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Leonhard Euler

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# EXPLICATIO
# MOTVS OSCILLATORII
## MIRABILIS IN LIBRA MAIORE OBSERVATI.

Auctore

*L. EVLERO.*

§. 1.

Cum nuper in cauſam huius phaenomeni, quod illuſtris *Bernoullius* commemorat inquiſiſſem, et ſolum librae iugum circa ſuum centrum mobile aſſumſiſſem, luculenter demonſtraui, hinc motum iſtum mirabilem nullo modo oriri potuiſſe. Nullum igitur plane eſt dubium, quin cauſa iſtius phaenomeni in motu oſcillatorio totius librae ex ſcapo ſuſpenſae ſit quaerendum, quandoquidem talis motus a rudi illa concuſſione toti librae imprimi debuit. Quoniam igitur agitationes ſolius iugi vix quicquam in motu oſcillatorio lancium immutare valent, vti in ſuperiore diſſertatione oſtendi, hic iugum cum trutina firmiter connexum aſſumo, vt totum librae ſyſtema exceptis lancibus tanquam corpus rigidum ſpectari poſſit.

§. 2. Sit igitur O punctum ſuſpenſionis, circa quod tota libra eſt mobilis, per quod ducatur recta horizontalis $a O b$, tum vero etiam verticalis $O o$; Tab. III. Fig. 5.

 atque

atque elapso post initium tempore $= t$, quod semper in minutis secundis exprimo, teneat corpus librae situm vtcunque inclinatum $OAB$, vbi $A$ et $B$ sunt puncta ex quibus lances suspenduntur; ita vt recta $AB$ situ aequilibrii foret horizontalis, recta vero $OC$, ad eam perpendiculariter ducta, verticalis, in qua ergo dabitur centrum grauitatis totius librae $G$. Ductae concipiantur etiam rectae $OA$ et $OB$, quae vocentur $OA = OB = a$, et anguli $OAB = OBA = \alpha$; ita vt sit $OC = a \sin. \alpha$ et $AC = BC = a \cos. \alpha$: distantia autem centri grauitatis $G$ a puncto suspensionis $O$ ponatur $OG = c$. Praeterea sit massa seu pondus totius librae, lancibus exceptis $= M$, eiusque momentum inertiae $Mkk$, ita vt $\frac{kk}{c}$ foret longitudo penduli simplicis isochroni, si sola libra circa punctum $O$ oscillaret. Ad commoditatem autem calculi vocemus insuper $OC = e = a \sin. \alpha$ et $AC = BC = f = a \cos. \alpha$. Denique lances vt pondera simplicia ex punctis $A$ et $B$ suspensa spectemus, quorum massa sit $= L$; filorum autem longitudo $AP = BQ = b$.

§. 3. Praesenti autem statu ponatur declinatio librae seu angulus $GOo = \Phi$, eruntque anguli $AOa = \alpha - \Phi$ et $BOb = \alpha + \Phi$; vbi rectae $Aa$ et $Bb$ intelligendae sunt verticales. Hinc igitur erit

$$Aa = a \sin. (\alpha - \Phi) = e \cos. \Phi - f \sin. \Phi \text{ et}$$
$$Oa = a \cos. (\alpha - \Phi) = f \cos. \Phi + e \sin. \Phi;$$

ex altera autem parte

$$Bb = a \sin. (\alpha + \Phi) = e \cos. \Phi + f \sin. \Phi \text{ et}$$
$$Ob = a \cos. (\alpha + \Phi) = f \cos. \Phi - e \sin. \Phi.$$

§. 4. Porro pro praesenti statu declinet altera lanx P a situ verticali angulo $PA\alpha = \eta$; altera vero lanx Q angulo $QB\beta = \vartheta$; vnde, si ex punctis P et Q verticales agantur $Pp$ et $Qq$, ob vtramque longitudinem $AP = BQ = b$ habebimus

$$ap = b \sin.\eta \text{ et } Pp = A\alpha + b\cos.\eta = e\cos.\Phi - f\sin.\Phi + b\cos.\eta,$$

et ex altera parte

$$\beta q = b\sin.\vartheta \text{ et } Qq = B\beta + b\cos.\vartheta = e\cos.\Phi + f\sin.\Phi + b\cos.\vartheta.$$

