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Dynkin Diagrams and Fusion Algebras

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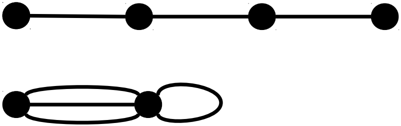
Graphs - a quick look

A graph $\mathcal{G}$ (undirected) has a set of vertices and a set of edges. Each edge joins two (possibly not distinct) vertices.
An edge from a vertex to itself is called a *loop*.
Two vertices are *connected* if there is a path of edges joining them. Connectedness is an equivalence relation that partitions the graph into *connected components*.
A graph is *connected* if it has one connected component.

Graphs - example



This graph has two connected components, one loop, and one set of multiple edges.

Dynkin diagrams

Dynkin diagrams encode the geometry of the root system of a semisimple Lie algebra & can be used to classify simple Lie algebras. Extended Dynkin diagrams perform the same role for affine Lie algebras. [cf. Kac]

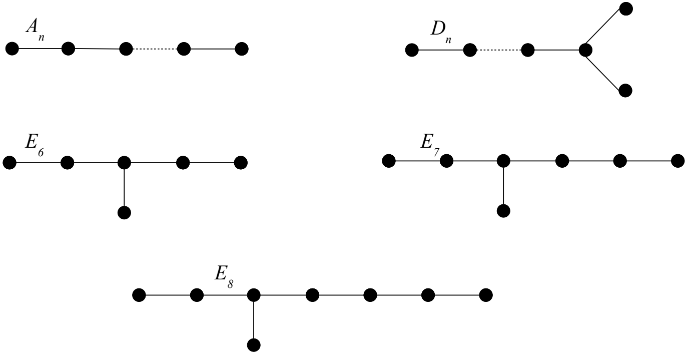
For type ADE simple Lie algebras, the Dynkin diagram:

- is connected, but has neither loops nor multiple edges,
- has no vertex of order 4, and
- has the property that removing a vertex gives another Dynkin diagram.

Extended Dynkin diagrams: same as above, except only one has a vertex of order 4, and it only has 5 vertices.

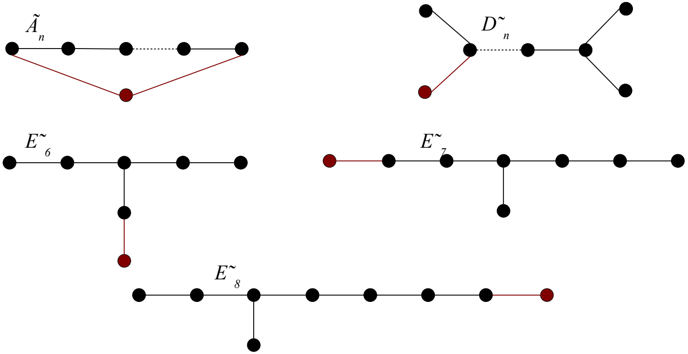
ADE Dynkin diagrams

(subscript = # of vertices)



Extended ADE Dynkin diagrams

(subscript $+1 = \#$ of vertices)



Fusion Algebras

$\mathcal{F}$ is a *fusion algebra* if $\mathcal{F}$ is a commutative, associative ring with a $\mathbb{Z}_{\geq 0}$-basis $\{B_1 = 1_{\mathcal{F}}, B_2, \ldots, B_\ell\}$ that is closed under $*$, a "dual" isomorphism of $\mathcal{F}$. Let $B_i^* = B_{i^*}$. Moreover, if the $n_{i,j}^k \in \mathbb{Z}_{\geq 0}$ satisfy

$$B_i \cdot B_j = \sum_k n_{i,j}^k B_k, \quad \text{then} \quad n_{i,j}^k = n_{i,k^*}^{j^*}.$$

The motivating example of a fusion algebra is the representation theory of finite groups over $\mathbb{C}$ (with $*$ as contragredient), but fusion algebras also arise in the representation theory of other algebraic objects, like finite-dimensional (quasi-)Hopf algebras.

