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De integratione formulae ∫ $(dx \log x)/\sqrt{(1-xx)}$ ab $x=0$ ad $x=1$ extensa

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DE
INTEGRATIONE FORMVLAE
$\int \frac{dx\, lx}{\sqrt{(1-xx)}}$,
AB $x = 0$ AD $x = 1$ EXTENSA.

Auctore
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§. 1.

Methodus maxime naturalis huiusmodi formulas $\int p\, dx\, lx$ tractandi in hoc consistit, vt eae ad alias huiusmodi formas $\int q\, dx$ reducantur, in quibus littera q sit functio algebraica ipsius x; quandoquidem regulae integrandi potissimum ad tales formulas sunt accommodatae. Huiusmodi autem reductio nulla prorsus laborat difficultate, quando functio p ita est comparata, vt integrale $\int p\, dx$ algebraice exhiberi queat. Si enim fuerit $\int p\, dx = P$, ita vt formula proposita sit $\int dP\, lx$, ea sponte reducitur ad hanc expressionem: $P\, lx - \int \frac{P\, dx}{x}$, sicque iam totum negotium ad integrationem huius formulae $\int \frac{P\, dx}{x}$ est perductum. Quando vero formula $\int p\, dx$ integrationem algebraicam non admittit, quemadmodum euenit in nostra formula proposita $\int \frac{dx\, lx}{\sqrt{1-xx}}$, talis reductio successu penitus

 caret;

caret. Cum enim sit $\int \frac{dx}{\sqrt{1-xx}} = $ A. sin. x, ista reductio daret

$$\int \frac{dx\, lx}{\sqrt{1-xx}} = \text{A sin. } x\, lx - \int \frac{dx}{x} \text{A sin. } x$$

sicque post signum integrationis noua quantitas transcendens A sin. x occurreret, cuius integratio aeque est abscondita ac ipsius propositae. Quare cum nuper singulari methodo inuenissem esse

$$\int \frac{dx\, lx}{\sqrt{(1-xx)}} \left[\begin{smallmatrix} ab & x = 0 \\ ad & x = 1 \end{smallmatrix} \right] = -\tfrac{1}{2} \pi\, l\, 2$$

expressio integralis eo maiori attentione digna est censenda, quod eius inuestigatio neutiquam est obuia; vnde operae pretium esse duxi eius veritatem etiam ex aliis fontibus ostendisse, ante quam ipsam methodum, quae me eo perduxit, exponerem.

Prima demonstratio integrationis propositae.

§. 2. Quoniam hic potissimum ad series infinitas est recurrendum, formula autem lx talem resolutionem simplicem respuit, adhibeamus substitutionem $\sqrt{(1-xx)} = y$ vnde fit $x = \sqrt{1-yy}$, hincque porro

$$lx = -\frac{yy}{2} - \frac{y^4}{4} - \frac{y^6}{6} - \frac{y^8}{8} - \text{etc.}$$

hoc igitur modo formula integralis proposita $\int \frac{dx\, lx}{\sqrt{1-xx}}$ transformatur in sequentem formam:

$$\int \frac{dy}{\sqrt{1-yy}} \left(\frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \frac{y^8}{8} + \text{etc.} \right)$$

vbi, cum sit $y = \sqrt{(1-xx)}$, notetur integrationem extendi debere ab $y = 1$ vsque ad $y = 0$; quare si hos terminos integrationis permutare velimus, signum totius formae mutari oportet.

§. 3. Quo autem minus tali signorum mutatione confundamur, designemus valorem quaesitum littera S, vt sit

$$S = \int \frac{dx\, lx}{\sqrt{1-xx}} \left[\begin{smallmatrix} ab & x = 0 \\ ad & x = 1 \end{smallmatrix} \right]$$

atque

quae facta ſubſtitutione $y = \sqrt{1 - xx}$, habebimus, vti modo monuimus

$$S = -\int \frac{dy}{\sqrt{1-yy}} \left(\frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \text{etc.} \right) \left[\begin{matrix} ab\ y = 0 \\ ad\ y = 1 \end{matrix} \right].$$

ſub his autem integrationis terminis, ſcilicet ab $y = 0$ ad $y = 1$, iam ſatis notum eſt, ſingulas partes, quae hic occurrunt, ad ſequentes valores reduci:

$$\int \frac{yy\,dy}{\sqrt{(1-yy)}} = \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\int \frac{y^4\,dy}{\sqrt{(1-yy)}} = \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\pi}{2}$$

$$\int \frac{y^6\,dy}{\sqrt{(1-yy)}} = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{\pi}{2}$$

$$\int \frac{y^8\,dy}{\sqrt{(1-yy)}} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{\pi}{2}$$

$$\int \frac{y^{10}\,dy}{\sqrt{(1-yy)}} = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} \cdot \frac{\pi}{2}$$

vbi nimirum eſt $\frac{\pi}{2} = \int \frac{dy}{\sqrt{(1-yy)}}$, ita vt $1 : \pi$ exprimat rationem diametri ad peripheriam circuli.

§. 4. Quodſi ergo ſingulos iſtos valores introducamus, pro valore quaeſito S impetrabimus ſequentem ſeriem infinitam:

$$S = -\frac{\pi}{2} \left(\frac{1}{2^2} + \frac{1 \cdot 3}{2 \cdot 4^2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6^2} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8^2} + \text{etc.} \right)$$

ſicque nunc totum negotium eo eſt reductum, vt iſtius ſeriei infinitae ſumma inueſtigetur; qui labor fortaſſe haud minus operoſus videri poteſt, quam id ipſum, quod nobis exequi eſt propoſitum. Interim tamen ad cognitionem ſummae huius ſeriei haud difficulter ſequenti modo nobis pertingere licebit.

§. 5. Cum ſit

$$\frac{1}{\sqrt{(1-zz)}} = 1 + \frac{1}{2} zz + \frac{1 \cdot 3}{2 \cdot 4} z^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} z^6 + \text{etc.}$$

ſi vtrinque per $\frac{dz}{z}$ multiplicemus et integremus, obtinebimus

$$\int \frac{dz}{z\sqrt{1-zz}} = lz + \frac{1}{2^2} zz + \frac{1 \cdot 3}{2 \cdot 4^2} z^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6^2} z^6 + \text{etc.}$$

ſicque ad ipſam ſeriem noſtram ſumus perducti, cuius ergo valor quaeri debet ex hac expreſſione: $\int \frac{dz}{z\sqrt{1-zz}} - lz$, integrali ſcilicet ita ſumto, vt euaneſcat poſito $z = 0$, quo facto ſtatuatur $z = 1$, ac prodibit ipſa ſeries

$$\frac{1}{2^2} + \frac{1\cdot 3}{2\cdot 4^2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6^2} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8^2} + \text{etc.}$$

Hoc igitur modo totum negotium perductum eſt ad iſtam formulam integralem $\int \frac{dz}{z\sqrt{1-zz}}$, quae poſito $\sqrt{1-zz} = v$ tranſit in hanc formam: $\frac{-dv}{1-vv}$ cuius integrale conſtat eſſe $-\frac{1}{2} l \frac{1+v}{1-v} = -l \frac{1+v}{\sqrt{(1-vv)}}$. Quodſi loco v reſtituatur valor $\sqrt{1-zz}$, tota expreſſio, qua indigemus, ita ſe habebit: $\int \frac{dz}{(z\sqrt{1-zz})} - lz = -l\frac{(1+\sqrt{1-zz})}{z} - lz + C = C - l(1+\sqrt{1-zz})$, vbi conſtans ita accipi debet, vt valor euaneſcat, poſito $z = 0$, ideoque erit $C = l2$. Quamobrem, poſito $z = 1$, ſumma ſeriei quaeſita erit $l2$, hincque valor ipſius formulae integralis propoſitae erit

$$\int \frac{dx\, lx}{\sqrt{(1-xx)}} = S = -\frac{\pi}{2} l2$$

prorſus vti longe alia methodo inueneram, ex quo iam ſatis intelligitur, iſtam veritatem vtique altioris eſſe indaginis, ideoque attentione Geometrarum maxime dignam.

Alia demonſtratio integrationis propoſitae.

