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# Observationes circa novum et singulare progressionum genus

Leonhard Euler

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# OBSERVATIONES CIRCA NOVVM ET SINGVLARE PROGRESSIONVM

GENVS.

Auctore

L. EVLERO.

Inter res saepe numero, quae attentione nostra haud dignae videantur, observantur quaedam, quae satis profundam investigationem requirunt, ac non parum sublimibus speculationibus occasionem praebent. Quod cum plurimis exemplis confirmari possit, tum nuper etiam ipse sum expertus, dum quaestionem illam tyronibus notissimam, attentius contemplerem, qua quindecim Christiani totidemque Iudaei ita ordine sunt collocandi, ut si, numerandi initio in dato loco sumto, nonus quisque vel decimus in mare sit eiciendus, haec poena in solos Iudaeos sit casura. Quae quaestio etiam si in se nihil habeat difficultatis, tamen mox vidi, si in genere de hominum numero quocunque, ex quibus non nonus sed secundum alium quemvis numerum quotusquisque sit eiciendus, proponatur, difficillimum fore, ordinem eorum, qui continuo eicientur, assignare. Neque adeo methodus constat hoc in genere

Q 2

nere

neri praestandi, tamen quovis casu oblato, dum numeratio actu instituitur, solatio facillime obtinetur. Ex hoc genere haud parum curiosa mihi videtur questio, si v. gr. ex plurium sortium numero is solus sit supplicio afficiendus, qui, postquam nonus quisque vel secundum alium numerum ex ordine fuerit exemptus, tandem ultimo solus sit remansurus; hic scilicet maxime intererit, ante nosse illum fatalem locum, in quo numeratio illa ultimo terminabitur.

2. Quo omnia quae hic investiganda occurrunt, clarius perspiciantur, casum illum perpendamus, quo ex serie 30 notarum nona quaeque expungitur, quod negotium numeratione actu instituta ita commodissime repraesentatur:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
23	20	28	24	14	4	7	12	13	19	16	10	26	27	25	5	28	12	8	19	30	23	11	22	6	3	21	17	9	

Hic superiores numeri indicant, quoto loco a primo computando quaeque nota sit posita, inferiores vero numeri ostendunt, quando quaeque eliciatur, dum scilicet continuo nona quaeque expungitur: ita patet, primo nonam, secundo decimam octavam, tertio vicesimam septimam, quarto sextam, quinto decimam sextam et ita porro expungi, donec ultimo delenda super sit sola vicesima prima, qui adeo foret locus ille fatalis ante memoratus. Si indices electorum ordine disponantur, indicesque notarum subscribantur, haec series prodibit.

