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De oscillationibus minimis penduli quotunque ponduscululis onusti

Leonhard Euler

DE OSCILLATIONIBVS MINIMIS PENDVLI QVOTCVNQVE PONDVSCVLIS ONVSTI.

Auctore

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Problema.

Si filo tenuissimo siue grauitatis experti quotcunque ponduscula A, B, C, D in datis a se inuicem interuallis fuerint alligata, idque ex puncto O suspensum et vtcunque ad motum concitatum oscillationes minimas peragat, eius statum et motum ad quoduis tempus definire.

Solutio.

§. 1. Ex puncto suspensionis O ducatur recta verticalis O V, et quicunque motus pendulo primum fuerit impressus elapso tempore $= t$ pendulum teneat situm in figura expressum O A B C D etc. et ex singulis ponduscalis ad verticalem O V agantur normales A P, B Q, C R, D S etc. Iam quia singula ponduscula dantur, eorum massae seu pondera designentur litteris A, B, C, D etc. et quia eorum interualla etiam dantur ponamus distantias Tab. III. Fig. 1.

$$OA = a; \; AB = b; \; BC = c; \; CD = d \text{ etc.}$$

 Porro

Porro pro ſingulis ponduſculis ſtatuantur coordinatae

$OP = x$; $OQ = x'$; $OR = x''$; $OS = x'''$ etc.

$PA = y$; $QB = y'$; $RC = y''$; $SD = y'''$ etc.

Tum vero ductis verticalibus Aq, Br; Cs etc. vocentur anguli quibus ſingula interualla a ſitu verticali declinant

$AOp = p$; $BAq = q$; $CBr = r$; $DCs = s$ etc.

ex quibus illae coordinatae ita determinantur vt ſit

$x = a\,\mathrm{cof.}\,p$	$y = a\,\mathrm{fin.}\,p$
$x' = a\,\mathrm{cof.}\,p + b\,\mathrm{cof.}\,q$	$y' = a\,\mathrm{fin.}\,p + b\,\mathrm{fin.}\,q$
$x'' = a\,\mathrm{cof.}\,p + b\,\mathrm{cof.}\,q + c\,\mathrm{cof.}\,r$	$y'' = a\,\mathrm{fin.}\,p + b\,\mathrm{fin.}\,q + c\,\mathrm{fin.}\,r$
$x''' = a\,\mathrm{cof.}\,p + b\,\mathrm{cof.}\,q + c\,\mathrm{cof.}\,r + d\,\mathrm{cof.}\,s$	$y''' = a\,\mathrm{fin.}\,p + b\,\mathrm{fin.}\,q + c\,\mathrm{fin.}\,r + d\,\mathrm{fin.}\,s$
etc.	etc.

§. 2. His poſitis, pro motu determinando vocetur

tenſio fili $OA = P$

tenſio fili $AB = Q$

tenſio fili $BC = R$

tenſio fili $CD = S$

atque hinc, ſi tempus t in minutis ſecundis exprimatur eiusque differentiale dt pro conſtante habeatur, altitudo autem ex qua grauia vno minuto ſecundo libere delabuntur notetur littera g, principia mechanica ſequentes ſuppeditant aequationes

$\frac{ddx}{2gdt^2}$

$\frac{ddx}{2g\,dt^2} = 1 - \frac{P\,cos.\,p}{A} + \frac{Q\,cos.\,q}{A}$	$\frac{ddy}{2g\,dt^2} = -\frac{P\,sin.\,p}{A} + \frac{Q\,sin.\,q}{A}$
$\frac{ddx'}{2g\,dt^2} = 1 - \frac{Q\,cos.\,q}{B} + \frac{R\,cos.\,r}{B}$	$\frac{ddy'}{2g\,dt^2} = -\frac{Q\,sin.\,q}{B} + \frac{R\,sin.\,r}{B}$
$\frac{ddx''}{2g\,dt^2} = 1 - \frac{R\,cos.\,r}{C} + \frac{S\,cos.\,s}{C}$	$\frac{ddy''}{2g\,dt^2} = -\frac{R\,sin.\,r}{C} + \frac{S\,sin.\,s}{C}$
$\frac{ddx'''}{2g\,dt^2} = 1 - \frac{S\,cos.\,s}{D}$	$\frac{ddy'''}{2g\,dt^2} = -\frac{S\,sin.\,s}{D}$
etc. etc.	etc. etc.

harum aequationum numerus, qui duplo maior eſt quam numerus pendulorum, ſufficit tam ad ſingulas tenſiones P, Q, R, S etc. quam ad angulos p, q, r, s etc. determinandos pro quouis tempore t.

§. 3. Haec ita ſe habent in genere quantaecunque etiam fuerint oſcillationes, quo autem caſu vix vlterius progredi licet, quam ob rem cogimur inueſtigationes noſtras tantum ad eos caſus accommodare, quibus oſcillationes ſunt quam minimae, vti in problemate enunciatur. Quin igitur hoc caſu omnes anguli p, q, r, s eſſe debent quam minimi, pro eorum coſinibus ſcribere licebit vnitatem; pro ſinibus autem ipſos angulos p, q, r, s etc. Hinc igitur ſingulae abſciſſae et applicatae ſortientur valores

$x = a$	$y = ap$
$x' = a+b$	$y' = ap+bq$
$x'' = a+b+c$	$y'' = ap+bq+cr$
$x''' = a+b+c+d$	$y''' = ap+bq+cr+ds$
etc.	etc.

