Euler’s Algebraic Creativity and Undergraduate Math

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Christopher Goff

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SVCCM, Sac City College, 9 March 2013
Leonhard Euler (1707-1783) (Swiss)
Worked in St. Petersburg and Berlin
Had 13 kids
By 1735, blind in right eye – went totally blind later, but kept writing (secretary)
Published 530 books and papers in his life, and many more after his death (including the one we will consider)
Very prolific and successful, but also not always rigorous

Graphic from http://sebastianiaguirre.wordpress.com/2011/04/12/project-euler/
Some of Euler’s Mathematics

1. Notation: $f(x), e, (a, b, c), \sum, i$
2. $e^{ix} = \cos x + i \sin x \quad [e^{i\pi} + 1 = 0]$
3. $V - E + F = 2$
4. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{6}$
5. Euler line (geometry)
6. Euler’s method (ODEs)
7. Eulerian path (graph theory)
Our Problem

- Find six (6) positive integers satisfying the following properties.
  - Their sum is a square of an integer.
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  - The sum of their squares is the fourth power of an integer.
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- One answer was implicitly given by Euler in an 1824 paper entitled: “On finding three or more numbers, the sum of which is a square and the sum of the squares of which is a fourth power.”
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- Find six (6) positive integers satisfying the following properties.
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- One answer was implicitly given by Euler in an 1824 paper entitled: “On finding three or more numbers, the sum of which is a square and the sum of the squares of which is a fourth power.”
- Our goal: understand this original paper by Euler and use it to solve our problem.

Graphic from http://eulerarchive.maa.org/
Where did this problem come from?

- Diophantus (c.200 - c.284) of Alexandria, number theorist, advanced algebraic notation

"Diophantine" problems require integer and rational solutions. Find a right triangle whose legs add up to a perfect square and whose hypotenuse is also a perfect square. That is, if the legs are $x$ and $y$, and the hypotenuse is $z = N^2$, then $x + y = M^2$ and $x^2 + y^2 = z^2 = N^4$. Also solved by Pierre de Fermat (1601-1665) and Joseph-Louis Lagrange (1736-1813), before Euler.

My translations are NOT literal, but get the point across.
Diophantus (c.200 - c.284) of Alexandria, number theorist, advanced algebraic notation

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  - Euclid’s generating formulas: \(a = p^2 - q^2; \ b = 2pq; \ c = p^2 + q^2\).

\[
\begin{align*}
a^2 + b^2 &= (p^2 - q^2)^2 + (2pq)^2 \\
&= p^4 - 2p^2q^2 + q^4 + 4p^2q^2 \\
&= p^4 + 2p^2q^2 + q^4 \\
&= (p^2 + q^2)^2 = c^2. \quad \Box
\end{align*}
\]
How did Euler solve it?

First, some algebraic preliminaries:

- **Perfect squares**
  - \((s + t)^2 = s^2 + 2st + t^2\)
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- **Pythagorean triples \((a, b, c)\)**
  - Euclid’s generating formulas: \(a = p^2 - q^2; b = 2pq; c = p^2 + q^2\).
    \[
    a^2 + b^2 = (p^2 - q^2)^2 + (2pq)^2 = p^4 - 2p^2q^2 + q^4 + 4p^2q^2 = p^4 + 2p^2q^2 + q^4 = (p^2 + q^2)^2 = c^2. \square
    \]
  - (Euclid: EVERY primitive Pyth. triple can be put in this format.)
Find $x, y$ so that $x + y = M^2$ and $x^2 + y^2 = N^4$. 
Euler’s solution to Diophantus’ problem: §5

Find $x, y$ so that $x + y = M^2$ and $x^2 + y^2 = N^4$.