Indices

## Indices electionis

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4709, 4718, 4727, 4736, 4745, 4754, 4763, 4772, 4781, 4790, 4800, 4809, 4818, 4827, 4836, 4845, 4854, 4863, 4872, 4881, 4890, 4900, 4909, 4918, 4927, 4936, 4945, 4954, 4963, 4972, 4981, 4990, 5000, 5009, 5018, 5027, 5036, 5045, 5054, 5063, 5072, 5081, 5090, 5100, 5109, 5118, 5127, 5136, 5145, 5154, 5163, 5172, 5181, 5190, 5200, 5209, 5218, 5227, 5236, 5245, 5254, 5263, 5272, 5281, 5290, 5300, 5309, 5318, 5327, 5336, 5345, 5354, 5363, 5372, 5381, 5390, 5400, 5409, 5418, 5427, 5436, 5445, 5454, 5463, 5472, 5481, 5490, 5500, 5509, 5518, 5527, 5536, 5545, 5554, 5563, 5572, 5581, 5590, 5600, 5609, 5618, 5627, 5636, 5645, 5654, 5663, 5672, 5681, 5690, 5700, 5709, 5718, 5727, 5736, 5745, 5754, 5763, 5772, 5781, 5790, 5800, 5809, 5818, 5827, 5836, 5845, 5854, 5863, 5872, 5881, 5890, 5900, 5909, 5918, 5927, 5936, 5945, 5954, 5963, 5972, 5981, 5990, 6000, 6009, 6018, 6027, 6036, 6045, 6054, 6063, 6072, 6081, 6090, 6100, 6109, 6118, 6127, 6136, 6145, 6154, 6163, 6172, 6181, 6190, 6200, 6209, 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11945, 11954, 11963, 11972, 11981, 11990, 12000, 12009, 12018, 12027, 12036, 12045, 12054, 12063, 12072, 12081, 12090, 12100, 12109, 12118, 12127, 12136, 12145, 12154, 12163, 12172, 12181, 12190, 12200, 12209, 12218, 12227, 12236, 12245, 12254, 12263, 12272, 12281, 12290, 12300, 12309, 12318, 12327, 12336, 12345, 12354, 12363, 12372, 12381, 12390, 12400, 12409, 12418, 12427, 12436, 12445, 12454, 12463, 12472, 12481, 12490, 12500, 12509, 12518, 12527, 12536, 12545, 12554, 12563, 12572, 12581, 12590, 12600, 12609, 12618, 12627, 12636, 12645, 12654, 12663, 12672, 12681, 12690, 12700, 12709, 12718, 12727, 12736, 12745, 12754, 12763, 12772, 12781, 12790, 12800, 12809, 12818, 12827, 12836, 12845, 12854, 12863, 12872, 12881, 12890, 12900, 12909, 12918, 12927, 12936, 12945, 12954, 12963, 12972, 12981, 12990, 13000, 13009, 13018, 13027, 13036, 13045, 13054, 13063, 13072, 13081, 13090, 13100, 13109, 13118, 13127, 13136, 13145, 13154, 13163, 13172, 13181, 13190, 13200, 13209, 13218, 13227, 13236, 13245, 13254, 13263, 13272, 13281, 13290, 13300, 13309, 13318, 13327, 13336, 13345, 13354, 13363, 13372, 13381, 13390, 13400, 13409, 13418, 13427, 13436, 13445, 13454, 13463, 13472, 13481, 13490, 13500, 13509, 13518, 13527, 13536, 13545, 13554, 13563, 13572, 13581, 13590, 13600, 13609, 13618, 13627, 13636, 13645, 13654, 13663, 13672, 13681, 13690, 13700, 13709, 13718, 13727, 13736, 13745, 13754, 13763, 13772, 13781, 13790, 13800, 13809, 13818, 13827, 13836, 13845, 13854, 13863, 13872, 13881, 13890, 13900, 13909, 13918, 13927, 13936, 13945, 13954, 13963, 13972, 13981, 13990, 14000, 14009, 14018, 14027, 14036, 14045, 14054, 14063, 14072, 14081, 14090, 14100, 14109, 14118, 14127, 14136, 14145, 14154, 14163, 14172, 14181, 14190, 14200, 14209, 14218, 14227, 14236, 14245, 14254, 14263, 14272, 14281, 14290, 14300, 14309, 14318, 14327, 14336, 14345, 14354, 14363, 14372, 14381, 14390, 14400, 14409, 14418, 14427, 14436, 14445, 14454, 14463, 14472, 14481, 14490, 14500, 14509, 14518, 14527, 14536, 14545, 14554, 14563, 14572, 14581, 14590, 14600, 14609, 14618, 14627, 14636, 14645, 14654, 14663, 14672, 14681, 14690, 14700, 14709, 14718, 14727, 14736, 14745, 14754, 14763, 14772, 14781, 14790, 14800, 14809, 14818, 14827, 14836, 14845, 14854, 14863, 14872, 14881, 14890, 14900, 14909, 14918, 14927, 14936, 14945, 14954, 14963, 14972, 14981, 14990, 15000, 15009, 15018, 15027, 15036, 15045, 15054, 15063, 15072, 15081, 15090, 15100, 15109, 15118, 15127, 15136, 15145, 15154, 15163, 15172, 15181, 15190, 15200, 15209, 15218, 15227, 15236, 15245, 15254, 15263, 15272, 15281, 15290, 15300, 15309, 15318, 15327, 15336, 15345, 15354, 15363, 15372, 15381, 15390, 15400, 15409, 15418, 15427, 15436, 15445, 15454, 15463, 15472, 15481, 15490, 15500, 15509, 15518, 15527, 15536, 15545, 15554, 15563, 15572, 15581, 15590, 15600, 15609, 15618, 15627, 15636, 15645, 15654, 15663, 15672, 15681, 15690, 15700, 15709, 15718, 15727, 15736, 15745, 15754, 15763, 15772, 15781, 15790, 15800, 15809, 15818, 15827, 15836, 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ordinem harum differentiarum auctarum vix assignare liceat; generatim certe hic nihil omnino definiri posse videtur. Circa finem autem imprimis haec series electionis ita fit irregularis, ut nulli prorsus legi adstricta videatur. Eum in finem autem hanc seriem hic exposui, quo clarius omnes difficultates, quibus perscrutatio eius impeditur, perspiciantur, haecque ipsa series ex ludicro principio enata attentione nostra non indigna videatur.

