1776

Solutio quorundam problematum Diophanteorum

Leonhard Euler

SOLVTIO QVORVNDAM PROBLEMATVM DIOPHANTAEORVM.

Auctore

L. EVLERO.

Problema 1.

Inuenire duo quadratorum paria xx, yy et tt, uu, ita vt tam $(xx+yy)(ttxx+uuyy)$ quam $(xx+yy)(uuxx+ttyy)$ fiat numerus quadratus.

Analysis.

1. Primo patet, quicunque bini numeri tam pro x, y quam pro t, u fuerint inuenti, eorum aeque multipla veluti αx, αy et βt, βu quaesito aeque satisfacere; sicque problema ita restringi conueniet, vt tam x et y quam t et u sint numeri primi inter se.

2. Incipiamus a formula priori $(xx+yy)(ttxx+uuyy)$, quae posita huic quadrato $(xx+yy)^2 xxyy(pp+qq)^2$ aequalis fit

$$ttxx+uuyy=xxyy(xx+yy)((pp-qq)^2+(2pq)^2)$$

vnde concluditur

$$tx=xy(x(pp-qq)+2pqy);\quad uy=xy(y(pp-qq)-2pqx)$$

sicque erit

$$t=xy(pp-qq)+2pqyy;\quad u=xy(pp-qq)-2pqxx.$$

3.

3. Iam pro altera formula, cum sit

$$ty = xyy(pp-qq) + 2pqy^3; \quad ux = xxy(pp-qq) - 2pqx^3.$$

fiet

$$ttyy + uuxx = xxy^4(pp-qq)^2 + 4pqxy^5(pp-qq) + 4ppqqy^6$$
$$+ x^4yy(pp-qq)^2 - 4pqx^5y(pp-qq) + 4ppqqx^6$$

quae forma, quia manifesto per $xx+yy$ est divisibilis, abit in

$$(xx+yy)(xxyy(pp-qq)^2 - 4pqxy(xx-yy)(pp-qq)$$
$$+ 4ppqq(x^4 - xxyy + y^4)).$$

4. Cum nunc haec forma per $xx+yy$ multiplicata numerum quadratum praebere debeat, habebimus sequentem expressionem ad quadratum reducendam:

$$4ppqqx^4 - 4pq(pp-qq)x^3y + (p^4 - 6p^2q^2 + q^4)x^2y^2$$
$$+ 4pq(pp-qq)xy^3 + 4ppqqy^4$$

quae quidem manifesto fit quadratum, si $x = y$; verum hunc casum vtpote facillimum hinc merito excludimus; siquidem tota quaestio huc rediret, vt $2(tt+uu)$ quadratum efficeretur.

5. At ponendo illam formulam aequalem huic quadrato

$$(2pqxx - (pp-qq)xy + 2pqyy)^2$$

deletis terminis paribus fit

$$(p^4 - 6ppqq + q^4)x^2y^2 + 4pq(pp-qq)xy^3 = (p^4 + 6ppqq + q^4)x^2y^2$$
$$- 4pq(pp-qq)xy^3$$

hincque $8pq(pp-qq)y = 12ppqqx$

vnde colligitur haec solutio problematis:

$$x = 2(pp-qq); \; y = 3pq; \text{ hincque porro}$$
$$t = 6pq(p^4 + ppqq + q^4) \text{ et } u = -2pq(pp-qq)^2.$$

6. En ergo solutionem prima infinite patentem, quoniam numeros p et q ad arbitrium capere licet; reductis scilicet numeris t et u ad minimos terminos, et quia perinde est siue sint positiui siue negatiui habebimus

$$x = 2(pp-qq);\quad t = 3(pp^4+ppqq+q^4) = \tfrac{3}{4}xx+yy$$
$$y = 3pq;\quad u = (pp-qq)^2 = \tfrac{1}{4}xx$$

hincque reperitur

$$xx+yy = 4p^4+ppqq+4q^4$$
$$ttxx+uuyy = xxyy(xx+yy)(pp+qq)^2 = xx(xx+yy)(\tfrac{9}{16}xx+yy)$$
$$uuxx+ttyy = 4ppqq(xx+yy)(p^4+7ppqq+q^4)^2 = (xx+yy)(\tfrac{1}{4}xx+yy)^2.$$

7. Vt alias solutiones inueniamus, ponamus superioris formae radicem quadratam:

$$2ppxx-(pp-qq)xy-2pqyy+Ayy$$

cuius quadrato illi aequali posito prodibit aequatio:

$$(AA-4Apq)yy-2A(pp-qq)xy+(4Apq-4ppqq)xx=0$$

hic si $A=4pq$ prodit solutio praecedens; at posito $A=pq$ fit

$$+3pqy+2(pp-qq)x=0,$$

quae cum illa pariter congruit.