§. 5. Nunc pro motu determinando statuantur pro binis punctis P et Q coordinatae orthogonales

$$Op = x = e\sin.\Phi + f\cos.\Phi + b\sin.\eta \text{ et}$$
$$Pp = y = e\cos.\Phi - f\sin.\Phi + b\cos.\eta;$$

et ex altera parte

$$Oq = x' = f\cos.\Phi - e\sin.\Phi - b\sin.\vartheta \text{ et}$$
$$Qq = y' = e\cos.\Phi + f\sin.\Phi + b\cos.\vartheta.$$

His positis ponatur tensio fili $AP = P$ et fili $BQ = Q$; quibus lances P et Q propemodum sursum trahuntur, dum ob propriam grauitatem deorsum vrgentur vi $= L$. At tensio P praebet vim directe sursum trahentem $= P\cos.\eta$, et vim horizontalem dextrorsum $= P\sin.\eta$: pro altera vero lance tensio Q praebet vim verticalem sursum vrgentem $= Q\cos.\vartheta$ et horizontalem dextrorsum $= Q\sin.\vartheta$.

§. 6. Definitis istis viribus, quibus lances sollicitantur, principia motus pro lance P suppeditant has duas aequationes:

$$\text{I. } \frac{ddx}{2g\,dt^2} = \frac{P\sin.\eta}{L} \text{ et II. } \frac{ddy}{2g\,dt^2} = \frac{L - P\cos.\eta}{L}$$

pro

pro altera autem lance Q habebimus has:

$$\text{III. } \frac{ddx'}{2g\,dt^2} = \frac{Q\,\text{ſin.}\,\vartheta}{L} \quad \text{et} \quad \text{IV. } \frac{ddy'}{2g\,dt^2} = \frac{L - Q\,\text{coſ.}\,\vartheta}{L}$$

atque ex his quatuor aequationibus motus vtriusque lancis determinari debet.

§. 7. Iam motum angularem ipſius librae aggrediamur; ac primo quidem momenta omnium virium quibus ſollicitatur reſpectu puncti ſuſpenſionis O colligi debent. Hic igitur primo occurrit proprium librae pondus M, quod in centro grauitatis G vnitum producit momentum $= MOG\,\text{ſin.}\,\Phi = Mc\,\text{ſin.}\,\Phi$. Deinde vero tenſio P momentum producit $= Pa\,\text{ſin.}\,OAP$. Eſt vero hic angulus $OAP = 90° + \alpha - \Phi + \eta$ cuius ſinus $= \text{coſ.}\,(\alpha - \Phi + \eta)$, ita vt hoc momentum ſit

$$Pa\,\text{coſ}\,(\alpha - \Phi + \eta) = P(f\,\text{coſ.}\,(\eta - \Phi) - e\,\text{ſin.}\,(\eta - \Phi)).$$

Quod momentum pariter ac primum tendit ad angulum $\Phi$ imminuendum. Ex altera parte tenſio Q momentum exerit $= Q.\,OB\,\text{ſin.}\,OBQ$: eſt vero angulus $OBQ = 90° + \alpha + \Phi - \vartheta$, cuius ſinus eſt $\text{coſ.}(\alpha + \Phi - \vartheta) = \text{coſ.}\,\alpha\,\text{coſ.}\,(\Phi - \vartheta) - \text{ſin.}\,\alpha\,\text{ſin.}\,(\Phi - \vartheta)$ vnde iſtud momentum erit

$$Q.\,(f\,\text{coſ.}\,(\Phi - \vartheta) - e\,\text{ſin.}\,(\Phi - \vartheta)),$$

quod ad declinationem $\Phi$ augendam tendit. Hinc igitur per principia motus deducitur haec aequatio:

$$\text{V. } \frac{dd\Phi}{2g\,dt^2} = -\frac{Mc\,\text{ſin.}\,\Phi - P(f\,\text{coſ.}\,(\eta - \Phi) - e\,\text{ſin.}\,(\eta - \Phi)) + Q(f\,\text{coſ.}\,(\Phi - \vartheta) - e\,\text{ſin.}\,(\Phi - \vartheta))}{Mkk}.$$

