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Diophantine Geometry, Eulerian Number Theory, and Undergraduates

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Diophantine Geometry, Eulerian Number Theory, and Undergraduates

Christopher Goff
University of the Pacific
San Diego JMM 2013
History of Geometry Session
Thursday, January 10, 3pm
How this talk began...

- JMM 2012, Boston: Mini-course to translate Latin
- I signed up to translate a paper from the Euler Archive: E763.

[Disclaimer: I am not a math historian.]
How Euler’s paper began…

• “Famous is the problem once proposed by Fermat, and recently studied by … Lagrange …., in which are sought two positive whole numbers, the sum of which is a square, while the sum of the squares is a fourth power.”

\[ x + y = A^2; \quad x^2 + y^2 = B^4 \]
Euler’s Solution (part 1)

- Repeated use of identities, such as:
  \[ x = a^2 - b^2; \quad y = 2ab \Rightarrow x^2 + y^2 = (a^2 + b^2)^2 \]
  \[ a = p^2 - q^2; \quad b = 2pq \Rightarrow x^2 + y^2 = (p^2 + q^2)^4 \]

- Algebraic manipulation, kind of like perturbation theory, combined with guess-and-check.

- Finds that \( p = 3 \) and \( q = 2 \) is a solution, but that gives \( x = -119 \). So let \( q = 2 \) and \( p = 3 + \nu \). Then

\[ x + y = 1 + 148\nu + 102\nu^2 + 20\nu^3 + \nu^4 \]
Euler’s Solution (part 2)

• Guess that \( 1 + 148v + 102v^2 + 20v^3 + v^4 \) is the square of \( 1+74v – v^2 \).

\[
(1 + 74v – v^2)^2 = 1 + 148v + 5474v^2 – 148v^3 + v^4
\]

• Then \( 5372v^2 = 168v^3 \), whence \( v = 1343/42 \), and \( p = 1469/42 \).

• So to clear denominators, let \( p = 1469 \) and \( q = 84 \). Then find \( a \) and \( b \), and then \( x \) and \( y \).
Euler’s Answer

“...which, though they exceed a billion, are nevertheless the smallest satisfying the problem:

\[
\begin{align*}
  x &= 4,565,486,027,761 \\
  y &= 1,061,652,293,520 \\
\end{align*}
\]

which are the same that Fermat, and others after him, found. The sum of them is the square of the number 2,372,159, while the sum of the squares is the fourth power of the number 2,165,017.”
Euler’s Answer

“...which, though they exceed a billion, are nevertheless the smallest satisfying the problem:

\[ x = 4,565,486,027,761 \]
\[ y = 1,061,652,293,520 \]

which are the same that Fermat, and others after him, found. The sum of them is the square of the number 2,372,159, while the sum of the squares is the fourth power of the number 2,165,017.”
But…

• What about Lagrange’s and Fermat’s solutions?
• Where to look?
  ▪ École Polytechnique archives
  ▪ Bibliothèque Nationale
  ▪ Internet Archive
• Fortunately, I had some very specific numbers to look for.
Joseph Louis Lagrange (1736 – 1813)
Lagrange’s Solution (part 1: a different problem)

- Fermat’s Method of Descent can show a difference of fourth powers is never a square.
- But the difference of twice a fourth power and a fourth power being a square also “descends,” but has a basic solution.
  \[ 2x^4 - y^4 = z^2; \quad x = y = z = 1. \]
- From this initial solution, others can be built by “ascension.”
- Lagrange finds: \( x = 13, y = 1; \) \( x = 2,165,017, y = 2,372,159. \)
Lagrange’s Solution (part 2: connections)

• Lagrange ties the previous equation to a Diophantine geometry problem, also considered by Fermat: to find a right triangle whose legs add up to a perfect square, and whose hypotenuse is also a perfect square.

\[ p + q = y^2; \quad p^2 + q^2 = x^4 \]

• Lagrange noticed that if \( p \) and \( q \) satisfy the above equations, then:

\[
2x^4 - y^4 = 2(p^2 + q^2) - (p + q)^2 = (p - q)^2 = z^2
\]

• Conversely, if you know \( y \) and \( z \) above, then:

\[
p = \frac{y^2 + z}{2}; \quad q = \frac{y^2 - z}{2}
\]
Lagrange’s Answer

From solutions to the first equation, Lagrange finds:

- $p = 1$ and $q = 0$;
- $p = 120$ and $q = -119$;
- $p = 1,061,652,293,520$ and $q = 4,565,486,027,761$ as solutions to the right triangle problem.
Pierre de Fermat (1601 - 1665)
Sir Thomas Heath (1861 - 1940)

  (2nd edition 1910, pp. 297-300)
Heath’s Explanation

• Gives a solution.
• Explains Lagrange’s justification of the minimality of Fermat’s solution.
• Gives Euler’s solution as a footnote.
• (Gives essentially an annotated translation of Euler’s solution.)
Back to Euler

- Euler did not stop with two numbers; he found three numbers:
  \[ x + y + z = A^2; \quad x^2 + y^2 + z^2 = B^4 \]
  namely, \( x = 409, \ y = 152, \) and \( z = 64 \) (among other solutions).
- Then Euler found four such numbers: 193, 104, 48, and 16.
- Then he found five such numbers: 89, 72, 32, 16, and 16.
- Then he pointed out that he had found a pattern in his solution technique, so that the general problem had been solved.
Last spring, I assigned a project in which students had to engage with a primary source or a translation of a primary source.

One student chose this paper.

She found six numbers that had the same property. Namely: 97, 112, 64, 64, 64, and 128.

Their sum is $529 = 23^2$, and the sum of their squares is $50625 = 15^4$. 
What I Learned

- Always cite your sources.
- Become friends with your librarian.
- Translating can be fun.
- Diophantine geometry can lead to interesting mathematics, even for undergraduates.

Thank You!