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Considerationes super problemate astronomico in tomo commentarior. veter. IV. pertractato. Auctore L. Eulero

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CONSIDERATIONES SVPER PROBLEMATE ASTRONOMICO IN TOMO COMMENTARIOR. VETER. IV. PERTRACTATO.

Auctore

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§. 1.

Cum illo tempore calculus angulorum adhuc parum esset excultus; solutiones ibi traditae huius problematis non satis sunt dilucide ac plerumque per longas ambages erutae; vnde haud abs re fore arbitror hoc idem problema retractare, quandoquidem plures egregias obsertiones nunc adiicere licebit, quibus istud argumentum multo magis illustrabitur.

§. 2. Requiritur autem in hoc problemate, *vt ex tribus eiusdem stellae fixae obseruatis altitudinibus, vna cum temporis interuallis inter obseruationes elapsis tam eleuatio poli eius loci, vbi obseruationes sunt factae, quam declinatio ipsius stellae, seu eius distantia a polo difiniatur.* Sit igitur P polus et Z zenith eius loci, vbi obseruationes sunt institutae et OABCV parallelus, quem stella motu diurno percurrit, eritque PZ complementum altitudinis poli quaesitae et arcus PA, PB et PC referent distantiam stellae a polo, siue complementum eius declinationis, quae pariter desideratur; vnde has duas incognitas vocemus $PZ = x$ et $PA = PB = PC = y$. Tab. X. Fig. 1.

§. 3.

§. 3. Iam quae ſunt data contemplemur, ac primo quidem tempore obſeruationis primae fuerit ſtella in A, voceturque arcus $ZA = a$, qui erit complementum altitudinis obſeruatae, poſtquam ſcilicet per refractionem fuerit correcta. Deinde elapſo quodam tempore cognito ſtella fuerit in B, voceturque arcus $ZB = b$, ac denuo elapſo tempore quodam cognito peruenerit ſtella in C, voceturque arcus $ZC = c$. Ex datis autem temporis interuallis innoteſcent anguli ad Polum APB et BPC, quos ponamus $APB = \alpha$ et $BPC = \beta$, atque hae ſunt quinque quantitates cognitae, ex quibus binas incognitas x et y determinari oportet.

§. 4. Euidens autem eſt quamlibet harum trium obſeruationum tanquam primam ſpectari poſſe. Ita ſi tempore primae obſeruationis ſtella fuerit in B; tempore ſecundae erit in C, exiſtente angulo $BPC = \beta$; tertia vero poſtridie eueniet in A, elapſo tempore, cui reſpondet angulus horarius $= 360^\circ - \alpha - \beta$, ſiue quia in huiusmodi calculis totam peripheriam 360° negligere licet; iſte angulus erit $-\alpha - \beta$, quem breuitatis gratia ponamus $= \gamma$, ita vt ſit $\alpha + \beta + \gamma = 0$. Hi ergo anguli, ſi prima obſeruatio in A ſtatuatur, ordinem tenent α, β, γ; ſin autem prima ſit in B, ordo angulorum erit β, γ, α; ſumpto denique prima obſeruatione in C ordo angulorum erit γ, α, β.

§. 5. Ad ſolutionem autem perficiendam neceſſe eſt inſuper angulum ZPA in calculum introducere. Ponamus igitur $ZPA = \Phi$, atque ob analogiam ſtatuamus $ZPB = \Phi'$ et $ZPC = \Phi''$, eritque igitur $\Phi' = \Phi + \alpha$, $\Phi'' = \Phi' + \beta$ et $\Phi = \Phi'' - \alpha - \beta$, ſiue $\Phi = \Phi'' + \gamma$ ita vt hi tres anguli Φ, Φ', Φ'' pari ordine procedant atque anguli α, β, γ; vnde permutato obſeruationum ordine ſimilis permutatio locum habebit tam in angulis α, β, γ quam in angulis Φ, Φ', Φ''. Haec ideo notaſſe iuuabit, vt formulae pro vno caſu inventae facili ad reliquos caſus transferri poſſint.

