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De motu oscillatorio penduli cuiuscunque, dum arcus datae amplitudinis absolvit

Leonhard Euler

DE MOTV OSCILLATORIO PENDVLI CVIVSCVNQVE, DVM ARCVS DATAE AMPLITVDINIS ABSOLVIT.

Auctore

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§. 1.

Quando vulgo doctrina de motu pendulorum in elementis Mechanicae tractatur, potiſſimum ſpectari ſolent pendula ſimplicia, quae, dum oſcillantur, excurſiones infinite paruas peragunt; vnde longitudo penduli ſimplicis ſingulis minutis oſcillantis ſollicite determinari ſolet. Cum autem pendula, quibus experimenta inſtitui ſolent, neque ſint ſimplicia, neque oſcillationes, quantumuis fuerint exiguae, tanquam infinite paruae ſpectari queant, illa theoria duplici correctione indiget, quarum altera per centrum oſcillationis remedium affertur, dum ſcilicet, propoſito pendulo quocunque, longitudo penduli ſimplicis quaeritur, quod paribus temporibus oſcillationes ſuas infinite paruas abſoluat: altera vero correctio, quam oſcillationes per arcus maiores factae exigunt, etſi a Geometris iam omni ſtudio eſt definita, tamen non ita in vulgus eſt cognita, vt ad quosuis caſus facile accommodari queat. Praeterea vero etiam iſta poſterior correctio tantum ad pendula ſimplicia reſtringi ſolet; vnde non inutile videtur iſtud argumentum ita in genere pertractare, vt inde ſine vllo labore correctiones pro pendulis vtcunque compoſitis et pro quauis arcuum

arcuum descriptorum amplitudine peti queant; in quo quidem negotio animum a resistentia aëris aliisque impedimentis abstrahemus, quandoquidem oscillationes inde oriundae iam satis accurate a Geometris sunt inuestigatae.

Tab. III. Fig. 1. §. 2. Denotet igitur in figura punctum A axem horizontalem, circa quem pendulum quodcunque AG oscillationes peragat, vbi quidem planum tabulae verticale est concipiendum, ad quod axis A sit normalis, ita vt pendulum in hoc plano verticali oscillationes suas absoluat, in quo porro concipiatur recta verticalis AB, quae situm naturalem penduli, in quo acquiescere possit, referat; id quod continget, quando centrum grauitatis totius penduli in ista recta verticali AB extiterit. Tum vero angulus $BAG = \zeta$ repraesentet excursionem maximam, ad quam pendulum in motu suo oscillatorio a recta verticali AB vtrinque alternatim digrediatur; ita vt iam nobis incumbat tempus definire, quo pendulum ex situ obliquo AG ad rectam verticalem AB sit peruenturum, quippe quod bis sumptum indicabit tempus vnius oscillationis. Per se autem manifestum est, rectam AG, vnde angulus excursionis $BAG = \zeta$ aestimatur, per centrum grauitatis totius penduli ab axe A duci debere.

§. 3. Sit igitur punctum G centrum grauitatis totius penduli, vtcunque fuerit compositum, ac ponatur eius distantia ab axe $AG = a$; tota vero penduli massa indicetur littera M, qua simul eius pondus designetur; vnde vis grauitatis in hoc pendulum ita aget, ac si ipsi in ipso centro grauitatis G applicata esset vis $= M$, in directione verticali GH sollicitans. Praeterea vero principia motus postulant, vt innotescat momentum inertiae totius penduli respectu axis per centrum grauitatis G ducti et axi gyrationis

nis A paralleli, quod reperitur, si singula penduli elementa per quadrata distantiarum suarum ab isto axe per G ducto multiplicentur atque omnia ista producta in vnam summam colligantur. Statuatur igitur istud momentum inertiae $= Mkk$, quandoquidem semper assignare licebit eiusmodi longitudinem k, vt productum Mkk aequetur summae omnium memoratorum productorum elementarium. Cognito autem isto momento inertiae respectu puncti G ex Mechanica constat, eius momentum inertiae respectu axis gyrationis A fore $= M(aa + kk)$.

§. 4. Consideremus nunc situm penduli quemcunque, quem inter oscillandum teneat, vbi recta $AG = a$, cum recta verticali AB constituat angulum $BAG = \Phi$, qui igitur est variabilis, dum a situ verticali, vbi $\Phi = 0$, alternatim vtrinque vsque ad $\Phi = \zeta$ excrescere potest, siquidem ζ amplitudinem excursionis maximae denotat. Cum igitur pendulum a sola vi grauitatis vrgeri statuatur, vim sustinebit centro grauitatis G in directione verticali GH applicatam $= M$, cuius ergo momentum respectu axis gyrationis A erit $= Ma \sin.\Phi$, cuius actio tendit ad angulum $BAG = \Phi$ diminuendum. Vnde, si pendulum ad situm verticalem AB accedat, eius motus ab ista vi accelerabitur; contra vero, si pendulum a situ naturali recedat, eius motus ab hac vi tantundem retardabitur; ex quo intelligitur, tam accessiones quam recessiones aequalibus temporibus absolui debere.

§. 5. Concipiamus igitur pendulum a situ naturali AB recedere, et elapso tempore t perueniisse in situm AG, confecto angulo $BAG = \Phi$. Sumamus porro tempus t in minutis secundis exprimi, et quoniam pendulum elemento Tab. III. Fig. 2.

 tem-

temporis dt conficiet angulum $GAg = d\Phi$, eius celeritas angularis erit $\frac{d\Phi}{dt}$, ideoque eius differentiale per dt diuisum dabit accelerationem $= \frac{dd\Phi}{dt^2}$, sumpto scilicet elemento dt constante, quae acceleratio secundum principia motus proportionalis est momento vis sollicitantis diuiso per momentum inertiae totius corporis respectu axis gyrationis, qui est in A. Vidimus autem momentum vis sollicitantis esse $= Ma \sin.\Phi$, quod, quia tendit ad motum retardandum, negatiue accipi debet; tum vero ostendimus momentum inertiae esse $= M(aa + kk)$. Quod si iam littera g denotet altitudinem, ex qua grauia vno minuto secundo delabuntur, praecepta Mechanicae nobis suppeditant istam aequationem differentialem secundi gradus:

$$\frac{dd\Phi}{dt^2} = -\frac{2gMa \sin.\Phi}{M(aa+kk)} = -\frac{2ga \sin.\Phi}{aa+kk}$$

ex qua totum penduli motum deriuare oportet.

§. 6. Multiplicemus hanc aequationem vtrinque per $2d\Phi$, et quia $\int d\Phi \sin.\Phi = -\cos.\Phi$, hinc integrando consequimur hanc aequationem: $\frac{d\Phi^2}{dt^2} = \frac{4ga\cos.\Phi}{aa+kk} + C$; vbi ad constantem rite determinandam notetur, formulam $\frac{d\Phi^2}{dt^2}$ exprimere quadratum celeritatis angularis, quae cum euanescere debeat, quando pendulum ad maximam excursionem pertigerit, hoc est, casu, quo fit $\Phi = \zeta$, haec constans ita definietur, vt fit $C = -\frac{4a\cos.\zeta}{aa+kk}$ ita vt iam aequatio motum penduli definiens fit $\frac{d\Phi^2}{dt^2} = \frac{4ga(\cos.\Phi - \cos.\zeta)}{aa+kk}$. Ponamus breuitatis gratia $\frac{aa+kk}{a} = b$ et habebimus

$$\frac{d\Phi^2}{dt^2} = \frac{4g}{b}(\cos.\Phi - \cos.\zeta)$$

vbi notetur, longitudinem $b = a + \frac{kk}{a}$ exprimere distantiam centri oscillationis ab axe suspensionis, siue, quod eodem redit, longitudinem penduli simplicis isochroni, quod scilicet eundem motum oscillatorium esset recepturum, siquidem arcus eiusdem amplitudinis absolueret. Hinc igitur

tur patet, ipso initio, quo pendulum ex situ verticali AB processit, quadratum eius celeritatis fuisse $= \frac{4g}{b}(1 - \cos.\zeta)$ quae formula simul exprimit altitudinem huic celeritati debitam.

