1774

Solutio problematis de inveniendo triangulo, in quo rectae ex singulis angulis latera opposita bisecantes sint rationales

Leonhard Euler

ubi primi in partibus decimis, secunda in millesimis, tertia in centies millesimis, et quarta in centies centenis millesimis aberrat.

Coroll. 3.

52. Talis lex progressionis etiam in formulis generalibus pro e^x deprehenditur: Si enim nostras fractiones ponamus:

$e^x = \frac{A}{\mathfrak{A}};\ \frac{B}{\mathfrak{B}};\ \frac{C}{\mathfrak{C}};\ \frac{D}{\mathfrak{D}};\ \frac{E}{\mathfrak{E}}$ etc. sumtis $A=1$ et $\mathfrak{A}=1$, erit:

$B=2+x;\ C=6B+Axx;\ D=10C+Bxx;\ E=14D+Cxx;$ etc.

$\mathfrak{B}=2-x;\ \mathfrak{C}=6\mathfrak{B}+\mathfrak{A}xx;\ \mathfrak{D}=10\mathfrak{C}+\mathfrak{B}xx;\ \mathfrak{E}=14\mathfrak{D}+\mathfrak{C}xx;$ etc.

unde series tam numeratorum, quam denominatorum facile continuatur.

SOLUTIO PROBLEMATIS DE INVENIENDO TRIANGULO IN QUO RECTAE EX SINGULIS ANGULIS LATERA OPPOSITA BISECANTES SINT RATIONALES.

Auctore

L. EULERO.

1.

Vocatis ternis lateribus $2a$, $2b$, $2c$ et rectis haec latera bisecantibus f, g, h: quaestio reducitur ad resolutionem trium sequentium formularum

$$2bb+2cc-aa=ff$$
$$2cc+2aa-bb=gg$$
$$2aa+2bb-cc=hh.$$

2. Hinc differentiis sumendis sequitur fore:

$3(bb-aa)=ff-gg;\ 3(cc-bb)=gg-hh;\ 3(cc-aa)=ff-hh$

seu $ff+3aa=gg+3bb=hh+3cc$.

Cum autem sit $ff=2bb+2cc-aa$, habebimus,

$$ff+3aa=gg+3bb=hh+3a=2(aa+bb+cc).$$

3. Summa porro nostrarum trium formularum praebet:

$$2(ff+gg+hh)=3ff+9aa=3gg+9bb=3hh+9cc$$

ita ut hinc istae ternae formulae resultent:

$$2gg + 2hh - ff = 9aa$$
$$2hh + 2ff - gg = 9bb$$
$$2ff + 2gg - hh = 9cc.$$

4. Quae cum similes sint ipsis propositis, concludimus si pro lateribus $2a$, $2b$, $2c$ sint rectae bisecantes f, g, h tum pro lateribus $2f$, $2g$, $2h$ fore rectas bisecantes $3a$, $3b$, $3c$, ideoque pro lateribus f, g, h rectas bisecantes $\frac{3}{2}a$, $\frac{3}{2}b$, $\frac{3}{2}c$. Quare invento uno hujusmodi triangulo, si rectae bisecantes pro lateribus novi trianguli accipiantur, hoc eadem gaudebit proprietate, quia in hoc rectae bisecantes sunt tres quadrantes laterum praecedentis.

5. His observatis solutionem quaestionis sequenti modo aggredior. Primo binis tantum formulis satis facturus eas ita exhibeo:

$$(b-c)^2 + (b+c)^2 - aa = (b-c)^2 + (b+c+a)(b+c-a) = ff$$
$$(a-c)^2 + (a+c)^2 - bb = (a-c)^2 + (a+c+b)(a+c-b) = gg$$

statuo igitur:

$$f = b - c + (b+c+a)p \quad \text{et} \quad g = a - c + (a+c+b)q$$

ut facta substitutione divisio per $a+b+c$ succedat, hoc modo obtinetur:

$$b + c - a = 2(b-c)p + (b+c+a)pp$$
$$a + c - b = 2(a-c)q + (a+c+b)qq.$$

6. Ex utraque aequatione definiatur valor ipsius c:

$$c = \frac{a(1+pp) - b(1-2p-pp)}{1+2p-pp} = \frac{b(1+qq) - a(1-2q-qq)}{1+2q-qq}$$

unde fit

$$a + b + c = \frac{2a(1+p) + 4bp}{1+2p-pp} = \frac{2b(1+q) + 4aq}{1+2q-qq}$$

ex quo duplici valore ratio inter numeros a et b colligitur:

$$a(1+p)(1+2q-qq) - 2aq(1+2p-pp) =$$
$$b(1+q)(1+2p-pp) - 2bp(1+2q-qq)$$

quamobrem statuo:

$$a = 1 + q - pp - 2pq - ppq + 2pqq$$
$$b = 1 + p - qq - 1pq - pqq + 2ppq$$

hincque fit

$$\frac{a+b+c}{2} = \frac{1 + 3p + q + pp - pq - 7ppq - p^3 + 3p^3q}{1+2p-pp}$$

seu $a + b + c = 2p + 2q - 6pq$.