§. 6.

§. 6. Consideremus nunc triangulum sphaericum AZP, ex cuius lateribus $ZA = a$, $ZP = x$ et $PA = y$ angulus $ZPA = \Phi$ ita determinatur, vt sit $\cos.\Phi = \frac{\cos.a - \cos.x\cos.y}{\sin.x\sin.y}$ siue $\cos.\Phi\sin.x\sin.y = \cos.a - \cos.x\cos.y$. Simili modo ex triangulo ZBP colligemus hanc determinationem:

$$\cos.\Phi'\sin.x\sin.y = \cos.b - \cos.x\cos.y$$

ac tertio similiter ex triangulo ZCP habebimus

$$\cos.\Phi''\sin.x\sin.y = \cos.c - \cos.x\cos.y,$$

atque ex his tribus aequationibus totam solutionem erui oportet.

§. 7. Quo autem calculum subleuemus ponamus, breuitatis gratia $\cos.x\cos.y = p$ et $\sin.x\sin.y = q$, ita vt hinc fiat $p + q = \cos.(x - y)$ et $p - q = \cos.(x + y)$: inuentis ergo litteris p et q facillime ambo anguli quaesiti x et y innotescent; vbi imprimis notari meretur, binos angulos x et y inter se esse permutabiles, namque si facta permutatione posuissemus $PZ = y$ et $PA = PB = PC = x$, ad easdem aequationes perueníssemus; sicque pro x et y quouis casu bini reperientur anguli, quorum alterum pro arcu PZ, alterum vero pro arcu PA accipere licebit, scilicet ex ipsa quaestionis natura eleuatio poli ac declinatio stellae semper inter se commutari poterunt.

§. 8. His igitur litteris p et q introductis tres nostrae aequationes ita se habebunt:

$$\text{I. } q\cos.\Phi = \cos.a - p$$

$$\text{II. } q\cos.\Phi' = \cos.b - p$$

$$\text{III. } q\cos.\Phi'' = \cos.c - p.$$

Hinc igitur primo facillime eliminabimus quantitatem incognitam p; si enim quamlibet harum aequationum a sequente subtrahamus, obtinebimus has aequationes:

I. q

I. $q(\cos.\Phi' - \cos.\Phi) = \cos. b - \cos. a$

II. $q(\cos.\Phi'' - \cos.\Phi') = \cos. c - \cos. b$

III. $q(\cos.\Phi - \cos.\Phi'') = \cos. a - \cos. c$,

quarum autem duas tantum euoluisse sufficiet. Quodsi iam porro breuitatis gratia ponamus

$\cos. b - \cos. a = A$; $\cos. c - \cos. b = B$ et $\cos. a - \cos. c = C$;

ita vt sit $A + B + C = 0$. impetrabimus has aequationes maxime succinctas:

I. $\cos.\Phi' - \cos.\Phi = \frac{A}{q}$

II. $\cos.\Phi'' - \cos.\Phi' = \frac{B}{q}$

III. $\cos.\Phi - \cos.\Phi'' = \frac{C}{q}$.

§. 9. Nunc etiam facile erit alteram incognitam q eliminare; diuidamus enim primam harum postremarum aequalitatum per tertiam, et nanciscemur istam $\frac{\cos.\Phi' - \cos.\Phi}{\cos.\Phi - \cos.\Phi''} = \frac{A}{C}$; quare cum sit $\Phi' = \Phi + \alpha$ et $\Phi'' = \Phi - \gamma$, erit

