,2p - 3\mu\nu\,\mathrm{fin.}\,2r - \tfrac{3}{2}\nu\nu\,\mathrm{fin.}(4r-2p)$$
$$-\tfrac{3}{2}\mu\mu x\,\mathrm{fin.}\,2p - 3\mu\nu x\,\mathrm{fin.}\,2r - \tfrac{3}{2}\nu\nu x\,\mathrm{fin.}(4r-2p)$$
$$-y - \lambda\lambda y + \tfrac{3}{2}(\mu\mu+\nu\nu)y - \tfrac{3}{2}\mu\mu y\,\mathrm{fin.}\,2p - \tfrac{3}{2}\nu\nu y\,\mathrm{fin.}(4r-2p)$$
$$+3\mu\nu y\,\mathrm{cof.}(2r-2p) - \mu\nu y\,\mathrm{cof.}\,2r$$
$$-\tfrac{3}{2}\mu z\,\mathrm{fin.}\,\iota\,\mathrm{cof.}\,r + \tfrac{3}{2}\nu z\,\mathrm{fin.}\,\iota\,\mathrm{cof.}\,r + \tfrac{3}{2}\mu z\,\mathrm{fin.}\,\iota\,\mathrm{cof.}(r-2p)$$
$$-\tfrac{3}{2}\nu z\,\mathrm{fin.}\,\iota\,\mathrm{cof.}(3r-2p).$$

§. 59. Eiusdem aequationis pars finiftra, eodem modo quo fupra difpofita, ita referatur:

$$\tfrac{1}{2}\beta\beta \text{ fin. } 2r$$
$$+\frac{ddy}{dt^2}+\frac{2(l-\alpha)dx}{dt}+\frac{2\beta \text{ fin. } r.dz}{dt}$$
$$+\tfrac{1}{2}\beta\beta x \text{ fin. } 2r-(l-\alpha)^2 y-\tfrac{1}{2}\beta\beta y+\tfrac{1}{2}\beta\beta y \text{ cof. } 2r+\alpha\beta z \text{ cof. } r$$

quo circa, fi loco $(l-\alpha)^2$ fcribamus eius valorem

$$1+\lambda\lambda-\tfrac{3}{2}(\mu\mu+\nu\nu)-\tfrac{1}{2}\beta\beta.$$

et aequationem vt ante inftruamus, dum fcilicet omnes termini incogniti ad finiftram, cogniti vero ad dextram, difponuntur, aequatio noftra fecunda hanc induet formam:

$$\frac{ddy}{dt^2}+2(l-\alpha)\frac{dx}{dt}+2\beta \text{ fin. } r.\frac{dz}{dt}$$
$$+\tfrac{1}{2}\beta\beta x \text{ fin. } 2r+\tfrac{3}{2}\mu\mu x \text{ fin. } 2p+3\mu\nu x \text{ fin. } 2r+\tfrac{3}{2}\nu\nu x \text{ fin. } (4r-2p)$$
$$+\tfrac{1}{2}\beta\beta y \text{ cof. } 2r+\tfrac{3}{2}\mu\mu \text{ fin. } 2p+\tfrac{3}{2}\nu\nu y \text{ fin. } (4r-2p)-3\mu\nu y \text{ cof. } (2r-2p)$$
$$+\mu\nu y \text{ cof. } 2r+\alpha\beta z \text{ cof. } r+\tfrac{3}{3}\mu z \text{ fin. } \iota \text{ cof. } r-\tfrac{3}{2}\nu z \text{ fin. } \iota \text{ cof. } r$$
$$+\tfrac{3}{2}\mu z \text{ fin. } \iota \text{ cof. } (3r-2p)$$
$$+\tfrac{2}{3}\nu z \text{ fin. } \iota \text{ cof. } (3r-2p)$$
$$=-\tfrac{1}{2}\beta\beta \text{ fin. } 2r-\tfrac{3}{2}\mu\mu \text{ fin. } 2p-3\mu\nu \text{ fin. } 2r-\tfrac{3}{2}\nu\nu \text{ fin. } (4r-2p).$$