4. Haec autem series electionis specialis duabus rebus determinatur, quarum altera in numero notarum, qui est 30, altera vero numeratore, qui est 9 continetur. Quocirca in genere quaestio huc redit: ut dato notarum numero una cum numeratore ipsa series electionis exhibeatur, cuius solutionem cum in genere sperare nequeamus, in casibus particularibus attentionem nostram exerceri conveniet, num forte legem quampiam detegere videamus. Ac primo quidem patet, si numerator fuerit unitas, seriem electionis ipsam fore seriem numerorum naturalium 1, 2, 3, 4 etc. quoniam enim primus quisque elicitur, primo loco primus terminus, secundo secundus, tertio tertius et ita porro elicitur, ita ut ultima nota simul sit terminorum electorum ultimus.

5. Sit igitur numerator  $= 2$ , ita ut secundus quisque eliciatur, seu electio secundum alternos instituat, ac pro notarum numero series electionis ita se habere deprehenduntur:

numerus

numerus notarum	series electionis pro numeratore 2
1	1
2	2, 1
3	2, 1, 3
4	2, 4, 3, 1
5	2, 4, 1, 5, 3
6	2, 4, 6, 3, 1, 5
7	2, 4, 6, 1, 5, 3, 7
8	2, 4, 6, 8, 3, 7, 5, 1
9	2, 4, 6, 8, 1, 5, 9, 7, 3
10	2, 4, 6, 8, 10, 3, 7, 1, 9, 5
11	2, 4, 6, 8, 10, 1, 5, 9, 3, 11, 7
12	2, 4, 6, 8, 10, 12, 3, 7, 11, 5, 1, 9
13	2, 4, 6, 8, 10, 12, 1, 5, 9, 13, 7, 3, 11
14	2, 4, 6, 8, 10, 12, 14, 3, 7, 11, 1, 9, 5, 13
15	2, 4, 6, 8, 10, 12, 14, 1, 5, 9, 13, 3, 11, 7, 15
16	2, 4, 6, 8, 10, 12, 14, 16, 3, 7, 11, 15, 5, 13, 9, 1.

Hoc schema inspicienti facile erit pluribus modis ordinem quendam obseruare. Vltimi scilicet termini manifesto tenent progressionem arithmeticam binario crescentem, dummodo termini qui numerum notarum essent superaturi, infra eum deprimantur, numero scilicet notarum inde detracto. Ita cum primo habeatur 1, pro secunda serie vltimus, qui foret 3, binario subtracto ad vnitatem reducitur; hunc sequitur 3, et sequens 5 numerum notarum vnitatem superans ad vnitatem reducitur, et ita porro. Simili lege progrediuntur termini penultimi, tum vero etiam antepenultimi, atque adeo omnes

ab

ab ultimis aequidistantes. Quoniam igitur omnes rectae obliquae ei, quae per terminos ultimos transit parallelae, per huiusmodi progressionem arithmeticas pro numero notarum mutilatas transeunt, hinc istae series quousque lubuerit facile continuantur.

6. Exponamus simili modo series electionis pro numeratore  $= 3$ , ac lex progressionis multo magis abscondita prodibit

numerus notarum	series electionis pro numeratore 3
1	1
2	1, 2
3	3, 1, 2
4	3, 2, 4, 1
5	3, 1, 5, 2, 4
6	3, 6, 4, 2, 5, 1
7	3, 6, 2, 7, 5, 1, 4
8	3, 6, 1, 5, 2, 8, 4, 7
9	3, 6, 9, 4, 8, 5, 2, 7, 1
10	3, 6, 9, 2, 7, 1, 8, 5, 10, 4
11	3, 6, 9, 1, 5, 10, 4, 11, 8, 2, 7
12	3, 6, 9, 12, 4, 8, 1, 7, 2, 11, 5, 10
13	3, 6, 9, 12, 2, 7, 11, 4, 10, 5, 1, 8, 13
14	3, 6, 9, 12, 1, 5, 10, 14, 7, 13, 8, 4, 11, 2
15	3, 6, 9, 12, 15, 4, 8, 13, 2, 10, 1, 11, 7, 14, 5
16	3, 6, 9, 12, 15, 2, 7, 11, 16, 5, 13, 4, 14, 10, 1, 8,
	etc.