$\cos.\Phi' = \cos.\Phi \cos.\alpha - \sin.\Phi \sin.\alpha$ et

$\cos.\Phi'' = \cos.\Phi \cos.\gamma + \sin.\Phi \sin.\gamma$;

quibus valoribus substitutis aequatio hanc induet formam:

$$\frac{\cos.\Phi \cos.\alpha - \sin.\Phi \sin.\alpha - \cos.\Phi}{\cos.\Phi - \cos.\Phi \cos.\gamma - \sin.\Phi \sin.\gamma} = \frac{A}{C}$$

quae sponte redigitur ad hanc formam:

$$\frac{\cos.\alpha - \sin.\alpha\, tg.\Phi - 1}{1 - \cos.\Phi - \sin.\gamma\, tg.\Phi} = \frac{A}{C},$$

vnde igitur commodissime deducitur angulus Φ, cum sit $\tang.\Phi = \frac{A(1 - \cos.\gamma) + C(1 - \cos.\alpha)}{A \sin.\gamma - C \sin.\alpha}$. Hoc igitur modo angulus Φ per meras quantitates cognitas determinatur, quo inuento porro colligitur fore $q = \frac{A}{\cos.\Phi' - \cos.\Phi}$, hincque denique $p = \cos. a - q \cos.\Phi$; sicque problema perfecte erit solutum.

§. 10. Interim tamen operae pretium erit singulas formulas, ad quas hoc modo peruenietur, accuratius euoluere, vere, ac primo quidem, ſi obſeruatio prima ex A in B vel C transferatur, eodem modo perueniemus ad has aequationes:

$$\text{tang.}\, \Phi' = \frac{B(1 - \text{cof.}\, \alpha) + A(1 - \text{cof.}\, \beta)}{B\, \text{fin.}\, \alpha - A\, \text{fin.}\, \beta} \text{ et}$$

$$\text{tang.}\, \Phi'' = \frac{C(1 - \text{cof.}\, \beta) + B(1 - \text{cof.}\, \gamma)}{C\, \text{fin.}\, \beta - B\, \text{fin.}\, \gamma}.$$

§. 11. Inuenta tangente anguli Φ quaeramus quoque eius tam ſinum quam coſinum, ac reperiemus

$$\text{fin.}\, \Phi = \frac{A(1 - \text{cof.}\, \gamma) + C(1 - \text{cof.}\, \alpha)}{\sqrt{2AA(1 - \text{cof.}\gamma) + 2CC(1 - \text{cof.}\alpha) + 2AC(1 - \text{cof.}\gamma - \text{cof.}\alpha + \text{cof.}\gamma\, \text{cof.}\alpha - \text{fin.}\gamma\, \text{fin.}\alpha)}}$$

Hic autem pro denominatore notetur eſſe

$$\text{cof.}\, \alpha\, \text{cof.}\, \gamma - \text{fin.}\, \alpha\, \text{fin.}\, \gamma = \text{cof.}\, (\alpha + \gamma) = \text{cof.}\, \beta$$

ſicque iſte denominator habebit hanc formam:

$$\sqrt{(2AA(1 - \text{cof.}\gamma) + 2CC(1 - \text{cof.}\alpha) + 2AC(1 - \text{cof.}\gamma - \text{cof.}\alpha + \text{cof.}\beta))}$$

hunc ergo denominatorem ſi breuitatis gratia deſignemus per Δ erit

$$\text{fin.}\, \Phi = \frac{A(1 - \text{cof.}\, \gamma) + C(1 - \text{cof.}\, \alpha)}{\Delta} \text{ et cof.}\, \Phi = \frac{A\, \text{fin.}\, \gamma - C\, \text{fin.}\, \alpha}{\Delta}.$$

§. 12. Hic autem imprimis notari meretur pro quantitate irrationali Δ perpetuo eundem valorem reſultare, etiamſi litterae a, b, c; A, B, C et α, β, γ ordine praeſcripto inter ſe permutentur. Cum enim ſit

$$\tfrac{1}{2}\Delta^2 = AA(1 - \text{cof.}\gamma) + CC(1 - \text{cof.}\alpha) + AC(1 - \text{cof.}\gamma - \text{cof.}\alpha + \text{cof.}\beta)$$

ſingulis terminis ſecundum ternos coſinus, cof. α, cof. β et cof. γ diſponendis, erit

$$\tfrac{1}{2}\Delta^2 = AA + AC + CC - (AC + CC)\text{cof.}\alpha + AC\, \text{cof.}\beta - (A^2 + AC)\text{cof.}\gamma$$