§. 8. Nunc igitur ternos noſtros angulos declinationis $\eta$, $\vartheta$ et $\Phi$ tanquam infinite paruos ſpe-

ctemus

ctemus, vt eorum cosinus vnitati, sinus vero ipsis angulis aequales reputari queant; tum igitur erit

$$x = e\Phi + f + b\eta; \quad y = e - f\Phi + b$$
$$x' = f + e\Phi - b\vartheta; \quad y' = e + f\Phi + b$$

et nunc quinque nostrae aequationes induent sequentes formas:

$$\text{I. } \frac{e\,dd\Phi + b\,dd\eta}{2g\,dt^2} = -\frac{P\eta}{L}; \quad \text{II. } \frac{-f\,dd\Phi}{2g\,dt^2} = 1 - \frac{P}{L}$$

$$\text{III. } \frac{-e\,dd\Phi - b\,dd\vartheta}{2g\,dt^2} = \frac{Q\vartheta}{L}; \quad \text{IV. } \frac{+f\,dd\Phi}{2g\,dt^2} = 1 - \frac{Q}{L}$$

$$\text{V. } \frac{dd\Phi}{2g\,dt^2} = -\frac{Mc\Phi - P(f - e(\eta - \Phi)) + Q(f - e(\Phi - \vartheta))}{Mkk}.$$

§. 9. Ex his aequationibus primo tensiones P et Q definiri opportet, id quod commodissime fit ex aequatione secunda et quarta, vnde, quia membra priora $\frac{f\,dd\Phi}{2g\,dt^2}$ sunt quasi infinite parua erit $P = L$ et $Q = L$, qui valores proxime veri pro reliquis aequationibus sufficiunt, ita vt nobis relinquantur tres sequentes aequationes:

$$\text{I. } \frac{e\,dd\Phi + b\,dd\eta}{2g\,dt^2} = -\eta; \quad \text{III. } -\frac{e\,dd\Phi - b\,dd\vartheta}{2g\,dt^2} = \vartheta \text{ et}$$

$$\text{V. } \frac{dd\Phi}{2g\,dt^2} = -\frac{Mc\Phi - 2Le\Phi + Le(\Phi + \vartheta)}{Mkk}$$

hic ergo ternae nostrae variabiles $\eta$, $\vartheta$ et $\Phi$ maxime inter se sunt permixtae, ita vt non parum difficultatis in earum separatione occurrat.

§. 10. Quoniam autem in his aequationibus ternae variabiles vbique vnicam dimensionem habent, recurramus ad eam methodum, qua iam saepius optimo cum successu sum vsus, scilicet, constituatur talis aequatio differentialis fecundi gradus $\frac{ddz}{2g\,dt^2} = -\frac{z}{b}$, qua motus penduli fimplicis longitudinis $= b$ exprimitur, ac ftatuamus $\eta = Az$; $\vartheta = Bz$ et $\phi = Cz$ fietque

$$\frac{dd\eta}{2g\,dt^2} = -\frac{Az}{b};\ \frac{dd\vartheta}{2g\,dt^2} = -\frac{Bz}{b} \text{ et } \frac{dd\phi}{2g\,dt^2} = -\frac{Cz}{b},$$

quibus valoribus in noftris aequationibus fubftitutis et littera $z$ per diuifionem fublata nancifcimur fequentes aequationes:

$$\text{I.}\ -\frac{Ce}{b} - \frac{Ab}{b} = -A;\ \text{III.}\ \frac{Ce}{b} + \frac{Bb}{b} = B$$

$$\text{V.}\ -\frac{C}{b} = -\frac{MCc - 2LCe + Le(A+B)}{Mkk}.$$

§. 11. Ex harum aequationum duabus prioribus ftatim elicimus

$$A = \frac{-Ce}{b-b} \quad \text{et} \quad B = \frac{-Ce}{b-b};$$

qui valores in quinta fubftituti praebent

$$-\frac{CMkk}{b} = -C(Mc + 2Le) - \frac{b\,CLee}{b-b}$$

quae reducitur ad hanc

$$+bb(Mc+2Le) - b(Mbc + 2Lbe + 2Lee + Mkk) + Mbkk = 0$$

cuius binas radices ponamus breuitatis gratia effe vnam $b = p$ et alteram $b = q$, ita vt $p$ vel $q$ fequenti forma fit expreffum

$$\frac{\frac{1}{2}b(Mc+2Le) + Lce + \frac{1}{2}Mkk}{Mc+2Le}$$

$$\frac{\pm\sqrt{(\frac{1}{4}bb(Mc+2Le)^2 + b(Mc+2Le)Lee - \frac{1}{2}b(Mc+2Le)Mkk + LLe^4 + LMeekk + \frac{1}{4}M^2k^4)}}{Mc+2Le}.$$