§. 6. Cum ſit $\frac{dx}{\sqrt{1-xx}}$ elementum arcus circuli cuius ſinus $= x$, ponamus iſtum angulum $= \phi$, ita vt ſit $x = \text{ſin.}\,\phi$ et $\frac{dx}{\sqrt{1-xx}} = d\phi$, atque facta hac ſubſtitutione valor quantitatis S, in quem inquirimus, ita repraeſentabitur: $S = \int d\phi\, l\,\text{ſin.}\,\phi \left[\begin{smallmatrix} a\ \phi = 0 \\ ad\ \phi = 90^\circ \end{smallmatrix}\right]$. Cum enim ante termini fuiſſent $x = 0$ et $x = 1$, iis nunc reſpondent $\phi = 0$ et $\phi = 90^\circ$ ſiue $\phi = \frac{\pi}{2}$. Hic igitur totum negotium eo redit, vt formula $l\,\text{ſin.}\,\phi$ commode in ſeriem infinitam conuertavertatur. Hunc in finem ponamus $l\,\text{fin.}\,\Phi = s$ eritque $ds = \frac{d\Phi\,\text{cof.}\,\Phi}{\text{fin.}\,\Phi}$. Nouimus autem esse

$$\frac{\text{cof.}\,\Phi}{\text{fin.}\,\Phi} = 2\,\text{fin.}\,2\Phi + 2\,\text{fin.}\,4\Phi + 2\,\text{fin.}\,6\Phi + 2\,\text{fin.}\,8\Phi + \text{etc.}$$

Si enim vtrinque per fin. Φ multiplicemus, ob

$$2\,\text{fin.}\,n\Phi\,\text{fin.}\,\Phi = \text{cof.}\,(n-1)\Phi - \text{cof.}\,(n+1)\Phi,$$

vtrinque prodit

$$\text{cof.}\,\Phi = \text{cof.}\,\Phi + \text{cof.}\,3\Phi + \text{cof.}\,5\Phi + \text{cof.}\,7\Phi + \text{cof.}\,9\Phi + \text{etc.}$$
$$- \text{cof.}\,3\Phi - \text{cof.}\,5\Phi - \text{cof.}\,7\Phi - \text{cof.}\,9\Phi - \text{etc.}$$

Hac igitur ferie pro $\frac{\text{cof.}\,\Phi}{\text{fin.}\,\Phi}$ in vfum vocata erit

$$s = C - \text{cof.}\,2\Phi - \tfrac{1}{2}\text{cof.}\,4\Phi - \tfrac{1}{3}\text{cof.}\,6\Phi - \tfrac{1}{4}\text{cof.}\,8\Phi - \tfrac{1}{5}\text{cof.}\,10\Phi - \text{etc.}$$

vbi cum fit $s = l\,\text{fin.}\,\Phi$ ideoque $s = 0$, quando fin. $\Phi = 1$ ideoque $\Phi = \frac{\pi}{2}$, conftantem C ita definire oportet, vt pofito $\Phi = \frac{\pi}{2} = 90°$ euadat $s = 0$, ex quo colligitur fore

$$C = -1 + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{5} + \text{etc.} = -l\,2.$$

§. 7. Cum igitur fit

$$l\,\text{fin.}\,\Phi = -l\,2 - \text{cof.}\,2\Phi - \tfrac{1}{2}\text{cof.}\,4\Phi - \tfrac{1}{3}\text{cof.}\,6\Phi - \tfrac{1}{4}\text{cof.}\,8\Phi - \text{etc.}$$

erit valor formulae propofitae

$$\int d\Phi\,l\,\text{fin.}\,\Phi = C - \Phi\,l\,2 - \tfrac{1}{2}\text{fin.}\,2\Phi - \tfrac{1}{8}\text{fin.}\,4\Phi - \tfrac{1}{18}\text{fin.}\,6\Phi$$
$$- \tfrac{1}{32}\text{fin.}\,8\Phi - \tfrac{1}{50}\text{fin.}\,10\Phi - \text{etc.}$$

quae expreffio cum euanefcere debeat pofito $\Phi = 0$, conftans hic ingreffa erit $C = 0$, ita vt iam in genere fit

$$\int d\Phi\,l\,\text{fin.}\,\Phi = -\Phi\,l\,2 - \frac{2\,\text{fin.}\,2\Phi}{2^2} - \frac{2\,\text{fin.}\,4\Phi}{4^2} - \frac{2\,\text{fin.}\,6\Phi}{6^2} - \frac{2\,\text{fin.}\,8\Phi}{8^2}$$
$$- \frac{2\,\text{fin.}\,10\Phi}{10^2} - \frac{2\,\text{fin.}\,12\Phi}{12^2} - \text{etc.}$$

Quodfi iam hic capiatur $\Phi = 90° = \frac{\pi}{2}$, omnium angulorum 2Φ; 4Φ; 6Φ; 8Φ etc. qui hic occurrunt finus euanefcunt, ideoque valor quaefitus erit

$$S = \int d\Phi\,l\,\text{fin.}\,\Phi \left[\begin{smallmatrix} a & \Phi = 0 \\ ad & \Phi = 90° \end{smallmatrix}\right] = -\tfrac{\pi}{2}\,l\,2$$

quemadmodum etiam in priore demonftratione oftendimus.

§. 8.

§. 8. Ista autem demonstratio praecedenti ideo longe antecellit, quod nobis non solum valorem formulae propositae exhibeat casu quo $\phi = 90^\circ$, sed etiam verum eius valorem ostendat, quicunque angulus pro ϕ accipiatur, id quod ad ipsam formulam propositam $\int \frac{dx\, lx}{\sqrt{1-xx}}$ transferri poterit, cuius adeo valorem pro quolibet valore ipsius x assignare poterimus. Quodsi enim istius formulae valorem desideremus ab $x = 0$ vsque ad $x = a$, quaeratur angulus α cuius sinus sit aequalis ipsi a atque semper habebitur

$$\int \frac{dx\, lx}{\sqrt{1-xx}} \left[\begin{smallmatrix} ab\ x = 0 \\ ad\ x = a \end{smallmatrix} \right] = -\alpha\, l\, 2 - \frac{2\,\text{fin.}\, 2\alpha}{2^2} - \frac{2\,\text{fin.}\, 4\alpha}{4^2} - \frac{2\,\text{fin.}\, 6\alpha}{6^2} - \frac{2\,\text{fin.}\, 8\alpha}{8^2} - \text{etc.}$$

Vnde patet quoties fuerit $\alpha = \frac{i\pi}{2}$, denotante i numerum integrum quemcunque, quoniam omnes sinus euanescunt, valor formulae his casibus finite exprimi per $-\frac{i\pi}{2}\, l\, 2$; aliis vero casibus valor nostrae formulae per seriem infinitam satis concinnam exprimetur. Ita si capiatur $a = \frac{1}{\sqrt{2}}$, vt sit $\alpha = \frac{\pi}{4}$ valor nostrae formulae erit

$$-\frac{\pi}{4}\, l\, 2 - \frac{2}{2^2} + \frac{2}{6^2} - \frac{2}{10^2} + \frac{2}{14^2} - \frac{2}{18^2} + \frac{2}{22^2} - \text{etc.}$$

quae series elegantius ita exprimitur:

$$-\frac{\pi}{4}\, l\, 2 - \frac{1}{2}\left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + \text{etc.}\right)$$

sicque hic occurrit series satis memorabilis

$$1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \frac{1}{81} - \frac{1}{121} + \text{etc.}$$

cuius summam nullo adhuc modo ad mensuras cognitas reuocare licuit.

§. 9. Quoniam tam egregia series hic se quasi praeter exspectationem obtulit, etiam alios casus euoluamus notabiliores, sumamusque $a = \frac{1}{2}$, vt sit $\alpha = 30^\circ = \frac{\pi}{6}$ atque nostrae formulae hoc casu valor erit

$$-\frac{\pi}{6}\, l\, 2 - \frac{\sqrt{3}}{2^2} - \frac{\sqrt{3}}{4^2} + \frac{\sqrt{3}}{8^2} + \frac{\sqrt{3}}{10^2} - \frac{\sqrt{3}}{14^2} - \frac{\sqrt{3}}{16^2} + \text{etc.}$$

quae expressio ita exhiberi potest

$$-\frac{\pi}{6}\, l\, 2 - \frac{\sqrt{3}}{4}\left(1 + \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} - \frac{1}{10^2} - \frac{1}{11^2} + \text{etc.}\right)$$

in

in qua ferie quadrata multiplorum ternarii deficiunt. Sumamus nunc fimili modo $a = \frac{\sqrt{3}}{2}$, vt fit $\alpha = 60^\circ = \frac{\pi}{3}$, ac valor noftrae formulae hoc cafu prodibit

$$[illegible] = \frac{\pi}{3} l 2 - \frac{\sqrt{3}}{2^2} + \frac{\sqrt{3}}{4^2} - \frac{\sqrt{5}}{8^2} + \frac{\sqrt{3}}{10^2} - \frac{\sqrt{3}}{14^2} + \frac{\sqrt{3}}{16^2} - \text{etc.}$$

fiue hoc modo exprimetur:

$$[illegible] \frac{\pi}{3} l 2 - \frac{\sqrt{3}}{4}\left(1 - \frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{10^2} - \frac{1}{11^2} + \text{etc.}\right)$$

Adhuc alia demonftratio integrationis propofitae.

