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De valore formulae integralis $\int (x^{a-1} dx)/(\log x) \cdot (1-x^b)(1-x^c)/(1-x^n)$ a termino $x=0$ usque ad $x=1$ extensae

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DE VALORE FORMULAE INTEGRALIS

$$\int \frac{x^{a-1} dx}{lx} \cdot \frac{(1-x^b)(1-x^c)}{1-x^n}$$

A TERMINO $x=0$ VSQVE AD $x=1$ EXTENSAE.

Auctore

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§. 1.

Quae non ita pridem de integratione eiusmodi formularum differentialium, in quarum denominatore occurrit $1/x$, in medium attuli, ubi ostendi, valorem huius formulae integralis: $\int \frac{x^{a-1} - x^{b-1}}{lx} dx$ ab $x=0$ ad $x=1$ extensum esse $= l \frac{a}{b}$, non solum summa attentione digna, sed etiam quasi nouum campum in methodo integrandi aperire sunt visa; propterea quod huiusmodi formularum integratio prorsus singularia artificia postulat, at ex principiis etiam nunc parum cognitis erat deducta. Tunc quidem temporis ista inuestigatio non admodum late patere videbatur, dum praeter formulam modo allegatam ad paucas alias eam mihi quidem extendere licuit, nunc autem, postquam hoc argumentum accuratius sum perscrutatus, deprehendi, formulam multo generaliorem, eam scilicet quae hic in titulo conspicitur, pari successu expediri posse. Quin etiam methodus, quam hic sum expositurus, etiam ad formulas adhuc generales facili extendi potest; vnde haud contemnenda incrementa in vniuersam Analysin redundare videntur.

§. 2. Designemus igitur littera S valorem formulae propositae, quem scilicet induit, si eius integratio a termino $x=0$ vsque ad $x=1$ extendatur, ita ut sit

$$S = \int \frac{x^{a-1} dx}{1-x^n} \cdot \frac{(1-x^b)(1-x^c)}{1-x^n} \quad \left[\begin{array}{l} \text{ab } x=0 \\ \text{ad } x=1 \end{array} \right]$$

ad quem valorem inuestigandum ante omnia obseruari conuenit, fractionem $\frac{(1-x^b)(1-x^c)}{1-x^n}$ ita esse comparatam,

ut posito $x=1$ penitus euanescat. Cum enim in nume-
ratore tam $(1-x^b)$ quam $(1-x^c)$ factorem $(1-x)$
inuoluat, ideoque totus numerator factorem habeat $(1-x)^2$,
dum in denominatore tantum factor simplex $1-x$ inest,
euidens est, posito $x=1$ totam fractionem euanescere de-
bere, id quod etiam inde intelligitur, quod casu $x=1$
tam numerator quam denominator euanescit; unde, si iuxta
regulam notissimam tam loco numeratoris, qui euolutus est
 $1-x^b-x^c+x^{b+c}$, quam loco denominatoris vtriusque diffe-
rentialia scribantur, prodit ista fractio:

illius $\frac{bx^{b-1}-cx^{c-1}+(b+c)x^{b+c-1}}{1-nx^{n-1}}$ apud
illius aequalis casu $x=1$, posito autem $x=1$, ista fractio
abit in hanc: $\frac{-b-c+b+c}{-n}$, quae manifesto est $=0$.

§. 3. Cum numerator fractionis modo consideratae
sit $1-x^b-x^c+x^{b+c}$, si is per $1-x^n$ diuidatur, ex qua-
ternis terminis orientur quatuor sequentes series geometri-
cae infinitae:

- I. $1+x^n+x^{2n}+x^{3n}+x^{4n}+x^{5n}+\text{etc.}$
- II. $-x^b-x^{n+b}-x^{2n+b}-x^{3n+b}-x^{4n+b}-x^{5n+b}-\text{etc.}$
- III. $-x^c-x^{n+c}-x^{2n+c}-x^{3n+c}-x^{4n+c}-x^{5n+c}-\text{etc.}$
- IV. $x^{b+c}+x^{n+b+c}+x^{2n+b+c}+x^{3n+b+c}+x^{4n+b+c}+x^{5n+b+c}+\text{etc.}$

Harum

Hanc igitur seriem singulos terminos duci oportet in formulam $\frac{x^m}{1-x}$; tum enim omnium integralia, ab $x=0$ ad $x=1$ extensa, si in vnam summam colligantur, dabunt valorem quaesitum littera S. designatum.

S. 4. Hoc ergo modo totum negotium reducitur ad integrationem talis formulae $\frac{x^m dx}{1-x}$, ab $x=0$ ad $x=1$ extendendam. Haec autem formula continet fundamentum principale, unde omnia, quae olim de hoc argumento commentatus, sunt deducta; tum autem ad eius integrale inueniendum vsus sum doctrina circa functiones duarum variorum versante, quam ad praesens institutum non satis commodum applicare liceret; quamobrem hic aliam methodum in medium sum allaturus, cuius beneficio ista integratio, qua indigemus, multo facilius et clarius institui poterit, et qua simul omnia, quae huc pertinent, haud mediocriter illustrabuntur.

S. 5. Cum sit $1x^m = m 1x$, si littera e denotet numerum, cuius logarithmus hyperbolicus unitati aequatur, posito breuitatis gratia $m 1x = y$ erit $1x^m = y = y e$, hinc facillime fiet $1x^m = e^y = e^{m 1x}$. Cum igitur per seriem geometricam sit

$$e^y = 1 + y + \frac{y^2}{1 \cdot 2} + \frac{y^3}{1 \cdot 2 \cdot 3} + \frac{y^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.}$$

erit pro nostro casu

$x^m = 1 + \frac{m 1x}{1} + \frac{m m}{1 \cdot 2} (1x)^2 + \frac{m m m}{1 \cdot 2 \cdot 3} (1x)^3 + \frac{m m m m}{1 \cdot 2 \cdot 3 \cdot 4} (1x)^4 + \text{etc.}$
 hac igitur serie in vsum vocata erit

$$\frac{x^m}{1-x} = \frac{1}{1-x} + \frac{m}{1} \frac{1x}{1-x} + \frac{m m}{1 \cdot 2} \frac{(1x)^2}{1-x} + \frac{m m m}{1 \cdot 2 \cdot 3} \frac{(1x)^3}{1-x} + \frac{m m m m}{1 \cdot 2 \cdot 3 \cdot 4} \frac{(1x)^4}{1-x} + \text{etc.}$$