§. 12. Cum igitur fit $\frac{ddz}{2g\,dt^2} = -\frac{z}{b}$, erit vti conftat

$$z = \text{fin.}\left(\gamma + t\sqrt{\tfrac{2g}{b}}\right),$$ vnde

vnde quia prodit $A = B = \frac{Ce}{b-b}$ ſolutio ita ſe habebit:

$$\Phi = C\,\text{ſin.}(\gamma + t\sqrt{\tfrac{2g}{b}}) \text{ et } \eta = \vartheta = \frac{Ce}{b-b}\,\text{ſin.}(\gamma + t\sqrt{\tfrac{2g}{b}}).$$

Nunc autem ſi loco $b$ ſcribamus binos valores inventos $p$ et $q$, motus inde oriundos etiam coniungere licet, vt motus prodeat oſcillatorius ex binis ſimplicibus mixtus, qui vt fiat generalis, pro altero loco conſtantium C et $\gamma$ ſcribamus D et $\delta$; ficque obtinebitur ſequens ſolutio generalior

$$\Phi = C\,\text{ſin.}(\gamma + t\sqrt{\tfrac{2g}{p}}) + D\,\text{ſin.}(\delta + t\sqrt{\tfrac{2g}{q}}) \text{ et}$$

$$\eta = \vartheta = \frac{Ce}{p-\gamma}\,\text{ſin.}(\gamma + t\sqrt{\tfrac{2g}{p}}) + \frac{De1}{q-b}\,\text{ſin.}(\delta + t\sqrt{\tfrac{2g}{q}}).$$

§. 13. Quanquam autem ob quatuor conſtantes arbitrarios C, $\gamma$ et D, $\delta$ infinita varietas motuum, quos libra cum lancibus recipere poteſt, contineatur: tamen alia motuum genera non complectitur, nifi in quibus ambae lances aequali motu cientur. Haec igitur ſolutio neutiquam pro completa haberi poteſt, tamen vtique vſu venire poteſt vt ambae lances motu diuerſo agitentur, quemadmodum in phaenomeno obſeruato reuera accidit. Quin etiam ipſa analyſis declarat, hanc ſolutionem non eſſe generalem, ob id ipſum quod tantum quatuor conſtantes arbitrarias contineat. Quia enim tres habebamus aequationes differentiales ſecundi gradus, integratio completa adeo ſex quantitates arbitrarias inuoluere debet, ad quam eliciendam nouis artificiis erit opus; quod igitur negotium omni cura ſuſcipiamus.

## RESOLVTIO GENERALIS problematis propositi.

§. 14. Omni attentione igitur perpendamus tres aequationes differentiales ſecundi gradus, quibus ſolutio continetur

I. $\frac{e\,dd\Phi + b\,dd\eta}{2g\,dt^2} + \eta = 0$

III. $\frac{e\,dd\Phi + b\,dd\vartheta}{2g\,dt^2} + \vartheta = 0$

V. $\frac{Mkk\,dd\Phi}{2g\,dt^2} + (Mc + 2Le)\Phi - Le(\eta + \vartheta) = 0$

vbi, quia in duabus prioribus anguli $\eta$ et $\vartheta$ tantum permutantur, ſtatuamus $\eta = s + u$ et $\vartheta = s - u$, vt prodeat ſumma $\eta + \vartheta = 2s$ et $\eta - \vartheta = 2u$, vnde conficietur

I + III. $\frac{e\,dd\Phi + b\,dds}{2g\,dt^2} + s = 0$ et

I − III. $\frac{b\,ddu}{2g\,dt^2} + u = 0$

quae poſtrema aequatio, quia vnicam variabilem $u$ cum tempore $t$ inuoluit, ſtatim praebet integrale

$u = E\,\text{ſin.}\left(\varepsilon + t\sqrt{\frac{2g}{b}}\right)$;

quod in praecedente ſolutione erat praetermiſſum.

§. 15. Pro duabus reliquis aequationibus fiat vt ſupra $\Phi = Cz$ et $s = Az$ exiſtente $\frac{ddz}{2g\,dt^2} = -\frac{z}{b}$ ideoque $z = \text{ſin.}\left(\gamma + t\sqrt{\frac{2g}{b}}\right)$ eritque