§. 7. Quaeramus nunc ex hac aequatione valorem elementi temporis dt, qui erit $dt = \frac{d\Phi\sqrt{b}}{2\sqrt{g(\cos.\Phi - \cos.\zeta)}}$ cuius ergo formulae integrale erit inuestigandum. Hunc in finem ponamus $\sin.\frac{1}{2}\zeta = c$ et $\sin.\frac{1}{2}\Phi = z$, fietque

$$\cos.\zeta = 1 - 2cc \text{ et } \cos.\Phi = 1 - 2zz;$$

deinde vero, cum sit $\cos.\frac{1}{2}\Phi = \sqrt{1 - zz}$, sumptis differentialibus erit $\frac{1}{2}d\Phi\cos.\frac{1}{2}\Phi = dz$; vnde concluditur $d\Phi = \frac{2dz}{\sqrt{1-zz}}$; quibus valoribus substitutis erit nostrum elementum temporis

$$dt = \frac{dz\sqrt{b}}{\sqrt{2g(cc-zz)(1-zz)}} \text{ siue } \frac{dt\sqrt{2g}}{\sqrt{b}} = \frac{dz}{\sqrt{(cc-zz)(1-zz)}}$$

cuius integrale, vt totum tempus ascensus a $\Phi = 0$ vsque ad $\Phi = \zeta$ exprimat, extendi debet a $z = 0$ vsque ad $z = c$, hocque modo obtinebitur tempus vnius dimidiae oscillationis; vbi notetur, quantitatem c vnitatem nunquam superare posse atque adeo in exiguis oscillationibus fore fractionem valde paruam.

§. 8. Vt ambo termini integrationis ad terminos 0 et 1 reuocentur, statuatur $z = cy$, et aequatio nostra induet hanc formam: $\frac{dt\sqrt{2g}}{\sqrt{b}} = \frac{dy}{\sqrt{(1-yy)(1-ccyy)}}$. Quo nunc hinc integrale per seriem infinitam expressum elicere queamus, postremum factorem $\frac{1}{\sqrt{(1-ccyy)}} = (1 - ccyy)^{-\frac{1}{2}}$ in seriem euoluamus, quae erit

$$1 + \tfrac{1}{2}ccyy + \tfrac{1.3}{2.4}c^4y^4 + \tfrac{1.3.5}{2.4.6}c^6y^6 + \tfrac{1.3.5.7}{2.4.6.8}c^8y^8 + \text{etc.}$$

ita vt nunc habeamus

$$\frac{dt\sqrt{2g}}{\sqrt{b}} = \frac{dy}{\sqrt{1-yy}}\left(1 + \tfrac{1}{2}ccyy + \tfrac{1.3}{2.4}c^4y^4 + \tfrac{1.3.5}{2.4.6}c^6y^6 + \text{etc.}\right)$$

vbi singularum partium integralia ab $y = 0$ vsque ad $y = 1$ sunt funt extendenda. Nunc primo ftatim patet effe $\frac{dy}{\sqrt{1-yy}} = \frac{\pi}{2}$, denotante π peripheriam circuli, cuius diameter $= 1$; pro reliquis vero, cum in genere pro iisdem terminis integrationis fit:

$$\int \frac{y^{n+1}\,dy}{\sqrt{(1-yy)}} = \frac{n}{n+1}\int \frac{y^{n-1}\,dy}{\sqrt{(1-yy)}} \text{ erit}$$

$$\int \frac{yy\,dy}{\sqrt{(1-yy)}} = \frac{1}{2}\cdot\frac{\pi}{2}$$

$$\int \frac{y^4\,dy}{\sqrt{(1-yy)}} = \frac{1.3}{2.4}\cdot\frac{\pi}{2}$$

$$\int \frac{y^6\,dy}{\sqrt{(1-yy)}} = \frac{1.3.5}{2.4.6}\cdot\frac{\pi}{2}$$

$$\int \frac{y^8\,dy}{\sqrt{(1-yy)}} = \frac{1.3.5.7}{2.4.6.8}\cdot\frac{\pi}{2}$$

etc. etc.

His igitur valoribus fubftitutis integratio fingularum partium nos perducit ad fequentem feriem:

$$\frac{t\sqrt{2g}}{\sqrt{b}} = \frac{\pi}{2}\left(1 + \frac{1}{2^2}cc + \frac{1^2.3^2}{2^2.4^2}c^4 + \frac{1^2.3^2.5^2}{2^2.4^2.6^2}c^6 + \frac{1^2.3^2.5^2.7^2}{2^2.4^2.6^2.8^2}c^8 + \text{etc.}\right)$$

§. 9. Hinc igitur innotefcit tempus t pro dimidia ofcillatione, quod duplicatum praebebit tempus vnius ofcillationis integrae, quod fi indicetur littera T, erit

$$T = \frac{\pi\sqrt{b}}{\sqrt{2g}}\left(1 + \frac{1^2}{2^2}cc + \frac{1^2.3^2}{2^2.4^2}c^4 + \frac{1^2.3^2.5^2}{2^2.4^2.6^2}c^6 + \frac{1^2.3^2.5^2.7^2}{2^2.4^2.6^2.8^2}c^8 + \text{etc.}\right)$$

quod tempus adeo exprimitur in minutis fecundis, fiquidem g denotet altitudinem, per quam grauia vno minuto fecundo libere delabuntur. Atque hinc ftatim patet, fi ofcillationes fuerint infinite paruae, quo cafu fit $\zeta = 0$ ideoque $c = 0$, tempus cuiusque ofcillationis futurum effe $T = \frac{\pi\sqrt{b}}{\sqrt{2g}}$, vbi b defignat longitudinem penduli fimplicis ifochroni, fiue diftantiam centri ofcillationis ab axe gyrationis A. Vnde fi velimus, vt pendulum fingulas ofcillationes vno minuto fecundo abfoluat, fieri debet $\frac{\pi\sqrt{b}}{\sqrt{2g}} = 1$, vnde colligitur longitudo talis penduli fimplicis $b = \frac{2g}{\pi\pi}$.

Quare

Quare cum in pedibus Rhenanis sit $g = 15\frac{5}{8}$ ped. ideoque $2g = 31\frac{1}{4}$ ped. erit longitudo talis penduli $b = 3,16621$. Sin autem longitudo penduli fuerit maior vel minor, tum tempora oscillationum, vti in vulgus est notum, sequentur rationem subduplicatam longitudinis penduli b; vbi autem probe notandum, oscillationes hic considerari infinite paruas.