Huius igitur seriei singulos terminos in dx ductos integrari proponet, unde quidem ex termino primo orietur formula $\int \frac{dx}{1-x}$, cuius valorem, ab $x=0$ ad $x=1$ extensum, esse infi-

infinitum ostendi, cuius loco hic ubique scribamus characterem Δ ; tum vero ex termino secundo oritur integrale

§. 6. Pro integralibus ex reliquis terminis oriundis ex elementis calculi integralis satis liquet, si integralia ab $x=0$ ad $x=1$ extendantur, fore ut sequitur:
 $\int dx \log x = -1$; $\int dx (1/x)^2 = +1/2$; $\int dx (1/x)^3 = -1/2 \cdot 3$;
 $\int dx (1/x)^4 = +1/2 \cdot 3 \cdot 4$; $\int dx (1/x)^5 = -1/2 \cdot 3 \cdot 4 \cdot 5$ etc.
 his igitur valoribus substitutis reperiemus fore

$$\int \frac{x^m dx}{1/x} = \Delta + m - \frac{m \cdot m}{2} + \frac{m^2}{3} - \frac{m^3}{4} + \frac{m^4}{5} - \frac{m^5}{6} + \frac{m^6}{7} - \text{etc.}$$

Ex doctrina autem logarithmorum constat esse

$$\log(1 + m) = m - \frac{m \cdot m}{2} + \frac{m^3}{3} - \frac{m^4}{4} + \text{etc.}$$

quo valore substituto habebimus

$$\int \frac{x^m dx}{1/x} = \Delta + \log(1 + m)$$

qui ergo est valor huius formulae integralis a termino $x=0$ ad $x=1$ extensae, quos terminos in sequentibus semper subintelligi oportet, unde eos non amplius commemorabimus.

§. 7. Iste quidem valor integralis insigni incommodo laborare videtur, propterea quod characterem Δ implicat, cuius valor non solum est incognitus, sed adeo infinitus; verum quia pro omnibus huiusmodi formulis perpetuo idem manet, ita ut sit

$$\int \frac{x^n dx}{1/x} = \Delta + \log(1 + n)$$

evidens est, si harum formularum altera ab altera subtrahatur, istum characterem penitus ex calculo egredi, ac prodire

$$\int \frac{x^m - x^n}{1/x} dx = \log \frac{1+m}{1+n}, \text{ qui est ille ipse casus, ad quem}$$

pri-

primo initio sum perductus. Quo autem clarius appareat, quibusnam casibus iste character Δ penitus ex calculo sit excellendus, contemplemur hanc formam indefinitam:

$$X = A x^{\alpha} + B x^{\beta} + C x^{\gamma} + D x^{\delta} + E x^{\epsilon} + \text{etc.}$$

se per integrale illud inuentum erit

$$\int \frac{x^{\Delta} dx}{1-x} = A + B \Delta + C \Delta + D \Delta + \text{etc.} \\ + A/(1+\alpha) + B/(1+\beta) + C/(1+\gamma) + D/(1+\delta) + \text{etc.}$$

Quocirca, si coefficientes A, B, C, D etc. ita fuerint comparati, ut sit $A + B + C + D + \text{etc.} = 0$, semper istud integrale ita exprimetur:

$$\int \frac{x^{\Delta} dx}{1-x} = A/(1+\alpha) + B/(1+\beta) + C/(1+\gamma) + D/(1+\delta) + \text{etc.}$$

perinde ac si formula canonica fuisset $\int \frac{x^m dx}{1-x} = l(1+m)$,

reiecto characterē Δ .

§. 6. Quoties igitur fuerit

$$X = A x^{\alpha} + B x^{\beta} + C x^{\gamma} + D x^{\delta} + \text{etc.}$$

existente $A + B + C + D + \text{etc.} = 0$, tum integrale $\int \frac{x^{\Delta} dx}{1-x}$, non amplius characterē Δ inquinabitur, atque singulas integrationes ita instituire licebit, quasi reuera foret

$$\int \frac{x^m dx}{1-x} = l(1+m).$$

Cum igitur tenes $A + B + C + D + \text{etc.}$ exhibeat valorem ipsius X , si ponatur $x=1$, manifestum est, istam integrationem perpetuo succedere, si X eiusmodi exprimat functionem ipsius x , ut posito $x=1$ ea in nihilum abeat. Quare cum formula, quam hic tractare suscepimus

$$X = \frac{x^{a-1} (1-x^b) (1-x^c)}{1-x^n}$$

vis iam obseruamus ad nihilum redigitur posito $x=1$, eius integrationem rite absoluere licebit ope formulae canonicae $\int \frac{x^m dx}{1-x} = l(1+m)$, nullo scilicet respectu habito ad characterem Δ initio introductum.

§. 9. Quoniam igitur iam supra perducti sumus ad quatuor series infinitas, quas per formulam $\frac{x^n - dx}{1-x}$ multiplicari, tum vero integrari oportet, si hanc operationem in singulis terminis instituamus, valor quaesitus S per sequentes quatuor series infinitas expressus reperiatur:

$$\begin{aligned} & \text{I. } l(a) + l(a+n) + l(a+2n) + l(a+3n) + l(a+4n) + \text{etc.} \\ & \text{II. } -l(a+b) - l(a+b+n) - l(a+b+2n) - l(a+b+3n) \\ & \quad - l(a+b+4n) - \text{etc.} \\ & \text{III. } -l(a+c) - l(a+c+n) - l(a+c+2n) - l(a+c+3n) \\ & \quad - l(a+c+4n) - \text{etc.} \\ & \text{IV. } l(a+b+c) + l(a+b+c+n) + l(a+b+c+2n) + l(a+b+c+3n) \\ & \quad + l(a+b+c+4n) + \text{etc.} \end{aligned}$$

Hoc igitur modo tota quaestio huc est reducta, ut expressiones finitae investigentur, quae istis logarithmorum seriebus infinitis sint aequales.