$\frac{dd\Phi}{2g\,dt^2} = -\frac{Cz}{b}$ et $\frac{dds}{2g\,dt^2} = -\frac{Az}{b}$

quibus ſubſtitutis nanciſcemur ſequentes duas aequationes

$-\frac{Ce}{b} - \frac{Ab}{b} + A = 0$ et

$-\frac{CMkk}{b} + C(Mc + 2Le) - 2ALe = 0$

ex quarum priore fit $A = \frac{Ce}{b-b}$. Sicque reſtabit haec vnica aequatio

$$-\frac{Mkk}{b} + Mc + 2Le - \frac{2Lee}{b-b} = 0.$$

§. 16. Quo hanc aequationem commodiorem reddamus, ponamus breuitatis gratia

$$\frac{Mkk}{Mc+2Le} = m \text{ et } \frac{2Lee}{Mc+2Le} = l$$

et noſtra aequatio hanc induet formam:

$$-\frac{m}{b} + 1 - \frac{l}{b-b} = 0$$

quae reducitur ad hanc formam quadraticam

$$bb - b(b+l+m) + bm = 0$$

cuius binae radices erunt

$$b = \tfrac{1}{2}(b+l+m) \pm \sqrt{\tfrac{1}{4}(b+l+m)^2 - bm}$$

quae etiam ita exprimi poſſunt

$$b = \tfrac{1}{2}(b+l+m) \pm \sqrt{\tfrac{1}{4}(l+m-b)^2 + bl}$$

vnde patet quantitatem poſt ſignum radicale poſitam ſemper eſſe poſitiuam, et quidem maiorem quam $\frac{1}{4}(l+m-b)^2$; quare ſi hanc radicem quadratam ponamus $= \frac{1}{2}(l+m-b) + \omega$, erunt bini valores ipſius $b$, quos iterum deſignemus literis $p$ et $q$ iſti

$$p = l + m + \omega \text{ et } q = b - \omega.$$

§. 17. Quod ſi iam iſtos valores inuentos retenta primo littera $b$ ſubſtituamus reperiemus hos valores

$$\Phi = C\,\text{ſin.}\left(\gamma + t\sqrt{\tfrac{2g}{b}}\right) \text{ et } s = \frac{Ce}{b-b}\,\text{ſin.}\left(\gamma + t\sqrt{\tfrac{2g}{b}}\right).$$

Erat vero $u = E\,\text{ſin.}\left(\varepsilon + t\sqrt{\tfrac{2g}{b}}\right)$ vnde porro colligimus

$$\eta = \frac{Ce}{b-b}\,\text{fin.}(\gamma + t\sqrt{\tfrac{2g}{b}}) + \text{E}\,\text{fin.}(\varepsilon + t\sqrt{\tfrac{2g}{b}})$$
$$\vartheta = \frac{Ce}{b-b}\,\text{fin.}(\gamma + t\sqrt{\tfrac{2g}{b}}) - \text{E}\,\text{fin.}(\varepsilon + t\sqrt{\tfrac{2g}{b}}).$$

§. 18. Hinc igitur ſolutionem completam obtinebimus, ſi loco $b$ ſucceſſiue ſcribamus tam $p$ quam $q$, et valores inde reſultantes in vnum ſummam colligamus. Dum autem pro $b$ ſcribimus $q$, loco conſtantium C et $\gamma$ ſcribamus D et $\delta$; quo facto ſolutio completa ita ſe habebit

$$\text{I.}\quad \Phi = \text{C}\,\text{fin.}(\gamma + t\sqrt{\tfrac{2g}{p}}) + \text{D}\,\text{fin.}(\delta + t\sqrt{\tfrac{2g}{q}})$$
$$\text{II.}\quad \eta = \frac{Ce}{p-b}\,\text{fin.}(\gamma + t\sqrt{\tfrac{2g}{p}}) + \frac{De}{q-b}\,\text{fin.}(\delta + t\sqrt{\tfrac{2g}{q}}) + \text{E}\,\text{fin.}(\varepsilon + t\sqrt{\tfrac{2g}{b}})$$
$$\text{III.}\quad \vartheta = \frac{Ce}{p-b}\,\text{fin.}(\gamma + t\sqrt{\tfrac{2g}{p}}) + \frac{De}{q-b}\,\text{fin.}(\delta + t\sqrt{\tfrac{2g}{q}}) - \text{E}\,\text{fin.}(\varepsilon + t\sqrt{\tfrac{2g}{b}})$$

quae formulae ſex conſtantes arbitrarias inuoluunt, ſcilicet C, $\gamma$; D, $\delta$; E, $\varepsilon$; in quo character veritatis conſpicitur.