8. Ponamus $A=-2pp$ prodibitque haec aequatio

$$(pp+2pq)yy+(pp-qq)xy-(2pq+qq)xx=0$$

quae per $x+y$ diuisa dat

$$(pp+2pq)y-(2pq+qq)x=0$$

vnde

vnde fit

$$x = p(p + 2q) \text{ et } y = q(q + 2p) \text{ tum vero}$$
$$u = pqq(p + 2q)(qq + 2pq + 3pp).$$

9. En ergo aliam ſolutionem a praecedente diuerſam, et infinite patentem, qua numeris t et u ad minimos terminos reductis fit

$$x = p(p + 2q);\ t = p(q + 2p)(pp + 2pq + 3qq)$$
$$y = q(q + 2p);\ u = q(p + 2q)(qq + 2pq + 3pp)$$

hincque reperitur:

$$xx + yy = p^4 + 4p^3q + 8ppqq + 4pq^3 + q^4$$
$$ttxx + uuyy = (p + 2q)^2(q + 2p)^2(pp + qq)^2(xx + yy)$$
$$uuxx + ttyy = ppqq(5pp + 8pq + 5qq)^2(xx + yy).$$

10. Poſito $A = 2pp$ prodit $(pp - 2pq)yy - (pp - qq)xy + (2pq - qq)xx = 0$ quae per $y - x$ diuiſa dat $(pp - 2pq)y - (2pq - qq)x = 0$ ideoque

$$x = p(p - 2q);\ t = p(2p - q)(pp - 2pq + 3qq)$$
$$y = q(2p - q);\ u = q(p - 2q)(qq - 2pq + 3qq)$$
$$xx + yy = p^4 - 4p^3q + 8ppqq - 4pq^3 + q^4$$
$$ttxx + uuyy = (p - 2q)^2(2p - q)^2(pp + qq)^2(xx + yy)$$
$$uuxx + ttyy = ppqq(5pp - 8pq + 5qq)^2(xx + yy).$$

Haec autem ſolutio a praecedente non differt; neque poſitiones $A = 2qq$ et $A = -2qq$ ſolutionis diuerſas praebent.

11. Conſtant methodi, quarum beneficio ex vna ſolutione inuenta aliae erui poſſunt; verum eae

 ad

ad calculos nimium intricatos deducunt. Ita reperire licet

$$x = q(pp - qq)(3pp - qq)$$
$$y = (pp + qq)(p(pp + qq) \pm q(3pp + qq))$$

conuenientes vero valores pro t et u paragr. 2 suppeditat.

Solutio I.

In hac solutione ratio numerorum x et y est $\frac{x}{y} = \frac{1}{2} \cdot \frac{pp - qq}{2pq}$ vnde ex cathetis trianguli rectanguli inueniuntur; tum vero ratio $\frac{t}{u} = \frac{3xx + 4yy}{xx}$; vnde solutiones simpliciores sunt:

1°. $x = 5$; $y = 2$; $t = 91$; $u = 25$
2°. $x = 7$; $y = 5$; $t = 247$; $u = 49$
3°. $x = 5$; $y = 9$; $t = 399$; $u = 25$
4°. $x = 3$; $y = 10$; $t = 427$; $u = 9$
5°. $x = 11$; $y = 14$; $t = 1147$; $u = 121$
6°. $x = 15$; $y = 7$; $t = 871$; $u = 225$
7°. $x = 16$; $y = 5$; $t = 217$; $u = 64$
8°. $x = 16$; $y = 9$; $t = 273$; $u = 64$
9°. $x = 7$; $y = 18$; $t = 1443$; $u = 49$
10°. $x = 13$; $y = 20$; $t = 2107$; $u = 169$.

Solutio II.