§. 10. Cum igitur valor quaesitus S infinitis logarithmis aequalis sit inuentus, eum ipsum tanquam logarithmum spectari conveniet, quamobrem statuamus $S = lO$ atque a logarithmis ad numeros regrediendo valor ipsius O sequenti modo per factores exprimi deprehendetur:

$$O = \frac{a(a+b+c)}{(a+b)(a+c)} \cdot \frac{(c+n)(c+n+n)}{(c+n+n)(c+n+n)} \cdot \frac{(a+2n)(a+b+c+2n)}{(a+b+2n)(a+c+2n)} \cdot \frac{(a+3n)(c+n+c+3n)}{(a+b+3n)(a+c+3n)} \text{ etc.}$$

quam expressionem in membra puncto separata distinximus, quorum quodlibet continet binos factores in numeratore, totidemque in denominatore, qui factores in singulis membris ita sunt comparati, ut summa factorum numeratoris semper aequalis sit summae factorum denominatoris. Praeterea vero notetur, sumendo i pro numero infinito membrum infinitesimum esse $\frac{(a+in)(a+b+c+in)}{(a+b+in)(a+c+in)}$, quod evolutum praebet $\frac{a(a+b+c) + in(a+b+c) + in^2}{(a+b)(a+c) + in(a+b+c) + in^2}$, cuius valor ob partes primas finitas evanescens manifesto unitati aequatur.

unde intelligitur, hanc expressionem valorem finitum determinatam esse habituram, et quo plura membra sunt, et si unicum ducantur, eo propius continuo ad valorem illius O appropinquatum iri, quandoquidem membra illa, remota continuo minus ab unitate discrepant.

Sed nunc in verum valorem litterae O inquiramus, in quodvis vocemus insignis Lemma, cuius veritatem iam in calculo integrali fufius demonftravi, quod ita se habet: Si ponatur

$$P = \int x^{m-n} dx (1-x^n)^{-n} \text{ et}$$

$$Q = \int x^{m-n} dx (1+x^n)^{-n}$$

$$P = \frac{(m-p)q}{(p+n)(m+n)} + \frac{(p+n)(q+n)}{(p+2n)(m+2n)} + \frac{(m+p+3n)(q+3n)}{(p+3n)(m+q+3n)} \text{ etc.}$$

quae expressio pariter ex infinitis membris constat, in quorum singulis tam numerator quam denominator etiam binis factoribus constat, prout in nostra expressio pro O tenetur, unde haud difficulter litterae p , q et m ita defini poterunt, ut prodeat $O = \frac{P}{Q}$, si quidem littera n vtriusque eundem significatum retinet, hocque modo valor litterae O saltem ad formulas integrales ordinarias P et Q reducitur.

Hic autem probe est recordandum, singulas litteras p , q , m et n numeros positivos designare debere, id quod etiam de nostris litteris a , b et c est tenendum, quandoquidem formula nostra canonica $\int \frac{x^m dx}{1+x} = \log(1+x)$ cum

veritate consistere nequit, nisi $1+m$ fuerit numerus positivus, quia alioquin logarithmi numerorum negativorum huc prodeuntes forent imaginarii.

§. 12. Ad hanc conformitatem $\frac{P}{Q}$ et O constituentem, sufficet membra prima, quae sunt $\frac{a(q+c)}{(a+b)(a+c)}$ et $\frac{(m+p)q}{p(m+q)}$, ad

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iden-

identitatem perduxisse, propterea quod deinceps omnia sequentia membra sponte inter se conuenient. Ista autem identitas duplici modo obtineri poterit: sumto enim $q=a$ vel statui poterit $m+q=a+b$ vel $m+q=a+c$, ita ut priori modo sit $m=b$, posteriori vero modo $m=c$; at vero tum pro priori modo erit $p=a+c$, unde sponte fiet $m+p=a+b+c$; pro posteriori vero modo, quo $m=c$, sumi debet $p=a+b$, unde denuo sponte fit $m+p=a+b+c$; quam ob rem hinc geminos valores pro p et q nanciscemur, unde etiam geminae solutiones orientur, quae sunt:

$$\text{I. Solutio} \quad \begin{cases} P = \int x^{a+c-1} dx (1-x^n)^{\frac{b-n}{n}} \\ Q = \int x^{a-1} dx (1-x^n)^{\frac{b-n}{n}} \end{cases}$$

$$\text{II. Solutio} \quad \begin{cases} P = \int x^{a+b-1} dx (1-x)^{\frac{c-n}{n}} \\ Q = \int x^{a-1} dx (1-x)^{\frac{c-n}{n}} \end{cases}$$

utrinque enim erit $O = \frac{P}{Q}$, et cum sit $S = IO$, erit $S = IP - IQ$ ficque valorem ipsius S per formulas finitas expressum inuenimus.

§. 13. Circa valores autem litterarum p et q duos casus imprimis memorabiles notari conuenit, quibus eos adeo absolute exhibere licet: alter enim praebet

$$\text{I. } \int x^{n-1} dx (1-x^n)^{\frac{m-n}{n}} = \frac{1}{m}$$

alter vero in hoc consistit ut sit

$$\text{II. } \int x^{n-m-1} dx (1-x^n)^{\frac{m-n}{n}} = \frac{\pi}{n \sin \frac{m\pi}{n}}$$

ubi π denotat 180° , siue semiperipheriam circuli, cuius radius $= 1$. Quare cum pro nostra solutione priore sit $m=b$

$n=b$, videamus, utrum p et q ad istos valores absolutos reducere liceat. Hoc autem evenit quando $b=c$ et insuper $a=n=b$, quo casu ambae solutiones inter se congruunt, quem ergo casum seorsim evoluisse operae pretium est.

Evolutio casus quo $c=b$ et $a=n=b$.

§. 14. Hoc igitur casu erit formula proposita

$$S = \int \frac{x^{n-b} dx (1-x^n)^{\frac{b-n}{n}}}{1-x^n}$$

cum vero vidimus esse

$$P = \int x^{n-1} dx (1-x^n)^{\frac{b-n}{n}} = \frac{1}{b} \text{ et}$$

$$Q = \int x^{n-b-1} dx (1-x^n)^{\frac{b-n}{n}} = \frac{\pi}{n \sin \frac{b\pi}{n}}$$

quam ob rem, cum sit $S = P - Q = \frac{1}{b} - \frac{\pi}{n \sin \frac{b\pi}{n}}$, erit his valoribus

substitutis $S = \frac{1}{b} - \frac{\pi}{n \sin \frac{b\pi}{n}}$, vbi evidens est esse debere $b < n$,

unde sequentia exempla considerare iuvabit.