Quia igitur abſciſſae hoc caſu fiunt conſtantes, earum differentialia euaneſcent; ex quibus naſcentur ſequentes aequationes:

$$0 = A$$

$$o = A - P + Q;\quad o = B - Q + R;\quad o = C - R + S;\quad o = D - S$$

ex quibus ſtatim ſingulae tenſiones facillime definiuntur, ſcilicet

$S = D$; $R = C + D$; $Q = B + C + D$ et $P = A + B + C + D$; etc.

hinc ad calculum contrahendum ponamus breuitatis gratia

$$\frac{P}{A} = 1 + \frac{B+C+D}{A} = \alpha \text{ hinc erit } \frac{Q}{A} = \frac{B+C+D}{A} = \alpha - 1$$

$$\frac{Q}{B} = 1 + \frac{C+D}{B} = \beta \qquad \frac{R}{B} = \frac{C+D}{B} = \beta - 1$$

$$\frac{R}{C} = 1 + \frac{D}{C} = \gamma \qquad \frac{S}{C} = \frac{D}{C} = \gamma - 1$$

$$\frac{S}{D} = 1 = \delta \qquad \text{etc.} \quad \text{etc.} \quad \text{etc.}$$

§. 4. Quod ſi iam pro applicatis y, y', y'' itemque pro tenſionibus P, Q, R, S etc. ſuos ſcribamus valores, adipiſcemur ſequentes aequationes differentiales ſecundi gradus:

I. $\frac{a\,ddp}{2g\,dt^2} = -\alpha p + (\alpha - 1)q$

II. $\frac{a\,ddp + b\,ddq}{2g\,dt^2} = -\beta q + (\beta - 1)r$

III. $\frac{a\,ddp + b\,ddq + c\,ddr}{2g\,dt^2} = -\gamma r + (\gamma - 1)s$

IV. $\frac{a\,ddp + b\,ddq + c\,ddr + d\,dds}{2g\,dt^2} = -\delta s = -s.$

Sicque totum negotium ad reſolutionem harum aequationum differentio-differentialium reducitur, quae vtique artificia prorſus ſingularia poſtulat.

§. 5. Quia in omnibus his aequationibus variabiles p, q, r, s etc tantum vnicam tenent dimenſionem, euidens eſt, his aequationibus ſatisfieri poſſe, ſi ſi inter quantitates p, q, r, s certae rationes conſtantes ſtatuantur. Sit igitur

$$p = \mathfrak{A} z;\ q = \mathfrak{B} z;\ r = \mathfrak{C} z;\ s = \mathfrak{D} z$$

ſic enim illae aequationes ſequentes induent formas:

$$\text{I. } \frac{\mathfrak{A} a\, ddz}{2g\, dt^2} = -\alpha \mathfrak{A} z + (\alpha - 1)\mathfrak{B} z$$

$$\text{II. } \frac{(\mathfrak{A} a + \mathfrak{B} b)\, ddz}{2g\, dt^2} = -\beta \mathfrak{B} z + (\beta - 1)\mathfrak{C} z$$

$$\text{III. } \frac{(\mathfrak{A} a + \mathfrak{B} b + \mathfrak{C} c)\, ddz}{2g\, dt^2} = -\gamma \mathfrak{C} z + (\gamma - 1)\mathfrak{D} z$$

$$\text{IV. } \frac{(\mathfrak{A} a + \mathfrak{B} b + \mathfrak{C} c + \mathfrak{D} d)\, ddz}{2g\, dt^2} = -\delta \mathfrak{D} z + = -\mathfrak{D} z$$

quae aequationes cum omnes inter ſe conuenire debeant, ſingulas ad hanc formam reuocemus:

$$\frac{ddz}{2g\, dt^2} = -\frac{z}{k}$$

quo valore in ſingulis ſubſtituto nanciſcemur ſequentes quatuor aequationes inter meras quantitates conſtantes, ſcilicet

$$\text{I. } -\frac{\mathfrak{A} a}{k} = -\alpha \mathfrak{A} + (\alpha - 1)\mathfrak{B}$$

$$\text{II. } -\frac{\mathfrak{A} a - \mathfrak{B} b}{k} = -\beta \mathfrak{B} + (\beta - 1)\mathfrak{C}$$

$$\text{III. } -\frac{\mathfrak{A} a - \mathfrak{B} b - \mathfrak{C} c}{k} = -\gamma \mathfrak{C} + (\gamma - 1)\mathfrak{D}$$

$$\text{IV. } -\frac{\mathfrak{A} a - \mathfrak{B} b - \mathfrak{C} c - \mathfrak{D} d}{k} = -\mathfrak{D}.$$

§. 6. Ex his iam aequationibus determinare licebit coëfficientes aſſumtos $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. Ex prima enim erit

$\mathfrak{B} = \frac{\mathfrak{A}}{\alpha - 1}\left(\alpha - \frac{a}{k}\right)$; ex ſecunda erit

$\mathfrak{C} = \frac{1}{\beta - 1}\left(\beta \mathfrak{B} - \frac{\mathfrak{A} a - \mathfrak{B} b}{k}\right)$ ſiue $\mathfrak{C} = \frac{\mathfrak{B}}{\beta - 1}\left(\beta - \frac{b}{k}\right) - \frac{\mathfrak{A}}{\beta - 1}\frac{a}{k}$

eodem modo ex tertia elicimus

$$\mathfrak{D} = \frac{\mathfrak{C}}{\gamma - 1}\left(\gamma - \frac{c}{k}\right) - \frac{\mathfrak{B}}{\gamma - 1}\frac{b}{k} - \frac{\mathfrak{A}}{\gamma - 1}\frac{a}{k}$$