§. 19. Ex his valoribus iam ſatis eſt perſpicuum, mirandum illud phaenomenon, quo ambae lances alternatim quieſcere et moueri ſunt obſeruatae, vtique euenire poſſe. Cum enim ambo anguli $\eta$ et $\vartheta$ talibus formulis definiantur $\eta = P + Q$ et $\vartheta = P - Q$, vbi literae P et Q cum tempore certo modo varientur, ſi eueniat vt quodam tempore fiat $Q = P$ angulus $\eta$ maximum obtineret valorem, $\vartheta$ vero ad nihilum redigeretur, vnde prior lanx notabilem habebit motum gyratorium, poſterior vero quaſi quieſcet. Deinceps vero veniet tempus quo erit $Q = -P$, ac tum prior lanx ad quietem redigetur, poſterior vero maximas peraget oſcillationes: quae viciſſitudo ſaepius occurret ſtatis temporibus, donec motus fuerit prorſus

ſus extinctus. Omnes autem circumſtantiae talis motus reciproci a conſtantibus arbitrariis pendebunt, ita vt etiam hic infinita varietas locum habere poſſit.

§. 20. Accuratius autem hanc ſolutionem perpendamus, ſimulque mutationem momentaneam ſingulorum angulorum $\Phi$, $\eta$ et $\vartheta$ ſiue eorum celeritates angulares definiamus; quod, quo facilius fieri poſſit ponamus breuitatis gratia

$\sqrt{\frac{2g}{p}} = \mu$; $\sqrt{\frac{2g}{q}} = \nu$ et $\sqrt{\frac{2g}{b}} = \lambda$ vt habeamus

$$\Phi = C \sin.(\gamma + \mu t) + D \sin.(\delta + \nu t)$$

$$\eta = \frac{Ce}{p-b} \sin.(\gamma + \mu t) + \frac{De}{q-b} \sin.(\delta + \nu t) + E \sin.(\varepsilon + \lambda t)$$

$$\vartheta = \frac{Ce}{p-b} \sin.(\gamma + \mu t) + \frac{De}{q-b} \sin.(\delta + \nu t) - E \sin.(\varepsilon + \lambda t)$$

vnde pro celeritatibus reperiemus

$$\frac{d\Phi}{dt} = \mu C \cos.(\gamma + \mu t) + \nu D \cos.(\delta + \nu t)$$

$$\frac{d\eta}{dt} = \frac{\mu Ce}{p-b} \cos.(\gamma + \mu t) + \frac{\nu De}{q-b} \cos.(\delta + \nu t) + \lambda E \cos.(\varepsilon + \lambda t)$$

$$\frac{d\vartheta}{dt} = \frac{\mu Ce}{p-b} \cos.(\gamma + \mu t) + \frac{\nu De}{q-b} \cos.(\delta + \nu t) - \lambda E \cos.(\varepsilon + \lambda t).$$

§. 21. In talibus igitur librae agitationibus tres motus oſcillatorii ſimplices inter ſe permiſcentur, quorum primus reſpondet pendulo ſimplici longitudinis $= p$, quod ſingulas oſcillationes ſuas peraget tempore $= \frac{\pi}{\mu}$ min. ſec. Alter motus ſimplex reſpondet pendulo longitudinis $= q$, quod ſingulas oſcillationes peragit tempore $\frac{\pi}{\nu}$ ſec. tertius vero motus conuenit cum pendulo longitudinis $b$, quod ſingulas oſcillationes facit tempore $= \frac{\pi}{\lambda}$ ſec. Vbi quidem notandum, motum ipſius librae tantum ex duobus prioribus

ribus generibus componi, dum in lancibus omnes tres motus inter se permisceri possunt, vnde sine dubio maxima irregularitas in his motibus, atque adeo admirabilia phaenomena locum habere possunt.

§. 22. Quoniam sex habemus constantes arbitrarias, solutionem nostram ad omnes status initiales accommodare poterimus, quando scilicet pro tempore $t = 0$ tam ipsi anguli $\Phi$, $\eta$ et $\vartheta$ quam etiam eorum celeritates $\frac{d\Phi}{dt}$, $\frac{d\eta}{dt}$ et $\frac{d\vartheta}{dt}$ vtcunque dantur; tum igitur iucundum erit singulos casus per experimenta comprobare. Quia autem in hac solutione iugo librae peculiarem motum non tribuimus, si experimenta instituere voluerimus, scapum seu trutinam cum lingula seu examine firmiter connectere debemus. Tales igitur aliquos casus pro statu initiali dato euoluamus.