Exemplum I. quo $b=1$ et $n=2$.

§. 15. Hoc ergo casu erit $\sin \frac{b\pi}{n} = 1$, hincque $S = \frac{1}{b} - \frac{\pi}{n}$, quam ob rem, si formula proposita fuerit $S = \int \frac{x^{n-b} dx (1-x^n)^{\frac{b-n}{n}}}{1-x^n}$, erit $S = \frac{1}{b} - \frac{\pi}{n}$; at vero valorem ipsius S per logarithmos evoluendo, ut supra fecimus, ob $a=1$, $b=c=1$ et $n=2$ prodibit

$$S = \begin{cases} 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \text{etc.} \\ - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \frac{1}{10} - \text{etc.} \\ + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{12} + \text{etc.} \end{cases}$$

quibus in ordinem reductis erit

$$S = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} \text{ etc.}$$

Vicissim igitur, si proponatur ista series
 $1 - \frac{1}{2^\pi} + \frac{1}{3^\pi} - \frac{1}{4^\pi} + \frac{1}{5^\pi} - \frac{1}{6^\pi} + \frac{1}{7^\pi} \text{ etc.}$
 eius summa assignari poterit. Cum enim sit $S_{-2} s = -1 = 0$,
 ob $S = \frac{1}{\pi}$ erit $s = \frac{1}{\pi} = \sqrt{\frac{1}{\pi}}$; siue cum sit $\pi > 2$ erit
 $s = -\sqrt{\frac{1}{\pi}}$; ista scilicet summa, erit negativa.

Exemplum II, quo $b = 1$ et $n = 3$.

§. 17. Hoc igitur casu, quo $a = 2$, formula integra proposita erit

$$S = \int \frac{x dx}{1-x^2} = \frac{(1-x^2)^{-1/2}}{-1/2} = -\frac{2}{1-x^2} = -\frac{2}{1-x^2}$$
 deinde cum sit fin. $\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, valor quaesitus erit $S = \frac{1}{2\pi} \sqrt{3}$; at
 vero idem valor S per seriem logarithmorum expressus ob
 $a=2$; $b=c=1$ et $n=3$ erit
 $\frac{1}{2\pi} \left(\frac{1}{2} + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \frac{1}{216} + \frac{1}{343} + \frac{1}{512} + \frac{1}{729} + \frac{1}{1000} + \frac{1}{1331} + \frac{1}{1728} + \frac{1}{2197} + \frac{1}{2744} + \frac{1}{3375} + \frac{1}{4096} + \frac{1}{5000} + \frac{1}{6000} + \frac{1}{7000} + \frac{1}{8000} + \frac{1}{9000} + \frac{1}{10000} + \frac{1}{11000} + \frac{1}{12000} + \frac{1}{13000} + \frac{1}{14000} + \frac{1}{15000} + \frac{1}{16000} + \frac{1}{17000} + \frac{1}{18000} + \frac{1}{19000} + \frac{1}{20000} + \frac{1}{21000} + \frac{1}{22000} + \frac{1}{23000} + \frac{1}{24000} + \frac{1}{25000} + \frac{1}{26000} + \frac{1}{27000} + \frac{1}{28000} + \frac{1}{29000} + \frac{1}{30000} + \frac{1}{31000} + \frac{1}{32000} + \frac{1}{33000} + \frac{1}{34000} + \frac{1}{35000} + \frac{1}{36000} + \frac{1}{37000} + \frac{1}{38000} + \frac{1}{39000} + \frac{1}{40000} + \frac{1}{41000} + \frac{1}{42000} + \frac{1}{43000} + \frac{1}{44000} + 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\frac{1}{100000} + \frac{1}{101000} + \frac{1}{102000} + \frac{1}{103000} + \frac{1}{104000} + \frac{1}{105000} + \frac{1}{106000} + \frac{1}{107000} + \frac{1}{108000} + \frac{1}{109000} + \frac{1}{110000} + \frac{1}{111000} + \frac{1}{112000} + \frac{1}{113000} + \frac{1}{114000} + \frac{1}{115000} + \frac{1}{116000} + \frac{1}{117000} + \frac{1}{118000} + \frac{1}{119000} + \frac{1}{120000} + \frac{1}{121000} + \frac{1}{122000} + \frac{1}{123000} + \frac{1}{124000} + \frac{1}{125000} + \frac{1}{126000} + \frac{1}{127000} + \frac{1}{128000} + \frac{1}{129000} + \frac{1}{130000} + \frac{1}{131000} + \frac{1}{132000} + \frac{1}{133000} + \frac{1}{134000} + \frac{1}{135000} + \frac{1}{136000} + \frac{1}{137000} + \frac{1}{138000} + \frac{1}{139000} + \frac{1}{140000} + \frac{1}{141000} + \frac{1}{142000} + \frac{1}{143000} + \frac{1}{144000} + \frac{1}{145000} + \frac{1}{146000} + \frac{1}{147000} + \frac{1}{148000} + \frac{1}{149000} + \frac{1}{150000} + \frac{1}{151000} + 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\frac{1}{256000} + \frac{1}{257000} + \frac{1}{258000} + \frac{1}{259000} + \frac{1}{260000} + \frac{1}{261000} + \frac{1}{262000} + \frac{1}{263000} + \frac{1}{264000} + \frac{1}{265000} + \frac{1}{266000} + \frac{1}{267000} + \frac{1}{268000} + \frac{1}{269000} + \frac{1}{270000} + \frac{1}{271000} + \frac{1}{272000} + \frac{1}{273000} + \frac{1}{274000} + \frac{1}{275000} + \frac{1}{276000} + \frac{1}{277000} + \frac{1}{278000} + \frac{1}{279000} + \frac{1}{280000} + \frac{1}{281000} + \frac{1}{282000} + \frac{1}{283000} + \frac{1}{284000} + \frac{1}{285000} + \frac{1}{286000} + \frac{1}{287000} + \frac{1}{288000} + \frac{1}{289000} + \frac{1}{290000} + \frac{1}{291000} + \frac{1}{292000} + \frac{1}{293000} + \frac{1}{294000} + \frac{1}{295000} + \frac{1}{296000} + \frac{1}{297000} + \frac{1}{298000} + \frac{1}{299000} + \frac{1$