Interim tamen etsi secundum lineas horizontales et verticales ordo magis est abstrusus, tamen in ultimis

mis iterum progressio arithmetica se prodit secundum ternarium crescens; haecque eadem lex quoque in penultimis et antepenultimis ut ante deprehenditur, ex quo et has series facillime continuare licet.

7. Circa hanc legem in terminis ultimis locum habentem dubitare amplius non poterimus, dum ea adhuc pro numeratore 4 obseruetur. Pari ergo modo series electionis inde erectas repraesentemus

numerus notarum	series electionis pro numeratore 4
1	1
2	2, 1
3	1, 3, 2
4	4, 1, 3, 2
5	4, 3, 5, 2, 1
6	4, 2, 1, 3, 6, 5
7	4, 1, 6, 5, 7, 3, 2
8	4, 8, 5, 2, 1, 3, 7, 6
9	4, 8, 3, 9, 6, 5, 7, 2, 1
10	4, 8, 2, 7, 3, 10, 9, 1, 6, 5
11	4, 8, 1, 6, 11, 7, 3, 2, 5, 10, 9
12	4, 8, 12, 5, 10, 3, 11, 7, 6, 9, 2, 1
13	4, 8, 12, 3, 9, 1, 7, 2, 11, 10, 13, 6, 5
14	4, 8, 12, 2, 7, 13, 5, 11, 6, 1, 14, 3, 10, 9
15	4, 8, 12, 1, 6, 11, 2, 9, 15, 10, 5, 3, 7, 14, 13
16	4, 8, 12, 16, 5, 10, 15, 6, 13, 3, 14, 9, 7, 11, 2, 1
	etc.

Hinc ergo lex illa in seriebus oblique descendentes

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R

pror-



prorsus confirmatur, quae scilicet hic sunt arithmeticae quaternario crescentes, dum termini numerum nostrum superantes infra eum deprimuntur. In seriebus autem horizontalibus et verticalibus ordo fit continuo intricatio. Quin etiam ipsa rei natura in seriebus horizontalibus nullam progressionis legem patitur, propterea quod eae, cum omnes numeros notarum numero non maiores fuerint complexae, ulteriori continuationi aduersantur, ita ut continuatio tanquam imaginaria sit spectanda.

8. En ergo insignem legem, cuius ope pro quouis numeratore et notarum numero, nota ultimo eiicienda assignari potest. Existente scilicet numeratore  $= n$ , si pro notarum numero  $\nu$  ultima eiicienda sit  $z$ , seu indici  $z$  respondeat, tum pro numero notarum  $\nu + 1$ , ultima eiicienda erit  $z + n$ , siquidem non sit  $z + n > \nu + 1$ ; at si  $z + n > \nu + 1$ , ultima erit  $z + n - \nu - 1$  vel  $z + n - 2 (\nu + 1)$  vel  $z + n - 3 (\nu + 1)$ , vel generatim diuidendo  $z + n$  per  $\nu + 1$ , residuum ex diuisione relictum dabit indicem ultimae notae eiiciendae. Vbi notetur, si diuisio nihil reliquat, tum pro residuo 0 scribi notarum numerum  $\nu + 1$ . Cum ergo pro numero notarum  $n$  cognita fuerit ultimo eiecta, pro omnibus notarum numeris maioribus ultimo eiecta facile per hanc regulam assignabitur. Perpetuo autem si unica fuerit nota, eadem quoque erit ultimo eiecta, seu si fuerit  $\nu = 1$ , erit  $z = 1$ , vnde sequentes omnes sine villo negotio reperiuntur. Quae regula eo magis est notatu digna, quod sine electionis ordine

dine cognito statim ultimo eliciendam exhibeat, etiam si ea manifesto ab ordine ante electarum pendeat. Quamobrem haec regula merito tanquam insignis Theorema spectari debet, in cuius demonstrationem inquirere omnino operae erit pretium.