Hic ratio numerum x et y est $\frac{x}{y} = \frac{p(p + 2q)}{q(q + 2p)}$ numerorum t et u vero $\frac{t}{u} = \frac{p(q + 2p)(pp + 2pq + 3qq)}{q(p + 2q)(qq + 2pq + 3pp)}$ si numeros x et y vt datos spectemus, ob

$$ppy + 2pqy = qqx + 2pqx\text{, reperitur}$$
$$\frac{p}{q} = \frac{x - y \pm \sqrt{(xx - xy + yy)}}{y};$$

vnde

vnde numerorum x et y character in hoc consistit, vt $xx - xy + yy$ sit quadratum; cuiusmodi numeri cum facile inueniantur; sit $xx - xy + yy = zz$; eritque

$$\frac{p}{q} = \frac{x - y + z}{y} = \frac{x}{y - x + z}; \text{ seu } \frac{p}{q} = \frac{z - y}{x - z};$$

hinc fit

$$\frac{p + 2q}{q + 2p} = \frac{2x - y - z}{z - 2y + x} = \frac{z + y}{z + x} \text{ et } \frac{p(q + 2p)}{q(p + 2q)} = \frac{(z - y)(z + x - 2y)}{(z - x)(z + y - 2x)} \text{ vt}$$

$$(z - y)(z + x - 2y) = (z + x - y)(2z - x - y)(z - x)(z + y - 2x)$$
$$= z + y - x)(2z - x - y)$$

vnde $\frac{p(q + 2p)}{q(p + 2q)} = \frac{z + x - y}{z + y - x}$.

Deinde est

$$\frac{pp + 2pq + 3qq}{qq + 2pq + 3pp} = \frac{(x - y)^2 + 2(x - z)^2}{(x - y)^2 + 2(z - y)^2} = \frac{2z + y - x}{2z + x - y};$$

hincque tandem elicitur

$$\frac{t}{u} = \frac{(z + x - y)(2z - x + y)}{(z + y - x)(2z + x - y)}.$$

Sicque pro x et y eiusmodi numeris inuentis, vt sit rationaliter $\sqrt{(xx - xy + yy)} = z$ capiatur:

$$t = (z + x - y)(2z - x + y) = xx + yy + (x - y)z$$
$$u = (z + y - x)(2z - y + x) = xx + yy - (x - y)z.$$

hinc obtinetur

$$ttxx + uuyy = (xx + yy)(xx - 2xy + yy + (x + y)z)^2$$
$$uuxx + ttyy = (xx + yy)(xx - 2xy + yy - (x + y)z)^2$$

vel etiam hoc modo

$$ttxx + uuyy = \tfrac{1}{9}(xx + yy)(x + y + z)^2(4z - x - y)^2$$
$$uuxx + ttyy = \tfrac{1}{9}(xx + yy)(x + y - z)^2(4z + x + y)^2.$$

Num autem quo facilius valores pro x et y idoneos

 repe-

reperiamus, ſpectemus x vt datum ac ponamus $z = y - v$ eritque

$$xx - xy = -2yv + vv \text{ et } y = \frac{xx - vv}{-2v + x} = \frac{vv - xx}{2v - x};$$

vbi pro quouis valore ipſius x aſſumto caſus integri pro y ſunt eruendi: notandum vero eſt, pro x numerum impariter parem aſſumi non poſſe, quia y quoque fieret par:

x	y	z	t	u
3	− 5	7	45	11
3	+ 8	7	19	54
5	+ 8	7	34	55
5	− 16	19	340	59
5	+ 21	19	81	385
7	− 8	13	154	41
7	+ 15	13	85	189
7	− 33	37	1309	171
7	+ 40	37	214	1435
8	+ 15	13	99	190
9	− 56	61	3591	374
9	+ 65	61	445	3861
11	− 24	31	891	194
11	+ 35	31	301	1045
11	− 85	91	8041	695
11	+ 96	91	801	8536
13	− 35	43	1729	335
13	+ 48	43	484	1989
13	− 120	127	15730	1161
13	+ 133	127	1309	16549

Problema

Problema 2.

Inuenire duo quadratorum paria xx, yy et tt, uu, vt $(ttxx+uuyy)(uuxx+ttyy)$ fit numerus quadratus.

Solutio.

Hoc problema eandem fortitur folutionem, quod praecedensi, idemque quaterni numeri pro x, y, t, u inuenti fatisfaciunt. Inde ergo folutio fimpliciffima eft

$$x=2\,;\; y=5\,;\; t=11\,;\; u=45$$

ex qua fit

$$ttxx+uuyy=34.9.169\,;\; uuxx+ttyy=34.625$$

ideoque

$$(ttxx+uuyy)(uuxx+ttyy)=34^2.39^2.5^2.$$

Ceterum haec folutio non folum ob eam caufam tantum eft particularis, ob quam talis erat, fed etiam hoc problema infinitas folutiones admittere videtur, quae praecedenti non conueniant. Fieri enim poteft, vt haec formula

$$(ttxx+uuyy)(uuxx+ttyy)$$

fit quadratum, etiamfi neutra praecedentium

$(xx+yy)(ttxx+uuyy)$, et $(xx+yy)(uuxx+ttyy)$ fuerit quadratum, cuius rei vnicum exemplum dediffe fufficiat:

$$x=973\,;\; y=263:\; t=973\,;\; u=1841$$

eft enim

$uuxx+ttyy=2.25.263^2.973^2$ quadratum duplicatum
$ttxx+uuyy=2.25.141793^2$ quadratum duplicatum.