§. 10. Quod autem ad oscillationes per arcus maiores attinet, in genere quidem patet ex nostra formula, quo maiores fuerint hi arcus, quoniam quantitas $c = \sin.\frac{1}{2}\zeta$ etiam augetur, tempora oscillationum fieri aliquanto minora; quod quo clarius ob oculos ponamus, sumamus quantitatem cc tam esse paruam, vt eius altiores potestates c^4, c^6 etc. negligere liceat, ac tum tempus vnius oscillationis erit $T = \frac{\pi\sqrt{b}}{\sqrt{2g}}(1 + \frac{1}{4}cc)$, vnde, si Θ denotet tempus vnius oscillationis infinite paruae eiusdem penduli, ob $\Theta = \frac{\pi\sqrt{b}}{\sqrt{2g}}$ erit nunc $T = \Theta(1 + \frac{1}{4}cc)$, siue erit $\Theta : T = 1 : 1 + \frac{1}{4}cc$, id quod valet pro oscillationibus satis exiguis: si enim arcus maiores absoluantur, etiam plures terminos seriei inuentae assumi oportebit. Vt nunc hanc formulam ad vsum practicum accommodemus, consideremus nostrum pendulum A G O in excursione maxima, ita vt cum recta verticali A B faciat angulum $BAO = \zeta$ sitque G, vt hactenus, centrum grauitatis penduli, punctum O vero centrum oscillationis, ita vt sit $AO = b = \frac{aa+kk}{a}$. Iam ducta ex O horizontali OD erit $OD = b\sin.\zeta$ et $AD = b\cos.\zeta$, vnde fiet sagitta $BD = b(1 - \cos.\zeta) = 2b\sin.\frac{1}{2}\zeta^2$; erit ergo hoc interuallum $BD = 2bcc$. Quoniam igitur, data amplitudine siue angulo ζ, hoc interuallum BD facile metiri licet, vocemus id $BD = d$, eritque $cc = \frac{d}{2b}$; quam ob rem pro Tab. III. Fig. 3.

oscil-

oscillationibus satis exiguis erit $\Theta : T = 1 : 1 + \frac{d}{8b}$ siue $\Theta : T = 8b : 8b + d$, hoc est vt $8\,AB : 8\,AB + BD$ qua regula iam passim in experimentando vti solent.

§. 11. Sin autem oscillationes per maiores arcus peragantur, atque Θ vt ante denotet tempus vnius oscillationis infinite paruae penduli propositi, ob

$$\Theta = \frac{\pi \sqrt{b}}{\sqrt{2g}} \text{ et } cc = \frac{d}{2b}$$

tempus vnius oscillationis pro angulo $BAO = \zeta$ ita exprimetur vt sit

$$T = \Theta\left(1 + \frac{1^2}{2^2} \cdot \frac{d}{2b} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \cdot \frac{dd}{4bb} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \cdot \frac{d^3}{8b^3} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} \cdot \frac{d^4}{16b^4} + \text{etc.}\right)$$

vnde si pendulum inter oscillandum totum semicirculum percurrat, ita vt angulus ζ euadat $= 90°$ erit $b = d$ ideoque hoc casu tempus vnius oscillationis erit

$$T = \Theta\left(1 + \frac{1^2}{2^2} \cdot \frac{1}{2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \cdot \frac{1}{4} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \cdot \frac{1}{8} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} \cdot \frac{1}{16} + \text{etc.}\right)$$

cuius valor in fractionibus decimalibus computatus colligitur proxime $T = 1,1805\,\Theta$, quae ratio proxime est vt $5 : 6$. Si arcus percursi adeo maiores euadant quam semicirculus, tempora oscillationum continuo magis increscent; atque adeo si pendulum totam peripheriam percurrere debeat, tempus vsque in infinitum augetur; postquam enim pendulum in locum supremum fuerit perductum, situm tenebit verticalem et nunquam ex eo delabetur; vnde mirum non est calculum tempus infinite magnum ostendere. Caeterum, quia hoc casu fit $d = 2b$, series supra inuenta abibit in hanc:

$$1 + \frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} + \text{etc.}$$

cuius summam infinitam esse ex prima formula integrali manifesto liquet, quae, ob $c = 1$, erit $\int \frac{dz}{1 - zz}$ a $z = 0$ ad $z = 1$ extendenda; eius vero integrale est $\frac{1}{2} l \frac{1+z}{1-z}$, qui valor, posito $z = 1$, manifesto fit infinitus.

Addita-

Additamentum ad dissertationem de motu pendulorum.

§. 12. Cum circa finem superioris dissertationis ostendissem tempus oscillationis in infinitum augeri, si angulus ζ vsque ad 180 grad. excrescat, quaestio hic se offert non parum curiosa, quantum futurum sit tempus oscillationis, quando angulus ζ propemodum ad 180 grad. augetur, ita vt quantitas $c = \sin.\frac{1}{2}\zeta$ tantum non vnitati fiat aequalis, siue quam minime ab ea deficiat, tum enim series inuenta

$$T = \frac{\pi\sqrt{b}}{\sqrt{2g}}\left(1 + \frac{1^2}{2^2}cc + \frac{1^2.3^2}{2^2.4^2}c^4 + \frac{1^2.3^2.5^2}{2^2.4^2.6^2}c^6 + \text{etc.}\right)$$

etsi summam habet finitam, tamen eius termini nimis lente conuergunt, quam vt eius verum valorem saltem proxime inde determinare liceat; neque etiam vlla via patere videtur, istam seriem ita transformandi, vt eius summa satis exacte inde definiri queat.

§. 13. Constituto igitur axe gyrationis in A, circa eum radio $AB = Ab = b$ describatur circulus, in quo diameter Bb situm teneat verticalem. Iam sursum ducatur radius Ag parum a situ verticali discrepans, vnde pendulum per peripheriam circuli descendere incipiat, ita vt angulus bAg sit valde exiguus, quem vocemus $bAg = \eta$. Quare cum in calculo praecedenti littera ζ designasset angulum BAg, erit nunc $\zeta = 180 - \eta$, hincque $c = \sin.(90 - \frac{1}{2}\eta) = \cos.\frac{1}{2}\eta$ propemodum vnitati aequabitur. Ponamus nunc elapso tempore $= t$ pendulum ex g descendisse in Z et vocemus angulum $bAZ = \psi$ ita vt sit angulus $BAZ = \Phi = 180 - \psi$, ideoque, cum posuissemus $z = \sin.\frac{1}{2}\Phi$ erit nunc $z = \cos.\frac{1}{2}\psi$; quam ob rem aequatio pro motu penduli supra inuenta Tab. III. Fig. 4.

$$\frac{dt\sqrt{2g}}{\sqrt{b}} = \frac{dz}{\sqrt{(1-zz)(cc-zz)}}$$

per istos nouos valores euolui debebit.

§. 14.

§. 14. Cum igitur fit $cc = \operatorname{cof.} \frac{1}{2}\eta^2 = 1 - \operatorname{fin.} \frac{1}{2}\eta^2$ et $zz = \operatorname{cof.} \frac{1}{2}\psi^2 = 1 - \operatorname{fin.} \frac{1}{2}\psi^2$ fiet, $\frac{dz}{\sqrt{1-zz}} = \frac{1}{2} d\psi$ cui fignum $+$ tribuimus, quia hic corpus defcendere affumimus, dum ante afcenfus fuiffet confideratus. His igitur valoribus fubftitutis noftra aequatio differentialis erit

$$\frac{dt\sqrt{2g}}{\sqrt{b}} = \frac{d\psi}{2\sqrt{(\operatorname{fin.} \frac{1}{2}\psi^2 - \operatorname{fin.} \frac{1}{2}\eta^2)}}$$

vbi probe notetur terminum $\operatorname{fin.} \frac{1}{2}\eta^2$ effe quam minimum.