“Let us begin with the second condition. First, the formula $xx + yy$ [sic] shall be made a square, by placing $x = a^2 - b^2$ and $y = 2ab$, for then $x^2 + y^2 = (a^2 + b^2)^2$. 
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In addition, $a^2 + b^2$ should be a square, which happens in the same way by setting $a = p^2 - q^2$ and $b = 2pq$: from here, it follows that $x^2 + y^2 = (a^2 + b^2)^2 = (p^2 + q^2)^4$, and thus the latter condition has now been fully satisfied. [**]
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Then, it remains to satisfy the first condition, namely that $x + y$ be a square.”
“From these facts it is found that

\[ x = a^2 - b^2 = p^4 - 6p^2q^2 + q^4 \quad \text{and} \quad y = 4p^3q - 4pq^3; \]

and so the following formula \([x + y]\) ought to be a square

\[ p^4 + 4p^3q - 6p^2q^2 - 4pq^3 + q^4, \ldots \]

[with \(p > q > 0\) and \(a > b\)].”
Euler’s solution: §7

“The formula is solved by setting $\sqrt{x + y} = p^2 - 2pq + q^2, \ldots$”
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Wait a minute. What is Euler doing? He’s guessing! Let’s check:
Euler’s solution: §7

“The formula is solved by setting $\sqrt{x + y} = p^2 - 2pq + q^2, \ldots$”

Wait a minute. What is Euler doing? He’s guessing! Let’s check:

$$(p^2 - 2pq + q^2)^2 = p^4 - 4p^3q + 6p^2q^2 - 4pq^3 + q^4,$$

which doesn’t quite equal $p^4 + 4p^3q - 6p^2q^2 - 4pq^3 + q^4$, as he claimed.
But he is close. Three of the terms are identical, and the other two just have different signs. So, let’s set the two expressions equal and see what happens.
But he is close. Three of the terms are identical, and the other two just have different signs. So, let’s set the two expressions equal and see what happens.

\[ p^4 - 4p^3q + 6p^2q^2 - 4pq^3 + q^4 = p^4 + 4p^3q - 6p^2q^2 - 4pq^3 + q^4 \]
Euler’s solution: §7 (cont.)

But he is close. Three of the terms are identical, and the other two just have different signs. So, let’s set the two expressions equal and see what happens.

\[ p^4 - 4p^3 q + 6p^2 q^2 - 4pq^3 + q^4 = p^4 + 4p^3 q - 6p^2 q^2 - 4pq^3 + q^4 \]
\[ 12p^2 q^2 = 8p^3 q \]
But he is close. Three of the terms are identical, and the other two just have different signs. So, let’s set the two expressions equal and see what happens.

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\]
\[
12p^2q^2 = 8p^3q
\]
\[
3q = 2p, \text{ or } \frac{p}{q} = \frac{3}{2}.
\]

Euler doesn’t need the formula to be a square IDENTICALLY. He just needs to find values of \( p \) and \( q \) making the formula equal to a square.
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- “The formula is solved by setting \( \sqrt{x + y} = p^2 - 2pq + q^2 \), from which \( \frac{p}{q} = \frac{3}{2} \), or \( p = 3 \) and \( q = 2 \). [**]
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- “The formula is solved by setting \( \sqrt{x + y} = p^2 - 2pq + q^2 \), from which \( \frac{p}{q} = \frac{3}{2} \), or \( p = 3 \) and \( q = 2 \). [**]
- But then \( a = 5 \), and \( b = 12 \), and so \( x < 0 \), and this solution is rejected.” [\( x = -119; y = 120 \)]
“On account of this, a new method must be established . . . and so we keep \( q = 2 \) but at the same time we put \( p = 3 + \nu \), from which we deduce the following values:
“On account of this, a new method must be established . . . and so we keep \( q = 2 \) but at the same time we put \( p = 3 + \nu \), from which we deduce the following values:

\[
\begin{align*}
    p^4 &= 81 + 108\nu + 54\nu^2 + 12\nu^3 + \nu^4, \\
    4p^3q &= 216 + 216\nu + 72\nu^2 + 8\nu^3, \\
    6p^2q^2 &= 216 + 144\nu + 24\nu^2, \\
    4pq^3 &= 96 + 32\nu, \\
    q^4 &= 16.
\end{align*}
\]