 Qui

Qui valores in quarta substituti producent aequationem algebraicam, ex qua quantitatem incognitam k determinari opportebit; vbi aequatio tot inuoluat radices, quot dantur pondufcula: ita vt pro k totidem diuersi valores sint prodituri. Quarta autem aequatio quae hic est vltima hac forma repraesentatur:

$$\mathfrak{A}a + \mathfrak{B}b + \mathfrak{C}c + \mathfrak{D}d - \mathfrak{D}k = 0.$$

§. 7. Substituamus nunc successiue valores ex prioribus aequationibus inuentos in posterioribus; et quia erat:

$$\mathfrak{B} = \frac{\mathfrak{A}}{\alpha - 1}\left(\alpha - \frac{a}{k}\right) \text{ fiet:}$$

$$\mathfrak{C} = \frac{\mathfrak{A}}{(\alpha - 1)(\beta - 1)}\left(\alpha - \frac{a}{k}\right)\left(\beta - \frac{b}{k}\right) - \frac{\mathfrak{A}}{\beta - 1}\frac{a}{k}$$

$$\mathfrak{D} = \frac{\mathfrak{A}}{(\alpha - 1)(\beta - 1)(\gamma - 1)}\left(\alpha - \frac{a}{k}\right)\left(\beta - \frac{b}{k}\right)\left(\gamma - \frac{c}{k}\right) - \frac{\mathfrak{A}}{(\alpha - 1)(\gamma - 1)}\left(\alpha - \frac{a}{k}\right)\frac{b}{k} - \frac{\mathfrak{A}}{\gamma - 1}\frac{a}{k} - \frac{\mathfrak{A}}{(\beta - 1)(\gamma - 1)}\left(\gamma - \frac{c}{k}\right)\frac{a}{k}$$

qui valores ad sequentes formas reducuntur

$$(\alpha - 1)\frac{\mathfrak{B}}{\mathfrak{A}} = \alpha - \frac{a}{k}$$

$$(\alpha - 1)(\beta - 1)\frac{\mathfrak{C}}{\mathfrak{A}} = \alpha\beta - \frac{a(\alpha + \beta - 1) - \alpha b}{k} + \frac{ab}{kk}$$

$$(\alpha - 1)(\beta - 1)(\gamma - 1)\frac{\mathfrak{D}}{\mathfrak{A}} = \alpha\beta\gamma - \frac{a(\alpha\beta + \alpha\gamma + \beta\gamma - \alpha - \beta - \gamma + 1) - \alpha b(\beta + \gamma - 1) - c\alpha\beta}{k} + \frac{ab(\beta + \gamma - 1) + ac(\alpha + \beta - 1) + bc\alpha}{kk} - \frac{abc}{k^3}.$$

§. 8. Quod si iam istos valores in aequatione inuenta substituamus, pro determinatione quantitatis k prodibit aequatio quarti gradus, ad quam commodius inueniendam illam aequationem multiplicemus per $\frac{(\alpha - 1)(\beta - 1)(\gamma - 1)}{\mathfrak{A}k}$ vt habeatur ista aequatio:

$$\frac{(\alpha - 1)(\beta - 1)(\gamma - 1)a}{k} + \frac{(\alpha - 1)(\beta - 1)(\gamma - 1)\mathfrak{B}b}{\mathfrak{A}k} + \frac{(\alpha - 1)(\beta - 1)(\gamma - 1)\mathfrak{C}c}{\mathfrak{A}k}$$

$$+ \frac{(\alpha - 1)(\beta - 1)(\gamma - 1)\mathfrak{D}d}{\mathfrak{A}k} - \frac{(\alpha - 1)(\beta - 1)(\gamma - 1)\mathfrak{D}}{\mathfrak{A}} = 0$$

facto

facto autem calculo aequatio ista biquadratica ita reperietur expressa:

$$\alpha\beta\gamma . k^4 - \alpha\beta\gamma(a+b+c+d)k^3 + \left.\begin{matrix} + ab.\beta\gamma \\ + bc.\alpha\gamma \\ + cd.\alpha\beta \\ + ac.\gamma(\alpha+\beta-1) \\ + bd.\alpha(\beta+\gamma-1) \\ + ad(\alpha\beta+\alpha\gamma+\beta\gamma-\alpha-\beta-\gamma+1) \end{matrix}\right\} kk$$

$$\left.\begin{matrix} - bcd.\alpha \\ - abc.\gamma \\ - abd(\beta+\gamma-1) \\ - acd(\alpha+\beta-1) \end{matrix}\right\} k + abcd = 0$$

vbi obseruasse iuuabit, primo litteras a, b, c, d semper denotare distantias positiuas; tum vero litteras α, β, γ, esse numeros positiuos atque adeo vnitate maiores. Hinc enim ratio intelligi poterit, cur omnes quatuor radices huius aequationis proditurae sint reales: eas autem omnes esse positiuas permutatio signorum declarat.

§. 9. Ipsi resolutioni huius aequationis hic non immoramus, quando quidem si numerus pondusculorum esset maior a tali inuestigatione prorsus abstinere cogeremur: designemus igitur quatuor huius aequationis radices litteris k, k', k'' et k''' ex quarum singulis peculiares valores pro litteris $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ et $\mathfrak{D}$ colligemus, quos pariter hoc modo designemus $\mathfrak{A}'$ $\mathfrak{B}'$, $\mathfrak{C}'$ $\mathfrak{D}'$; $\mathfrak{A}''$, $\mathfrak{B}''$, $\mathfrak{C}''$, $\mathfrak{D}''$ et $\mathfrak{A}'''$, $\mathfrak{B}'''$, $\mathfrak{C}'''$, $\mathfrak{D}'''$; vbi quidem patet, litteras $\mathfrak{A}$, $\mathfrak{A}'$, $\mathfrak{A}''$, $\mathfrak{A}'''$ arbitrio nostro penitus relinqui; ita vt hoc modo quatuor habeamus quantitates pro lubitu accipiendas.