§. 60. Pro tertia denique aequatione noftra membra ad dextram partem pofita ita fe habebunt:

$$Fz = z+\lambda\lambda z \text{ et}$$
$$Gu = 3AC+3ACx-3BCy-3CCz$$

vbi notetur effe

$$BC=-\tfrac{1}{2}\mu \text{ fin. } \iota \text{ cof. } r+\tfrac{1}{2}\nu \text{ fin. } \iota \text{ cof. } r+\tfrac{1}{2}\mu \text{ fin. } \iota \text{ cof. } (r-2p)$$
$$-\tfrac{1}{2}\nu \text{ fin. } \iota \text{ cof. } (3r-2p)$$
$$AC=\tfrac{1}{2}\mu \text{ fin. } \iota \text{ fin. } r+\tfrac{1}{2}\mu \text{ fin. } \iota \text{ fin. } (r-2p)+\tfrac{1}{2}\nu \text{ fin. } \iota \text{ fin. } (3r-2p)$$
$$-\tfrac{1}{2}\nu \text{ fin. } \iota \text{ fin. } r$$
$$CC=\tfrac{1}{2}\text{ fin. } \iota-\tfrac{1}{2}\text{ fin. } \iota^2 \text{ cof. } (2r-2p)$$

quibus valoribus fubftitutis erit

 Pars

Pars dextra.

$$-\tfrac{3}{2}\mu\,\text{fin.}\,\iota\,\text{fin.}\,r-\tfrac{3}{2}\mu\,\text{fin.}\,\iota\,\text{fin.}(r-2p)-\tfrac{3}{2}\nu\,\text{fin.}\,\iota\,\text{fin.}(3r-2p)$$
$$+\tfrac{3}{2}\nu\,\text{fin.}\,\iota\,\text{fin.}\,r$$
$$-\tfrac{3}{2}\mu x\,\text{fin.}\,\iota\,\text{fin.}\,r-\tfrac{3}{2}\mu x\,\text{fin.}\,\iota\,\text{fin.}(r-2p)-\tfrac{3}{2}\nu x\,\text{fin.}\,\iota\,\text{fin.}(3r-2p)$$
$$+\tfrac{3}{2}\nu x\,\text{fin.}\,\iota\,\text{fin.}\,r$$
$$-\tfrac{3}{2}\mu y\,\text{fin.}\,\iota\,\text{cof.}\,r+\tfrac{3}{2}\nu y\,\text{fin.}\,\iota\,\text{cof.}\,r+\tfrac{3}{2}\mu y\,\text{fin.}\,\iota\,\text{cof.}(r-2p)$$
$$-\tfrac{3}{2}\nu y\,\text{fin.}\,\iota\,\text{cof.}(3r-2p)$$
$$-z-\lambda\lambda z+\tfrac{3}{2}z\,\text{fin.}\,\iota^2-\tfrac{3}{2}\,\text{fin.}\,\iota^2\,\text{cof.}(2r-2p).$$

§. 61. Eodem modo quo ante pars finiftra ita referatur:

$$-2\beta l\,\text{fin.}\,r+\alpha\beta\,\text{fin.}\,r$$
$$+\frac{ddz}{dt^2}+2\beta\,\text{cof.}\,r.\frac{dx}{dt}-2\beta\,\text{fin.}\,r\frac{dy}{dt}$$
$$+2\beta ly\,\text{cof.}\,r+\alpha\beta y\,\text{cof.}\,r-\beta\beta z$$
$$-2\beta lx\,\text{fin.}\,r+\alpha\beta x\,\text{fin.}\,r.$$