§. 10. Introducatur in formulam noftram angulus Φ, cuius cofinus fit $= x$, fiue fit $x = \text{cof.}\,\Phi$ et formula noftra induet hanc formam: $-\int d\Phi\, l\,\text{cof.}\,\Phi$, quod integrale a $\Phi = 90^\circ$ vsque ad $\Phi = 0$ erit extendendum. Quodfi autem hos terminos permutemus, valor S, quem quaerimus, ita exprimetur:

$$S = \int d\Phi\, l\,\text{cof.}\,\Phi \left[\begin{matrix} a & \Phi = 0 \\ ad & \Phi = 90^\circ \end{matrix} \right].$$

Vt hic $l\,\text{cof.}\,\Phi$ in feriem idoneam conuertamus, ftatuamus vt ante $s = l\,\text{cof.}\,\Phi$, eritque $ds = -\frac{d\Phi\,\text{fin.}\,\Phi}{\text{cof.}\,\Phi}$. Conftat autem per feriem effe

$$\frac{\text{fin.}\,\Phi}{\text{cof.}\,\Phi} = 2\,\text{fin.}\,2\Phi - 2\,\text{fin.}\,4\Phi + 2\,\text{fin.}\,6\Phi - 2\,\text{fin.}\,8\Phi + \text{etc.}$$

Cum enim in genere fit

$$2\,\text{fin.}\,n\Phi\,\text{cof.}\,\Phi = \text{fin.}\,(n+1)\Phi + \text{fin.}\,(n-1)\Phi$$

fi vtrinque per cof. Φ multiplicemus orietur

$$\begin{aligned} \text{fin.}\,\Phi = {} & \text{fin.}\,3\Phi - \text{fin.}\,5\Phi + \text{fin.}\,7\Phi - \text{fin.}\,9\Phi \\ & + \text{fin.}\,\Phi - \text{fin.}\,3\Phi + \text{fin.}\,5\Phi - \text{fin.}\,7\Phi + \text{fin.}\,9\Phi \end{aligned} \;\text{etc.}$$

quare cum fit $ds = -\frac{d\Phi\,\text{fin.}\,\Phi}{\text{cof.}\,\Phi}$, erit nunc

$$S = C + \frac{\text{cof.}\,2\Phi}{1} - \frac{\text{cof.}\,4\Phi}{2} + \frac{\text{cof.}\,6\Phi}{3} - \frac{\text{cof.}\,8\Phi}{4} + \frac{\text{cof.}\,10\Phi}{5} - \text{etc.}$$

Quia igitur eft $s = l\,\text{cof.}\,\Phi$, euidens eft pofito $\Phi = 0$ fieri debere $s = 0$, vnde colligitur

$$C = -1 + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{5} + \text{etc.} = -l\,2$$

 ficque

ſicque erit $l\cos.\Phi = -l2 + \frac{\cos. 2\Phi}{1} - \frac{\cos. 4\Phi}{2} + \frac{\cos. 6\Phi}{3} - \frac{\cos. 8\Phi}{4} + \text{etc.}$ quae ſeries ducta in $d\Phi$ et integrata praebet

$$S = \int d\Phi l \cos.\Phi = C - \Phi l 2 + \frac{\sin. 2\Phi}{2} - \frac{\sin. 4\Phi}{8} + \frac{\sin. 6\Phi}{18} - \frac{\sin. 8\Phi}{32} + \frac{\sin. 10\Phi}{50} - \text{etc.}$$

quae expreſſio quia ſponte euaneſcit poſito $\Phi = 0$, inde patet fore $C = 0$, ſicque habebimus

$$\int d\Phi l \cos.\Phi = -\Phi l 2 + \tfrac{1}{2}\left(\frac{\sin. 2\Phi}{1} - \frac{\sin. 4\Phi}{2^2} + \frac{\sin. 6\Phi}{3^2} - \frac{\sin. 8\Phi}{4^2} + \frac{\sin. 10\Phi}{5^2} - \text{etc.}\right)$$

Sumto igitur $\Phi = \frac{\pi}{2} = 90°$, oritur vt ante $S = -\frac{\pi}{2} l 2$. Praeterea vero etiam hinc integrale ad quemuis terminum vsque extendere licet.

§. 11. Quodſi formulam poſteriorem a praecedente ſubtrahamus, adipiſcemur in genere hanc integrationem:

$$\int d\Phi l \,\text{tang.}\,\Phi = -\sin. 2\Phi - \tfrac{1}{3^2}\sin. 6\Phi - \tfrac{1}{5^2}\sin. 10\Phi - \text{etc.}$$

vnde patet hoc integrale euaneſcere caſibus $\Phi = 90°$ et in genere $\Phi = i\frac{\pi}{2}$. Poſtquam igitur iſtam integrationem triplici modo demonſtrauimus, ipſam Analyſin, quae me primum huc perduxit, hic delucide ſum expoſiturus.

Analyſis ad integrationem formulae $\int \frac{dx\, lx}{\sqrt{1-xx}}$ aliarumque ſimilium perducens.

§. 12. Tota haec Analyſis innititur ſequenti lemmati a me iam olim demonſtrato: Poſito breuitatis gratia $(1-x^n)^{\frac{m-n}{n}} = X$, ſi hinc duae formulae integrales formentur $\int X x^{p-1} dx$ et $\int X x^{q-1} dx$, quae a termino $x = 0$ vſque ad terminum $x = 1$ extendantur, ratio horum valorum ſequenti modo ad productum ex infinitis factoribus conflatum reduci poteſt

$$\frac{\int X x^{p-1} dx}{\int X x^{q-1} dx} = \frac{(m+p)q}{p(m+q)} \cdot \frac{(m+p+n)(q+n)}{(p+n)(m+q+n)} \cdot \frac{(m+p+2n)(q+2n)}{(p+2n)(m+q+2n)} \text{ etc.}$$

vbi

vbi scilicet singuli factores tam numeratoris, quam denominatoris continuo eadem quantitate n augentur. Hic autem probe tenendum est, veritatem istius lemmatis subsistere non posse, nisi singulae litterae m, n, p et q denotent numeros positiuos, quos tamen semper tanquam integros spectare licet.

§. 13. Circa has duas formulas integrales, a termino $x = 0$ vsque ad $x = 1$ extensas, duo casus imprimis seorsim notari merentur, quibus integratio actu succedit, verusque valor absolute assignari potest. Prior casus locum habet, si fuerit $p = n$, ita vt formula sit $\int X x^{n-1} dx$. Posito enim $x^n = y$ fiet $X = (1-y)^{\frac{m-n}{n}}$ et $x^{n-1} dx = \frac{1}{n} dy$ sicque ista formula euadet $\frac{1}{n} \int dy (1-y)^{\frac{m-n}{n}}$ pariter a termino $y = 0$ vsque ad $y = 1$ extendenda, quae porro posito $1 - y = z$, abit in hanc formulam $-\frac{1}{n} \int z^{\frac{m-n}{n}} dz$ a termino $z = 1$ vsque ad $z = 0$ extendendam; eius ergo integrale manifesto est $-\frac{1}{m} z^{\frac{m}{n}} + \frac{1}{m}$, vnde facto $z = 0$ valor erit $= \frac{1}{m}$. Consequenter pro casu $p = n$ habebimus

$$\int X x^{n-1} dx \left[\begin{smallmatrix} ab\ x = 0 \\ ad\ x = 1 \end{smallmatrix} \right] = \frac{1}{m}$$

sicque si fuerit vel $p = n$ vel $q = n$, integrale absolute innotescit.