$$S = \left\{ \begin{array}{l} 12 + 15 + 18 + 21 + 24 + 27 + \dots \\ 213 + 216 + 219 + 2112 + 2115 + \dots \\ + 14 + 17 + 110 + 113 + 116 + 119 + \dots \end{array} \right.$$

fique ergo erit

$$S = 12 - 213 + 14 + 15 - 216 + 17 + 18 - 219 + 110 + 111 - 2112 + 113 + 114 \text{ etc.}$$

curus ergo feriei satis regularis summa est $S = \frac{\pi \sqrt{3}}{2\pi}$.

Exemplum III. quo $b = 2$ et $n = 3$ et $c = 2$ sit.

§. 18. Hoc igitur casu erit $a = 1$ et formula nos

fra integralis fiet

$$S = \int \frac{(1-x^2)(2x+3x^2)^2}{(1+x^2)^2} dx = \int \frac{(1-x^2)(4x^2+12x^3+9x^4)}{(1+x^2)^2} dx$$

cuius ergo valor erit $S = \sqrt{\frac{1+x}{1-x}}$, at vero idem valor S pe-

feriem logarithmorum expressus ob $a=1$, $b=c=2$ e

$$n = 3, \text{ erit}$$

$$C = 71 + 74 + 77 + 79 + 713 + \text{etc.}$$

$$S = \begin{cases} 1/1 + 1/4 + 1/9 + 1/16 + \dots \\ -2/3 - 2/6 - 2/9 - 2/12 - \dots \\ 1/5 + 1/8 + 1/11 + 1/14 + \dots \end{cases} \quad \text{sic}$$

21. $(+15 + 18 + 111 + 114 + \text{etc.})$

ergo erit $S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} + \frac{1}{2048} + \frac{1}{4096} + \frac{1}{8192} + \frac{1}{16384} + \frac{1}{32768} + \frac{1}{65536} + \frac{1}{131072} + \frac{1}{262144} + \frac{1}{524288} + \frac{1}{1048576} + \frac{1}{2097152} + \frac{1}{4194304} + \frac{1}{8388608} + \frac{1}{16777216} + \frac{1}{33554432} + \frac{1}{67108864} + \frac{1}{134217728} + \frac{1}{268435456} + \frac{1}{536870912} + \frac{1}{1073741824} + \frac{1}{2147483648} + \frac{1}{4294967296} + \frac{1}{8589934592} + \frac{1}{17179869184} + \frac{1}{34359738368} + \frac{1}{68719476736} + \frac{1}{137438953472} + \frac{1}{274877906944} + \frac{1}{549755813888} + \frac{1}{1099511627776} + \frac{1}{2199023255552} + \frac{1}{4398046511104} + \frac{1}{8796093022208} + \frac{1}{17592186044416} + \frac{1}{35184372088832} + \frac{1}{70368744177664} + \frac{1}{140737488355328} + \frac{1}{281474976710656} + \frac{1}{562949953421312} + \frac{1}{1125899906842624} + \frac{1}{2251799813685248} + \frac{1}{4503599627370496} + \frac{1}{9007199254740992} + \frac{1}{18014398509481984} + \frac{1}{36028797018963968} + \frac{1}{72057594037927936} + \frac{1}{144115188075855872} + \frac{1}{288230376151711744} + \frac{1}{576460752303423488} + \frac{1}{1152921504606846976} + \frac{1}{2305843009213693952} + \frac{1}{4611686018427387904} + \frac{1}{9223372036854775808} + \frac{1}{18446744073709551616} + \frac{1}{36893488147419103232} + \frac{1}{73786976294838206464} + \frac{1}{147573952589676412928} + \frac{1}{295147905179352825856} + \frac{1}{590295810358705651712} + \frac{1}{1180591620717411303424} + \frac{1}{2361183241434822606848} + \frac{1}{4722366482869645213696} + \frac{1}{9444732965739290427392} + \frac{1}{18889465931478580854784} + \frac{1}{37778931862957161709568} + \frac{1}{75557863725914323419136} + \frac{1}{151115727451828646838272} + \frac{1}{302231454903657293676544} + \frac{1}{604462909807314587353088} + \frac{1}{1208925819614629174706176} + \frac{1}{2417851639229258349412352} + \frac{1}{4835703278458516698824704} + \frac{1}{9671406556917033397649408} + \frac{1}{19342813113834066795298816} + \frac{1}{38685626227668133590597632} + \frac{1}{77371252455336267181195264} + \frac{1}{154742504910672534362390528} + \frac{1}{309485009821345068724781056} + \frac{1}{618970019642690137449562112} + \frac{1}{1237940039285380274899124224} + \frac{1}{2475880078570760549798248448} + \frac{1}{4951760157141521099596496896} + \frac{1}{9903520314283042199192993792} + \frac{1}{19807040628566084398385987584} + \frac{1}{39614081257132168796771975168} + \frac{1}{79228162514264337593543950336} + \frac{1}{158456325028528675187087900672} + \frac{1}{316912650057057350374175801344} + \frac{1}{633825300114114700748351602688} + \frac{1}{1267650600228229401496703205376} + \frac{1}{2535301200456458802993406410752} + \frac{1}{5070602400912917605986812821504} + \frac{1}{10141204801825835211973625643008} + \frac{1}{20282409603651670423947251286016} + \frac{1}{40564819207303340847894502572032} + \frac{1}{81129638414606681695789005144064} + \frac{1}{162259276829213363391578010288128} + \frac{1}{324518553658426726783156020576256} + \frac{1}{649037107316853453566312041152512} + \frac{1}{1298074214633706907132624082305024} + \frac{1}{2596148429267413814265248164610048} + \frac{1}{5192296858534827628530496329220096} + \frac{1}{10384593717069655257060992658440192} + \frac{1}{20769187434139310514121985316880384} + \frac{1}{41538374868278621028243970633760768} + \frac{1}{83076749736557242056487941267521536} + \frac{1}{166153499473114484112975882535043072} + \frac{1}{332306998946228968225951765070086144} + \frac{1}{664613997892457936451903530140172288} + \frac{1}{1329227995784915872903807060280344576} + \frac{1}{2658455991569831745807614120560689152} + \frac{1}{5316911983139663491615228241121378304} + \frac{1}{10633823966279326983230456482242756608} + \frac{1}{21267647932558653966460912964485513216} + \frac{1}{42535295865117307932921825928971026432} + \frac{1}{85070591730234615865843651857942052864} + \frac{1}{170141183460469231731687303715884105728} + \frac{1}{340282366920938463463374607431768211456} + \frac{1}{680564733841876926926749214863536422912} + \frac{1}{1361129467683753853853498429727072845824} + \frac{1}{2722258935367507707706996859454145691648} + \frac{1}{5444517870735015415413993718908291383296} + \frac{1}{10889035741470030830827987437816582766592} + \frac{1}{21778071482940061661655974875633165533184} + \frac{1}{43556142965880123323311949751266331066368} + \frac{1}{87112285931760246646623899502532662132736} + \frac{1}{174224571863520493293247799005065324265472} + \frac{1}{348449143727040986586495598010130648530944} + \frac{1}{696898287454081973172991196020261297061888} + \frac{1}{1393796574908163946345982392040522594123776} + 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\frac{1}{926336713898529563388567880069503262826159877325124512315660672063305037119488} + \frac{1}{1852673427797059126777135760139006525652319754650249024631321344126610074238976} + \frac{1}{3705346855594118253554271520278013051304639509300498049262642688253220148477952} + \frac{1}{741069371118823650710854304055602610260927901860099$