quocirca aequatio tertia modo ante tradito difpofita erit

$$\frac{ddz}{dt^2}+2\beta\,\text{cof.}\,r.\frac{dx}{dt}-2\beta\,\text{fin.}\,r.\frac{dy}{dt}$$
$$-2\beta lx\,\text{fin.}\,r+\alpha\beta x\,\text{fin.}\,r+\tfrac{3}{2}\mu x\,\text{fin.}\,\iota\,\text{fin.}\,r+\tfrac{3}{2}\mu x\,\text{fin.}\,\iota\,\text{fin.}(r-2p)$$
$$+\tfrac{3}{2}\nu x\,\text{fin.}\,\iota\,\text{fin.}(3r-2p)-\tfrac{3}{2}\nu x\,\text{fin.}\,\iota\,\text{fin.}\,r$$
$$-2\beta ly\,\text{cof.}\,r+\alpha\beta y\,\text{cof.}\,r+\tfrac{3}{2}\mu y\,\text{fin.}\,\iota\,\text{cof.}\,r-\tfrac{3}{2}\nu y\,\text{fin.}\,\iota\,\text{cof.}\,r$$
$$-\tfrac{3}{2}\mu y\,\text{fin.}\,\iota\,\text{cof.}(r-2p)+\tfrac{3}{2}\nu y\,\text{fin.}\,\iota\,\text{cof.}(3r-2p)$$
$$+z-\beta\beta z+\lambda\lambda z-\tfrac{3}{2}z\,\text{fin.}\,\iota^2+\tfrac{3}{2}z\,\text{fin.}\,\iota^2\,\text{cof.}(2r-2p)=$$
$$=2\beta l\,\text{fin.}\,r-\alpha\beta\,\text{fin.}\,r-\tfrac{3}{2}\mu\,\text{fin.}\,\iota\,\text{fin.}\,r+\tfrac{3}{2}\mu\,\text{fin.}\,\iota\,\text{fin.}(r-2p)$$
$$-\tfrac{3}{2}\nu\,\text{fin.}\,\iota\,\text{fin.}(3r-2p)+\tfrac{3}{2}\nu\,\text{fin.}\,\iota\,\text{fin.}\,r.$$

§. 62. In huius aequationis parte dextra potiffimum confiderari debet terminus fin. r, coefficiente:

$$2\beta l-\alpha\beta-\tfrac{3}{2}\mu\,\text{fin.}\,\iota+\tfrac{3}{2}\nu\,\text{fin.}\,\iota$$

affectus, qui nifi penitus abfit, id erit indicio, inclinationem orbitae lunaris ι non recte effe affumtam. Tum enim valor literae z neceffario inuolueret huiusmodi terminum:

k fin. r

x sin. r; quare cum per hypothesin inclinatio ι sit rite assumta, necesse est, vt ille coefficiens euanescat, ita vt fiat

$$2\beta l - \alpha\beta - \tfrac{1}{2}(\mu - \nu)\,\text{sin.}\,\iota = 0.$$

nisi forte euolutione terminorum sinistrae partis pariter eiusmodi termini resultent, qui cum isto simul sumti euanescere deberent: manifestum autem est hos terminos semper fore quam minimos. Hinc igitur patet, dari certam quandam relationem inter motum lineae nodorum, a qua literae α et β pendent, et ipsam inclinationem ι, quam adeo ex Theoria definire liceat.

SECTIO X.

De valoribus numericis quantitatum constantium l, m, α, β, μ, ν, hincque λ cum inclinatione ι.

§. 62. Ante quam euolutionem trium aequationum in sectione praecedente exhibitarum suscipiamus, conueniet ex tabulis mediorum motuum Lunae valores literarum l, m, α, β, μ, ν cum angulo ι excerpere, quo facilius vera cuiusque quantitas cognosci et diiudicari possit, quaenam prae reliquis tam sint exiguae, vt in calculo negligi queant. Hunc in finem ex tabulis angulorum ζ, θ, η, ☊ et π, qui tempori sunt proportionales, incrementa, quae certo temporis interuallo accipiunt, depromere necesse est, quippe quibus eorundem angulorum differentialia sunt proportionalia.