9. Sequenti modo autem eius demonstratio commodissime adstrui videtur. Consideretur notarum numerus  $\nu + 1$ , unde secundum numeratorem  $n$  prima fiat electio, quae cadet, in notam  $n$ , siquidem fuerit  $n < \nu + 1$ , vel in notam  $n - \alpha(\nu + 1)$ , qui indices autem omnes indici  $n$  aequivalent. Expungatur ergo haec nota, uti haec punctorum series A indicat

A;  $\overset{1}{\cdot} \overset{2}{\cdot} \overset{3}{\cdot} \dots \overset{n}{\cdot} \overset{n+1}{\cdot} \overset{n+2}{\cdot} \dots \overset{\nu+1}{\cdot}$

ac notae praecedentes 1, 2, 3, ...,  $(n-1)$  ad finem adiungantur indicibus numero  $\nu + 1$  auctis, ut prodeat ista punctorum series

B  $\dots \overset{n+1}{\cdot} \overset{n+2}{\cdot} \overset{n+3}{\cdot} \dots \overset{\nu+1}{\cdot} \overset{\nu+2}{\cdot} \dots \overset{\nu+2}{\cdot}$

in qua notarum numerus est  $\nu$ , quaeque series ab ea, ubi numerus notarum est  $\nu$ , quam ita repraesento,

C  $\overset{1}{\cdot} \overset{2}{\cdot} \overset{3}{\cdot} \overset{4}{\cdot} \dots \overset{\nu-1}{\cdot} \overset{\nu}{\cdot}$

aliter non differt, nisi quod ibi indices numeratore  $n$  sunt aucti. Vtrinque ergo electiones secundum numeratorem  $n$  factae in eadem ordine notas cadent, ac si electio ultima in serie C incidat in notam cuius index est  $z$ , ea in serie B incidet in notam cuius

R 2 ius

ius index est  $n + 2$ ; id quod etiam in serie notarum A, quarum numerus est  $v + 1$  eveniet. Quo ipso veritas nostri Theorematis euincitur. Simul autem inde patet, quod hic de notis ultimo electis est demonstratum, idem de penultimis, antepenultimis, omnibusque ordinibus ab ultimis aequidistantibus valere.

10. Huius igitur regulae ope statim pro quovis numeratore series electionis formare poterimus, cuius specimen pro numeratore 5 hic appono

numerus notarum	series electionis pro numeratore 5
1	1
2	1, 2
3	2, 3, 1
4	1, 3, 4, 2
5	5, 1, 3, 4, 2
6	5, 4, 6, 2, 3, 1
7	5, 3, 2, 4, 7, 1, 6
8	5, 2, 8, 7, 1, 4, 6, 3
9	5, 1, 7, 4, 3, 6, 9, 2, 8
10	5, 10, 6, 2, 9, 8, 1, 4, 7, 3
11	5, 10, 4, 11, 7, 3, 2, 6, 9, 1, 8
12	5, 10, 3, 9, 4, 12, 8, 7, 11, 2, 6, 1

12. Et si autem series notarum ultimo loco electarum tam simplicem ac facilem legem sequitur: tamen hoc maxime mirabile usu venit, quod in genere hanc seriem nullo modo exhibere liceat. Vultu

Inti si pro numeratore series ultimo electorum ita repræsentetur

num. notarum 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. . . .

series . . . 1. A B C D E F G H I . . . N

nouimus quidem fore A vel 1 vel 2, seu  $A = n - 2i$  vero

$B = A + n - 3i$ ,  $C = B + n - 4i$ ,  $D = C + n - 5i$  etc.

verum tamen hinc generaliter terminum N assignare non valemus, propterea quod in singulis littera  $i$  determinatum numerum denotat, tantum scilicet, vt terminus indicem supra scriptum non superet. Hinc etsi determinatio  $D = C + n - 5i$  nihil habet difficultatis, tamen si velimus pro C suum valorem  $B + n - 4i$  ponere, vt prodeat  $D = B + 2n - 4i - 5i$  hinc nihil concludere possumus, quandoquidem geminae litterae  $i$  valores non innotescunt. Causa igitur, cur in genere circa hanc seriem nihil definire liceat, in hoc consistit, quod continuo terminorum reductio ad alios numeros sit instituenda. Facilius hoc intelligetur, si perpendamus, nullum terminum ex præcedente absolute determinari, sed ad plures interdum conditiones esse respiciendum: Scilicet si quartus detur C, quintus erit vel  $C + n$  nisi  $C + n > 5i$

vel erit  $C + n - 5$  nisi  $C + n > 10$

vel erit  $C + n - 10$  nisi  $C + n > 15$

etc.