§. 15. Quoniam defcenfus ex puncto g incipere affumitur, exiftente angulo $b\,A\,g = \eta$, euidens eft, integrale euanefcere pofito $\psi = \eta$; quo obferuato valor litterae t dabit tempus in minutis fecundis expreffum, quo pendulum ex fitu initiali $A\,g$ peruenerit in fitum quemcunque alium $A\,z$. Quoniam igitur hic quantitas $\operatorname{fin.} \frac{1}{2}\eta^2$ mox prae termino $\operatorname{fin.} \frac{1}{2}\psi^2$ euadet quam minima, formula differentialis $\frac{d\psi}{2\sqrt{(\operatorname{fin.} \frac{1}{2}\psi^2 - \operatorname{fin.} \frac{1}{2}\eta^2)}}$ commode in hanc feriem euoluetur

$$\tfrac{1}{2} d\psi \left(\frac{1}{\operatorname{fin.} \frac{1}{2}\psi} + \frac{1 \cdot \operatorname{fin.} \frac{1}{2}\eta^2}{2 \cdot \operatorname{fin.} \frac{1}{2}\psi^3} + \frac{1 \cdot 3 \operatorname{fin.} \frac{1}{2}\eta^4}{2 \cdot 4 \operatorname{fin.} \frac{1}{2}\psi^5} + \frac{1 \cdot 3 \cdot 5 \operatorname{fin.} \frac{1}{2}\eta^6}{2 \cdot 4 \cdot 6 \operatorname{fin.} \frac{1}{2}\psi^7} + \text{etc.} \right)$$

vbi integrale primi termini ftatim per logarithmos ita exprimi poteft, vt fit

$$\int \frac{d\psi}{2 \operatorname{fin.} \frac{1}{2}\psi} = l \frac{\operatorname{tg.} \frac{1}{4}\psi}{\operatorname{tg.} \frac{1}{4}\eta}$$

vnde intelligitur, fi angulus η plane euanefceret, valorem huius integralis fore infinitum; vnde ftatim patet, quo minor accipiatur angulus η, eo minus prodire debere tempus; quam ob rem, fi praeter primum terminum fequentes negligere liceret, iam haberemus

$$\frac{t\sqrt{2g}}{\sqrt{b}} = l \frac{\operatorname{tg.} \frac{1}{4}\psi}{\operatorname{tg.} \frac{1}{4}\eta} \text{ ideoque } t = \frac{\sqrt{b}}{\sqrt{2g}}\, l \frac{\operatorname{tg.} \frac{1}{4}\psi}{\operatorname{tg.} \frac{1}{4}\eta}.$$

§. 16.

§. 16. Manifestum autem est, ex hac formula tempus totius descensus, quo pendulum ex situ initiali Ag vsque ad situm infimum AB perueniet, definiri posse, quandoquidem posito $\psi = 180^\circ$ tempus istud fiet

$$t = \frac{\sqrt{b}}{\sqrt{2g}}\, l \frac{1}{tg \frac{1}{4}\eta} = \frac{\sqrt{b}}{\sqrt{2g}}\, l \cot. \tfrac{1}{4}\eta.$$

Quanquam autem haec expressio primam tantum partem totius valoris continet, tamen ex ea iam satis exacte tempus totius descensus definiri poterit; quem in finem operae pretium erit inuestigare, quantam correctionem sequentes termini seriei supra inuentae producere valeant.

§. 17. Quod quo commodius fieri possit, statuamus $\frac{1}{2}\psi = \omega$ et $\frac{1}{2}\eta = \alpha$, vnde integrale primi termini prodit $= l \frac{tg \frac{1}{2}\omega}{tg \frac{1}{2}\alpha}$; Pro secundo autem termino habebimus differentiale $\frac{d\,\omega \sin.\alpha^2}{\sin.\omega^3}$, ad cuius integrale inueniendum fingamus $\int \frac{d\omega}{\sin.\omega^3} = \frac{A \cos.\omega}{\sin.\omega^2} + B \int \frac{d\omega}{\sin.\omega}$: vnde sumptis differentialibus erit $\frac{1}{\sin.\omega^3} = -\frac{A}{\sin.\omega} - \frac{2A\cos.\omega^2}{\sin.\omega^3} + \frac{B}{\sin.\omega}$, quae aequatio, loco $\cos.\omega^2$ substituto valore $1 - \sin.\omega^2$, transit in hanc:

$$\frac{1}{\sin.\omega^3} = \frac{A}{\sin.\omega} - \frac{2A}{\sin.\omega^3} + \frac{B}{\sin.\omega}$$

ex qua patet, sumi debere $2A = -1$ et $B + A = 0$, ita vt sit $A = -\frac{1}{2}$ et $B = \frac{1}{2}$, vnde ergo colligitur

$$\int \frac{d\omega}{\sin.\omega^3} = -\frac{\cos.\omega}{2\sin.\omega^2} + \tfrac{1}{2} l\, tg \tfrac{1}{2}\omega + C$$

vbi ad constantem inueniendam statuatur $\omega = \alpha$, fierique debet $0 = -\frac{\cos.\alpha}{2\sin.\alpha^2} + \frac{1}{2} l\, tg \frac{1}{2}\alpha + C$, ideoque $C = \frac{\cos.\alpha}{2\sin.\alpha^2} - \frac{1}{2} l\, tg \frac{1}{2}\alpha$ quo substituto erit

$$\int \frac{d\omega}{\sin.\omega^3} = \frac{\cos.\alpha}{2\sin.\alpha^2} - \frac{\cos.\omega}{2\sin.\omega^2} + \tfrac{1}{2} l \frac{tg \frac{1}{2}\omega}{tg \frac{1}{2}\alpha}$$

 quo-

quo circa, binis partibus prioribus coniungendis, consequemur hunc valorem:

$$\frac{t\sqrt{2g}}{\sqrt{b}} = (1 + \tfrac{1}{4}\,\text{fin.}\,\alpha^2)\, l\,\frac{\text{tg.}\,\frac{1}{2}\omega}{\text{tg.}\,\frac{1}{2}\alpha} + \tfrac{1}{4}\,\text{cof.}\,\alpha - \frac{\text{cof.}\,\omega\,\text{fin.}\,\alpha^2}{4\,\text{fin.}\,\omega^2}$$

Hinc pro toto descensu, ponendo $\omega = 90°$, fiet

$$\frac{t\sqrt{2g}}{\sqrt{b}} = (1 + \tfrac{1}{4}\,\text{fin.}\,\alpha^2)\, l\,\text{cot.}\,\tfrac{1}{2}\alpha + \tfrac{1}{4}\,\text{cof.}\,\alpha,$$

vnde patet ob secundum terminum imprimis accessisse quantitatem satis notabilem $\frac{1}{4}$ cof. α, cuius valor propemodum est $\frac{1}{4}$; quam ob rem eo magis necesse est etiam sequentes terminos exsequi.

§. 18. Haec autem operatio vt in genere institui queat, lemma generale praemittamus pro integratione formulae $\int \frac{d\omega}{\text{fin.}\,\omega^{n+1}}$, quem in finem ponamus

$$\int \frac{d\omega}{\text{fin.}\,\omega^{n+1}} = \frac{A\,\text{cof.}\,\omega}{\text{fin.}\,\omega^n} + B \int \frac{d\omega}{\text{fin.}\,\omega^{n-1}}$$

quae forma differentiata praebet

$$\frac{1}{\text{fin.}\,\omega^{n+1}} = -\frac{A}{\text{fin.}\,\omega^{n-1}} - \frac{n\,A\,\text{cof.}\,\omega^2}{\text{fin.}\,\omega^{n+1}} + \frac{B}{\text{fin.}\,\omega^{n-1}}$$

quae ob $\text{cof.}\,\omega^2 = 1 - \text{fin.}\,\omega^2$ abit in hanc:

$$\frac{1}{\text{fin.}\,\omega^{n+1}} = \frac{(n-1)\,A}{\text{fin.}\,\omega^{n-1}} - \frac{n\,A}{\text{fin.}\,\omega^{n+1}} + \frac{B}{\text{fin.}\,\omega^{n-1}}$$

vnde patet esse debere $A = -\frac{1}{n}$ & $B = \frac{n-1}{n}$, ita vt, introducta constante debita, qua integrale euanescat posito $\omega = \alpha$, habeamus in genere hanc reductionem:

$$\int \frac{d\omega}{\text{fin.}\,\omega^{n+1}} = \frac{\text{cof.}\,\alpha}{n\,\text{fin.}\,\alpha^n} - \frac{\text{cof.}\,\omega}{n\,\text{fin.}\,\omega^n} + \frac{n-1}{n} \int \frac{d\omega}{\text{fin.}\,\omega^{n-1}},$$

quod postremum integrale tanquam cognitum spectare licet.

§. 19.