§. 14. Alter casus notatu dignus est quo $p = n - m$, ita vt formula integranda sit $\int X x^{n-m-1} dx$; tum enim, si ponatur $x (1 - x^n)^{\frac{-1}{n}}$ siue $\frac{x}{(1-x^n)^{\frac{1}{n}}} = y$, posito $x = 0$ fiet $y = 0$, at posito $x = 1$ fiet $y = \infty$; tum autem erit

$$y^{n-m} = \frac{x^{n-m}}{(1 - x^n)^{\frac{n-m}{n}}} = X x^{n-m}$$

 vnde

vnde formula integranda erit $\int y^{n-m}\frac{dx}{x}$. Cum igitur sit $\frac{x}{(1-x^n)^{\frac{1}{n}}} = y$, erit $\frac{x^n}{1-x^n} = y^n$, vnde colligitur $x^n = \frac{y^n}{1+y^n}$, ideoque $n\,l\,x = n\,l\,y - l\,(1+y^n)$, cuius differentiatio praebet

$$\frac{dx}{x} = \frac{dy}{y(1+y^n)}$$

quo valore substituto formula nostra integranda erit $\int\frac{y^{n-m-1}\,dy}{1+y^n}$ a termino $y = 0$ vsque ad $y = \infty$ extendenda, quae formula ideo est notatu digna, quod ab omni irrationalitate est liberata.

§. 15. Quoniam igitur hoc casu ad formulam rationalem sumus perducti, ex elementis calculi integralis constat, eius integrationem semper per logarithmos et arcus circulares absolui posse, tum vero pro hoc casu non ita pridem ostendi, huius formulae $\int\frac{x^{m-1}\,dx}{1+x^n}$ integrale ab $x = 0$ vsque ad $x = \infty$ extensum reduci ad valorem $\frac{\pi}{n\,\text{sin.}\,\frac{m\pi}{n}}$; facta igitur applicatione pro nostro casu habebimus

$$\int\frac{y^{n-m-1}\,dy}{1+y^n} = \frac{\pi}{n\,\text{sin.}\,\frac{(n-m)\pi}{n}} = \frac{\pi}{n\,\text{sin.}\,\frac{m\pi}{n}}$$

quamobrem pro casu $p = n - m$ valor integralis sequenti modo absolute exprimi potest, eritque

$$\int X\,x^{n-m-1}\,dx\left[\begin{matrix}\text{ab } x = 0\\ \text{ad } x = 1\end{matrix}\right] = \frac{\pi}{n\,\text{sin.}\,\frac{m\pi}{n}},$$

quod idem manifesto tenendum est, si fuerit $q = n - m$.

§. 16. His praemissis ponamus porro breuitatis gratia

$$\int X\,x^{p-1}\,dx\left[\begin{matrix}ab\ x = 0\\ ad\ x = 1\end{matrix}\right] = P \text{ et}$$

$$\int X\,x^{q-1}\,dx\left[\begin{matrix}ab\ x = 0\\ ad\ x = 1\end{matrix}\right] = Q$$

atque

atque lemma allatum nobis praebet hanc aequationem

$$\frac{P}{Q} = \frac{(m+p)q}{p(m+q)} \cdot \frac{(m+p+n)(q+n)}{(p+n)(m+q+n)} \cdot \frac{(m+p+2n)(q+2n)}{(p+2n)(m+q+2n)} \text{ etc.}$$

Hinc igitur sumendis logarithmis deducimus

$$lP - lQ = l(m+p) - lp + l(m+p+n) - l(p+n) + l(m+p+2n) - l(p+2n) \text{ etc.}$$
$$+ lq - l(m+q) + l(q+n) - l(m+q+n) + l(q+2n) - l(m+q+2n) \text{ etc.}$$

haecque aequalitas semper locum habebit, quicunque valores litteris m, n, p et q tribuantur, dummodo fuerint positiui.

§. 17. Cum igitur haec aequalitas in genere subsistat, etiam veritati erit consentanea, quando quaepiam harum litterarum m, n, p et q infinite parum immutantur, siue tanquam variabiles spectantur. Hanc ob rem consideremus solam quantitatem p tanquam variabilem, ita vt reliquae litterae m, n et q maneant constantes, ideoque etiam quantitas Q erit constans, dum altera P variabitur; ex quo differentiando nanciscemur hanc aequationem:

$$\frac{dP}{P} = \frac{dp}{m+p} - \frac{dp}{p} + \frac{dp}{m+p+n} - \frac{dp}{p+n} + \frac{dp}{m+p+2n} - \frac{dp}{p+2n}$$
$$+ \frac{dp}{m+p+2n} - \frac{dp}{p+3n} + \text{etc.}$$

vbi totum negotium eo redit, quemadmodum differentiale formulae P, quae est integralis, exprimi oporteat.

§. 18. Cum igitur P sit formula integralis solam quantitatem x tanquam variabilem inuoluens, quandoquidem in eius integratione exponens p vt constans tractari debet, demum post integrationem ipsam quaentitatem P tanquam functionem duarum variabilium x et p spectare licebit, vnde quaestio huc redit, quomodo valorem, hoc charactere $\left(\frac{dP}{dp}\right)$ exprimi solitum inuestigari oporteat, qui si indicetur littera Π, aequatio ante inuenta hanc induet formam:

$$\frac{\Pi}{P} = \frac{1}{m+p} - \frac{1}{p} + \frac{1}{m+p+n} - \frac{1}{p+n} + \frac{1}{m+p+2n} - \frac{1}{p+2n} + \text{etc.}$$

 Hanc

Hanc vero feriem infinitam haud difficulter ad expreffionem finitam reuocare licebit hoc modo: Ponatur

$$s = \frac{v^{m+p}}{m+p} - \frac{v^p}{p} + \frac{v^{m+p+n}}{m+p+n} - \frac{v^{p+n}}{p+n} + \frac{v^{m+p+2n}}{m+p+2n} - \frac{v^{p+2n}}{p+2n} + \text{etc.}$$

ita vt facto $v = 1$ littera s nobis exhibeat valorem quaefitum $\frac{\Pi}{P}$; at vero differentiatio nobis dabit

$$\frac{ds}{dv} = v^{m+p-1} - v^{p-1} + v^{m+p+n-1} - v^{p+n-1} + v^{m+p+2n-1} - v^{p+2n-1} + \text{etc.}$$

cuius feriei infinitae fumma manifefto eft

$$\frac{v^{m+p-1} - v^{p-1}}{1 - v^n} = \frac{v^{p-1}(v^m - 1)}{1 - v^n}.$$

Hinc igitur viciffim concludimus fore

$$S = \int \frac{v^{p-1}(v^m - 1)\,dv}{1 - v^n}$$

quae formula integralis a $v = 0$ vsque ad $v = 1$ eft extendenda; ficque habebimus

$$\frac{\Pi}{P} = \int \frac{v^{p-1}(v^m - 1)\,dv}{1 - v^n} \left[\begin{array}{l} \text{a } v = 0 \\ \text{ad } v = 1 \end{array}\right].$$

§. 19. Ad valorem autem $\left(\frac{dP}{dp}\right)$, quem hic littera Π indicauimus, inueftigandum, ex principiis calculi integralis ad functiones duarum variabilium applicati iam fatis notum eft differentiale formulae integralis $P = \int X x^{p-1} dx$ ex fola variabilitate ipfius p oriundum obtineri, fi formula poft fignum integrationis pofita $X x^{p-1}$ ex fola variabilitate ipfius p differentietur atque elementum dp figno integrationis praefigatur; at vero quia X non continet p, hic vt conftans tractari debet: poteftatis vero x^{p-1} differentiale hinc natum erit $x^{p-1} dp\, lx$; quam ob rem ex hac differentiatione orietur $dP = dp \int X x^{p-1} dx\, lx$, ita vt tantum poft fignum integrationis factor lx accefferit, ex quo manifeftum manifestum est, fore

$$\Pi = \int X x^{p-1} dx\, lx \left[{}^{ab\ x=0}_{ad\ x=1} \right].$$

Hinc igitur sequens theorema generale constituere licebit.

Theorema generale.

§. 20. Posito breuitatis gratia $X = (1 - x^n)^{\frac{m-n}{n}}$ si sequentes formulae integrales omnes a termino $x = 0$ ad terminum $x = 1$ extendantur, sequens aequalitas semper erit veritati consentanea:

$$\int \frac{X x^{p-1} dx\, lx}{\int X x^{p-1} dx} = \int \frac{x^{p-1}(x^m - 1) dx}{1 - x^n}$$

nihil enim obstabat, quo minus loco v scriberemus x, quandoquidem isti valores tantum a terminis integrationis pendent.