## EVOLVTIO CASVS

### quo initio erat $\Phi = 0$; $\eta = \alpha$ et $\vartheta = 0$ $\frac{d\Phi}{dt} = 0$; $\frac{d\eta}{dt} = 0$ et $\frac{d\vartheta}{dt}$.

§. 23. Posito igitur $t = 0$ sequentibus sex conditionibus erit satisfaciendum

I. $0 = C \sin. \gamma + D \sin. \delta$

II. $\alpha = \frac{Ce}{p-b} \sin. \gamma + \frac{De}{q-b} \sin. \delta + E \sin. \varepsilon$

III. $0 = \frac{Ce}{p-b} \sin. \gamma + \frac{De}{q-b} \sin. \delta - E \sin. \varepsilon$

IV. $0 = \mu C \cos. \gamma + \nu D \cos. \delta$

V. $0 = \frac{\mu Ce}{p-b} \cos. \gamma + \frac{\nu De}{q-b} \cos. \delta + \lambda E \cos. \varepsilon$

VI. $0 = \frac{\mu Ce}{p-b} \cos. \gamma + \frac{\nu De}{q-b} \cos. \delta - \lambda E \cos. \varepsilon.$

§. 24. Hic euidens eſt, tribus poſterioribus conditionibus ſatisfieri his tribus determinationibus:

$1^\circ.\ \gamma = 90;\quad 2^\circ.\ \delta = 90;\quad 3^\circ.\ \epsilon = 90$

hincque tres priores aequationes euadent

I. $0 = C + D$

II. $\alpha = \frac{Ce}{p-b} + \frac{De}{q-b} + E$ et

III. $0 = \frac{Ce}{p-b} + \frac{De}{q-b} - E$

ex quarum prima fit $D = -C$. Deinde, quia II + III dat

$$\alpha = \frac{2Ce}{p-b} + \frac{2De}{q-b} = \frac{2Ce}{p-b} - \frac{2Ce}{q-b}$$

inde fit $\alpha(p-b)(q-b) = 2Ce(q-p)$

vnde colligitur

$$C = \frac{\alpha(p-b)(q-b)}{2e(q-p)} \text{ et } D = -\frac{\alpha(p-b)(q-b)}{2e(q-b)}$$

Denique ob II − III ſeu $\alpha = 2E$ erit $E = \frac{1}{2}\alpha$.

§. 25. Si igitur pro $\frac{\alpha(p-b)(q-b)}{2e(q-p)}$ ſcribamus $C$, pro motu ſecuturo habebimus ſex ſequentes determinationes:

I. $\Phi = C \cos.\ \mu t - C \cos.\ \nu t$

II. $\eta = \frac{Ce}{p-b} \cos.\ \mu t - \frac{Ce}{q-b} \cos.\ \nu t + \frac{1}{2}\alpha \cos.\ \lambda t$

III. $\vartheta = \frac{Ce}{p-b} \cos.\ \mu t - \frac{Ce}{q-b} \cos.\ \nu t - \frac{1}{2}\alpha \cos.\ \lambda t$

IV. $\frac{d\Phi}{dt} = -\mu C \sin.\ \mu t + \nu C \sin.\ \nu t$

V. $\frac{d\eta}{dt} = -\frac{\mu Ce}{p-b} \sin.\ \mu t + \frac{\nu Ce}{q-b} \sin.\ \nu t - \frac{1}{2}\alpha\lambda \sin.\ \lambda t$

VI. $\frac{d\vartheta}{dt} = -\frac{\mu Ce}{p-b} \sin.\ \mu t + \frac{\nu Ce}{q-b} \sin.\ \nu t + \frac{1}{2}\alpha\lambda \sin.\ \lambda t$

vnde ad quoduis tempus $t$ tam ſtatus librae cum

 lanci-

lancibus, quam finguli motus angulares definiri poterunt.

§. 26. Ponamus exempli gratia $b = 4$, $l = 3$ et $m = 1$, fietque $p = 6$ et $q = 2$; tum vero fumamus $e = 2$, eritque

$$\mu = \sqrt{\tfrac{g}{3}};\ \nu = \sqrt{g};\ \lambda = \sqrt{\tfrac{g}{2}} \text{ et } C = \tfrac{\alpha}{4}.$$

Pro hoc ergo cafu determinationes noftrae erunt

I. $\Phi = \frac{1}{4}\alpha \operatorname{cof.} t\sqrt{\tfrac{g}{3}} - \frac{1}{4}\alpha \operatorname{cof.} t\sqrt{g}$

II. $\eta = \frac{1}{4}\alpha \operatorname{cof.} t\sqrt{\tfrac{g}{3}} + \frac{1}{4}\alpha \operatorname{cof.} t\sqrt{g} + \frac{1}{2}\alpha \operatorname{cof.} t\sqrt{\tfrac{g}{2}}$