R 3

quem-

quemlibet autem terminum ad suam debitam formam deprimi oportet, antequam ex eo sequentem ope regulae demonstratae eliciamus.

13. Pro casibus autem particularibus ad terminos valde remotos per saltus progredi licet, ut non sit opus omnes intermedios euoluisse. Scilicet si pro numeratore  $n$ , indici  $\nu$ , qui hic notarum numerum significat, respondeat terminus  $a$ , tum indici  $\nu + x$  respondebit terminus  $a + nx$ , dum sit  $a + nx < \nu + x$  seu  $x < \frac{\nu - a}{n - 1}$ : quin adhuc hic terminus recte se habet, si  $x$  unitate augeatur, hoc est si  $x > \frac{\nu - a}{n - 1}$ , ut excessus unitate sit minor, tumque indici  $\nu + x$  respondebit terminus  $(n - 1)x - \nu + a$ . Simili modo ab hoc per saltum ad remotiorem terminum pervenire licet, saltus autem continuo fiunt maiores: per singulos autem saltus termini in progressionem arithmetica secundum numeratorem  $n$  crescente procedunt. Ab initio quidem singuli termini seorsim sunt definiendi, statim autem atque ad indices numeratore maiores pervenitur, calculus per saltus commodius instituitur, cuius specimen pro numeratore 9 opponam, ubi perpetuo numerum  $\nu - a$  per 8 ita dividi oportet, ut quotus nimis magnus accipiatur: tum enim ipse quotus dabit valorem ipsius  $x$ , et residuum erit terminus per hunc saltum sequens:

Series

## Series pro numeratore 9

Indices.	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18
termini.	1, 2, 2, 3, 2, 5, 7, 8, 8, 7, 5, 2, 11, 6, 15, 8, 17, 8
Saltus	
indices	20, 22, 25, 28, 32, 36, 40, 45, 51, 57, 64, 72, 81
termini.	6, 2, 4, 3, 7, 7, 3, 3, 6, 3, 2, 2, 2
Saltus	
indices	91, 103, 116, 130, 146, 165, 185, 208, 234, 264, 297, 334
termini.	1, 6, 7, 3, 1, 7, 2, 1, 1, 7, 7, 6
Saltus	
indices	376, 423, 475, 535, 602, 677, 762, 857, 964, 1084
termini.	8, 8, 1, 6, 7, 5, 8, 6, 5, 1
Saltus	
indices	1220, 1372, 1544, 1737, 1954, 2198, 2473, 2782, 3130
termini.	5, 1, 5, 5, 4, 2, 4, 3, 5
	etc.

14. Hanc ergo seriem facili labore ultra ter-  
mille terminos continuauimus, ac si vltius pro-  
gredi velimus, ex numeris postremis 3130 et 5  
calculum ita instituiamus:

ab 3130 Hinc saltus per 391 terminos porri-  
subtr. 5 gitur, indeque terminus cuius index  
est 3521 erit vt residuum indicat  
8)3125 3. Porro

391 (3)

ab 3521 Hic saltus fit per 440 terminos, vnde  
subtr. 3 oritur index 3521 + 440 = 3961,  
cui respondet terminus 2 residio  
8)3518 indicatus.

440 (2)

ab

$$\begin{array}{r}
 \text{ab} \quad 3961 \\
 \text{subtr.} \quad 2 \\
 \hline
 8)39759 \\
 \hline
 495^{(1)}
 \end{array}$$

Hinc colligitur pro indice 4456  
terminus 1.