§. 21. Hoc igitur modo deducti sumus ad integrationem huiusmodi formularum $\int X x^{p-1} dx\, lx$ in quibus quantitas logarithmica lx post signum integrationis tanquam factor inest, quarum valorem exprimere licuit per binas formulas integrales ordinarias, cum sit

$$\int X x^{p-1} dx\, lx = \int X x^{p-1} dx. \int \frac{x^{p-1}(x^m - 1) dx}{1 - x^n}$$

integralibus scilicet ab $x = 0$ ad $x = 1$ extensis, vbi breuitatis gratia posuimus $(1 - x^n)^{\frac{m-n}{n}} = X$. Hinc igitur pro binis casibus memorabilibus supra expositis bina theoremata particularia deriuemus.

Theorema particulare I, quo $p = n$.

§. 22. Quoniam supra vidimus casu $p = n$ fieri $\int X x^{n-1} dx = \frac{1}{m}$, hoc valore substituto habebimus istam aequationem satis elegantem:

$$\int X x^{n-1} dx\, lx = \frac{1}{m} \int \frac{x^{n-1}(x^m - 1) dx}{1 - x^n}$$

dum

dum scilicet ambo integralia ab $x = 0$ ad $x = 1$ extenduntur.

Theorema particulare II, quo $p = n - m$.

§. 23. Quoniam pro hoc casu, quo $p = n - m$ supra ostendimus esse

$$\int X x^{n-m-1} dx = \frac{\pi}{n \sin. \frac{m\pi}{n}}$$

nunc deducimur ad sequentem integrationem maxime notata dignam:

$$\int X x^{n-m-1} dx\, lx = \frac{\pi}{n \sin. \frac{m\pi}{n}} \int \frac{x^{n-m-1}(x^m - 1)dx}{1 - x^n}$$

siquidem haec ambo integralia ab $x = 0$ vsque ad $x = 1$ extendantur; vbi meminisse oportet esse

$$X = (1 - x^n)^{\frac{m-n}{n}}.$$

§. 24. Hic probe notetur theorema generale latissime patere, propterea quod in eo insunt tres exponentes indefiniti, scilicet m, n et p, qui penitus arbitrio nostro relinquuntur, quos ergo infinitis modis pro lubitu definire licet, dummodo singulis valores positiui tribuantur, ita vt semper valor huius formulae integralis $\int X x^{p-1} dx\, lx$, quam ob factorem lx tanquam transcendentem spectari oportet, per formulas integrales ordinarias exprimi queat, quae cum sint generalissima, operae pretium erit non nullos casus speciales euoluere.

I. Euolutio casus quo $m = 1$ et $n = 2$.

§. 25. Hoc igitur casu erit $X = \frac{1}{\sqrt{1-xx}}$, vnde pro hoc casu theorema generale ita se habebit

$$\int \frac{x^{p-1} dx\, lx}{\sqrt{1-xx}} = -\int \frac{x^{p-1} dx}{\sqrt{1-xx}} \cdot \int \frac{x^{p-1} dx}{1+x}$$

si-

siquidem singula haec integralia ab $x=0$ ad $x=1$ extendantur. Quoniam igitur hic tantum exponens p arbitrio nostro relinquitur, hinc sequentia exempla perlustremus.

Exemplum I. quo $p=1$.

§. 26. Hoc igitur casu aequatio superior hanc induet formam:

$$\int \frac{dx\, lx}{\sqrt{1-xx}} = -\int \frac{dx}{\sqrt{1-xx}} \cdot \int \frac{dx}{1+x}$$

vbi, integralibus ab $x=0$ ad $x=1$ extensis, notum est fieri

$$\int \frac{dx}{\sqrt{1-xx}} = \frac{\pi}{2} \text{ et } \int \frac{dx}{1+x} = l\,2$$

ita vt iam habeamus

$$\int \frac{dx\, lx}{\sqrt{1-xx}} \left[\begin{matrix} ab & x=0 \\ ad & x=1 \end{matrix} \right] = -\frac{\pi}{2} l\,2,$$

quae est ea ipsa formula, quam initio huius dissertationis tractauimus et cuius veritatem iam triplici demonstratione corroborauimus.

§. 27. Eundem valorem elicere licet ex theoremate particulari secundo, quo erat $p=n-m$, siquidem nunc ob $n=2$ et $m=1$ erit $p=1$; inde enim ob $X=\frac{1}{\sqrt{1-xx}}$ istud theorema praebet

$$\int \frac{dx\, lx}{\sqrt{1-xx}} = \frac{\pi}{2\,\text{fin.}\frac{\pi}{2}} \int \frac{dx}{1+x} = -\frac{\pi}{2} l\,2.$$

Exemplum II. quo $p=2$.

§. 28. Hoc igitur casu aequatio superior hanc induet formam:

$$\int \frac{x\,dx\, lx}{\sqrt{1-xx}} = -\int \frac{x\,dx}{\sqrt{1-xx}} \int \frac{x\,dx}{1+x}.$$

Iam vero integralibus ab $x=0$ ad $x=1$ extensis notum est fore

$$\int \frac{x\,dx}{\sqrt{1-xx}} = 1 \text{ et } \int \frac{x\,dx}{1+x} = 1 - l\,2$$

ita vt habeamus

$$\int \frac{x\,dx\,lx}{\sqrt{1-xx}}\left[\begin{smallmatrix} ab\ x=0 \\ ad\ x=1 \end{smallmatrix}\right] = l\,2 - 1.$$

§. 29. Quoniam in hac formula integrale $\int \frac{x\,dx}{\sqrt{1-xx}}$ algebraice exhiberi potest, cum sit $= 1 - \sqrt{1-xx}$, valor quaesitus etiam per reductiones consuetas erui potest, cum sit

$$\int \frac{x\,dx\,lx}{\sqrt{1-xx}} = (1-\sqrt{1-xx})\,lx - \int \frac{dx}{x}(1-\sqrt{1-xx})$$

positoque $x = 1$ erit

$$\int \frac{x\,dx\,lx}{\sqrt{1-xx}} = -\int \frac{dx}{x}(1-\sqrt{1-xx})$$

ad quam formam integrandam fiat $1 - \sqrt{1-xx} = z$, vnde colligitur $xx = 2z - zz$, ergo $2\,lx = lz + l(2-z)$ sicque fiet $\frac{dx}{x} = \frac{dz(1-z)}{z(2-z)}$, quibus valoribus substitutis erit

$$+\int \frac{dx}{x}(1-\sqrt{1-xx}) = +\int \frac{dz(1-z)}{2-z}$$

qui ergo valor erit $= C - z - l(2-z)$. Quia igitur posito $x = 0$ fit $z = 0$, constans erit $C = +l\,2$; facto igitur $x = 1$, quia tum fit $z = 1$, iste valor integralis erit $l\,2 - 1$, prorsus vt ante.

§. 30. Eundem valorem suppeditat theorema prius supra allatum, quo erat $p = n = 2$; inde enim statim fit $\int \frac{x\,dx\,lx}{\sqrt{1-xx}} = \int - \frac{x\,dx}{1+x}$. Ante autem vidimus esse $\int \frac{x\,dx}{1+x} = 1 - l\,2$ ita vt etiam hinc prodeat valor quaesitus $l\,2 - 1$.

Exemplum III. quo $p = 3$.

§. 31. Hoc igitur casu aequatio in theoremate generali allata hanc induet formam:

$$\int \frac{xx\,dx\,lx}{\sqrt{1-xx}} = -\int \frac{xx\,dx}{\sqrt{1-xx}} \cdot \int \frac{xx\,dx}{1+x}.$$

Per reductiones autem notissimas constat esse

$$\int \frac{xx\,dx}{\sqrt{1-xx}}\left[\begin{smallmatrix} ab\ x=0 \\ ad\ x=1 \end{smallmatrix}\right] = \frac{1}{2}\cdot\frac{\pi}{2}$$

at

tum vero fractio spuria $\frac{xx}{1+x}$ resoluitur in has partes $x-1+\frac{1}{1+x}$, vnde erit $\int\frac{xx\,dx}{1+x}=\frac{1}{2}xx-x+l(1+x)$, quod integrale iam euanescit posito $x=0$; facto ergo $x=1$ eius valor erit $=-\frac{1}{2}+l2$; quamobrem integrale quod quaerimus erit

$$\int\frac{xx\,dx\,lx}{\sqrt{1-xx}}\left[\begin{matrix}ab\ x=0\\ ad\ x=1\end{matrix}\right]=-\frac{\pi}{4}\left(l2-\frac{1}{2}\right).$$

Exemplum IV. quo $p=4$.