En

En adhuc aliam ſolutionem latius patentem

$$x = 3n^4 + 6mmnn - m^4 \; ; \; t = mx$$
$$y = 3m^4 + 6mmnn - n^4 \; ; \; u = ny,$$

cuius inuentionis ratio facile intelligitur, poſita enim $t = mx$ et $u = ny$ fit

$ttxx + uuyy = mmx^4 + nny^4$ et $uuxx + ttyy = xxyy(mm + nn)$

ſicque ad quadratum reducenda eſt haec formula

$$(mm + nn)(mmx^4 + nny^4)$$

quae facto $x = v + z$ et $y = v - z$ ad iſtam ſolutionem perducit: hinc autem praecedentes ſolutiones non obtinentur.

Problema 3.

Inuenire duo quadratorum paria $xxyy$ et $ttuu$, vt tam hic numerus $ttxx + uuyy$ quam iſte $ttyy + uuxx$ fiat quadratus.

Solutio.

Ex modo tradita ſolutione problematis praecedentis ſolutio huius facile adornatur, pro m et n eiusmodi numeris ſumendis, vt $mm + nn$ fiat quadratum. Sic ſi fiat $m = 4$ et $n = 3$ reperitur

$$x = 851, \quad t = 3404$$
$$y = 1551, \quad u = 3653.$$

At ex problemate primo multo concinniores ſolutiones impetrantur, quibus adeo praeter binas praeſcriptas conditiones et haec tertia adimpletur; vt $xx + yy$ fiat etiam quadratum. At ſolutio ſecunda primi problematis vnum praebet caſum, quo $xx + yy$ fit quadratum ſcilicet

$$x = 8 \; ; \; y = 15 \; ; \; t = 99 \; ; \; u = 190$$

vnde

vnde fit

$$xx + yy = 17^2$$
$$ttxx + uuyy = 2^2 . 3^2 . 17^2 . 29^2$$
$$ttyy + uuxx = 5^2 . 5^2 . 5^2 . 17^2 .$$

Si insuper addita fuisset haec conditio, vt etiam $xx - xy + yy$ foret quadratum eadem solutio negotium conficeret. Huiusmodi autem solutiones elicentur quaerendis numeris x et y vt haec expressio

$$x^4 - x^3y + 2xxyy - xy^3 + y^4 \text{ fiat quadratum}$$

ad quos porro vt ante numeros t et u inuestigari oportet.

Occasionem hoc problema Diophantaeum tractandi praebuit problema Geometricum a Schotenio propositum, quo datis in triangulo basi, perpendiculo et ratione laterum ipsa latera quaerentur. Problema hoc geminam admittit solutionem, quarum vtraque vt praebeat latera rationaliter expressa, negotium ad problema istud Diophantaeum perducitur. Si enim basis trianguli ponatur $= a$, perpendiculum $= b$, et laterum ratio $m:n$; vocatis ipsis lateribus mz et nz; primo necesse est a et b ita exprimi

$$a = (mm - nn)(xx + yy) \text{ et } b = 2mnxy,$$

tum vero pro z haec duplex expressio reperitur:

$$z = \sqrt{(xx + yy)((m - n)^2 xx + (m + n)^2 yy)} \text{ et}$$
$$z = \sqrt{(xx + yy)((m + n)^2 xx + (m - n)^2 yy)}$$

quae vt ambae fiant rationales facto $m+n=t$; $m-n=u$ nascitur nostrum problema Diophantaeum. Cuius ergo casus simplicissimus erit sumto

$$x=3;\ y=5;\ t=45 \text{ et } u=11,$$

vnde haec nascuntur data:

ratio laterum $m:n=28:17$

basis trianguli $a=33$; et perpendiculum $=28$,

vnde reperiuntur ipsa latera:

vel $mz=\frac{140}{3}$ et $nz=\frac{85}{3}$

vel $mz=\frac{364}{5}$ et $nz=\frac{221}{5}$

siue in integris sumta

basis $a=495$ et perpendicula $b=420$

obtinebuntur latera rationem $28:17$ tenentia

vel $mz=700$ et $nz=425$

vel $mz=1092$ et $nz=663$.

SPECV-