Example: S_3 irreducibles

For us, $\mathcal{F}$ is a ring with $\mathbb{Z}$-basis consisting of self-dual $B_1 = 1_{\mathcal{F}}$, $B_2 = \text{sgn}$, and B_3, and with multiplication given by:

$$\begin{aligned} B_2 \cdot B_2 &= B_1 \\ B_2 \cdot B_3 &= B_3, \text{ and} \\ B_3 \cdot B_3 &= \sum_i B_i \end{aligned}$$

on the basis elements, and is extended linearly.

Example: $\mathbb{Z}_3$ irreducibles

Here, $\mathcal{F}$ is a ring with $\mathbb{Z}$-basis consisting of $B_1 = 1_{\mathcal{F}}$ (which is self-dual), B_2, and $B_3 = B_2^* = B_{2^*}$, and with multiplication given by:

$$\begin{aligned} B_2 \cdot B_2 &= B_3 \\ B_3 \cdot B_3 &= B_2 \\ B_2 \cdot B_3 &= B_1, \end{aligned}$$

extended linearly. Note that these three representations behave like $\mathbb{Z}_3$ itself.

Fusion Graphs

Given a fusion algebra $\mathcal{F}$ and a basis element $B_k \in \mathcal{F}$, define a (directed) fusion graph $\mathcal{G}_k$: the vertices of $\mathcal{G}_k$ are $\{v_1, \ldots, v_\ell\}$, and the number of edges from v_i to v_j is $n_{k,i}^j$. If $n_{k,i}^j = n_{k,j}^i$ for all i, j, then we consider $\mathcal{G}_k$ undirected.

Lemma: $\mathcal{G}_k$ is undirected if $k^* = k$.
Proof: $n_{k,i}^j = n_{k^*,i^*}^{j^*} = n_{k,j}^i$. $\square$

One can define $\mathcal{G}_B$ for any element B of the fusion algebra; $\mathcal{G}_B$ is undirected if $B = B^*$. We choose self-dual B from now on.

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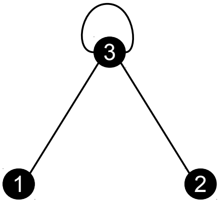
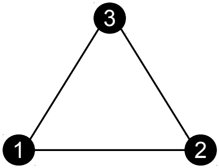
One can define $\mathcal{G}_B$ for any element B of the fusion algebra; $\mathcal{G}_B$ is undirected if $B = B^*$. We choose self-dual B from now on. Examples: $\mathcal{G}_3$ for S_3 and $\mathcal{G}_{2+3}$ for $\mathbb{Z}_3$.

Fusion Graphs - Examples

$\mathcal{G}_3$ for S_3



$\mathcal{G}_{2+3}$ for $\mathbb{Z}_3$



Fusion Graphs - Properties

Let $B = B_k$.

- $\mathcal{G}_B$ is *connected* if every basis element appears in some power of B. (For groups, true if B is faithful.)
- $\mathcal{G}_B$ has *loop* if there is a j such that $n^j_{k,j} > 0$.
- $\mathcal{G}_B$ has *multiple edges* if there exist i, j such that $n^j_{k,i} > 1$.

These can be easily extended to $B = \sum_k m_k B_k$.

The Question

Can each extended ADE Dynkin diagram arise as a fusion graph? If so, then which fusion graph (e.g., which group, which quasi-Hopf algebra)?

Finite Subgroups of $SO(3)$

Recall $SO(3) = \{M \in M_3(\mathbb{R}) \mid M^T = M^{-1}, \det(M) = 1\}$.
The nontrivial finite subgroups of $SO(3)$ are:

- cyclic ($\mathbb{Z}_n$) (order $n > 1$)
- dihedral (D_{2n}) (order $2n$, $n > 1$)
- A_4 (order 12) (tetrahedron)
- S_4 (order 24) (octahedron/cube)
- A_5 (order 60) (icosahedron/dodecahedron)

Finite Subgroups of $SU(2)$

Recall $SU(2) = \{U \in M_2(\mathbb{C}) \mid \bar{U}^T = U^{-1}, \det(U) = 1\}$, and that $1 \to \{\pm I_2\} \hookrightarrow SU(2) \to SO(3) \to 1$ is a short exact sequence. Thus the nontrivial finite subgroups of $SU(2)$ are:

- cyclic $\mathbb{Z}_n$ (order n, $n > 1$)
- binary dihedral $\mathbb{B}D_{2n}$ (order $4n$, $n > 1$)
- binary tetrahedral $\mathbb{BT}$ (order 24)
- binary octahedral $\mathbb{BO}$ (order 48)
- binary icosahedral $\mathbb{BI}$ (order 120)

[Klein, 1876]

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- cyclic $\mathbb{Z}_n$ (order n, $n > 1$) [n irreps]
- binary dihedral $\mathbb{B}D_{2n}$ (order $4n$, $n > 1$) [$n + 3$ irreps]
- binary tetrahedral $\mathbb{BT}$ (order 24) [7 irreps]
- binary octahedral $\mathbb{BO}$ (order 48) [8 irreps]
- binary icosahedral $\mathbb{BI}$ (order 120) [9 irreps]

[Klein, 1876]

Natural representation

For each finite subgroup of $SU(2)$, we wish to construct a fusion graph corresponding to the "natural" two-dimensional representation V (i.e., as elements of $SU(2)$). For each of the binary groups, V is irreducible and self-dual. For $\mathbb{Z}_n$, $V = W \oplus W^*$, where W is a generator of the fusion algebra.

McKay Correspondence

Let G be a finite subgroup of $SU(2)$ and let V be as above. Then the fusion graph $\mathcal{G}_V$ is an extended Dynkin diagram [McKay, '80]. Specifically:

group	$\mathbb{Z}_n$	$\mathbb{B}D_{2n}$	$\mathbb{BT}$	$\mathbb{BO}$	$\mathbb{BI}$
graph	$\tilde{A}_{n-1}$	$\tilde{D}_{n+2}$	$\tilde{E}_6$	$\tilde{E}_7$	$\tilde{E}_8$
# vertices	n	$n+3$	7	8	9

Quantum Double of a Finite Group [DPR, '90]

Let G be a finite group with identity element 1_G and let $e_g : G \to \mathbb{C}^*$ satisfy $e_g(h) = \delta_{g,h}$. The quantum double of a finite group, $D(G) = (\mathbb{C}G^* \otimes \mathbb{C}G, u, \Delta, \epsilon, S)$ is a Hopf algebra, where

$$
\begin{aligned}
(e_g \otimes x) \cdot (e_h \otimes y) &= \delta_{g,xhx^{-1}} (e_g \otimes xy) \\
u(1) &= \sum_{h \in G} (e_h \otimes 1_G) \\
\Delta (e_g \otimes x) &= \sum_{h \in G} (e_h \otimes x) \otimes (e_{h^{-1}g} \otimes x), \quad \text{and} \\
\epsilon (e_g \otimes x) &= \delta_{g,1_G} \\
S (e_g \otimes x) &= \left(e_{x^{-1}g^{-1}x} \otimes x^{-1}\right)
\end{aligned}
$$

for all $g, h, x, y \in G$.

Representations of $D(G)$

Let K be a conjugacy class of G, and let

$$D(K) = \text{span}\{(e_g \otimes x) \mid g \in K, x \in G\}.$$

Then each $D(K)$ is a two-sided ideal, and $D(G) = \bigoplus_K D(K)$.

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Rep'ns of $D(G)$ are induced from rep'ns of centralizers of G. Pick $g_K \in K$ and define $C_K := C_G(g_K)$. Let M be an irreducible C_K-module with character ρ. Then M can be induced up to $M(K, \rho)$, a $D(K)$-module.

Fusion algebra for $D(G)$

Theorem:

$$M(K,\rho) \otimes M(L,\psi) = \bigoplus_{J \subseteq KL} M(J,\sigma)$$

for those $\sigma \in C_J$ whose restriction to $Q = r^{-1}C_K r \cap s^{-1}C_L s \cap C_J$ is contained in $\rho^{(r)}\downarrow_Q \otimes \psi^{(s)}\downarrow_Q$, where $\rho^{(r)}(x) = \rho(rxr^{-1})$, and r, s are specific coset representatives of C_K, C_L satisfying $g_J = (r^{-1}g_K r)(s^{-1}g_L s)$.