§. 32. Hoc igitur casu aequatio superior hanc induet formam:

$$\int\frac{x^3\,dx\,lx}{\sqrt{1-xx}}=-\int\frac{x^3\,dx}{\sqrt{1-xx}}\cdot\int\frac{x^3\,dx}{1+x}.$$

Per reductiones autem notissimas constat esse

$$\int\frac{x^3\,dx}{\sqrt{1-xx}}\left[\begin{matrix}ab\ x=0\\ ad\ x=1\end{matrix}\right]=\frac{2}{3}$$

tum vero fractio spuria $\frac{x^3}{1+x}$ resoluitur in has partes: $xx-x+1-\frac{1}{x+1}$, vnde integrando fit

$$\int\frac{x^3\,dx}{1+x}=\frac{1}{3}x^3-\frac{1}{2}xx+x-l(1+x)$$

ex quo valor formulae erit $=\frac{5}{6}-l2$. His ergo valoribus substitutis adipiscimur hanc integrationem:

$$\int\frac{x^3\,dx\,lx}{\sqrt{1-xx}}\left[\begin{matrix}ab\ x=0\\ ad\ x=1\end{matrix}\right]=-\frac{2}{3}\left(\frac{5}{6}-l2\right).$$

Exemplum V. quo $p=5$.

§. 33. Hoc igitur casu aequatio superior hanc induet formam:

$$\int\frac{x^4\,dx\,lx}{\sqrt{1-xx}}=-\int\frac{x^4\,dx}{\sqrt{1-xx}}\cdot\int\frac{x^4\,dx}{1+x}.$$

Constat autem esse

$$\int\frac{x^4\,dx}{\sqrt{1-xx}}\left[\begin{matrix}ab\ x=0\\ ad\ x=1\end{matrix}\right]=\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2}$$

tum vero fractio spuria $\frac{x^4}{1+x}$ manifesto resoluitur in has partes: $x^3-xx+x-1+\frac{1}{x+1}$, vnde integrando fit

$$\int\frac{x^4\,dx}{1+x}=\frac{1}{4}x^4+\frac{1}{3}x^3+\frac{1}{2}xx-x+l(1+x)$$

 ex

ex quo valor formulae erit $= -\frac{7}{12} + l\,2$. His igitur valoribus substitutis prodibit ista integratio:

$$\int \frac{x^4\,dx\,lx}{\sqrt{1-xx}} \left[\begin{smallmatrix} ab & x=0 \\ ad & x=1 \end{smallmatrix}\right] = -\frac{1.3}{2.4} \cdot \frac{\pi}{2} \left(l\,2 - \frac{7}{12}\right).$$

Exemplum VI. quo $p = 6$.

§. 34. Hoc igitur casu aequatio superior induet hanc formam:

$$\int \frac{x^5\,dx\,lx}{\sqrt{1-xx}} = -\int \frac{x^5\,dx}{\sqrt{1-xx}} \cdot \int \frac{x^5\,dx}{1+x}.$$

Constat autem per reductiones notas esse

$$\int \frac{x^5\,dx}{\sqrt{1-xx}} \left[\begin{smallmatrix} ab & x=0 \\ ad & x=1 \end{smallmatrix}\right] = \frac{2.4}{3.5}$$

tum vero fractio spuria $\frac{x^5}{1+x}$ resoluitur in has partes:

$$x^4 - x^3 + xx - x + 1 - \frac{1}{x+1}$$

vnde integrando nanciscimur

$$\int \frac{x^5\,dx}{1+x} = \frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}xx + x - l(1+x)$$

ex quo valor huius formulae erit $= \frac{47}{60} - l\,2$; quibus valoribus substitutis prodibit ista integratio:

$$\int \frac{x^5\,dx\,lx}{\sqrt{1-xx}} \left[\begin{smallmatrix} ab & x=0 \\ ad & x=1 \end{smallmatrix}\right] = -\frac{2.4}{3.5}\left(\frac{47}{60} - l\,2\right).$$

II. Euolutio casus quo $m = 3$ et $n = 2$.

§. 35. Hic ergo erit $X = \sqrt{(1-xx)}$, vnde theorema nostrum generale nobis praebebit hanc aequationem:

$$\int x^{p-1}\,dx\,lx.\sqrt{1-xx} = \int x^{p-1}\,dx\sqrt{1-xx}.\int \frac{x^{p-1}(x^3-1)\,dx}{1-xx}$$

vbi cum sit

$$\frac{x^3-1}{1-xx} = \frac{-xx-x-1}{x+1} = -x - \frac{1}{x+1}.$$

erit postrema formula integralis

$$-\int x^p\,dx - \int \frac{x^{p-1}\,dx}{1+x}$$

quae

quae integrata ab $x=0$ ad $x=1$ dat

$$-\frac{1}{p+1}\int\frac{x^{p-1}dx}{1+x}$$

quam ob rem habebimus

$$\int x^{p-1}dx\,lx\,\sqrt{1-xx}=-\int x^{p-1}dx\sqrt{1-xx}\left(\frac{1}{p+1}+\int\frac{x^{p-1}dx}{1+x}\right).$$

Hinc igitur sequentia exempla notasse iuuabit.

Exemplum I. quo $p=1$.

§. 36. Pro hoc igitur casu postremus factor euadet, $\frac{1}{2}+l2$, ita vt sit

$$\int dx\,lx.\sqrt{1-xx}=-\left(\tfrac{1}{2}+l2\right)\int dx\sqrt{1-xx}.$$

Pro formula autem $\int dx\sqrt{1-xx}$ statuatur $\sqrt{1-xx}=1-vx$ fietque $x=\frac{2v}{1+vv}$ et $\sqrt{1-xx}=\frac{1-vv}{1+vv}$ atque $dx=\frac{2dv(1-vv)}{(1+vv)^2}$, vnde fiet $dx\sqrt{1-xx}=\frac{2dv(1-vv)^2}{(1+vv)^3}$, cuius integrale resoluitur in has partes: $\frac{2v}{(1+vv)^2}-\frac{v}{1+vv}+A\,\mathrm{tang.}\,v$; quae expressio, cum extendi debeat ab $x=0$ vsque ad $x=1$, prior terminus erit $v=0$, alter vero terminus est $v=1$; ita vt integrale illud a $v=0$ vsque ad $v=1$ extendi debeat. At vero illa expressio sponte euanescit posito $v=0$, facto autem $v=1$, valor integralis erit $=\frac{\pi}{4}$, quam ob rem habebimus

$$\int dx\,lx.\sqrt{1-xx}\left[\begin{smallmatrix}ab\ x=0\\ ad\ x=1\end{smallmatrix}\right]=-\tfrac{\pi}{4}\left(\tfrac{1}{2}+l2\right).$$

§. 37. Hic quidem calculum per longas ambages euoluimus, prouti reductio ad rationalitatem formulae $\sqrt{1-xx}$ manuduxit; at vero solus aspectus formulae $\int dx\sqrt{1-xx}$ statim declarat, eam exprimere aream quadrantis circuli, cuius radius $=1$, quem nouimus esse $=\frac{\pi}{4}$. Caeterum adhiberi potuisset ista reductio:

$$\int dx\sqrt{1-xx}=\tfrac{1}{2}x\sqrt{1-xx}+\tfrac{1}{2}\int\frac{dx}{\sqrt{1-xx}}$$

cuius valor ab $x=0$ ad $x=1$ extensus manifesto dat $\frac{\pi}{4}$.

Exemplum II. quo $p = 2$.