Quoniam vero eſt $A + B + C = 0$, primae parti huius expreſſionis adiiciatur formula $AB + BB + BC = 0$ atque prima pars euadet $AA + BB + CC + AB + AC + BC$, vbi ternae litterae manifeſto ſunt permutabiles. Deinde vero erit

$$AC + CC = C(A + C) = -BC \text{ et}$$
$$AA + AC = A(A + C) = -AB$$

quibus ſubſtitutis erit

$$\tfrac{1}{2}\Delta^2 = AA + BB + CC + AB + AC + BC + BC\,\text{coſ.}\,\alpha + AC\,\text{coſ.}\,\beta + AB\,\text{coſ.}\,\gamma,$$

vbi permutabilitas litterarum eſt manifeſta, ideoque pro omnibus obſeruationum ordinibus ſemper erit

$$\Delta = \sqrt{2(AA + BB + CC + AB + AC + BC + BC\,\text{coſ.}\,\alpha + AC\,\text{coſ.}\,\beta + AB\,\text{coſ.}\,\gamma)}$$

quae formula etiam hoc modo exhiberi poteſt

$$\Delta = \sqrt{(2A^2 + 2B^2 + 2C^2 + 4AB\,\text{coſ.}\tfrac{1}{2}\gamma^2 + 4AC\,\text{coſ.}\tfrac{1}{2}\beta^2 + 4BC\,\text{coſ.}\tfrac{1}{2}\alpha^2)}.$$

§. 13. Inuento iam iſto valore quantitatis irrationalis Δ tam ſinus quam coſinus angulorum Φ, Φ' et Φ'' ſequenti modo exprimentur:

$$\text{ſin.}\,\Phi = \frac{A(1-\text{coſ.}\,\gamma) + C(1-\text{coſ.}\,\alpha)}{\Delta};\quad \text{coſ.}\,\Phi = \frac{A\,\text{ſin.}\,\gamma - C\,\text{ſin.}\,\alpha}{\Delta};$$

$$\text{ſin.}\,\Phi' = \frac{B(1-\text{coſ.}\,\alpha) + A(1-\text{coſ.}\,\beta)}{\Delta};\quad \text{coſ.}\,\Phi' = \frac{B\,\text{ſin.}\,\alpha - A\,\text{ſin.}\,\beta}{\Delta};$$

$$\text{ſin.}\,\Phi'' = \frac{C(1-\text{coſ.}\,\beta) + B(1-\text{coſ.}\,\gamma)}{\Delta};\quad \text{coſ.}\,\Phi'' = \frac{C\,\text{ſin.}\,\beta - B\,\text{ſin.}\,\gamma}{\Delta}.$$

§. 14. Ex his iam formulis triplici modo valor litterae q elici poteſt, qui autem terni valores inter ſe perfecte congruere debebunt, id quod ſtatim ex primo valore $q = \frac{A}{\text{coſ.}\,\Phi' - \text{coſ.}\,\Phi}$ perſpicietur; cum enim ex formulis modo inuentis ſit

$$\text{coſ.}\,\Phi' - \text{coſ.}\,\Phi = \frac{B\,\text{ſin.}\,\alpha - A(\text{ſin.}\,\beta + \text{ſin.}\,\gamma) + C\,\text{ſin.}\,\alpha}{\Delta} \text{ ob}$$

$$B + C = -A \text{ erit } \text{coſ.}\,\Phi' - \text{coſ.}\,\Phi = -\frac{A(\text{ſin.}\,\alpha + \text{ſin.}\,\beta + \text{ſin.}\,\gamma)}{\Delta},$$

ideoque $q = -\frac{\Delta}{\text{ſin.}\,\alpha + \text{ſin.}\,\beta + \text{ſin.}\,\gamma}$, vbi permutabilitas in oculos incurrit ſimulque patet ex binis reliquis formulis

$$q = \frac{B}{\text{coſ.}\,\Phi'' - \text{coſ.}\,\Phi'} \text{ et } q = \frac{C}{\text{coſ.}\,\Phi - \text{coſ.}\,\Phi''},$$

prorſus eandem hanc expreſſionem reſultare debuiſſe.