Ab hoc autem saltus sequens ultra 5000 extenditur; neque tamen video, quomodo huius seriei terminus verbi gratia decies millesimus vel adeo centies millesimus nisi saltibus hoc modo continuendis, assignari possit: indices quidem per hos saltus crescentes secundum progressionem geometricam in ratione 8:9 proxime crescunt, sed quia hoc tantum proxime fit, hinc nullum subsidium pro continuatione obtinetur.

15. Hinc ergo pro quouis notarum numero, dummodo 5000 non longe superet, inquam electionis fors postremo cadet: ex serie scilicet hic per saltus exhibita is terminus quaeri debet, qui indici notarum numero aequali respondet. Perpetuo scilicet index proxime minor sumatur indeque progressis arithmetica usque ad indicem propositum per differentiam 9 continuetur, quod in nonnullis exemplis declarari expediet.

I. Quaceratur seriei illius terminus centesimus: Proxime inferior index per saltus inuentus est 91, cui conuenit terminus 1. Iam inde ad centissimum sunt loca 9, et nouies nouem seu 81 ad illum terminum 1 adiciendo prodit terminus centesimus 82. Quare si ex centum fontibus is sit supplicio afficiendus,

us, qui postquam reliqui per numerationem ad 9 fuerint libera, tandem solus relinquatur, haec poena in 182<sup>da</sup> ordine incidet.

II. Ut terminus 200<sup>mus</sup> reperiat, calculus ita instituitur:

200  
Index proximus 185 terminus 1  
15 per 9 dat 135

terminus quaesitus 137

III. Quaeratur terminus 500<sup>mus</sup>:

500  
Index proximus 475 terminus 1  
25 per 9 dat 225

terminus quaesitus 226

IV. Quaeratur terminus millesimus:

1000  
Index proximus 964 terminus 5  
36 per 9 dat 324

terminus quaesitus 329

V. Quaeratur terminus 5000<sup>mus</sup>:

5000  
Index proximus 4456 terminus 1  
544 per 9 dat 4896

terminus quaesitus 4897

17. Consideratio huiusmodi serierum tam facili negotio formandarum non solum est iucunda,

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S

fed



sed etiam non parum ad numerorum naturam tan-  
topere nobis adhuc absconditam feliciter perscrutan-  
dam conferre quicquam posse videtur. Eximium  
certe hoc est exemplum, et omni attentione dignum,  
quod series tam leui opera non solum formari sed  
etiam quousque libuerit, continuari possit, cum ta-  
men eius natura et vera indoles nobis maneat pror-  
sus incognita, neque ut aliae ad terminum genera-  
lem reuocari possit.

II. Simili modo etiam pro numeratore 6 ordines  
electionis subiungimus.

Numerus notarum	Series electionis pro numeratore 6
1.	1.
2.	2. 1
3.	3. 2. 1
4.	2. 1. 4. 3
5.	1. 3. 2. 5. 4
6.	6. 1. 3. 2. 5. 4
7.	6. 5. 7. 2. 1. 4. 3
8.	6. 4. 3. 5. 8. 7. 2. 1
9.	6. 3. 1. 9. 2. 5. 4. 8. 7
10.	6. 2. 9. 7. 5. 8. 1. 10. 4. 3
11.	6. 1. 8. 4. 2. 11. 3. 7. 5. 10. 9

16. Vicissim autem si detur ordo electorum,  
qui ab ultimo regrediendo fit:

$x, y, x, v, u, t, s, r$  etc.

ex eo pro quouis numeratore et quolibet notarum  
numero, initialis ordo notarum inuestigari poterit.

Quod

Quod quo clarius appareat, sit numerator = 4 et pro quolibet notarum numero ordo notarum sequenti modo se habebit:

multitudo notarum	ordo notarum initialis
1	zz
2	zyzy
3	xzy
4	xzyv
5	zyvux
6	vuxtzy
7	tzysvux
8	vuxrtzys
9	zysqvuxrt
10	xrtpzysqvu
11	qvuxrtpzys
12	zysnqvuxrt
13	rtpmzysnqvux
14	uoxlrtpmzysnqv
15	nqvk uoxlrtpmzys
16	zysinqvkuoxlrtpm
17	tpmkzysinqvkuoxlr
18	xlrgrtpmkzysinqvkuo.

Consideratio horum ordinum non solum eorum naturam satis luculenter declarat, sed etiam plures insignes speculationes suppeditare poterit.