Components of $D(G)$ fusion algebra

Corollary: If $K = \{1_G\} = 1$, then we have

$$M(1, \rho) \otimes M(L, \psi) = M(L, \rho{\downarrow_{C_L}} \otimes \psi).$$

Components of $D(G)$ fusion algebra

Corollary: If $K = \{1_G\} = 1$, then we have

$$M(1,\rho) \otimes M(L,\psi) = M(L, \rho\downarrow_{C_L} \otimes \psi).$$

Consequence: The $D(G)$ fusion graph for $M(1,\rho)$ has connected components labeled by conjugacy classes of G. Moreover, the connected component corresponding to L of the fusion graph is $\mathcal{G}_V$ for $V = \rho\downarrow_{C_L}$ in C_L.

"Orbifold" McKay Correspondence

Let $G \leq SU(2), |G| < \infty$.

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- If V is the natural two-dimensional rep'n of G, with character ρ, then $M(1, \rho) = V$ is (also) a $D(G)$-module.

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- $\mathcal{G}_V$ will thus be made of connected components, one for each conjugacy class.
- The component corresponding to the conjugacy class L is the extended Dynkin diagram for C_L.

Example: $D(\mathbb{BT})$

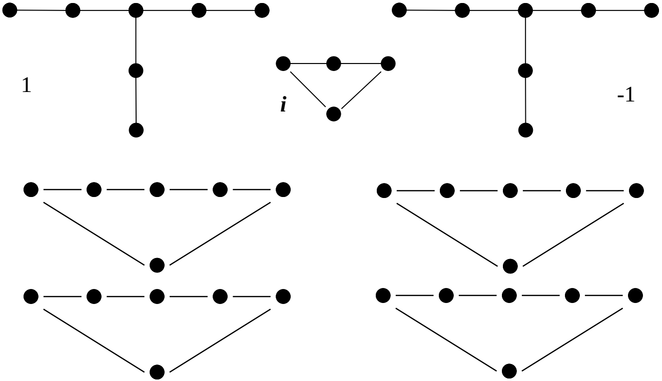
Here, we use

$$\mathbb{BT} \cong \{\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\}$$

where i, j, k are as in quaternions. (cf. Hurwitz quaternions)

g_K	1	-1	i	$\frac{1}{2}(1+i+j+k)$ (4 classes)
$\|K\|$	1	1	6	4 (4 times)
$C_G(g_K)$	G	G	$\mathbb{Z}_4$	$\mathbb{Z}_6$ (4 times)
diagram	$\tilde{E}_6$	$\tilde{E}_6$	$\tilde{A}_3$	$\tilde{A}_5$ (4 times)

Example: $D(\mathbb{BT})$



Generalized QDFG, $D(G,N)$ [G & Mason, '10]

Let G be a finite group with $N \lhd G$, and let $\bar{G} = G/N$. One can define a generalization of the quantum double of a finite group, $D(G,N) = (\mathbb{C}\bar{G}^* \otimes \mathbb{C}G, u, \Delta, \epsilon, S)$, via

$$\begin{aligned}
(e_{\bar{g}} \bowtie x) \cdot (e_{\bar{h}} \bowtie y) &= \delta_{\bar{g}, x\bar{h}x^{-1}} (e_{\bar{g}} \bowtie xy) \\
u(1) &= \sum_{\bar{h} \in \bar{G}} (e_{\bar{h}} \bowtie 1_G) \\
\Delta(e_{\bar{g}} \bowtie x) &= \sum_{\bar{h} \in \bar{G}} (e_{\bar{h}} \bowtie x) \otimes \left(e_{\bar{h}^{-1}\bar{g}} \bowtie x\right) \\
\epsilon(e_{\bar{g}} \bowtie x) &= \delta_{\bar{g}, 1_{\bar{G}}}, \quad \text{and} \\
S(e_{\bar{g}} \bowtie x) &= \left(e_{x^{-1}\bar{g}^{-1}x} \bowtie x^{-1}\right)
\end{aligned}$$

for all $\bar{g}, \bar{h} \in \bar{G}, x, y \in G$.