§. 10. Prosequamur igitur nostrum calculum pro sola radice k, cui respondent coëfficientes $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, quandoquidem quod pro hac radice fuerit comperatum facillime quoque ad reliquas radices applicatur. Cum igitur statuissemus hanc aequationem differentialem secundi gradus $\frac{ddz}{2g\,dt^2} = \frac{z}{k}$, ita vt sit $\frac{ddz}{dt^2} + \frac{2gz}{k} = 0$ si ponamus $\frac{2g}{k} = \lambda\lambda$ vt sit $\lambda = \sqrt{\frac{2g}{k}}$ si quidem k semper est quantitas realis positiua, notum est post duplicem integrationem prodire $z = f$ sin. $(\lambda t + \vartheta)$: vbi ϑ est angulus ab arbitrio nostro pendens. altera autem constans arbitraria f sine restrictione vnitati aequalis poni potest: propterea quod coëfficiens $\mathfrak{A}$ iam est arbitrarius. Hinc igitur ad quodvis tempus t singuli anguli p, q, r, s ita determinabuntur, vt sit

I. $p = \mathfrak{A}$ sin. $(\lambda t + \vartheta)$; $q = \mathfrak{B}$ sin. $(\lambda t + \vartheta)$; $r = \mathfrak{C}$ sin. $(\lambda t + \vartheta)$
et $s = \mathfrak{D}$ sin. $(\lambda t + \vartheta)$

similique modo si ex reliquis radicibus, k', k'', k''' ponamus

$$\lambda' = \sqrt{\tfrac{2g}{k'}},\ \lambda'' = \sqrt{\tfrac{2g}{k''}},\ \lambda''' = \sqrt{\tfrac{2g}{k'''}}$$

praeter illam solutionem adhuc habebimus tres sequentes

II. $p = \mathfrak{A}'$ sin $(\lambda' t + \vartheta')$; $q = \mathfrak{B}'$ sin. $(\lambda' t + \vartheta')$; $r = \mathfrak{C}'$ sin. $(\lambda' t + \vartheta')$;
$s = \mathfrak{D}'$ sin. $(\lambda' t + \vartheta')$

III. $p = \mathfrak{A}''$ sin. $(\lambda'' t + \vartheta'')$; $q = \mathfrak{B}''$ sin. $(\lambda'' t + \vartheta'')$; $r = \mathfrak{C}''$ sin. $(\lambda'' t + \vartheta'')$
$s = \mathfrak{D}''$ sin. $(\lambda'' t + \vartheta'')$

IV.

IV. $p = \mathfrak{A}''''\sin.(\lambda''''t + \vartheta'''')$; $q = \mathfrak{B}''''\sin.(\lambda''''t + \vartheta'''')$; $r = \mathfrak{C}''''\sin.(\lambda''''t + \vartheta'''')$; $s = \mathfrak{D}''''\sin.(\lambda''''t + \vartheta'''')$.

§. 11. Singulae autem hae quatuor ſolutiones maxime ſunt particulares: propterea quod duas tantum conſtantes arbitrarias inuoluunt, ſcilicet $\mathfrak{A}$ et ϑ, dum ſolutio generalis ob quatuor aequationes differentio-differentiales octo conſtantes arbitrarias complecti deberet. Qualis igitur motus ſingulis reſpondeat operae pretium erit accuratius inueſtigare; ac primo quidem quoniam pro qualibet caſu quatuor anguli p, q, r, s eandem perpetuo inter ſe ſeruant rationem, motus erit maxime regularis et pendulum perinde oſcillations ſuas pereget ac ſi eſſet ſimplex; atque quia elapſo tempore $t = \frac{2\pi}{\lambda}$ ſi loco t ſcribamus $t + \frac{2\pi}{\lambda}$ ſinguli anguli in eundem ſtatum reuertuntur, ideoque pendulum interea duos oſcillationes abſolviſſe cenſetur, ſicque tempus vnius oſcillationis erit $= \frac{\pi}{\lambda} = \frac{\pi\sqrt{k}}{\sqrt{2g}}$, quod adeo tempus in minutis ſecundis exprimitur. Eodem modo pro ſecunda radice k' erit tempus vnius cuiusque oſcillationis $= \frac{\pi\sqrt{k'}}{\sqrt{2g}}$; pro tertia radice $= \frac{\pi\sqrt{k''}}{\sqrt{2g}}$ et pro quarta $\frac{\pi\sqrt{k'''}}{\sqrt{2g}}$.