§. 64. Statuamus interuallum istud temporis, pro quo incrementa sunt definienda, 36 dierum. Ac primo ex tabulis solaribus longitudo Solis media ζ hoc tempore capit incrementum $29^\circ.\ 34'.\ 9,\ 9''$, vnde in minutis secundis erit in-

Incrementum longitudinis mediae Solis $\zeta = 106449,9$
vnde si subtrahatur motus apogaei qui est $= \quad 5,4$

prodibit incrementum anomaliae mediae $= 106444,5$
Hinc ergo erit increm. $t = 106444,5$, ad quod omnia reliqua incrementa deinceps referemus, sicque erit $\frac{d\zeta}{dt} = 1,000051$.

§. 65. Deinde ex tabulis lunaribus pro eodem tempore 30 dierum excerpatur motus Lunae medius, qui est $13^s. 5^\circ. 17'. 31''$, qui angulus in minuta secunda conuersus dabit

$$\text{incr. } \theta = 1422051, \text{ vnde fit } \frac{d\theta}{dt} = 13,390350.$$

Deinde ex eadem tabula motus retrogradus nodi colligitur $1^\circ. 35'. 19''$, hincque in minutis secundis incr. $\Omega = -5719$ vnde deducitur $\frac{d\Omega}{dt} = -0,053851$.

§. 66. Pro quantitate iusta inclinationis mediae ι etsi ea per elementa reliqua ex theoria determinetur: tamen, quia nondum constat, quantum partes adhuc neglectae eo conferre queant, eam ex meis tabulis lunaribus derivemus. Hinc igitur ex prima tabula pro quantitate z, cuius argumentum et angulus ι sumto $r = 90^\circ$ erit $z = 896400$ partibus decies millionesimis vnitatis, qui per distantiam mediam Solis a Terra diuisus dat tangentem inclinationis ι quam quaerimus. At vero in iisdem tabulis distantia media supponitur $= 9964129$ partium decies millonesimarum vnitatis, vnde colligitur tang. $\iota = \frac{896400}{9964129}$, hincque $\iota = 5^\circ.8'.26,2''$ hinc erit $\frac{\iota}{2} = 2^\circ. 34'. 13,1''$. Quare cum posuerimus

$$\mu = \text{cof.}\frac{\iota^2}{2} \text{ et } \nu = \text{fin.}\frac{\iota^2}{2}$$

statim habemus valores harum literarum

$$\mu = 0,998000 \qquad \nu = 0,002000$$
$$l\mu = 9,9991306 \quad l\nu = 7,3010300$$

ex

ex his porro deducitur

$$\mu\mu = 0,996004\ ;\ \mu\nu = 0,001996\ ;\ \nu\nu = 0,000004$$

$$l\mu\mu = 9,9981612;\ l\mu\nu = 7,3001606;\ l\nu\nu = 4,6068596.$$

§. 67. Deinde cum inuenerimus

$$\frac{d\text{☊}}{dt} = -0,058851$$

ſupra autem poſuerimus

$$\alpha = -\frac{d\text{☊}}{dt}\ \text{coſ.}\,\iota\ \text{et}\ \beta = -\frac{d\text{☊}}{dt}\ \text{ſin.}\,\iota,$$

valores harum literarum reperientur ſequentes:

$$\alpha = 0,053636\ ;\ \beta = 0,004825$$

$$l\alpha = 8,7294565;\ l\beta = 7,6835079$$

ex quibus deducuntur ſequentes valores

$$\alpha\alpha = 0,002877\ ;\ \alpha\beta = 0,000259\ ;\ \beta\beta = 0,000023$$

$$l\alpha\alpha = 7,4589130;\ l\alpha\beta = 6,4129644;\ l\beta\beta = 5,3670158.$$