§. 19. Hoc lemmate constituto sumamus primo $n = 2$ eritque

$$\int \frac{d\omega}{\text{sin.}\,\omega^3} = +\frac{\text{cos.}\,\alpha}{2\,\text{sin.}\,\alpha^2} - \frac{\text{cos.}\,\omega}{2\,\text{sin.}\,\omega^2} + \tfrac{1}{2}\,l\,\frac{\text{tag.}\,\frac{1}{2}\omega}{\text{tag.}\,\frac{1}{2}\alpha}$$

quemadmodum iam ante inuenimus. Nunc igitur ponamus porro $n = 4$ eritque pro tertio termino

$$\int \frac{d\omega}{\text{sin.}\,\omega^5} = \frac{\text{cos.}\,\alpha}{4\,\text{sin.}\,\alpha^2} - \frac{\text{cos.}\,\omega}{4\,\text{sin.}\,\omega^2} + \tfrac{3}{4}\int \frac{d\omega}{\text{sin.}\,\omega^3}.$$

Fiat porro $n = 6$, ac pro termino quarto habebimus

$$\int \frac{d\omega}{\text{sin.}\,\omega^7} = \frac{\text{cos.}\,\alpha}{6\,\text{sin.}\,\alpha^2} - \frac{\text{cos.}\,\omega}{6\,\text{sin.}\,\omega^2} + \tfrac{5}{6}\int \frac{d\omega}{\text{sin.}\,\omega^5}.$$

Simili modo, posito $n = 8$, pro termino quinto habebimus

$$\int \frac{d\omega}{\text{sin.}\,\omega^9} = \frac{\text{cos.}\,\alpha}{8\,\text{sin.}\,\alpha^2} - \frac{\text{cos.}\,\omega}{8\,\text{sin.}\,\omega^2} + \tfrac{7}{8}\int \frac{d\omega}{\text{sin.}\,\omega^7}.$$

Vlterius vero, ponendo $n = 10$, reperietur

$$\int \frac{d\omega}{\text{sin.}\,\omega^{11}} = \frac{\text{cos.}\,\alpha}{10\,\text{sin.}\,\alpha^2} - \frac{\text{cos.}\,\omega}{10\,\text{sin.}\,\omega^2} + \tfrac{9}{10}\int \frac{d\omega}{\text{sin.}\,\omega^{11}}.$$

etc. etc.

§. 20. Quoniam autem nobis imprimis propositum est in tempus totius descensus, quo fit $\omega = 90°$, inquirere, inuenta integralia ad hunc casum accommodata ita se habebunt:

$$\int \frac{d\omega}{\text{sin.}\,\omega} = l\,\text{cot.}\,\tfrac{1}{2}\alpha$$

$$\int \frac{d\omega}{\text{sin.}\,\omega^3} = \frac{\text{cos.}\,\alpha}{2\,\text{sin.}\,\alpha^2} + \tfrac{1}{2}\,l\,\text{cot.}\,\tfrac{1}{2}\alpha$$

$$\int \frac{d\omega}{\text{sin.}\,\omega^5} = \frac{\text{cos.}\,\alpha}{4\,\text{sin.}\,\alpha^4} + \frac{3\,\text{cos.}\,\alpha}{2.\,4.\,\text{sin.}\,\alpha^2} + \frac{1.\,3}{2.\,4}\,l\,\text{cot.}\,\tfrac{1}{2}\alpha$$

$$\int \frac{d\omega}{\text{sin.}\,\omega^7} = \frac{\text{cos.}\,\alpha}{6\,\text{sin.}\,\alpha^6} + \frac{5\,\text{cos.}\,\alpha}{4.\,6\,\text{sin.}\,\alpha^4} + \frac{3.\,5\,\text{cos.}\,\alpha}{2.\,4.\,6\,\text{sin.}\,\alpha^2} + \frac{1.\,3.\,5}{2.\,4.\,6}\,l\,\text{cot.}\,\tfrac{1}{2}\alpha$$

$$\int \frac{d\omega}{\text{sin.}\,\omega^9} = \frac{\text{cos.}\,\alpha}{8\,\text{sin.}\,\alpha^8} + \frac{7\,\text{cos.}\,\alpha}{6.\,8\,\text{sin.}\,\alpha^6} + \frac{5.\,7.\,\text{cos.}\,\alpha}{4.\,6.\,8\,\text{sin.}\,\alpha^4} + \frac{3.\,5.\,7\,\text{cos.}\,\alpha}{2.\,4.\,6.\,8\,\text{sin.}\,\alpha^2} + \frac{1.\,3.\,5.\,7}{2.\,4.\,6.\,8}\,l\,\text{cot.}\,\tfrac{1}{2}\alpha$$

etc. etc.

§. 21.

§. 21. Subſtituamus nunc ſingulos iſtos valores integrales in aequatione differentiali, quae erat

$$\frac{dt\sqrt{2g}}{\sqrt{b}} = d\omega \left(\frac{1}{\text{ſin.}\ \omega} + \frac{1.\ \text{ſin.}\ \alpha^2}{2.\ \text{ſin.}\ \omega^3} + \frac{1.3\ \text{ſin.}\ \alpha^4}{2.4\ \text{ſin.}\ \omega^5} + \text{etc.} \right)$$

atque integrale quaeſitum per totum arcum deſcenſus gzB extenſum erit:

$$\frac{t\sqrt{2g}}{\sqrt{b}} = \left\{ \begin{array}{l} l\ \text{cot.}\ \tfrac{1}{2}\alpha \\ + \frac{1^2}{2^2}\ \text{ſin.}\ \alpha^2\ l\ \text{cot.}\ \tfrac{1}{2}\alpha + \frac{1^2}{2^2}\ \text{coſ.}\ \alpha \\ + \frac{1^2.\ 3^2}{2^2.\ 4^2}\ \text{ſin.}\ \alpha^4\ l\ \text{cot.}\ \tfrac{1}{2}\alpha + \frac{1.\ 3}{2.\ 4^2}\ \text{coſ.}\ \alpha + \frac{1.\ 3^2}{2^2.\ 4^2}\ \text{coſ.}\ \alpha\ \text{ſin.}\ \alpha^2 \\ + \frac{1^2.\ 3^2.\ 5^2}{2^2.\ 4^2.\ 6^2}\ \text{ſin.}\ \alpha^6\ l\ \text{cot.}\ \tfrac{1}{2}\alpha + \frac{1.\ 3.\ 5}{2.\ 4.\ 6^2}\ \text{coſ.}\ \alpha + \frac{1.\ 3.\ 5^2}{2.\ 4^2.\ 6^2}\ \text{coſ.}\ \alpha\ \text{ſin.}\ \alpha^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \frac{1.\ 3^2.\ 5^2}{2^2.\ 4^2.\ 6^2}\ \text{coſ.}\ \alpha\ \text{ſin.}\ \alpha^4 \\ + \frac{1^2.\ 3^2.\ 5^2.\ 7^2}{2^2.\ 4^2.\ 6^2.\ 8^2}\ \text{ſin.}\ \alpha^8\ l\ \text{cot.}\ \tfrac{1}{2}\alpha + \frac{1.\ 3.\ 5.\ 7}{2.\ 4.\ 6.\ 8^2}\ \text{coſ.}\ \alpha + \frac{1.\ 3.\ 5.\ 7^2}{2.\ 4.\ 6^2.\ 8^2}\ \text{coſ.}\ \alpha\ \text{ſin.}\ \alpha^2 \\ \qquad\qquad\qquad + \frac{1.\ 3.\ 5^2.\ 7^2}{2.\ 4^2.\ 6^2.\ 8^2}\ \text{coſ.}\ \alpha\ \text{ſin.}\ \alpha^4 + \frac{1.\ 3^2.\ 5^2.\ 7^2}{2^2.\ 4^2.\ 6^2.\ 8^2}\ \text{coſ.}\ \alpha\ \text{ſin.}\ \alpha^6 \\ \qquad\qquad\qquad \text{etc.} \qquad\qquad\qquad\qquad\qquad \text{etc.} \end{array} \right.$$

vbi cum α denotet angulum valde paruum, ita vt propemodum ſit coſ. $\alpha = 1$, ſecunda columna, cuius omnes termini ſimpliciter continent coſ. α, prae reliquis valores notabiles exhibet, dum reliqui, qui continent vel ſin. α^2, vel ſin. α^4 vel ſin. α^6 etc. ſine ſenſibili errore negligi poſſunt.