§. 12. Cum igitur hae quatuor ſolutiones ſimplices problemati noſtro ſatisfaciant, quoniam in aequationibus differentio-differentialibus ad quas nos ſolutio perduxit ſingulae quantitates p, q, r, s vbique vnam tantum dimenſionem tenent, ſolutiones illae particulares quomodocunque inter ſe combinentur pro-

 blema-

blemati pariter satisfacient, vnde sequens solutio generalis conficitur:

$$p = \mathfrak{A}\,\text{sin.}(\lambda t + \vartheta) + \mathfrak{A}'\,\text{sin.}(\lambda' t + \vartheta') + \mathfrak{A}''\,\text{sin.}(\lambda'' t + \vartheta'') + \mathfrak{A}'''\,\text{sin.}(\lambda''' t + \vartheta''')$$
$$q = \mathfrak{B}\,\text{sin.}(\lambda t + \vartheta) + \mathfrak{B}'\,\text{sin.}(\lambda' t + \vartheta') + \mathfrak{B}''\,\text{sin.}(\lambda'' t + \vartheta'') + \mathfrak{B}'''\,\text{sin.}(\lambda''' t + \vartheta''')$$
$$r = \mathfrak{C}\,\text{sin.}(\lambda t + \vartheta) + \mathfrak{C}'\,\text{sin.}(\lambda' t + \vartheta') + \mathfrak{C}''\,\text{sin.}(\lambda'' t + \vartheta'') + \mathfrak{C}'''\,\text{sin.}(\lambda''' t + \vartheta''')$$
$$s = \mathfrak{D}\,\text{sin.}(\lambda t + \vartheta) + \mathfrak{D}\,\text{sin.}(\lambda' t + \vartheta') + \mathfrak{D}''\,\text{sin.}(\lambda'' t + \vartheta'') + \mathfrak{D}'''\,\text{sin.}(\lambda''' t + \vartheta''')$$

in his enim formulis octo occurrunt constantes arbitrariae, scilicet quatuor coëfficientes $\mathfrak{A}$, $\mathfrak{A}'$ $\mathfrak{A}''$, $\mathfrak{A}'''$ quippe per quos reliqui determinantur; tum quatuor anguli ϑ, ϑ', ϑ'', ϑ''', quemadmodum gemina integratio quatuor illarum aequationum postulat. Hinc igitur patet, principium Illustriss. D. *Bernoulli*, quo omnes huiusmodi oscillationes ex duobus vel pluribus motibus oscillatoriis simplicibus et regularibus componi statuit, omninò in primis motus principiis esse fundatum atque adeo ex iis immediate deduci posse.

§. 13. Ope harum igitur formularum ad quodvis tempus t singuli illi anguli p, q, r et s assignari hincque status penduli definiri poterit. Quin etiam horum angulorum variationes momentaneae celeritates praebebunt, quibus status penduli quouis temporis momento dt immutatur. Cum enim formulae

$$\frac{dp}{dt},\ \frac{dq}{dt},\ \frac{dr}{dt}\ \text{et}\ \frac{ds}{dt}$$

exprimant celeritates angulares, quibus isti anguli tempusculo dt augentur, hae celeritates ita se habebunt

$\frac{dp}{dt}$

$$\frac{dp}{dt} = \lambda\mathfrak{A}\operatorname{cos}(\lambda t+\vartheta)+\lambda'\mathfrak{A}'\operatorname{cos}(\lambda' t+\vartheta')+\lambda''\mathfrak{A}''\operatorname{cos}(\lambda'' t+\vartheta'')$$
$$+\lambda'''\mathfrak{A}'''\operatorname{cos}(\lambda''' t+\vartheta''')$$
$$\frac{dq}{dt} = \lambda\mathfrak{B}\operatorname{cos}(\lambda t+\vartheta)+\lambda'\mathfrak{B}'\operatorname{cos}(\lambda' t+\vartheta')+\lambda''\mathfrak{B}''\operatorname{cos}(\lambda'' t+\vartheta'')$$
$$+\lambda'''\mathfrak{B}'''\operatorname{cos}(\lambda''' t+\vartheta''')$$
$$\frac{dr}{dt} = \lambda\mathfrak{C}\operatorname{cos}(\lambda t+\vartheta)+\lambda'\mathfrak{C}'\operatorname{cos}(\lambda' t+\vartheta')+\lambda''\mathfrak{C}''\operatorname{cos}(\lambda'' t+\vartheta'')$$
$$+\lambda'''\mathfrak{C}'''\operatorname{cos}(\lambda''' t+\vartheta''')$$
$$\frac{ds}{dt} = \lambda\mathfrak{D}\operatorname{cos}(\lambda t+\vartheta)+\lambda'\mathfrak{D}'\operatorname{cos}(\lambda' t+\vartheta')+\lambda''\mathfrak{D}''\operatorname{cos}(\lambda'' t+\vartheta'')$$
$$+\lambda'''\mathfrak{D}'''\operatorname{cos}(\lambda''' t+\vartheta''')$$

ſicque omnia ſumus adepti, quae circa ſolutionem huius problematis deſiderari poſſunt.

Corollarium.

§. 14. Maxima igitur difficultas in reſolutione aequationis algebraicae ex qua omnes valores litterae k determinari oportet, occurrit; praecipue ſi pendulum pluribus pendusculis fuerit oneratum. Tum vero etiam quemadmodum pro ſingulis valoribus ipſius k coëfficientes $\mathfrak{B}$, $\mathfrak{C}$ et $\mathfrak{D}$ definiri commode queant nondum ſatis liquet pro pluribus quam quatuor ponduſculis. Quo igitur hanc inueſtigationem faciliorem reddamus, differentias inter binas aequationes ſe inſequentes (§. 5.) exhibitas conſideremus.