§. 38. Hoc ergo casu postremus factor fit

$$\tfrac{1}{3} + \int \frac{x\,dx}{1+x} = \tfrac{4}{3} - l\,2$$

sicque habebimus

$$\int x\,dx\,lx.\sqrt{1-xx} = -(\tfrac{4}{3} - l\,2)\int x\,dx\sqrt{1-xx}$$

perspicuum autem est esse

$$\int x\,dx\sqrt{1-xx} = C - \tfrac{1}{3}(1-xx)^{\frac{3}{2}}$$

qui valor ab $x = 0$ ad $x = 1$ extensus praebet $\frac{1}{3}$, ita vt habeamus

$$\int x\,dx\,lx.\sqrt{1-xx}\left[\begin{smallmatrix}ab\ x=0\\ ad\ x=1\end{smallmatrix}\right] = -\tfrac{1}{3}(\tfrac{4}{3} - l\,2).$$

III. Euolutio casus quo $m = 1$ et $n = 3$.

§. 39. Hoc igitur casu erit $X = \frac{1}{\sqrt[3]{(1-x^3)^2}}$, vnde theorema generale nobis praebet hanc aequationem:

$$\int \frac{x^{p-1}\,dx\,lx}{\sqrt[3]{(1-x^3)^2}} = \int \frac{x^{p-1}\,dx}{\sqrt[3]{(1-x^3)^2}} \cdot \int \frac{x^{p-1}(x-1)\,dx}{1-x^3}$$

vbi postrema formula reducitur ad hanc: $-\int \frac{x^{p-1}\,dx}{xx+x+1}$, ita vt habeamus

$$\int \frac{x^{p-1}\,dx\,lx}{\sqrt[3]{(1-x^3)^2}} = -\int \frac{x^{p-1}\,dx}{\sqrt[3]{(1-x^3)^2}} \cdot \int \frac{x^{p-1}\,dx}{xx+x+1}$$

sequentia igitur exempla adiungamus.

Exemplum I. quo $p = 1$.

§. 40. Hoc igitur casu postremus factor euadit $\int \frac{dx}{xx+x+1}$, cuius integrale indefinitum reperitur $\frac{2}{\sqrt{3}}\,A\,tg.\,\frac{x\sqrt{3}}{2+x}$ qui valor posito $x = 1$ abit in $\frac{\pi}{3\sqrt{3}}$; quocirca hoc casu habe-

habebimus

$$\int \frac{dx\,x}{\sqrt[3]{(1-x^3)^2}} = -\frac{\pi}{3\sqrt{3}} \int \frac{dx\,lx}{\sqrt[3]{(1-x^3)^2}}$$

at vero formula integralis $\int \frac{dx}{\sqrt[3]{(1-x^3)^2}}$ peculiarem quantitatem transcendentem inuoluit, quam neque per logarithmos, neque per arcus circulares explicare licet.

Exemplum II. quo $p = 2$.

§. 41. Hoc igitur casu postremus factor erit $\int \frac{xdx}{1+x+xx}$ qui in has partes resoluatur:

$$\int \frac{\frac{1}{2}(2xdx + dx)}{1+x+xx} - \frac{1}{2}\int \frac{dx}{1+x+xx}$$

vbi partis prioris integrale est

$$\tfrac{1}{2} l(1+x+xx) = \tfrac{1}{2} l\,3 \text{ (posito scilicet } x = 1)$$

alterius vero partis integrale est $-\frac{1}{2} \cdot \frac{\pi}{3\sqrt{3}}$, quo valore substituto habebimus

$$\int \frac{xdx\,lx}{\sqrt[3]{(1-x^3)^2}} = -\frac{1}{2}\left(l\,3 - \frac{\pi}{3\sqrt{3}}\right)\int \frac{x\,dx}{\sqrt[3]{(1-x^3)^2}}.$$

Nunc vero istam formulam integralem commode assignare licet per reductionem supra initio indicatam; cum enim hic sit $m = 1$ et $n = 3$, tum vero sumserimus $p = 2$, erit $p = n - m$. Supra autem §. 15. inuenimus, hoc casu integrale fore $= \frac{\pi}{n \sin. \frac{m\pi}{n}}$, qui valor nostro casu abit in $\frac{\pi}{3 \sin. \frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}}$. Hoc igitur valore substituto nostram formulam per meras quantitates cognitas exprimere poterimus, hoc modo:

$$\int \frac{xdx\,lx}{\sqrt[3]{(1-x^2)^2}} \left[\begin{smallmatrix} ab\ x = 0 \\ ad\ x = 1 \end{smallmatrix}\right] = -\frac{\pi}{3\sqrt{3}}\left(l\,3 - \frac{\pi}{3\sqrt{3}}\right).$$

IV.

IV. Euolutio casus quo $m = 2$ et $n = 3$.

§. 42. Hoc igitur casu erit $X = \frac{1}{\sqrt[3]{(1 - x^3)}}$ vnde theorema generale praebet istam aequationem:

$$\int \frac{x^{p-1} dx\, lx}{\sqrt[3]{(1 - x^3)}} = \int \frac{x^{p-1} dx}{\sqrt[3]{(1 - x^3)}} \cdot \int \frac{x^{p-1}(xx - 1) dx}{1 - x^3}.$$

vbi forma postrema transmutatur in hanc: $-\int \frac{x^{p-1} dx (1 + x)}{1 + x + xx}$; vnde fiet

$$\int \frac{x^{p-1} dx\, lx}{\sqrt[3]{(1 - x^3)}} = -\int \frac{x^{p-1} dx}{\sqrt[3]{(1 - x^3)}} \cdot \int \frac{x^{p-1} dx (1 + x)}{1 + x + xx}$$

vnde sequentia exempla expediamus.

Exemplum I. quo $p = 1$.

§. 43. Hoc ergo casu membrum postremum erit $\int \frac{dx(1 + x)}{1 + x + xx}$, cuius integrale in has partes distribuatur:

$$\frac{1}{2} \int \frac{2x dx + dx}{1 + x + xx} + \frac{1}{2} \int \frac{dx}{1 + x + xx}$$

vnde manifesto pro casu $x = 1$ prodit $\frac{1}{2}(l3 + \frac{\pi}{3\sqrt{3}})$; quamobrem nostra aequatio erit

$$\int \frac{dx\, lx}{\sqrt[3]{1 - x^3}} = -\frac{1}{2}(l3 + \frac{\pi}{3\sqrt{3}}) \int \frac{dx}{\sqrt[3]{(1 - 1x)^3}}.$$

In hac autem formula integrali, ob $m = 2$ et $n = 3$, quia sumsimus $p = 1$, erit $p = n - m$; pro hoc ergo casu per §. 15. valor istius formulae absolute exprimi poterit, eritque $\int \frac{dx}{\sqrt[3]{(1 - x^3)}} = \frac{2\pi}{3\sqrt{3}}$; consequenter etiam hoc casu per quantitates absolutas consequimur hanc formam:

$$\frac{dx\, lx}{\sqrt[3]{(1 - x^3)}} \left[\begin{matrix} ab\ x = 0 \\ ad\ x = 1 \end{matrix} \right] = -\frac{\pi}{3\sqrt{3}} (l3 + \frac{\pi}{3\sqrt{3}}).$$

§. 44. Quodsi hanc formam cum postrema casus praecedentis, quae itidem absolute prodiit expressa, combinemus,

nemus, earum summa primo dabit

$$\int \frac{x\,dx\,lx}{\sqrt[3]{(1-x^3)^2}} + \int \frac{dx\,lx}{\sqrt[3]{(1-x^3)}} = -\frac{2\pi l3}{3\sqrt{3}}$$

sin autem posterior a priore subtrahatur, orietur ista aequatio:

$$\int \frac{x\,dx\,lx}{\sqrt[3]{(1-x^3)^2}} - \int \frac{dx\,lx}{\sqrt[3]{(1-x)^3}} = \frac{2\pi\pi}{27}.$$

Quoniam hoc modo ad expressiones satis simplices sumus perducti, operae pretium erit ambas aequationes sub alia forma repraesentare, qua binae partes integrales commode in vnam coniungi queant; statuamus scilicet $\frac{x}{\sqrt[3]{(1-x^3)}} = z$, vnde fit $\frac{xx}{\sqrt[3]{(1-x^3)^2}} = zz$, sicque prior formula induet hanc speciem $\int \frac{zz\,dx\,lx}{x}$; posterior vero istam: $\int \frac{z\,dx\,lx}{x}$; tum vero habebimus $\frac{x^3}{1-x^3} = z^3$, vnde fit $x^3 = \frac{z^3}{1+z^3}$, ideoque

$$lx = lz - \tfrac{1}{3}l(1+z^3) = l\frac{z}{\sqrt[3]{(1+z^3)}},$$

hincque porro

$$\frac{dx}{x} = \frac{dz}{z} - \frac{zz\,dz}{1+z^3} = \frac{dz}{z(1+z^3)}.$$

quare his valoribus adhibitis prior formula integralis euadit $\int \frac{z\,dz}{1+z^3} \cdot l\frac{z}{\sqrt[3]{(1+z^3)}}$; altera vero formula erit $\int \frac{dz}{1+z^3} \cdot l\frac{z}{\sqrt[3]{(1+z^3)}}$.