§. 21. Praeter hos autem casus, quibus ambas formulas P et Q simul integrationem admittere obseruauimus, pro cento affirmare licet, nullos alios insuper dari, quibus hoc eveniat. Interim tamen dantur innumerabiles alii casus, quibus valor nostrae formulae integralis S absolute sine formulis integralibus assignari potest, etiamsi neutra formularum P et Q seorsim integrari queat, qui casus cum per se sint notatu dignissimi, illis inuestigandis sequens problema destinemus.

Problema.

Inuestigare casus, quibus formulae integralis propositae valorem S absolute sine formulis integralibus exprimere licet.

Solutio.

§. 22. Totum ergo negotium huc redit, vt eiusmodi relationes inter exponentes a, b, c et n eruantur, quibus fractio supra adhibita $\frac{P}{Q}$ absolute exprimi queat, quamuis neutra harum formularum seorsim integrationem admittat; tum enim formulae propositae valor quaesitus erit $S = \int \frac{P}{Q}$. Verum istam fractionem $\frac{P}{Q}$ vidimus designare istud productum in infinitum excurrere

$$\frac{P}{Q} = \frac{a(a+b+c)}{(a+b)(a+c)} \cdot \frac{(a+n)(a+b+c+n)}{(a+b+n)(a+c+n)} \cdot \frac{(a+2n)(a+b+c+2n)}{(a+b+2n)(a+c+2n)} \text{ etc.}$$

§. 23. Nunc vero meminisse iuuabit, tam sinus quam cosinus angulorum per huiusmodi producta infinita exprimi solere; cum enim sit

$$\sin. \frac{p\pi}{2r} = \frac{p\pi}{2r} \cdot \frac{4rr-pp}{4rr} \cdot \frac{16rr-pp}{16rr} \cdot \frac{36rr-pp}{36rr} \text{ etc.}$$

erit duabus huiusmodi expressionibus combinandis

$$\frac{\sin. \frac{p\pi}{2r}}{\sin. \frac{q\pi}{2r}} = \frac{p}{q} \cdot \frac{4rr-pp}{4rr-qq} \cdot \frac{16rr-pp}{16rr-qq} \cdot \frac{36rr-pp}{36rr-qq} \cdot \frac{64rr-pp}{64rr-qq} \text{ etc.}$$

Quare si superior expressio pro $\frac{P}{Q}$ inuenta ad hanc formam reuo-

reducatur, queat, tum utique erit

$$S = \int \sin \frac{p\pi}{2r} - \int \sin \frac{q\pi}{2r}$$

Quo autem ista reductio facilius succedat, posteriorem expressionem hac forma repraesentemus:

$$\frac{\sin \frac{p\pi}{2r}}{\sin \frac{q\pi}{2r}} = \frac{p(2r-p)(2r+p)(4r-p)(4r+p)(6r-p)}{q(2r-q)(2r+q)(4r-q)(4r+q)(6r-q)} \text{ etc.}$$

cuius expressionis membra manifesto ita progrediuntur, ut singuli factores tam numeratorum quam denominatorum continuo eodem incremento $2r$ augeantur. Quare cum in expressione $\frac{p}{q}$ singuli factores capiant incrementum n , statim debet $n = 2r$, quo notato sufficit prima membra ad conformitatem redigere, id quod eveniet sumendo

$$a = p; a + b + c = 2r - p; a + b = q; a + c = 2r - q$$

unde singulae litterae colliguntur

$$1^\circ. a = p; 2^\circ. b = q - p; 3^\circ. c = 2r - p - q$$

existente $n = 2r$. Hinc autem operae pretium erit notare, fore $2a + b + c = 2r = n$; ita ut formula nostra generalis ad casum hunc semper accommodari queat, si modo fuerit $n = 2a + b + c$: tum enim fit $p = a$; $q = a + b$ et $2r = 2a + b + c$.

§. 24. Quodsi vero formula nostra generalis enolvatur ac loco n scribatur iste valor $2a + b + c$, ea induet hanc formam:

$$S = \int \frac{dx}{x \log x} \cdot \left(\frac{x^a - x^{a+b} - x^{a+c} + x^{a+b+c}}{1 - x^{2a+b+c}} \right)$$

cuius ergo valor si loco p , q et r modo inuenti valores scribantur, erit

$$S = \int \frac{dx}{x} = \log \sin \frac{a\pi}{2a+b+c} - \log \sin \frac{(a+b)\pi}{2a+b+c}$$

quae formula utique ita est absoluta, ut nullam amplius formulam integram inuoluat, prorsus uti desideratur. Pa-

tet igitur casum ante tractatum in hoc casu non contineri: cum enim in illo fuisset $a = n - b$ et $r = b$, hinc fiet $2a + b + c = 2n$, cum praesenti casu fit $2a + b + c = n$.