I. $-\frac{\mathfrak{A}a}{k} = -\alpha\mathfrak{A}+(\alpha-1)\mathfrak{B}$ ſiue $0=\mathfrak{A}(\frac{a}{k}-\alpha)+(\alpha-1)\mathfrak{B}$

I–II. $\frac{\mathfrak{B}b}{k} = -\alpha\mathfrak{A}+\mathfrak{B}(\alpha+\beta-1)-\mathfrak{C}(\beta-1)$

II–III. $\frac{\mathfrak{C}c}{k} = -\beta\mathfrak{B}+\mathfrak{C}(\beta+\gamma-1)-\mathfrak{D}(\gamma-1)$

III–IV. $\frac{\mathfrak{D}d}{k} = -\gamma\mathfrak{C}+\gamma\mathfrak{D}$

vnde

vnde facile patet quomodo hae aequalitates ſint continuandae, ſi ponduſculorum numerus fuerit maior.

§. 15. Supra litterae $\mathfrak{B}$, $\mathfrak{C}$ et $\mathfrak{D}$ ex prima $\mathfrak{A}$ determinauimus; nunc autem a poſtrema incipientes ſingulas ex vltima $\mathfrak{D}$ deriuemus, vnde fit vt ſequitur

$$\gamma\,\mathfrak{C} = \mathfrak{D}\left(\gamma - \frac{d}{k}\right)$$
$$\beta\,\mathfrak{B} = \mathfrak{C}\left(\beta + \gamma - 1 - \frac{c}{k}\right) - \mathfrak{D}(\gamma - 1)$$
$$\alpha\,\mathfrak{A} = \mathfrak{B}\left(\alpha + \beta - 1 - \frac{b}{k}\right) - \mathfrak{C}(\beta - 1)$$

vnde reperimus

$$\frac{\mathfrak{C}}{\mathfrak{D}} = \frac{1}{\gamma}\left(\gamma - \frac{d}{k}\right)$$
$$\frac{\mathfrak{B}}{\mathfrak{D}} = \frac{1}{\beta\gamma}\left(\gamma - \frac{d}{k}\right)\left(\beta+\gamma-1-\frac{c}{k}\right) - \left(\frac{\gamma-1}{\beta}\right)$$
$$\frac{\mathfrak{A}}{\mathfrak{D}} = \frac{1}{\alpha\beta\gamma}\left(\gamma - \frac{d}{k}\right)\left(\beta+\gamma-1-\frac{c}{k}\right)\left(\alpha+\beta-1-\frac{b}{k}\right) - \frac{\gamma-1}{\alpha\beta}\left(\alpha+\beta-1-\frac{b}{k}\right) - \frac{\beta-1}{\alpha\gamma}\left(\gamma-\frac{d}{k}\right)$$

qui valores in prima aequatione ſubſtituti producent iſtam aequationem:

$$0 = -\frac{1}{\alpha\beta\gamma}\left(\alpha-\frac{a}{k}\right)\left(\alpha+\beta-1-\frac{b}{k}\right)\left(\beta+\gamma-1-\frac{c}{k}\right)\left(\gamma-\frac{d}{k}\right) + \frac{\gamma-1}{\alpha\beta}\left(\alpha-\frac{a}{k}\right)\left(\alpha+\beta-1-\frac{b}{k}\right) + \frac{\beta-1}{\alpha\gamma}\left(\alpha-\frac{a}{k}\right)\left(\gamma-\frac{d}{k}\right) + \frac{\alpha-1}{\beta\gamma}\left(\beta+\gamma-1-\frac{c}{k}\right)\left(\gamma-\frac{d}{k}\right) - \frac{(\alpha-1)(\gamma-1)}{\beta}$$

quae aequatio manifeſto aſcendit ad quartum ordinem, ex qua incognitae k quatuor valores inueſtigari oportet: hocque modo operatio inſtitui facile poterit, ſi ponduſculorum numerus fuerit maior.

Scho-

Scholion.

§ 16. Quamuis autem haec solutio sit maxime elegans, et problemati perfectissime satisfaciat, tamen maximae occurrunt difficultates, si eam ad casum determinatum applicare voluerimus. Quod si enim pro statu initiali vbi $t = 0$ singulis angulis p, q, r, s datos valores tribuere velimus, simulque singulis ponduscnlis datas celeritates angulares, ad octo aequationes perueniemus, quae similes erunt duabus aequationibus ex angulo p natis: si enim requiratur vt initio fuerit angulus $p = f$ eiusque celeritas angularis $= i$ hae duae obtinentur aequationes:

$$f = \mathfrak{A} \sin \vartheta + \mathfrak{A}' \sin. \vartheta' + \mathfrak{A}'' \sin. \vartheta'' + \mathfrak{A}''' \sin. \vartheta''' \text{ et}$$
$$i = \lambda \mathfrak{A} \cos. \vartheta + \lambda' \mathfrak{A}' \cos. \vartheta' + \lambda'' \mathfrak{A}'' \cos. \vartheta'' + \lambda''' \mathfrak{A}''' \cos. \vartheta'''$$

similesque binae aequationes obtinebuntur pro reliquis ponduscnlis. Nunc igitur requiritur vt ex his octo aequationibus octo illae constantes arbitrariae

$$\mathfrak{A}, \mathfrak{A}', \mathfrak{A}'', \mathfrak{A}''' \text{ et } \vartheta, \vartheta', \vartheta'', \vartheta'''$$

definiantur, quem sane laborem vix quisquam exsequetur: si modo corpusculorum numerus ternarium superauerit; quamobrem iam dudum non dubitaui asseuerare, solutionem hanc quantumuis elegantem et perfectam plane esse ineptam, vt ad casus determinatos, quibus penduli status initialis praescribitur adaptari possit. Ex quo manifesto sequitur, si problema ita proponatur, vt si pendulo datus status et motus initio imprimatur, motus deinceps secuturos definiri debeat, longe aliam solutionem requiri, quae proinde ab hac maxime discrepare debebit.