§. 45. Quoniam autem integralia ab $x=0$ ad $x=1$ extendi debent, notandum est, casu $x=0$ fieri $z=0$, at vero casu $x=1$ prodire $z=\infty$, ita vt nouas istas formas a $z=0$ ad $z=\infty$ extendi oporteat. Quo obseruato prior harum formularum dabit

$$\int \frac{z\,dz}{1+z^3} \cdot l\frac{z}{\sqrt[3]{(1+z)^3}} \left[\begin{matrix} a\ z=0 \\ ad\ z=\infty \end{matrix}\right] = -\frac{\pi l3}{3\sqrt{3}} + \frac{\pi\pi}{27}$$

posterior vero

$$\int \frac{dz}{1+z^3} \cdot l\frac{z}{\sqrt[3]{(1+z^3)}} \left[\begin{matrix} a\ z=0 \\ ad\ z=\infty \end{matrix}\right] = -\frac{\pi l3}{3\sqrt{3}} - \frac{\pi\pi}{27}.$$

Hinc igitur summa harum formularum erit

$$\int \frac{dz(1+z)}{1+z^3} \cdot l\frac{z}{\sqrt[3]{(1+z^3)}} = -\frac{2\pi l3}{3\sqrt{3}}$$

at vero differentia

$$\int \frac{dz(z-1)}{1+z^2} . l \frac{z}{\sqrt[3]{(1+z^3)}} = \frac{2\pi\pi}{27}.$$

§. 46. Hic non inutile erit obſeruaſſe, iſtum logarithmum $l \frac{z}{\sqrt[3]{(1+z^3)}}$ commode in ſeriem infinitam ſatis ſimplicem conuerti poſſe; cum enim ſit

$$l \frac{z}{\sqrt[3]{(1+z^3)}} = \frac{1}{3} l \frac{z^3}{1+z^3} = -\frac{1}{3} l \frac{1+z^3}{z^3}$$

erit per ſeriem

$$l \frac{z}{\sqrt[3]{1+z^3}} = -\frac{1}{3}\left(\frac{1}{z^3} - \frac{1}{2z^6} + \frac{1}{3z^9} - \frac{1}{4z^{12}} + \frac{1}{5z^{15}} \text{ etc.}\right)$$

verum iſta reſolutio nullum vſum praeſtare poteſt ad integralia haec per ſeries euoluenda, propterea quod poteſtates ipſius z in denominatoribus occurrunt, ideoque ſingulae partes non ita integrari poſſunt, vt euaneſcant poſito $z = 0$.

Exemplum II. quo $p = 2$.

§. 47. Hoc igitur caſu factor poſtremus euadit $\int \frac{xdx(1+x)}{1+x+xx}$, qui in has duas partes diſcerpitur: $\int dx - \int \frac{dx}{1+x+xx}$, cuius ergo integrale ab $x=0$ ad $x=1$ extenſum eſt $= 1 - \frac{\pi}{3\sqrt{3}}$. Hinc igitur deducimur ad hanc aequationem:

$$\int \frac{xdx\,lx}{\sqrt[3]{(1-x^3)}} = -\left(1 - \frac{\pi}{3\sqrt{3}}\right) \int \frac{xdx}{\sqrt[3]{(1-x^3)}}.$$

Hic autem notandum iſtam formulam integralem nullo modo abſolute exhiberi poſſe, ſed peculiarem quandam quantitatem tranſcendentem inuoluere.

V. Euolutio caſus, quo $m = 2$ et $n = 4$.

§. 48. Hoc igitur caſu erit $X = \frac{1}{\sqrt{(1-x^4)}}$, vnde theorema noſtrum generale nobis dabit hanc aequationem

$$\int \frac{x^{p-1}dx\,lx}{\sqrt{(1-x^4)}} = -\int \frac{x^{p-1}dx}{\sqrt{(1-x^4)}} . \int \frac{x^{p-1}dx}{1+xx}$$

at vero problema particulare prius pro hoc casu praebet $\int\frac{x^3 dx\, lx}{\sqrt{(1-x^4)}} = -\frac{1}{2}\int\frac{x^3 dx}{1+xx}$. Cum autem sit $\int\frac{x^3 dx}{1+xx} = \frac{1}{2} - \frac{1}{2} l 2$, erit absolute

$$\int\frac{x^3 dx\, lx}{\sqrt{(1-x^4)}}\left[\begin{matrix} ab\ x = 0 \\ ad\ x = 1 \end{matrix}\right] = -\frac{1}{4}(1 - l 2)$$

at vero hic casus congruit cum supra §. 28. tractato. Si enim hic ponamus $xx = y$, quo facto termini integrationis manent $y = 0$ et $y = 1$, erit $lx = \frac{1}{2} ly$ et $x dx = \frac{1}{2} dy$; quibus valoribus substitutis nostra aequatio abibit in hanc formam: $\frac{1}{4}\int\frac{y dy\, ly}{\sqrt{(1-yy)}} = -\frac{1}{4}(1 - l 2)$ siue $\int\frac{y dy\, ly}{\sqrt{(1-yy)}} = l 2 - 1$, prorsus vt supra.

§. 49. Alterum vero theorema particulare ad praesentem casum accommodatum dabit

$$\int\frac{x dx\, lx}{\sqrt{(1-x^4)}} = -\frac{\pi}{4}\int\frac{x dx}{1+xx}; \text{ est vero}$$

$$\int\frac{x dx}{1+xx} = l\sqrt{1+xx} = \frac{1}{2} l 2$$

ita vt habeamus

$$\int\frac{x dx\, lx}{\sqrt{(1-x^4)}}\left[\begin{matrix} ab\ x = 0 \\ ad\ x = 1 \end{matrix}\right] = -\frac{\pi}{8} l 2.$$

Quodsi vero hic vt ante statuamus $xx = y$, obtinebitur $\int\frac{dy\, ly}{\sqrt{(1-yy)}} = -\frac{\pi}{2} l 2$, qui est casus supra §. 26. tractatus. His duobus casibus exponens p erat numerus par, vnde casus impares euolui conueniet.

Exemplum I. quo $p = 1$.

§. 50. Hoc igitur casu formula integralis postrema fiet $\int\frac{dx}{1+xx} = \text{A tang.}\, x$, ita vt posito $x = 1$ prodeat $\frac{\pi}{4}$; tum vero aequatio nostra erit, $\int\frac{dx\, lx}{\sqrt{(1-x^4)}} = -\frac{\pi}{4}\int\frac{dx}{\sqrt{(1-x^4)}}$, integralibus scilicet ab $x = 0$ ad $x = 1$ extensis; vbi formula $\int\frac{dx}{\sqrt{(1-x^4)}}$ arcum curuae elasticae rectangulae exprimit, ideoque absolute exhiberi nequit.

Exemplum II. quo $p = 3$.

§. 51. Hoc ergo casu formula integralis postrema erit $\int \frac{xxdx}{1+xx} = \int dx - \int \frac{dx}{1+xx}$, cuius integrale posito $x = 1$ fit $= 1 - \frac{\pi}{4}$, ita vt nunc aequatio nostra euadat

$$\int \frac{xxdx\,lx}{\sqrt{(1-x^4)}} = -\left(1 - \frac{\pi}{4}\right) \int \frac{xxdx}{\sqrt{(1-x^4)}}$$

quae formula integralis pariter absolute exhiberi nequit; exprimit enim applicatam curuae elasticae rectangulae.

§. 52. Quanquam autem haec duo exempla ad formulas inextricabiles perduxerunt, tamen iam pridem demonstraui, productum horum duorum integralium $\int \frac{dx}{\sqrt{(1-x^4)}} \cdot \int \frac{xxdx}{\sqrt{(1-x^4)}}$ aequari areae circuli, cuius diameter $= 1$, siue esse $= \frac{\pi}{4}$; quam ob rem, binis exemplis coniungendis, hoc insigne theorema adipiscimur $\int \frac{dx\,lx}{\sqrt{(1-x^4)}} \cdot \int \frac{xxdx\,lx}{\sqrt{(1-x^4)}} = \frac{\pi^2}{16}\left(1 - \frac{\pi}{4}\right)$. Facile autem patet, innumera alia huiusmodi theoremata ex hoc fonte hauriri posse, quae, per se spectata, profundissimae indaginis sunt censenda.

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