When the terms are collected, the . . . formula adopts this form:”
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When the terms are collected, the ... formula adopts this form:

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x + y = 1 + 148\nu + 102\nu^2 + 20\nu^3 + \nu^4
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“On account of this, a new method must be established . . . and so we keep \( q = 2 \) but at the same time we put \( p = 3 + \nu \), from which we deduce the following values:

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\]

When the terms are collected, the . . . formula adopts this form:

\[ x + y = 1 + 148\nu + 102\nu^2 + 20\nu^3 + \nu^4 \]

Euler then guesses the root of this to be: \( 1 + 74\nu - \nu^2 \). Why?
More Algebraic Creativity (cont.)

Answer: it’s the same idea as before.
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\[(1 + 74\nu - \nu^2)^2 = 1 + 148\nu + 5474\nu^2 - 148\nu^3 + \nu^4,\]

which is not quite \(1 + 148\nu + 102\nu^2 + 20\nu^3 + \nu^4,\) but three of the terms are identical.
Answer: it’s the same idea as before.

\[(1 + 74v − v^2)^2 = 1 + 148v + 5474v^2 − 148v^3 + v^4,\]

which is not quite \(1 + 148v + 102v^2 + 20v^3 + v^4\), but three of the terms are identical. So, when setting them equal, several terms cancel, leaving:

\[
\begin{align*}
102v^2 + 20v^3 &= 5474v^2 − 148v^3 \\
168v^3 &= 5372v^2 \\
v &= \frac{5372}{168} = \frac{1343}{42}.
\end{align*}
\]
More Algebraic Creativity (cont.)

Answer: it’s the same idea as before.

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So \(p = 3 + v = \frac{1469}{42}\) and \(q = 2\). But Euler knew that he could multiply \(p\) and \(q\) by any constant and still have a perfect square.
Answer: it’s the same idea as before.

\[(1 + 74\nu - \nu^2)^2 = 1 + 148\nu + 5474\nu^2 - 148\nu^3 + \nu^4,\]

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\]

So \(p = 3 + \nu = \frac{1469}{42}\) and \(q = 2\). But Euler knew that he could multiply \(p\) and \(q\) by any constant and still have a perfect square. So he gets

\[p = 1469\quad \text{and} \quad q = 84.\]
Working backwards, Euler now gets
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\[ a = p^2 - q^2 = 1469^2 - 84^2 = 2,150,905 \]
Working backwards, Euler now gets

- \( a = p^2 - q^2 = 1469^2 - 84^2 = 2,150,905 \)
- \( b = 2pq = 2(1469)(84) = 246,792 \)
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- \( x = a^2 - b^2 = 4,565,486,027,761 \) and
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Working backwards, Euler now gets:

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“which are the same that Fermat, and others after him, found. The sum of them is the square of the number 2,372,159, while the sum of the squares is the fourth power of the number 2,165,017.”
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“which are the same that Fermat, and others after him, found. The sum of them is the square of the number 2,372,159, while the sum of the squares is the fourth power of the number 2,165,017.”

WOW!!! What does this mean for us?!?
Find $x$, $y$, $z$ so that

$$x + y + z = M^2 \quad \text{and} \quad x^2 + y^2 + z^2 = N^4.$$ 

Again, he begins with the second condition.
Euler’s solution to Euler’s problem

Find $x$, $y$, $z$ so that

$$x + y + z = M^2 \quad \text{and} \quad x^2 + y^2 + z^2 = N^4.$$

Again, he begins with the second condition.