§. 22. At vero terminus coſ. α multiplicatus reperitur per hanc ſeriem infinitam:

$$\frac{1}{2^2} + \frac{1.\ 3}{2.\ 4^2} + \frac{1.\ 3.\ 5}{2.\ 4.\ 6^2} + \frac{1.\ 3.\ 5.\ 7}{2.\ 4.\ 6.\ 8^2} + \frac{1.\ 3.\ 5.\ 7.\ 9}{2.\ 4.\ 6.\ 8.\ 10^2} + \frac{1.\ 3.\ 5.\ 7.\ 9.\ 11}{2.\ 4.\ 6.\ 8.\ 10.\ 12^2} + \text{etc.}$$

ad cuius ſummam inueſtigandam contemplemur in genere hanc ſeriem:

$$\frac{1}{\sqrt{(1 - vv)}} = 1 + \frac{1}{2} vv + \frac{1.\ 3}{2.\ 4} v^4 + \frac{1.\ 3.\ 5}{2.\ 4.\ 6} v^6 + \frac{1.\ 3.\ 5.\ 7}{2.\ 4.\ 6.\ 8} v^8 + \text{etc.}$$

quae diuiſa per v praebet

$$\frac{1}{v\sqrt{(1 - vv)}} = \frac{1}{v} + \frac{1}{2} v + \frac{1.\ 3}{2.\ 4} v^3 + \frac{1.\ 3.\ 5}{2.\ 4.\ 6} v^5 + \frac{1.\ 3.\ 5.\ 7}{2.\ 4.\ 6.\ 8} v^7 + \text{etc.}$$

ex

ex hac vero ducta in dv et integrata, prodit

$$\int\frac{dv}{v\sqrt{(1-vv)}} = lv + \frac{1}{2^2}vv + \frac{1.3}{2.4^2}v^4 + \frac{1.3.5}{2.4.6^2}v^6 + \frac{1.3.5.7}{2.4.6.8^2}v^8 + \text{etc.}$$

vnde patet, summam nostrae seriei resultare ex formula $\int\frac{dv}{v\sqrt{(1-vv)}} - lv$, si post integrationem statuatur $v=1$. Ponamus nunc $\sqrt{1-vv}=u$ eritque $vv=1-uu$, hinc $2lv = l\,\overline{1-uu}$ et $\frac{dv}{v} = -\frac{udu}{1-uu}$ vnde colligitur

$$\int\frac{dv}{v\sqrt{1-vv}} = -\int\frac{du}{1-uu} = -\frac{1}{2}l\frac{1+u}{1-u} = -l(1+u)+\frac{1}{2}l(1-uu)+C.$$

Iam loco u restituatur valor $\sqrt{1-vv}$ eritque

$$\int\frac{dv}{v\sqrt{1-vv}} = C - l(1+\sqrt{1-vv}) + lv$$

consequenter expressio proposita erit

$$\int\frac{dv}{v\sqrt{1-vv}} - lv = C - l(1+\sqrt{1-vv})$$

quod integrale quia ita sumi debet, vt euanescat posito $v=0$, dabit constantem $C=l2$; qua inuenta faciamus $v=1$ atque seriei nostrae

$$\frac{1}{2^2}+\frac{1.3}{2.4^2}+\frac{1.3.5}{2.4.6^2}+\frac{1.3.5.7}{2.4.5.8^2}+\text{etc.}$$

summa nunc nobis est cognita, scilicet $=l2$.

§. 23. Quodsi etiam termini tertiae columnae, qui singuli continent $\cos.\alpha\,\sin.\alpha^2$ aliqua attentione digni videantur, id quod euenit, quando initium descensus Ag aliquanto longius a situ summo Ab accipiatur: haud difficulter quoque summa seriei, quae est

$$\frac{1.3^2}{2^2.4^2}+\frac{1.3.5^2}{2.4^2.6^2}+\frac{1.3.5.7^2}{2.4.6^2.8^2}+\frac{1.3.5.7.9^2}{2.4.6.8^2.10^2}+\text{etc.}$$

inuestigari poterit. Cum enim iam inuenerimus

$$l2-l(1+\sqrt{1-vv})=\frac{1}{2^2}vv+\frac{1.3}{2.4^2}v^4+\frac{1.3.5}{2.4.6^2}v^6+\frac{1.3.5.7}{2.4.6.8^2}v^8+\text{etc.}$$

diuidamus vtrinque per v, vt habeamus

$$\frac{l2}{v}-\frac{l(1+\sqrt{1-vv})}{v}=\frac{1}{2^2}v+\frac{1.3}{2.4^2}v^3+\frac{1.3.5}{2.4.6^2}v^5+\frac{1.3.5.7}{2.4.6.8^2}v^7+\text{etc.}$$

quae

quae aequatio differentiata, pro parte finiſtra, omiſſo dv, praebet

$$-\frac{l\,2}{vv} + \frac{1}{vv} l(1 + \sqrt{1 - vv}) + \frac{1}{(1 + \sqrt{1 - vv})\sqrt{1 - vv}}$$

cuius poſtremum membrum facile mutatur in hanc formam: $\frac{1}{vv\sqrt{1 - vv}} - \frac{1}{vv}$; ita vt iam membrum finiſtrum ſit

$$-\frac{1 - l\,2}{vv} + \frac{1}{vv\sqrt{1 - vv}} + \frac{1}{vv} l(1 + \sqrt{1 - vv}).$$

Pro dextra autem parte habebimus

$$\frac{1}{2^2} + \frac{1.\,3^2}{2.\,4^2} vv + \frac{1.\,3.\,5^2}{2.\,4.\,6^2} v^4 + \frac{1.\,3.\,5.\,7^2}{2.\,4.\,6.\,8^2} v^6 + \text{etc.}$$

§. 24. Pro parte finiſtra ſcribamus ſimpliciter V, vt ſit

$$V = \frac{1}{2^2} + \frac{1.\,3^2}{2.\,4^2} vv + \frac{1.\,3.\,5^2}{2.\,4.\,6^2} v^4 + \frac{1.\,3.\,5.\,7^2}{2.\,4.\,6.\,8^2} v^6 + \text{etc.}$$

Hic valor ductus in $\frac{dv}{v}$ et integratus praebet

$$\int \frac{V\,dv}{v} = \frac{1}{2^2} l\,v + \frac{1.\,3^2}{2^2.\,4^2} vv + \frac{2.\,3.\,5^2}{2.\,4^2.\,6^2} v^4 + \frac{1.\,3.\,5.\,7^2}{2.\,4.\,6^2.\,8^2} v^6 + \text{etc.}$$

vnde patet ſummam noſtrae feriei fore $= \int \frac{V\,dv}{v} - \frac{1}{4} l\,v$, ſiquidem ſtatuatur $v = 1$. Iam quia V conſtat tribus partibus, quarum prima eſt $-\frac{1 - l\,2}{vv}$; ſecunda: $\frac{1}{vv\sqrt{1 - vv}}$ et tertia: $\frac{1}{vv} l(1 + \sqrt{1 - vv})$; ex prima parte eruitur

$$\int -\frac{dv}{v^3}(1 + l\,2) = \frac{1 + l\,2}{2vv}.$$

Ex ſecunda parte fit

$$\int \frac{dv}{v^3\sqrt{1 - vv}} = -\frac{\sqrt{1 - vv}}{2vv} - \frac{1}{2} l(1 + \sqrt{1 - vv}) + \frac{1}{2} l\,v.$$