Representations of $D(G, N)$

Rep'ns of $D(G, N)$ are still induced from rep'ns of centralizers of elements of $\bar{G}$ in G (i.e., stabilizers for the conjugation action; cf. abelian extensions of Hopf algebras [Kashina, Mason, Montgomery, '02]). Let $C_{\bar{L}} := C_G(\bar{g}_{\bar{L}})$. A similar formula,

$$M(\bar{1}, \rho) \otimes M(\bar{L}, \psi) = M(\bar{L}, \rho{\downarrow}_{C_{\bar{L}}} \otimes \psi),$$

still holds. So we can still calculate fusion with such an $M(\bar{1}, \rho)$ in the appropriate centralizer, and thus, the connected component corresponding to $\bar{L}$ of the fusion graph is $\mathcal{G}_V$ for $V = \rho{\downarrow}_{C_{\bar{L}}}$ in $C_{\bar{L}}$.

Orbifold McKay Correspondence

Let $G \leq SU(2), |G| < \infty$, with $-I_2 \in G$, and let $N = \{\pm I_2\} \lhd G$. Then G is one of the binary types listed earlier, or cyclic of even order, and $\bar{G}$ is cyclic, dihedral, A_4, S_4, or A_5.

Let ρ be the character of the natural two-dimensional representation of G. Then $M(\bar{1}, \rho)$ is a $D(G, N)$-module whose fusion graph will be made of connected components indexed by conjugacy classes of $\bar{G}$. The component corresponding to $\bar{L}$ is the extended Dynkin diagram corresponding to $C_{\bar{L}}$.

Example: $D(\mathbb{BT}, N)$, $\bar{G} \cong A_4$

$\bar{g}$	$\bar{1}$	(12)(34)	(123) (2 classes)
$\lvert\bar{K}\rvert$	1	3	4 (2 times)
$C_G(\bar{g})$	G	$\mathbb{B}D_4$	$\mathbb{Z}_6$ (2 times)
diagram	$\tilde{E}_6$	$\tilde{D}_4$	$\tilde{A}_5$ (2 components)

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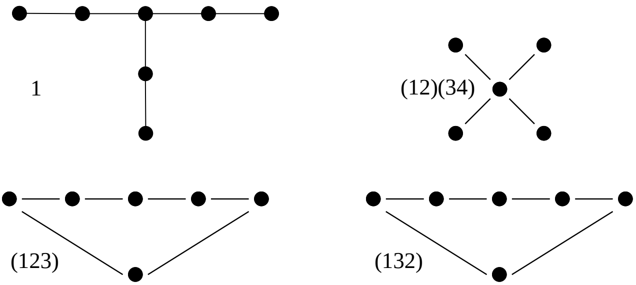
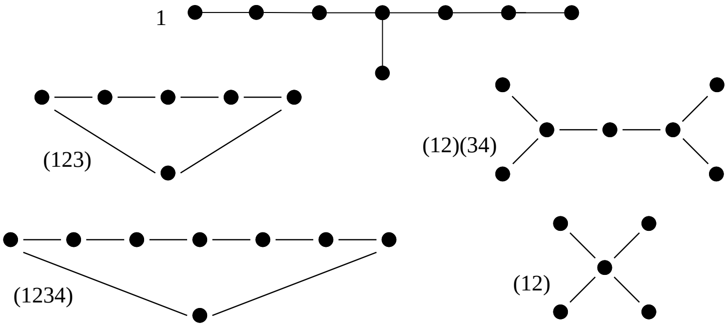


Example: $D(\mathbb{B}\mathbb{O}, N)$, $\bar{G} \cong S_4$ (from [GM, '10])

$\bar{g}$	$\bar{1}$	(12)	(12)(34)	(123)	(1234)
$\lvert\bar{K}\rvert$	1	6	3	8	6
$C_G(\bar{g})$	G	$\mathbb{B}D_4$	$\mathbb{B}D_8$	$\mathbb{Z}_6$	$\mathbb{Z}_8$
diagram	$\tilde{E}_7$	$\tilde{D}_4$	$\tilde{D}_6$	$\tilde{A}_5$	$\tilde{A}_7$

Example: $D(\mathbb{BO}, N)$, $\bar{G} \cong S_4$



More . . .

- Twisting to $D^{\omega}(G, N), \omega \in H^3(G, \mathbb{C}^*)$ [GM, '10]

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