- Let $x = a^2 + b^2 - c^2$; $y = 2ac$; $z = 2bc$. Then $x^2 + y^2 + z^2 =$
Find $x, y, z$ so that

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$$= \ (a^2 + b^2 - c^2)^2 + (2ac)^2 + (2bc)^2$$
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\[ = (a^2 + b^2 - c^2)^2 + (2ac)^2 + (2bc)^2 \]
\[ = a^4 + b^4 + c^4 + 2a^2b^2 - 2a^2c^2 - 2b^2c^2 + 4a^2c^2 + 4b^2c^2 \]
Euler’s solution to Euler’s problem

Find $x, y, z$ so that

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  $$= (a^2 + b^2 - c^2)^2 + (2ac)^2 + (2bc)^2$$

  $$= a^4 + b^4 + c^4 + 2a^2 b^2 - 2a^2 c^2 - 2b^2 c^2 + 4a^2 c^2 + 4b^2 c^2$$

  $$= a^4 + b^4 + c^4 + 2a^2 b^2 + 2a^2 c^2 + 2b^2 c^2$$
Euler’s solution to Euler’s problem

Find $x$, $y$, $z$ so that

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\[
= (a^2 + b^2 - c^2)^2 + (2ac)^2 + (2bc)^2
= a^4 + b^4 + c^4 + 2a^2b^2 - 2a^2c^2 - 2b^2c^2 + 4a^2c^2 + 4b^2c^2
= a^4 + b^4 + c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2
= (a^2 + b^2 + c^2)^2
\]
Find $x, y, z$ so that

$$x + y + z = M^2 \quad \text{and} \quad x^2 + y^2 + z^2 = N^4.$$  

Again, he begins with the second condition.

- Let $x = a^2 + b^2 - c^2; \ y = 2ac; \ z = 2bc$. Then $x^2 + y^2 + z^2 =$

  $$= (a^2 + b^2 - c^2)^2 + (2ac)^2 + (2bc)^2$$

  $$= a^4 + b^4 + c^4 + 2a^2b^2 - 2a^2c^2 - 2b^2c^2 + 4a^2c^2 + 4b^2c^2$$

  $$= a^4 + b^4 + c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2$$

  $$= (a^2 + b^2 + c^2)^2$$

- So now he needs to make $a^2 + b^2 + c^2$ into a perfect square, so that $x^2 + y^2 + z^2$ is a fourth power.
Euler’s solution to Euler’s problem (cont.)

Euler repeats the process.
Euler repeats the process.

- Let \( a = p^2 + q^2 - r^2 \); \( b = 2pr \); \( c = 2qr \). Then [**]
Euler repeats the process.

- Let $a = p^2 + q^2 - r^2$; $b = 2pr$; $c = 2qr$. Then [**]
  
  $$a^2 + b^2 + c^2 = (p^2 + q^2 + r^2)^2$$

  and so $x^2 + y^2 + z^2 = (a^2 + b^2 + c^2)^2 = (p^2 + q^2 + r^2)^4$. 
“Now let us express the letters \(x\), \(y\), \(z\) in terms of \(p\), \(q\), \(r\):

- \(x = p^4 + q^4 + r^4 + 2p^2q^2 + 2p^2r^2 - 6q^2r^2\)
- \(y = 4qr(p^2 + q^2 - r^2)\),
- \(z = 8pqr^2\).
"Now let us express the letters \( x, y, z \) in terms of \( p, q, r \):

- \( x = p^4 + q^4 + r^4 + 2p^2q^2 + 2p^2r^2 - 6q^2r^2 \)
- \( y = 4qr(p^2 + q^2 - r^2) \),
- \( z = 8pqr^2 \).

From here it is thus:

\[ x + y + z = p^4 + 2p^2(q + r)^2 + 8pqr^2 + q^4 + 4q^3r - 6q^2r^2 - 4qr^3 + r^4, \]

which must be a square."
“Now let us express the letters \( x, y, z \) in terms of \( p, q, r \):

- \( x = p^4 + q^4 + r^4 + 2p^2q^2 + 2p^2r^2 - 6q^2r^2 \)
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What is Euler’s guess for the square root?
“Now let us express the letters $x$, $y$, $z$ in terms of $p$, $q$, $r$:

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What is Euler’s guess for the square root? $p^2 + (q + r)^2$. 
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which must be a square.”