Pro parte tertia habemus $\int \frac{dv}{v^3} l(1 + \sqrt{1 - vv})$, quae reducta per formulam integralem notiſſimam praebet

$$\int \frac{dv}{v^3} l(1 + \sqrt{1 - vv}) = -\frac{1}{2vv} l(1 + \sqrt{1 - vv}) - \frac{1}{2}\int \frac{dv}{v^3} \frac{(1 - \sqrt{1 - vv})}{\sqrt{1 - vv}}.$$

Quia autem eſt

$$\int \frac{dv}{v^3\sqrt{1 - vv}} = -\frac{\sqrt{1 - vv}}{2vv} - \frac{1}{2} l(1 + \sqrt{1 - vv}) + \frac{1}{2} l\,v$$

ex

ex hac tertia parte nascitur ista quantitas:

$$-\frac{1}{2vv}l(1+\sqrt{1-vv})+\frac{\sqrt{1-vv}}{4vv}+\frac{1}{4}l(1+\sqrt{1-vv})-\frac{1}{4}lv-\frac{1}{4vv}.$$

His igitur partibus omnibus collectis totum membrum finitrum erit

$$\frac{1+2l2}{4vv}+\frac{1}{4}lv-\frac{\sqrt{1-vv}}{4vv}-\left(\frac{1}{4}+\frac{1}{2vv}\right)l(1+\sqrt{1-vv})+C.$$

§. 25. His inuentis et adiecta constante, qua tota expressio ad nihilum redigatur posito $v=0$, habebimus

$$\int\frac{v\,dv}{[illegible]}-\frac{1}{4}lv=C+\frac{1+2l2}{4vv}-\frac{\sqrt{1-vv}}{4vv}-\left(\frac{1}{4}+\frac{1}{2vv}\right)l(1+\sqrt{1-vv})$$

unde, si statuamus $v=1$, prodit valor seriei quaesitae. Quoniam vero constans C ita debet esse comparata, vt tota expressio euanescat posito $v=0$, insigne incommodum hic se offert, quod posito $v=0$ omnes partes in infinitum excrescant. Nullum autem est dubium, quin singula haec infinita se mutuo destruant, quandoquidem pro C valor determinatus prodire debet. Ad hoc incommodum euitandum recurrendum est ad remedium in omnibus huiusmodi casibus vsitatum, quo quantitas v non ipsi nihilo aequalis, sed infinite parua concipitur; tum enim vtique eueniet, vt omnes termini per vv diuisi se sponte tollant, vti hic clarius ostendemus.

§. 26. Tribuamus igitur litterae v valorem infinite paruum, ac primo habebimus $\sqrt{1-vv}=1-\frac{1}{2}vv$, vnde membrum $-\frac{\sqrt{1-vv}}{4vv}$ dabit $-\frac{1}{4vv}+\frac{1}{8}$; deinde vero erit

$$l(1+\sqrt{1-vv})=l\left(2-\frac{1}{2}vv\right)=l2+l\left(1-\frac{1}{4}vv\right).$$

Est vero

$$l\left(1-\frac{1}{4}vv\right)=-\frac{1}{4}vv$$

sicque habebimus

$$l(1+\sqrt{1-vv})=l2-\frac{1}{4}vv$$

quibus

quibus valoribus ſubſtitutis expreſſio noſtra hanc induet formam:

$$C + \frac{1+2l2}{4vv} - \frac{1}{4vv} + \frac{1}{8} - \left(\frac{1}{4} + \frac{1}{2vv}\right)\left(l2 - \frac{1}{4}vv\right)$$

ex qua, poſtremo membro euoluto, expreſſio noſtra fiet $C + \frac{1}{4} - \frac{1}{4}l2$, ad nihilum redigenda; vnde prodit $C = \frac{1}{4}l2 - \frac{1}{4}$.

§. 27. Definita igitur noſtra conſtante C debitus valor noſtrae expreſſionis $\int\frac{Vdv}{v} - \frac{1}{4}lv$ erit

$$= \frac{1}{4}l2 - \frac{1}{4} + \frac{1+2l2}{4vv} - \frac{\sqrt{1-vv}}{4vv} - \left(\frac{1}{4} + \frac{1}{2vv}\right)l\left(1+\sqrt{1-vv}\right)$$

quam ob rem, ſi hic ſtatuamus $v = 1$, prodibit valor ipſius ſeriei infinitae, cuius ſummam quaerimus, qui ergo erit $= \frac{3}{4}l2$; ita vt tertia columna, cuius ſinguli termini continent productum $\operatorname{cof}.\alpha \operatorname{fin}.\alpha^2$ abeat in hanc ſimplicem expreſſionem $\frac{3}{4}\operatorname{cof}.\alpha \operatorname{fin}.\alpha^2 l2$.

§. 28. Simili modo inueſtigare liceret ſummam ſeriei in quarta columna occurrentis; verum calculus requireretur adhuc multo magis operoſus ac taedioſus quam pro columna tertia, quo autem facile ſuperſedere poterimus, cum iſta columna contineat productum $\operatorname{cof}.\alpha \operatorname{fin}.\alpha^4$, quod, quia $\alpha = \frac{1}{2}\eta$, angulus vero η pro ſatis exiguo aſſumitur, ob poteſtatem quartam $\operatorname{fin}.\alpha^4$ tam paruum erit, vt tuto negligi queat. Caeterum ſatis probabile videtur, coefficientem huius termini pariter proditurum eſſe huius formae $\beta l2$, vbi β erit fractio minor quam $\frac{3}{4}$. Quibus obſeruatis ſequens problema alioquin difficillimum reſoluere poterimus.

Problema.

Si pendulum, dum circa axem A *oſcillatur, tam aſcendendo quam deſcendendo percurrat arcus parum a* 180 *gradibus deficientes, inuenire tempus cuiusque oſcillationis.*

Solutio.

Solutio.

§. 29. Denotet Θ tempus cuiusque oſcillationis, ſi idem pendulum arcus tantum infinite paruos percurreret, nunc autem ſit g punctum, a quo noſtrum pendulum deſcendere incipit. Huius declinationem a ſito verticali Ab poſuimus bA$g = \eta$; tum vero fecimus $\alpha = \frac{1}{2}\eta$. Hinc primo quaeratur valor ſeriei

$$1 + \frac{1^2}{2^2} \text{ſin.}\, \alpha^2 + \frac{1^2 . 3^2}{2^2 . 4^2} \text{ſin.}\, \alpha^4 + \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2} \text{ſin.}\, \alpha^6 + \text{etc.}$$

quae ob angulum α minimum vehementer conuergit, ita vt plerumque ſufficiat ternos quaternosue eius terminos ſumſiſſe. Tum vero tempus vnius oſcillationis quaeſitae, quod littera T indicauimus, erit

$$T = \Theta\, l \cot. \tfrac{1}{2}\alpha \left(1 + \tfrac{1}{4} \text{ſin.}\, \alpha^2 + \tfrac{1 . 9}{4 . 16} \text{ſin.}\, \alpha^4 + \tfrac{1 . 9 . 25}{4 . 16 . 36} \text{ſin.}\, \alpha^6 \right.$$
$$\left. + \text{coſ.}\, \alpha\, l\, 2 + \tfrac{3}{4} \text{coſ.}\, \alpha^2\, l\, 2 \right).$$

vbi meminiſſe oportet eſſe $\Theta = \frac{\pi \sqrt{b}}{\sqrt{2g}}$. Manifeſtum autem eſt ſemiſſem huius temporis praebere tempus cuiusque aſcenſus vel deſcenſus noſtri penduli.

Exemplum.

§. 30. Si pendulum ab angulo bA$g = 5°$ deſcendere incipiat, ideoque in ſequente aſcenſu ad eandem altitudinem aſſurgat, erit $\eta = 5°$ ideoque $\alpha = 2°, 30'$, vnde habebimus l coſ. $\alpha = 9,9995865$, l ſin. $\alpha = 8.6396796$ et

l ſin. $\alpha^2 = 7, 2793592$
l ſin. $\alpha^4 = 4, 5587184$
l ſin. $\alpha^6 = 1, 8380776$.