§. 25. Quodsi iam in hac expressione litteras p , q et r in calculum introducamus, formula nostra integralis ad hanc speciem reducetur:

$$S = \int \frac{dx}{x \log x} \cdot \frac{x^p - x^q - x^{2r-q} + x^{2r-p}}{1 - x^{2r}}$$

cuius igitur valor ab $x = 0$ ad $x = 1$ extensus erit

$$S = l \sin. \frac{p\pi}{2r} - l \sin. \frac{q\pi}{2r}$$

vbi manifestum est, hanc expressionem eandem manere, etiamsi loco p scribatur $2r - p$, loco q vero $2r - q$ propterea quod

$$\sin. \frac{(2r-p)\pi}{2r} = \sin. \frac{p\pi}{2r} \text{ et } \sin. \frac{(2r-q)\pi}{2r} = \sin. \frac{q\pi}{2r}$$

at vero ipsa formula integralis, facta sine alterutra substitutione, sine utraque coniunctim, prorsus non variatur.

§. 26. Quodsi loco p et q scribamus $r - p$ et $r - q$, illi sinus transmutantur in cosinus; tum autem ipsa formula integralis erit

$$S = \int \frac{dx}{x \log x} \cdot \frac{x^{r-p} - x^{r-q} - x^{r+q} + x^{r+p}}{1 - x^{2r}}$$

cuius valor nunc erit $= l \cos. \frac{p\pi}{2r} - l \cos. \frac{q\pi}{2r}$, vbi iterum manifestum est, nullam mutationem oriri, siue litterae p et q valores habeant positivos siue negativos.

Corollarium I.

§. 27. Cum igitur his casibus neutra formularum integralium P et Q integrationem actu admittat, eo magis notatu dignum hic occurrit, quod nihilo minus valor fractionis $\frac{P}{Q}$ absolute exprimi possit, cum per sinus f

Cum igitur hoc casu sit $a = p$; $b = q - p$;
 et $p = q$ et $n = 2r$ valores, integrales pro P et Q
 summa: §. 28. exhibita in sequentes labuntur formas:

$$P = \int \frac{x^{2r-p-1} dx}{(1-x^{2r})^{1+\frac{p-q}{2r}}} \quad \text{et} \quad Q = \int \frac{x^{2r-p-1} dx}{(1-x^{2r})^{1+\frac{p-q}{2r}}}$$

$$Q = \int \frac{x^{p-1} dx}{(1-x^{2r})^{1+\frac{p-q}{2r}}} = 9. I$$

Quicunque ergo valores exponentibus tribuantur, semper

$$P = \frac{\sin \frac{p\pi}{2r}}{\sin \frac{q\pi}{2r}}$$

Corollarium II.

§. 28. Quoniam hic loco p et q scribere licet $2r-p$
 et $2r-q$, hinc quaternas formulas integrales exhibere pos-
 sumus; ita ut pro singulis sit $\frac{P}{Q} = \frac{\sin \frac{p\pi}{2r}}{\sin \frac{q\pi}{2r}}$, qui quaterni va-
 lores ita se habebunt:

$$I. P = \int \frac{x^{2r-p-1} dx}{(1-x^{2r})^{1+\frac{p-q}{2r}}} \quad \text{et} \quad Q = \int \frac{x^{2r-p-1} dx}{(1-x^{2r})^{1+\frac{p-q}{2r}}}$$

$$II. P = \int \frac{x^{2r-p-1} dx}{(1-x^{2r})^{2-\frac{p-q}{2r}}} \quad \text{et} \quad Q = \int \frac{x^{2r-p-1} dx}{(1-x^{2r})^{2-\frac{p-q}{2r}}}$$

$$III. P = \int \frac{x^{q-1} dx}{(1-x^{2r})^{\frac{p+1}{2r}}} \quad \text{et} \quad Q = \int \frac{x^{p-1} dx}{(1-x^{2r})^{\frac{p+1}{2r}}}$$

$$IV. P = \int \frac{x^{q-1} dx}{(1-x^{2r})^{1+\frac{q-p}{2r}}} \quad \text{et} \quad Q = \int \frac{x^{2r-p-1} dx}{(1-x^{2r})^{1+\frac{q-p}{2r}}}$$

Corollarium III.

§. 29. Quodsi hic loco p et q scribamus $r-p$ et $r-q$, quo pacto sinus in cosinus transmutantur, quaternas impetrabimus formulas integrales pro P et Q , ita comparatas, ut pro omnibus sit $\frac{P}{Q} = \frac{\cos. \frac{p\pi}{2r}}{\cos. \frac{q\pi}{2r}}$, qui quaterni valores erunt:

$$\text{I. } P = \int \frac{x^{r+q-1} dx}{(1-x^{2r})^{1+\frac{q-p}{2r}}} \text{ et } Q = \int \frac{x^{r-p-1} dx}{(1-x^{2r})^{1+\frac{q-p}{2r}}}$$

$$\text{II. } P = \int \frac{x^{r+q-1} dx}{(1-x^{2r})^{1+\frac{p+q}{2r}}} \text{ et } Q = \int \frac{x^{r+p-1} dx}{(1-x^{2r})^{1+\frac{p+q}{2r}}}$$

$$\text{III. } P = \int \frac{x^{r-q-1} dx}{(1-x^{2r})^{1-\frac{p-q}{2r}}} \text{ et } Q = \int \frac{x^{r-p-1} dx}{(1-x^{2r})^{1-\frac{p-q}{2r}}}$$

$$\text{IV. } P = \int \frac{x^{r-q-1} dx}{(1-x^{2r})^{1+\frac{p-q}{2r}}} \text{ et } Q = \int \frac{x^{r+p-1} dx}{(1-x^{2r})^{1+\frac{p-q}{2r}}}$$

quae quaternae formulae tam pulchre inter se conspirant, ut aliter non discrepent, nisi ratione signorum, quibus litterae P et Q sunt affectae.