What is Euler’s guess for the square root? $p^2 + (q + r)^2$. So,

$$(p^2 + (q + r)^2)^2 = p^4 + 2p^2(q + r)^2 + (q + r)^4.$$
Euler’s answer

(lots of algebra) ...
Euler's answer

(lots of algebra) \ldots x + y + z is a perfect square if

\[ p = r + \frac{3}{2} q. \]
(lots of algebra) \( x + y + z \) is a perfect square if

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This is really the key equation:
Euler’s answer

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This is really the key equation:

- Can choose \( q \) and \( r \) more or less freely.
(lots of algebra) \( x + y + z \) is a perfect square if

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This is really the key equation:
- Can choose \( q \) and \( r \) more or less freely.
- This determines \( p \).
Euler’s answer

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This is really the key equation:
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- This determines \( p \).
- Then \( p, q, r \) determine \( a, b, c \).
(lots of algebra) \( x + y + z \) is a perfect square if

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This is really the key equation:
- Can choose \( q \) and \( r \) more or less freely.
- This determines \( p \).
- Then \( p, q, r \) determine \( a, b, c \).
- Then \( a, b, c \) determine \( x, y, z \).
An example

\[ p = r + \frac{3}{2} q \]

Let \( q = 2, \ r = 1 \).

Then

\[ a = 19, \ b = 8, \ c = 4. \]

Then

\[ x = 409, \ y = 152, \ z = 64. \]

So,

\[ x + y + z = 625 = 25^2 \]

and

\[ x^2 + y^2 + z^2 = 194, 481 = 441^2. \]

These are MUCH SMALLER than in the first problem.
An example

Let $q = 2$, $r = 1$.

Then $p = 4$. 

$$p = r + \frac{3}{2}q$$

So, $x = 409$, $y = 152$, $z = 64$.

Then $x^2 + y^2 + z^2 = 194$, $481 = 441^2 = 21^4$.

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- So, \( x + y + z = 625 = 25^2 \) and \( x^2 + y^2 + z^2 = 194,481 = 441^2 = 21^4 \).
- These are MUCH SMALLER than in the first problem.
Next, Euler finds four numbers \((x, y, z, v)\).

1. Set \(x = a^2 + b^2 + c^2 - d^2\), \(y = 2ad\), \(z = 2bd\), \(v = 2cd\).
Next, Euler finds four numbers \((x, y, z, v)\).

1. Set \(x = a^2 + b^2 + c^2 - d^2\), \(y = 2ad\), \(z = 2bd\), \(v = 2cd\).

2. Then set \(a = p^2 + q^2 + r^2 - s^2\), \(b = 2ps\), \(c = 2qs\), \(d = 2rs\).
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3. Then GUESS the square root of \(x + y + z + v\).
Finding a Pattern, I

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5. He then chooses \(r = 2\), \(q = s = 1\) to get \(p = 3\) and thus . . .
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4. ... Euler finds \(p = s + \frac{3}{2}r - q\).
5. He then chooses \(r = 2\), \(q = s = 1\) to get \(p = 3\) and thus ...
6. “\(x = 193; y = 104; z = 48; v = 16\), the sum of which is \(x + y + z + v = 361 = 19^2\); while the sum of the squares will be \(xx + yy + zz + vv = (pp + qq + rr + ss)^4 = 15^4\).”
Next, Euler finds five numbers \((x, y, z, v, w)\).

Set \(x = a^2 + b^2 + c^2 + d^2 - e^2\), \(y = 2ae\), \(z = 2be\), \(v = 2ce\), \(w = 2de\).
Next, Euler finds five numbers \((x, y, z, v, w)\).