Ex his igitur computemus primam ſeriem, qua l cot. $\frac{1}{2}\alpha$ afficitur, et quae erit $= 1,0004801$.

§. 31. Quoniam iſtam ſummam in logarithmum hyperbolicum ipſius l cot. $\frac{1}{2}\alpha$ duci oportet, quaeramus primo logarithmum vulgarem iſtius cotangentis $1^\circ, 15'$ qui reperitur $= 1,6611437$ et in logarithmum hyperbolicum conuertitur, ſi multiplicetur per $2,30258509$, hicque totum primum membrum noſtrae expreſſionis conſtabit his tribus factoribus:

$$1,0004801 . 2,3025851 . 1.6611437$$

qui per logarithmos euoluti primum membrum l cot. $\frac{1}{2}\alpha$ involuens praebent $= 3,82675$.

§. 32. Porro pro ſecundo membro notetur logarithmum hyperbolicum binarii eſſe $= 0,69314718$ qui ducatur in coſ. $\alpha =$ coſ. $2^\circ, 30'$ hoc modo

$$l\,0,6931472 = 9,8408253$$
$$\text{et } l \text{ coſ. } \alpha = 9,9995865$$

$$\text{ſumma} = 9,8404118 = l \text{ membr. II.}$$

ſicque erit ipſum membrum ſecundum $= 0,69249$. Hoc deinde ſecundum membrum, ſi ducatur in $\frac{3}{4}$ ſin. α^2, dabit membrum tertium, quod ergo ita reperitur:

l membr.

$$l \text{ membr. II.} = 9,8404118$$
$$l \tfrac{3}{4} = 9,8750613$$
$$l \text{ ſin. } \alpha^2 = 7,2793592$$
$$l \text{ membr. III.} = 6,9948323 \text{ ergo}$$
$$\text{membr. III.} = 0,00099$$

ex quo quia ne vnicam quidem partem milleſimam efficit, manifeſtum eſt ſequentia membra tuto omitti poſſe.

§. 33. His igitur partibus collectis tempus vnius integrae oſcillationis prodibit $T = 4,52023\,\Theta$; vnde ſi iſtud pendulum oſcillationis ſuas infinite paruas ſingulis minutis ſecundis abſoluat, pro eodem pendulo, dum motu ſuo arcum 350 grad. percurrit, tempus cuiusque oſcillationis erit circiter $4\frac{1}{2}$ ſecund. Quodſi pendulum arcus adhuc maiores et propius ad totam circuli peripheriam accedentes abſoluat, tempora oſcillationum vehementer inſuper augebuntur, dum pro tota peripheria, ſiue 360 grad. tempus adeo in infinitum excreſcit: vnde adhuc vnum exemplum euoluamus, quo talis arcus deſcriptus duobus tantum gradibus a peripheria deficit.

Exemplum.

§. 34. Si pendulum ab angulo $b\,A\,g = 1°$ deſcendere incipiat, ideoque in ſequente aſcenſu ad eandem altitudinem aſſurgat, erit $\eta = 1°$ ideoque $\alpha = 30'$; inde habebimus l coſ. $30' = 9,9999835$ et l ſin. $\alpha = 7,9408419$, vnde colligimus l ſin. $\alpha^2 = 5,8816838$; ſin. $\alpha^4 = 1,7633676$. Ex his igitur computemus primam ſeriem l cot. $\frac{1}{2}\alpha$ inuolventem, quae erit $= 1,00001904$.

§. 35. Quoniam istam summam in logarithmum hyperbolicum ipsius cot. $\frac{1}{2}\alpha$ duci oportet, quaeramus primo logarithmum vulgarem ipsius cot. $15'$ qui reperitur $= 2,3601799$, qui in logarithmum hyperbolicum conuertitur, si multiplicetur per $2,30258509$ sicque totum primum membrum nostrae expressionis constabit his partibus $1,00001904 \cdot 2,3601799 \cdot 2,30258509$ quae per logarithmos hunc in modum euoluuntur:

$$
\begin{array}{rl}
l\, 1,00001904 & = 0,0000082 \\
l\, 2,30258509 & = 0,3622157 \\
l\, 2,3601799 & = 0,3729452 \\
\hline
l \text{ membr. I.} & = 0,7351691
\end{array}
$$

ergo membrum I. $= 5,43461$.

§. 36. Porro pro secundo membro logarithmus hyperbolicus binarii ducatur in cos. α hoc modo

$$
\begin{array}{rl}
l\, 0,6931472 & = 9,8408253 \\
l \text{ cos. } \alpha & = 9,9999835 \\
\hline
l \text{ membri II.} & = 9,8408088
\end{array}
$$

ergo membrum II. $= 0,69312$.

Hoc membrum secundum si ducatur in $\frac{3}{4}$ sin. α^2 dabit membrum tertium, quod ergo ita reperietur

$$
\begin{array}{rl}
l \text{ membr. II.} & = 9,8408088 \\
l\, \frac{3}{4} & = 9,8750613 \\
l \text{ sin. } \alpha^2 & = 5,8816838 \\
\hline
l \text{ membri III.} & = 5,5975539
\end{array}
$$

ergo membrum III. $= 0,00004$.

His

His igitur partibus collectis tempus vnius integrae oscillationis prodibit $T = 6,12777\,\Theta$; vnde si oscillationes infinite paruae penduli singulis minutis secundis peragantur, tempus cuiusque oscillationis eiusdem penduli, dum arcus 358° absoluit, erit 6 secund.

§. 37. Hinc patet, quo minor accipiatur angulus α tempus vnius oscillationis non solum fieri maius, sed etiam minori opera assignari posse; cum contra, quo maior fuerit angulus α, inuestigatio temporis multo maiorem laborem requirat. Quin etiam, si angulus α tantus accipiatur, vt termini $\sin.\alpha^6$ inuoluentes non amplius negligi queant, ope huius methodi tempus ne quidem accurate assignare posset, propterea quod summae serierum quartae et sequentium columnarum nimis intricatos calculos postularent; neque vllum artificium Analyticum adhuc est inuentum, quo labor iste subleuari posset. Huiusmodi autem casibus series in ipsa dissertatione tradita negotium multo commodius conficiet: quoniam enim tum angulus $\zeta = 180 - \eta$ iam ita erit comparatus, vt quantitas

$$c = \sin.\tfrac{1}{2}\zeta = \cos.\tfrac{1}{2}\eta$$

satis notabiliter ab vnitate deficiat, series ibi inuenta

$$1 + \frac{1^2}{2^2}cc + \frac{1^2.3^2}{2^2.4^2}c^4 + \frac{1^2.3^2.5^2}{2^2.4^2.6^2}c^6 + \text{etc.}$$

satis conuerget, ita vt eius summa, pluribus terminis actu inuicem addendis, satis exacte assignari possit, quae deinde ducta in Θ tempus vnius oscillationis indicabit.

§. 38. Caeterum hoc additamentum circa oscillationes amplissimas, vbi totus arcus a pendulo descriptus propemodum ad totam circuli peripheriam augetur, eo maiori ſtudio pertractare eſt viſum, quod omnes, qui pendulorum motus ſunt perſcrutati, iſtum caſum plane non attigerunt. Interim tamen eſt manifeſtum, iſtum caſum ſummam attentionem mereri, propterea quod ſine ſingularibus artificiis, tam in ipſo calculo, quam integrationibus peragendis reſolutionem nullo modo exſpectare liceat, tum vero etiam ſerierum quartae columnae et ſequentium reſolutio, quam hic praetermittere ſumus coacti, Geometris anſam praebere poterit, iſtam egregiam partem Analyſeos vlterius promouendi.

PHYSICA.