Praeterea vero pro ſequentibus notaſſe iuuabit eſſe

$$l\ \text{coſ.}\,\iota = 9,9982596 \quad l\ \text{ſin.}\,\iota = 8,9523110.$$

§. 68. His valoribus definitis ad reliquos progrediamur; ac primo quidem, quia poſuimus $p = \theta - \zeta$ atque $\frac{dp}{dt} = m$, erit $m = \frac{d\theta}{dt} - \frac{d\zeta}{dt}$, quare ex valoribus ſupra inuentis habebimus

$$m = 12,390299\ \text{et}\ lm = 1,0930818.$$

Denique cum ſit

$r = \theta - \text{☊}$ et $l = \frac{dr}{dt}$ erit

$$l = \frac{d\theta}{dt} - \frac{d\text{☊}}{dt}$$

vnde ex valoribus ſupra inuentis colligemus

$$l = 13,444201\ \text{et Log.}\ l = 1,1285350.$$

§. 69. Ex his iam valoribus inuentis quaeramus etiam valorem literae λ ope huius aequationis:

$$\lambda\lambda = (l-\alpha)^2 + \tfrac{1}{2}\beta\beta + \tfrac{3}{4}(\mu\mu + \nu\nu) - 1.$$

Primo igitur habemus $l-\alpha = 13,390565$, hincque $(l-\alpha)^2 = 179,307250$; deinde vero est

$\tfrac{1}{2}\beta\beta = 0,0000011$; denique

$\tfrac{3}{4}(\mu\mu+\nu\nu) = 0,698004$; quibus inuentis concluditur fore

$$\lambda\lambda = 178,805265$$

hincque

$\lambda = 13,371806$ et $l\lambda = 1,1261901$

vbi notetur λ denotare celeritatem angularem, qua Luna in distantia $= 1$ circa Terram in circulo reuolueretur semota actione Solis.

§. 70. Tandem videamus quam prope valores inuenti conueniant cum conditione circa aequationem tertiam memorata, qua esse debebat

$$2\beta l - \alpha\beta - \tfrac{3}{2}(\mu-\nu)\,\text{sin.}\,i = 0 \text{ siue}$$
$$2l - \alpha - \tfrac{3}{2}\frac{(\mu-\nu)}{\beta}\,\text{sin.}\,i = 0$$

pro qua cum sit

$\mu - \nu = 0,996000$ et $2l - \alpha = 26,034766$, tum vero

$\frac{3(\mu-\nu)\,\text{sin.}\,i}{2\beta} = 27,743700$, debebat esse

$26,034766 - 27,743700 = 0 = -1,708934$

qui quidem error enormis videri posset; verum per β multiplicatus fit tantum $= 0,008246$, qui satis est exiguus, vt terminis in aequatione neglectis adscribi possit. Ex quo pro sequenti euolutione probe teneatur, omnes terminos formae k sin. r prorsus omitti debere. Praeterea vero etiam meminisse oportet, motum Lunae quoque ab actione Planetarum aliquantillum perturbari, ac fortasse etiam figura Lunae, quatenus a sphaerica discrepat, aliquid conferre potest.

§. 71. Tandem pro ſequentibus calculis, vbi excentricitas orbitae lunaris introducetur, ſtatuemus anomaliam mediam Lunae $= q$ et $\frac{dq}{dt} = n$, vnde cum $q = \theta -$ apogaeo, erit $n = \frac{d\theta}{dt} - \frac{d.\,apog.}{dt}$. Tum vero pro interuallo 30 dierum promouetur apogaeum per angulum $3^\circ.\,20'.\,32'' = 12032'$ ſicque erit $\frac{d.\,apog.}{dt} = 0,113296$, vnde colligitur

$$n = 13,277054 \text{ et } ln = 1,1221177.$$

§. 72. His omnibus praeparatis totum negotium ſimili fere modo abſolui poterit quo vſus ſum in *Theoria mea Lunae noua methodo pertractata*, dum ſcilicet inueſtigatio coordinatarum x, y, z in certos ordines diſtribuitur, ita vt in ſingulis geminae integrationes accurate expediri queant. Verum hoc opus tantae molis nunc quidem ſuſcipere vix auſim, vnde eius executionem vel in aliud tempus, differre, vel aliis, qui huiusmodi calculis delectantur relinquere cogor. Ceterum ex his iam perſpicitur tabulas lunares ſuper hac Theoria exſtructas longe aliam faciem eſſe habituras, quae fortaſſe ad vſum practicum magis erunt accommodatae.

CON-