1. Set \(x = a^2 + b^2 + c^2 + d^2 - e^2\), \(y = 2ae\), \(z = 2be\), \(v = 2ce\), \(w = 2de\).

2. Then set \(a = p^2 + q^2 + r^2 + s^2 - t^2\), \(b = 2pt\), \(c = 2qt\), \(d = 2rt\), \(e = 2st\).
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Next, Euler finds five numbers \((x, y, z, v, w)\).

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3. Then GUESS the square root of \(x + y + z + v + w\).

4. \(\ldots\) Euler finds \(p = t + \frac{3}{2}s - r - q\).

5. He then chooses \(s = 2, t = r = q = 1\) to get \(p = 2\) and thus \(\ldots\)

6. “\(x = 89; y = 72; z = 32; v = 16; w = 16\), the sum of which is \(x + y + z + v + w = 225 = 15^2\); while the sum of the squares will be \(x^2 + y^2 + z^2 + v^2 + w^2 = 11^4\).”
The Pattern

For 3 numbers, \( p = r + 3^2q \).

For 4 numbers, \( p = s + 3^2r - q \).

For 5 numbers, \( p = t + 3^2s - r - q \).

You try it!!

For 6 numbers, \( p = u + 3^2t - s - r - q \).
The Pattern

- For 3 numbers, \( p = r + \frac{3}{2}q \).
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The Pattern

- For 3 numbers, \( p = r + \frac{3}{2}q \).
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- For 3 numbers, \( p = r + \frac{3}{2}q \).
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The Pattern

- For 3 numbers, \( p = r + \frac{3}{2}q \).
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- You try it!!
- For 6 numbers, \( p = u + \frac{3}{2}t - s - r - q \).
Our turn

\[ p = u + \frac{3}{2} t - s - r - q \]

Can we find six numbers that add up to a perfect square and whose squares add up to a fourth power?
Our turn

\[ p = u + \frac{3}{2} t - s - r - q \]

Can we find six numbers that add up to a perfect square and whose squares add up to a fourth power?

- One solution: \( t = 2, \ u = s = r = q = 1 \). Then \( p = 1 \).
Our turn

\[ p = u + \frac{3}{2}t - s - r - q \]

Can we find six numbers that add up to a perfect square and whose squares add up to a fourth power?

- One solution: \( t = 2, u = s = r = q = 1 \). Then \( p = 1 \).
- Then \( a = 7, b = c = d = e = 2, f = 4 \).
Our turn

\[ p = u + \frac{3}{2}t - s - r - q \]

Can we find six numbers that add up to a perfect square and whose squares add up to a fourth power?

- One solution: \( t = 2, \ u = s = r = q = 1 \). Then \( p = 1 \).
- Then \( a = 7, \ b = c = d = e = 2, \ f = 4 \).
- Then \( x = 49, \ y = 56, \ z = v = w = m = 16 \).
\[ p = u + \frac{3}{2}t - s - r - q \]

Can we find six numbers that add up to a perfect square and whose squares add up to a fourth power?

- One solution: \( t = 2, u = s = r = q = 1 \). Then \( p = 1 \).
- Then \( a = 7, b = c = d = e = 2, f = 4 \).
- Then \( x = 49, y = 56, z = v = w = m = 16 \).
- Then \( x + y + z + v + w + m = 169 = 13^2 \), and \( x^2 + \ldots + m^2 = 6561 = 9^4 \).
Last spring, I taught Topics in the History of Mathematics.
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I assigned a project in which students had to engage with a primary source or a translation of a primary source.

One student chose this paper. She found six numbers that had the same property. Namely: 97, 112, 64, 64, 64, and 128.
Their sum is $23^2 = 529$, and the sum of their squares is $15^4 = 50625$. 
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Goff Euler's Creativity 24/25
Euler knew how to square things.
Euler was pretty creative when it came to algebra.
Even undergraduates can understand an original paper from a mathematical genius.

THANK YOU!!!

Goff
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