III. $\vartheta = \frac{1}{4}\alpha \operatorname{cof.} t\sqrt{\tfrac{g}{3}} + \frac{1}{4}\alpha \operatorname{cof.} t\sqrt{g} - \frac{1}{2}\alpha \operatorname{cof.} t\sqrt{\tfrac{g}{2}}$

IV. $\frac{d\Phi}{dt} = -\frac{1}{4}\alpha\sqrt{\tfrac{g}{3}} \operatorname{fin.} t\sqrt{\tfrac{g}{3}} + \frac{1}{4}\alpha\sqrt{g} \operatorname{fin.} t\sqrt{g}$

V. $\frac{d\eta}{dt} = -\frac{1}{4}\alpha\sqrt{\tfrac{g}{3}} \operatorname{fin.} t\sqrt{\tfrac{g}{3}} - \frac{1}{4}\alpha\sqrt{g} \operatorname{fin} t\sqrt{g} - \frac{1}{2}\alpha\sqrt{\tfrac{g}{2}} \operatorname{fin.} t\sqrt{\tfrac{g}{2}}$

VI. $\frac{d\vartheta}{dt} = -\frac{1}{4}\alpha\sqrt{\tfrac{g}{3}} \operatorname{fin.} t\sqrt{\tfrac{g}{3}} - \frac{1}{4}\alpha\sqrt{g} \operatorname{fin.} t\sqrt{g} + \frac{1}{2}\alpha\sqrt{\tfrac{g}{2}} \operatorname{fin.} t\sqrt{\tfrac{g}{2}}.$

## EVOLVTIO CASVS
### quo initio erat

$$\Phi = \alpha;\ \eta = 0;\ \vartheta = 0$$
$$\frac{d\Phi}{dt} = 0;\ \frac{d\eta}{dt} = 0;\ \frac{d\vartheta}{dt} = 0.$$

§. 27. Quia hic iterum tres celeritates funt nullae, habebimus vt ante $\gamma = \delta = \varepsilon = 90$, vnde tres priores conditiones dant

I. $\alpha = C + D$

II. $0 = \frac{Ce}{p-b} + \frac{De}{q-b} + E$

III. $0 = \frac{Ce}{p-b} + \frac{De}{q-b} - E.$

Quia nunc II - III dat $0 = 2E$ erit $E = 0$; tum vero

vero erit ex II$^{da}$ $D = -\frac{C(q-b)}{p-b}$, qui valor in prima ſubſtitutus praebet

$\alpha = C - \frac{C(q-b)}{p-b} = \frac{C(p-q)}{p-b}$, ideoque $C = \frac{\alpha(p-b)}{p-q}$ et

$D = -\alpha \frac{q-b}{p-q}$,

ſicque formulae noſtrae fient

I. $\Phi = \frac{\alpha(p-b)}{p-q} \operatorname{cof.} \mu t - \frac{\alpha(q-b)}{p-q} \operatorname{cof.} \nu t$

II. $\eta = \frac{\alpha e}{p-q} \operatorname{cof.} \mu t - \frac{\alpha e}{p-q} \operatorname{cof.} \nu t$

III. $\vartheta = \frac{\alpha e}{p-q} \operatorname{cof.} \mu t - \frac{\alpha e}{p-q} \operatorname{cof.} \nu t$.

ex quibus celeritates angulares facile deducuntur.

§. 28. Hoc igitur caſu quia $\eta = \vartheta$ ambae lances eundem motum oſcillatorium recipient, quemadmodum in noſtra prima ſolutione particulari vſu venit. Hoc igitur caſu tam motus librae ipſius quam vtriusque lancis ex duplici motu ſimplici erit compoſitus, qui hic neceſſario certo quodam modo coniunctus deprehenditur; ita vt iſte motus modo magis modo minus fiat irregularis: prouti tempora oſcillationum vtriusque generis plus vel minus a ſe inuicem diſcrepabunt. Superfluum autem foret plures huiusmodi caſus euoluere, cum iam ſatis ſuperque ſit demonſtratum, motus illos mirabiles, vtcunque primo aſpectui inextricabiles videantur, cum noſtra theoria ex primis mechanicae principiis deducta perfectiſſime conuenire.