EVOLVTIO CASVS

quo omnia ponduscula sunt aequalia eorumque interualla etiam aequalia $=a$; ponatur autem breuitatis gratia $\frac{a}{k}=u$.

§. 17. Sit primo pondusculorum numerus $=2$ erit $\alpha=2$ et $\beta=1$, vnde aequationes ex §. 13. erunt

$$0=-\mathfrak{A}(2-u)+\mathfrak{B} \text{ et } \mathfrak{B}u=-2\mathfrak{A}+2\mathfrak{B};$$

ex posteriore sequitur

$$\mathfrak{A}=\frac{\mathfrak{B}(2-u)}{2}$$

qui valor in priore substitutus dat

$$0=-\frac{\mathfrak{B}(2-u)^2}{2}+\mathfrak{B} \text{ siue } (2-u)^2-2=0$$

vnde statim deducitur

$$2-u=\pm\sqrt{2} \text{ ideoque } u=2\mp\sqrt{2}=\frac{a}{k},$$

tum vero $\mathfrak{A}$ per $\mathfrak{B}$ ita exprimitur vt sit $\mathfrak{A}=\frac{\mathfrak{B}(2-u)}{2}$ vbi $\mathfrak{B}$ pro lubitu accipi potest. Quare si bini valores ipsius k sint k et k', hisque respondeant litterae $\mathfrak{A}'$ et $\mathfrak{B}'$ solutio in his aequationibus continebitur posito $\lambda=\sqrt{\frac{2g}{k}}$ et $\lambda'=\sqrt{\frac{2g}{k'}}$

$$p=\mathfrak{A}\,\text{fin.}(\lambda t+\vartheta)+\mathfrak{A}'\,\text{fin.}(\lambda' t+\vartheta') \text{ et}$$
$$q=\mathfrak{B}\,\text{fin.}(\lambda t+\vartheta)+\mathfrak{B}'\,\text{fin.}(\lambda' t+\vartheta').$$

§. 18. Sit pondusculorum numerus $=3$, eritque $\alpha=3$, $\beta=2$ et $\gamma=1$, vnde aequationes nostrae erunt

$$0=-\mathfrak{A}(3-u)+2\mathfrak{B} \text{ fiue } 0=\mathfrak{A}(3-u)-2\mathfrak{B}$$
$$\mathfrak{B}u=-3\mathfrak{A}+4\mathfrak{B}-\mathfrak{C}$$
$$\mathfrak{C}u=-2\mathfrak{B}+2\mathfrak{C}$$

ex

Fx hac fit $\mathfrak{B} = \frac{\mathfrak{C}(2-u)}{2}$ tum vero experiore ;

$$\mathfrak{A} = \frac{\mathfrak{C}(2-u)(4-u)}{6} - \frac{\mathfrak{C}}{3} = \frac{\mathfrak{C}}{6}(6 - 6u + uu)$$

vnde aequatio prodit

$$\frac{(2-u)(3-u)(4-u)}{6} - \frac{(3-u)}{3} - (2-u) = 0 \text{ fiue}$$

$$6 - 18u + 9uu - u^3 = 0$$

cuius ergo dabuntur tres radices, ideoque valores pro k, k', k'', ex quibus tota folutio facile conficitur.

§. 19. Sit ponduſculorum numerus $= 4$, erit $\alpha = 4$, $\beta = 3$, $\gamma = 2$, $\delta = 1$, vnde noſtrae aequationes erunt

$$0 = \mathfrak{A}(4-u) - 3\mathfrak{B}$$

$$\mathfrak{B}u = -4\mathfrak{A} + 6\mathfrak{B} - 2\mathfrak{C}$$

$$\mathfrak{C}u = -3\mathfrak{B} + 4\mathfrak{C} - \mathfrak{D}$$

$$\mathfrak{D}u = -2\mathfrak{C} + 2\mathfrak{D}.$$

Ex vltima fit

$$\mathfrak{C} = \frac{\mathfrak{D}(2-u)}{2}$$

$$\mathfrak{B} = \frac{\mathfrak{D}(2-u)(4-u)}{6} - \frac{\mathfrak{D}}{3} = \frac{\mathfrak{D}}{6}(6 - 6u + uu)$$

$$\mathfrak{A} = \frac{\mathfrak{D}}{24}(24 - 36u + 12uu - u^3)$$

qui valores ſubſtituti hanc praebent aequationem

$$u^4 - 16u^3 + 72uu - 96u + 24 = 0$$

cuius quatuor radices quaeri oportet.

§. 20. Sit numerus ponduſculorum $= 5$, erit $\alpha = 5$, $\beta = 4$, $\gamma = 3$, $\delta = 2$, $\varepsilon = 1$, et aequationes noſtrae erunt