§. 15.

§. 15. Tantum igitur fupereft, vt etiam valorem litterae p hinc oriundum contemplemur, ad quod vtamur formula prima, qua fit $p = \cos. a - q \cos. \Phi$. Eft vero

$$q \cos. \Phi = -\frac{A \sin. \gamma + C \sin. \alpha}{\sin. \alpha + \sin. \beta + \sin. \gamma} \text{ ideoque}$$

$$p = \cos. a + \frac{A \sin. \gamma - C \sin. \alpha}{\sin. \alpha + \sin. \beta + \sin. \gamma},$$

verum quia hic cof. a non per litteras A, B, C exprimere licet, neceffe eft, vt loco litterarum A, B, C viciffim cofinus angulorum a, b, c in calculum introducantur; tum autem erit

$$q \cos. \Phi = -\frac{\cos. b \sin. \gamma + \cos. a \sin. \gamma + \cos. a \sin. \alpha - \cos. c \sin. \alpha}{\sin. \alpha + \sin. \beta + \sin. \gamma}$$

quo valore fubftituto reperietur

$$p = \frac{\cos. a \sin. \beta + \cos. b \sin. \gamma + \cos. c \sin. \alpha}{\sin. \alpha + \sin. \beta + \sin. \gamma}$$

vbi iterum permutabilitas fecundum ordinem litterarum per fe eft manifefta; vnde intelligitur ex omnibus tribus formulis litteram p continentibus evndem plane valorem refultare debere.

§. 16. Hic iam opere pretium erit etiam in valore litterae Δ loco litterarum A, B, C fuos valores fupra affignatos fubftituere, quo facto reperietur:

$$\begin{aligned}\tfrac{1}{2}\Delta^2 = {} & 2 \cos. a^2 \sin. \tfrac{1}{2}\beta^2 + 2 \cos. a \cos. b (\sin. \tfrac{1}{2}\alpha^2 - \sin. \tfrac{1}{2}\beta^2 - \sin. \tfrac{1}{2}\gamma^2) \\ & + 2 \cos. b^2 \sin. \tfrac{1}{2}\gamma^2 + 2 \cos. b \cos. c (\sin. \tfrac{1}{2}\beta^2 - \sin. \tfrac{1}{2}\gamma^2 - \sin. \tfrac{1}{2}\alpha^2) \\ & + 2 \cos. c^2 \sin. \tfrac{1}{2}\alpha^2 + 2 \cos. c \cos. a (\sin. \tfrac{1}{2}\gamma^2 - \sin. \tfrac{1}{2}\alpha^2 - \sin. \tfrac{1}{2}\beta^2)\end{aligned}$$

vnde colligitur fore

$$\Delta = 2\sqrt{\begin{cases} + \cos. a^2 \sin. \tfrac{1}{2}\beta^2 + \cos. a \cos. b (\sin. \tfrac{1}{2}\alpha^2 - \sin. \tfrac{1}{2}\beta^2 - \sin. \tfrac{1}{2}\gamma^2) \\ + \cos. b^2 \sin. \tfrac{1}{2}\gamma^2 + \cos. b \cos. c (\sin. \tfrac{1}{2}\beta^2 - \sin. \tfrac{1}{2}\gamma^2 - \sin. \tfrac{1}{2}\alpha^2) \\ + \cos. c^2 \sin. \tfrac{1}{2}\alpha^2 + \cos. c \cos. a (\sin. \tfrac{1}{2}\gamma^2 - \sin. \tfrac{1}{2}\alpha^2 - \sin. \tfrac{1}{2}\beta^2). \end{cases}}$$

vbi permutabilitas litterarum a, b, c item α, β, γ clariffime perfpicitur.

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