Corollarium IV.

§. 30. Hae autem formulae prorsus sunt diuersae ab illis quas supra in evolutione §. 14. habuimus, ubi erat $\frac{P}{Q} = \frac{n \sin. \frac{b\pi}{n}}{b\pi}$, quod discrimen quo clarius ob oculos ponatur, loco b et n scribamus p et $2r$, ut fiat $\frac{P}{Q} = \frac{2r \sin. \frac{p\pi}{2r}}{p\pi}$; tum autem fit

$$P = \int$$

$$P = \int \frac{x^{r-1} dx}{(1-x^{2r})^{1-\frac{p}{2r}}} \text{ et } Q = \int \frac{x^{2r-p-1} dx}{(1-x^{2r})^{1-\frac{p}{2r}}}$$

quae formulae actu integrationem admittent, dum colliguntur

$$P = \frac{1}{p} \text{ et } Q = \frac{\pi}{2r \sin \frac{p\pi}{2r}}$$

Corollarium V.

§. 31. Quodsi in formulis penultimi corollarii capiamus $q=0$, ut fiat $\frac{p}{Q} = \cos \frac{p\pi}{2r}$, binas tantum pro hoc casu diversas formulas pro P et Q nanciscemur, quae sunt

$$\text{I. } P = \int \frac{x^{r-1} dx}{(1-x^{2r})^{1-\frac{p}{2r}}} \text{ et } Q = \int \frac{x^{r-p-1} dx}{(1-x^{2r})^{1-\frac{p}{2r}}}$$

$$\text{II. } P = \int \frac{x^{r-1} dx}{(1-x^{2r})^{1+\frac{p}{2r}}} \text{ et } Q = \int \frac{x^{r+p-1} dx}{(1-x^{2r})^{1+\frac{p}{2r}}}$$

Sin autem in formulis antepenultimi corollarii statuamus $q=r$, ut predeat $\frac{p}{Q} = \sin \frac{p\pi}{2r}$, iterum prodibunt binae formulae pro P et Q, quae sunt:

$$\text{I. } P = \int \frac{x^{r-1} dx}{(1-x^{2r})^{\frac{1}{2}+\frac{p}{2r}}} \text{ et } Q = \int \frac{x^{p-1} dx}{(1-x^{2r})^{\frac{1}{2}+\frac{p}{2r}}}$$

$$\text{II. } P = \int \frac{x^{r-1} dx}{(1-x^{2r})^{\frac{1}{2}-\frac{p}{2r}}} \text{ et } Q = \int \frac{x^{2r-p-1} dx}{(1-x^{2r})^{\frac{1}{2}-\frac{p}{2r}}}$$

Corollarium VI.

§. 32. Quodsi in formulis Corollarii II. statuamus $q=r-p$, ut fiat $\sin \frac{q\pi}{2r} = \cos \frac{p\pi}{2r}$, habebitur $\frac{p}{Q} = \tan \frac{p\pi}{2r}$ et

quaterni valores pro formulis P et Q erunt

$$\text{I. } P = \int \frac{x^{r+p-1} dx}{(1-x^{2r})^{\frac{1}{2} + \frac{p}{r}}} \text{ et } Q = \int \frac{x^{p-1} dx}{(1-x^{2r})^{\frac{1}{2} + \frac{p}{r}}}$$

$$\text{II. } P = \int \frac{x^{r+p-1} dx}{(1-x^{2r})^{\frac{3}{2}}} \text{ et } Q = \int \frac{x^{2r-p-1} dx}{(1-x^{2r})^{\frac{3}{2}}}$$

$$\text{III. } P = \int \frac{x^{r-p-1} dx}{(1-x^{2r})^{\frac{1}{2}}} \text{ et } Q = \int \frac{x^{p-1} dx}{(1-x^{2r})^{\frac{1}{2}}}$$

$$\text{IV. } P = \int \frac{x^{r-p-1} dx}{(1-x^{2r})^{\frac{3}{2} - \frac{p}{r}}} \text{ et } Q = \int \frac{x^{2r-p-1} dx}{(1-x^{2r})^{\frac{3}{2} - \frac{p}{r}}}$$

Corollarium VII.

§. 33. Plurimum autem etiam intererit nosse, ipsam formulam integram S pro his casibus, quibus fit simpliciter vel $\frac{P}{Q} = \cos. \frac{p\pi}{2r}$, vel $\frac{P}{Q} = \sin. \frac{p\pi}{2r}$ vel, $\frac{P}{Q} = \tan. \frac{p\pi}{2r}$ fieri, pro primo:

$$S = \int \frac{dx}{x l x} \cdot \frac{x^{r-p} - 2x^r + x^{r+p}}{1-x^{2r}} = l \cos. \frac{p\pi}{2r}$$

pro secundo casu:

$$S = \int \frac{dx}{x l x} \cdot \frac{x^p - 2x^r + x^{2r-p}}{1-x^{2r}} = l \sin. \frac{p\pi}{2r}$$

pro tertio autem casu:

$$S = \int \frac{dx}{x l x} \cdot \frac{x^p - x^{r-p} - x^{r+p} + x^{2r-p}}{1-x^{2r}} = l \tan. \frac{p\pi}{2r}$$

quae postrema formula reducitur ad hanc:

$$\int \frac{dx}{x l x} \cdot \frac{x^p - x^{r-p}}{1+x^r} = l \tan. \frac{p\pi}{2r}$$

quae est eadem integratio, quam non ita pridem ex diversissimis principiis elicueram.

Scho-

Scholion.

§ 34. Postremo autem circa omnes has varias formulas integrales probe notetur, eas, in quibus exponens denominatoris reperitur unitate maior, vtpote incongruas reiciendas esse; propterea quod earum valores integrati posito $x=1$ euadant infiniti, quod quidem, cum in vtraque formula P et Q simul eueniat, non impedit, quo minus fractio $\frac{P}{Q}$ assignatum obtineat valorem; sed quia cum hinc definire non licet, etiam istiusmodi formulae optatum usum non praestant. Commode autem adueniunt plures formulae adsint, ex quibus valorem verum deducere liceat.