0 =

$$0 = \mathfrak{A}(5-u) - 4\mathfrak{B}$$
$$\mathfrak{B}u = -5\mathfrak{A} + 8\mathfrak{B} - 3\mathfrak{C}$$
$$\mathfrak{C}u = -4\mathfrak{B} + 6\mathfrak{C} - 2\mathfrak{D}$$
$$\mathfrak{D}u = -3\mathfrak{C} + 4\mathfrak{D} - \mathfrak{E}$$
$$\mathfrak{E}u = -2\mathfrak{D} + 2\mathfrak{E}$$

hic ex tribus posterioribus colligimus

$$\mathfrak{D} = \frac{\mathfrak{E}}{2}(2-u)$$
$$\mathfrak{C} = \frac{\mathfrak{E}}{6}(6 - 6u + uu)$$
$$\mathfrak{B} = \frac{\mathfrak{E}}{24}(24 - 36u + 12uu - u^3)$$

tum vero inuenitur

$$\mathfrak{A} = \frac{\mathfrak{E}}{120}(120 - 240u + 120uu - 20u^3 + u^4)$$

quibus valoribus substitutis aequatio sequens resultat

$$u^5 - 25u^4 + 200u^3 - 600uu + 600u - 120 = 0.$$

§. 21. Hinc generaliter si numerus ponduscu- lorum fuerit $= n$, aequatio ex qua u definiri debet per legitimam inductionem colligitur fore

$$0 = 1 - \frac{nu}{1^2} + \frac{n(n-1)uu}{1^2 \cdot 2^2} - \frac{n(n-1)(n-2)u^3}{1^2 \cdot 2^2 \cdot 3^2} + \frac{n(n-1)(n-2)(n-3)u^4}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2} \text{ etc.}$$

deinde vero si coëfficientium $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ vltimus fuerit $\mathfrak{N}$ singuli sequenti modo determinabuntur

$$\frac{\mathfrak{A}}{\mathfrak{N}} = 1 - \frac{n-1}{1^2 \cdot 2}u + \frac{(n-1)(n-2)}{1^2 \cdot 2^2 \cdot 3}u^2 - \frac{(n-1)(n-2)(n-3)}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4}u^3 + \frac{(n-1)(n-2)(n-3)(n-4)}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4^3 \cdot 5}u^4 \text{ etc.}$$
$$\frac{\mathfrak{B}}{\mathfrak{N}} = 1 - \frac{n-2}{1^2 \cdot 2}u + \frac{(n-2)(n-3)}{1^2 \cdot 2^2 \cdot 3}uu - \frac{(n-2)(n-3)(n-4)}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4}u^3 + \frac{(n-2)(n-3)(n-4)(n-5)}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2 \cdot 5}u^4 \text{ etc.}$$
$$\frac{\mathfrak{C}}{\mathfrak{N}} = 1 - \frac{n-3}{1^2 \cdot 2}u + \frac{(n-3)(n-4)}{1^2 \cdot 2^2 \cdot 3}uu - \frac{(n-3)(n-4)(n-5)}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4}u^3 + \frac{(n-3)(n-4)(n-5)(n-6)}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2 \cdot 5}u^5 \text{ etc.}$$
$$\frac{\mathfrak{D}}{\mathfrak{N}} = 1 - \frac{n-4}{1^2 \cdot 2}u + \frac{(n-4)(n-5)}{1^2 \cdot 2^2 \cdot 3}uu - \frac{(n-4)(n-5)(n-6)}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4}u^3 + \frac{(n-4)(n-5)(n-6)(n-7)}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2 \cdot 5}u^5 \text{ etc.}$$

etc. etc. etc.

Tantum

Tantum igitur restat methodus, cuius ope illius aequationis n radices elici queant, quippe quibus inuentis solutio completa huius casus habebitur.

§. 22. Aequatio illa ordinis n, ex qua valores litterae u definire oportet etiam hoc modo concinnius referri potest

$$0 = u^n - \frac{n^2}{1} u^{n-1} + \frac{n^2(n-1)^2}{1.\ 2} u^{n-2} - \frac{n^2(n-1)^2(n-2)^2}{1.\ 2.\ 3.} u^{n-3} + \frac{n^2(n-1)^2(n-2)^2(n-3)^2}{1.\ 2.\ 3.\ 4} u^{n-4} \text{ etc.}$$

hinc autem coëfficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. etiam hoc modo exhiberi possunt

$$\pm 1.\ 2.\ 3 \ldots n \frac{\mathfrak{A}}{\mathfrak{N}} = u^{n-1} - \frac{n(n-1)}{1} u^{n-2} + \frac{n(n-1)^2(n-2)}{1.\ 2} u^{n-3} - \frac{n(n-1)^2(n-2)^2(n-3)}{1.\ 2.\ 3} u^{n-4} \text{ etc.}$$

$$\mp 1.\ 2.\ 3 \ldots (n-1) \frac{\mathfrak{B}}{\mathfrak{N}} = u^{n-2} - \frac{n(n-1)(n-2)}{1} u^{n-3} + \frac{(n-1)(n-2)^2(n-3)}{1.\ 2} u^{n-4} - \frac{(n-1)(n-2)^2(n-3)^2(n-4)}{1.\ 2.\ 3} u^{n-5} \text{ etc.}$$

$$\pm 1.\ 2.\ 3 \ldots (n-2) \frac{\mathfrak{C}}{\mathfrak{N}} = u^{n-3} - \frac{(n-2)(n-3)}{1} u^{n-4} + \frac{(n-2)(n-3)^2(n-4)}{1.\ 2} u^{n-5} - \frac{(n-2)(n-3)^2(n-4)^2(n-5)}{1.\ 2.\ 3} u^{n-6} \text{ etc.}$$

$$\mp 1.\ 2.\ 3 \ldots (n-3) \frac{\mathfrak{D}}{\mathfrak{N}} = u^{n-4} - \frac{(n-3)(n-4)}{1} u^{n-5} + \frac{(n-3)(n-4)^2(n-5)}{1.\ 2} u^{n-6} - \frac{(n-2)(n-3)^2(n-4)^2(n-5)}{1.\ 2.\ 3} u^{n-7} \text{ etc.}$$

etc. etc.

vbi signorum ambiguorum superiora sunt accipienda si n fuerit numerus impar, inferiora autem si par.