7. Cum igitur sit:

$$a + b = 2 + p + q - pp - qq - 4pq + ppq + pqq$$

erit $\quad c = p + q + pp + qq - 2pq - ppq - pqq$

sicque binis formulis satisfit, numeris a, b, c sequentes valores tribuendo:

$$a = 1 + q - pp - 2pq - ppq + 2pqq$$
$$b = 1 + p - qq - 2pq - pqq + 2ppq$$
$$c = p + q + pp + qq - 2pq - ppq - pqq$$

unde cum fiat

$$a + b - c = 2 + 2p + 2q - 6pq$$
$$b - c = 1 - q - pp - 2qq + 3ppq$$
$$a - c = 1 - p - qq - 2pp + 3pqq$$

habebimus:

$$f=1+2p-q+pp-2qq+2pq-3ppq \quad \text{et} \quad b-c+f=2(1+q)(1+p-2q)$$
$$g=1+2q-p+qq-2pp+2pq-3pqq \qquad a-c+g=2(1+p)(1+q-2p).$$

8. Juvabit hinc etiam sequentes valores elicuisse:

$$a+b-c=2-2pp-2qq-2pq+2ppq+2pqq=2(1-p)(1-q)(1+p+q)$$
$$b+c-a=2p+2pp-2pq-4pqq+2ppq \qquad =2p(1+q)(1+p-2q)$$
$$a+c-b=2q+2qq-2pq-4ppq+2pqq \qquad =2q(1+p)(1+q-2p)$$

ubi cavendum est, ne harum ulla evanescat, quia alioquin triangulum periret, excluduntur ergo sequentes valores:

$$p=0,\ q=0,\ p=\pm 1,\ q=\pm 1,\ p+q=-1,\ q=\frac{p+1}{2},\ p=\frac{p+1}{2}.$$

Praeterea vero etiam excludi oportet $1+p+q+3pq$ ne summa laterum evanescat. Tum vero etiam notetur esse:

$$a-b=q+p+qq-pp+3pqq-3ppq=(q-p)(1+p+q+3pq)$$
$$g-f=3q-3p+3qq-3pp-3pqq+3ppq=3(q-p)(1+p+q-pq)$$

tandem vero est

$$aa+bb+cc=2(1-p-q+p-q+pp-pq+qq)(1+2(p+q)+(p+q)^2+3ppqq)$$

seu $aa+bb+cc-\frac{1}{2}((2-p-q)^2+3(p-q)^2)(1+p+q)^2+3ppqq).$

9. Superest igitur ut tertia conditio impleatur, quae in hac formula continetur:

$$hh=(a+b)^2+(a-b)^2-ac=(a-b)^2+(a+b+c)(a+b-c)$$

ubi si valores modo indicati substituantur, colligitur:

$$hh=(q-p)^2(1+p+q+3pq)^2+4(1-p)(1-q)(1+q)(1+p+q-3pq)$$

quae evolvitur in hanc formam:

$$hh=9ppqq(q-p)^2+6pq(p+q)(pp-4pq+qq)$$
$$+p^4+22p^3q+6ppqq+22pq^3+q^4$$
$$-2(p+q)^3-3(pp+6qq)+4(p+q)+4$$

quae secundum potestates ipsius q disposita fit

$$hh=(1+3p)^2q^4-2(1-11p+9pp+9p^3)q^3$$
$$-3(1+2p-2pp+6p^3-3p^4)q^2$$
$$+2(2-9p-3pp+11p^3+3p^4)q$$
$$+(2+p-pp)^2.$$

10. Alia methodus hanc aequationem resolvendi non patet, nisi ut more solito pro h ejusmodi expressio assumatur, qua substituta valor ipsius q per aequationem simplicem determinetur. Tum vero constat, quomodo uno valore invento ex eo continuo plures elici queant. Ad minores autem valores eruendos, generatim notetur si fuerit

$$hh=AAq^4+2Bq^3+Cqq2Dq+EE$$

sequentibus positionibus negotium confectum iri:

1°. si $h=Aqq+\frac{B}{A}q\pm E$ fit $q=\frac{2A(AD\mp BE}{BB-AA(C\mp 2AE)}$

2°. si $h=\pm Aqq+\frac{D}{E}q+E$ fit $q=\frac{DD-EE(C\mp 2AE)}{2E(BE\mp AD)}$

3°. si $h=Aqq+\frac{B}{A}q+\frac{C}{2A}-\frac{BB}{2A^3}$ fit $q=\frac{(BB-AAC)^2-4A^4EE}{4BA(B(BB-AAC)+2A4D)}$

4°. si $h=\frac{CEE-DD}{2E^3}qq+\frac{D}{E}q+E$ erit $q=\frac{4EE(D(DD-CEE)+2BE^4)}{(DD-CEE)^2-4AAE^4}$

11. Cum autem casus supra exclusi nostrae aequationi sponte satis faciant, et pro hh quadratum producant, ex iis novas formas similes elicere licet, unde deinceps novi valores idonei pro q erui queant.

Sit ergo primo $q = 1 + x$ eritque

$$bb = (1-p+x)^2(2+4p+(1+3p)x)^2 - 4x(1-p)(2+p+x)(2-p+(1-3p)x)$$

quae evoluta praebet hanc formam

$$\begin{aligned} hh = (1+3p)^2 x)^4 &+ 2(1+23p+9pp-9p^3)x^3 \\ &+ (-3+92p+10pp-72p^3+9p^4)xx \\ &+ 4(1-p)(-1+12p+10pp-6p^3)x^3 \\ &+ 4(1-p)^2(1+2p)^2 \end{aligned}$$

tum vero est

$$\begin{aligned} a+b-c &= -2x(1-p)(2+p+x) \\ b+c-a &= -2p(2+x)(1-p+2x) \\ a+c-b &= 2(1+p)(1+x)(2-2p+x). \end{aligned}$$

Praestabit autem quovis casu, quo loco p determinatus valor assumitur, substitutionem in priori forma facere, ac tum denique evolutionem instituere.

12. Sit igitur secundo $q = -1 - p + x$, eritque

$$hh = (1+2p-x)^2(3p(1+p) - 1 + 3p)x)^2 + 4x(1-p)(2+p-x)(3p(1+p) + (1-3p)x)$$

atque

$$\begin{aligned} a+b-c &= 2x(1-p)(2+p-x) \\ b+c-a &= -2p(p-x)(3+3p-2x) \\ a+c-b &= 2(1+p)(1+p-x)(3p-x). \end{aligned}$$

Sit tertio $q = -1 + x$ eritque

$$hh = (1+p-x)^2(2p-(1+3p)x)^2 + 4(1-p(2-x)(p+x(4p-(1-3p)x)$$

atque

$$\begin{aligned} a+b-c &= 2(1-p)(2-x)(p+x) \\ b+c-a &= 2px(3+p-2x) \\ a+c-b &= 2(1+p)(1-x)(2p-x). \end{aligned}$$

Sit quarto $q = \frac{1+p+x}{2}$ eritque

$$16hh = (1-p+x)^2(3(1+p)^2 + (1+3p)x)^2 + 8(1-p)(1-p-x)(3+3p+x)(3(1-pp)+(1-3p)x)$$

atque

$$\begin{aligned} a+b-c &= \tfrac{1}{2}(1-p)(1-p-x)(3(1+p)+x) \\ b+c-a &= -px(3+p+x) \\ a+c-b &= \tfrac{1}{2}(1+p)(1+p+x)(3(1-p)+x). \end{aligned}$$

Sit denique quinto $q = \frac{1+p+x}{3p-1}$ erit

$$(3p-1)^4 hh = ((1-p)(1+3p)+x)^2(6p(1+p)+(1+3p)x)^2 + 4(3p-1)^2 x(1-p)(2(1-p)+x)(3p(1+p)+x)$$

atque

$$\begin{aligned} a+b-c &= \frac{-2(1-p)(2(1-p)+x)(3p(1+p)+x)}{(3p-1)^2} \\ b+c-a &= \frac{-2p(4p+x)(3(1-pp)+2x)}{(3p-1)^2} \\ a+c-b &= \frac{2(1+p)(1+p+x)(6p(1-p)+x)}{(3p-1)^2} \end{aligned}$$

semper autem est

$$f = b - c + (a+b+c)p \text{ et } g = a - c + (a+b+c)q.$$

13. Hinc ergo satis patet innumerabiles solutiones nostri problematis inveniri posse. Invento enim pro q valore quocunque $q = n$, statuatur $q = n + x$, et aequatio resultans iterum hujusmodi formam habebit

$$hh = AAx^4 + 2Bx^3 + Cxx + 2Dx + EE$$

unde novos valores pro x et h eruere licet methodo ante indicata. Cum autem hic potissimum solutiones in minoribus numeris desiderentur, litterae p valores simpliciores tribuamus, unde quidem valores o et ± 1 excludi conveniet.

Casus 1. $p = -2$.

14. Ob $p = -2$, habemus:

$$a = -3 + q = 4qq; \quad f = 1 - 17q - 2qq$$
$$b = -1 + 12q + qq; \quad g = -5 - 2q + 7qq$$
$$c = 2 + q + 3qq; \quad a + b + c = -2 + 14q$$

unde fieri oportet

$$hh = (q+2)^2(5q+1)^2 - 12(q-1)^2(7q-1)$$

quae evoluta abit in hanc formam:

$$25q^4 + 26q^3 + 321qq - 64q + 16 = hh.$$

Hic igitur est A $= 5$, B $= 13$, C $= 321$, D $= -32$, et E $= 4$ ideoque sequentes solutiones nascuntur:

1°. si $h = 5qq + \frac{13}{5}q \pm 4$ fit $q = \frac{10(40 \pm 13)}{1964 \mp 250}$

2°. si $h = \pm 5qq - 8q + 4$ fit $q = \frac{-257 \pm 40}{26 \pm 80}$

ubi tertiam et quartam, quia numeros nimis magnos praebent, omitto.

15. Prioris solutionis signum superius praebet:

$$q = \frac{10.53}{1714} = \frac{5.53}{857},$$

unde nascuntur numeri nimis magni, signum vero inferius

$$q = \frac{10.27}{2214} = \frac{15}{123} = \frac{5}{41} \text{ ergo } h = \frac{-6066}{1641}.$$

Posterioris vero solutionis signum superius dat

$$q = \frac{-217}{105}$$

signum vero inferius:

$$q = \frac{-297}{-54} = \frac{11}{2} \text{ ergo } h = \frac{765}{4}$$

unde etiam reliquas litteras definiamus

$$a = -\frac{237}{2}; \ b = \frac{381}{4}; \ c = \frac{393}{4}$$
$$f = -153; \ g = \frac{783}{4}; \ h = \frac{765}{4}.$$

Hos numeros multiplicemus per 4 ac dividamus per 3 ut obtineamus hanc solutionem satis simplicem:

$$a = 158; \ b = 127; \ c = 131$$
$$f = 204; \ g = 261; \ h = 255$$

et quia litterae f, g, h quae communem habent divisorem 3, in locum litterarum a, b, c substitui possunt, prodibit haec solutio multo simplicior

$$a = 69; \ b = 87; \ c = 85$$
$$f = 158; \ g = 127; \ h = 131$$

unde fit $aa + bb + cc = 2.7.19.73$, qui factores utique sunt numeri formae $xx + 3yy$, uti natura rei postulat.

16. Cum loco q satisfaciat tam $+1$ quam -1, utamur hac substitutione $q = \frac{y+1}{y-1}$ fietque

$$\tfrac{1}{4}(y-1)^4 hh = 81y^4 + 54y^3 - 99yy - 36y + 100$$

unde ob

A $= 9$, B $= 27$, C $= -99$, D $= -18$, E $= 10$

habebimus has resolutiones:

1°. si $\frac{1}{2}(y-1)^2 h = 9yy + 3y \pm 10$, erit $y = \frac{-9(3 \pm 5)}{27(3 \pm 5)} = -\frac{1}{3}$

2°. si $\frac{1}{2}(y-1)^2 h = \pm 9yy - \frac{9}{2}y + 10$ erit $y = \frac{9 + 25(11 \pm 20)}{30(5 \pm 3)}$

quarum prior dat $q = -\frac{1}{2}$, qui est casus exclusus forma $q = \frac{p+1}{2}$; altera vero suppeditat sub signo superiori $y = \frac{784}{240} = \frac{49}{15}$ et $q = \frac{32}{17}$

sub signo inferiori $y = \frac{-24.9}{60} = \frac{-18}{5}$ et $q = \frac{13}{23}$.

17. Sit ergo $y = \frac{49}{15}$ et $q = \frac{32}{17}$ eritque

$$\frac{2.17^2}{15^2} h = \frac{2504}{25} \text{ hinc } h = \frac{9.1252}{289}, \text{ porro}$$

$$a = -3 + \frac{32}{17} - \frac{4096}{289} = -\frac{4419}{289}$$

$$b = -1 + \frac{12.32}{17} + \frac{1024}{289} = \frac{7263}{289}$$

$$c = 2 + \frac{32}{17} + \frac{3072}{289} = \frac{4194}{289}$$

$$f = 1 - \frac{17.32}{17} - \frac{2048}{289} = -\frac{11007}{298}$$

$$g = -5 - \frac{2.32}{17} + \frac{7.1024}{289} = \frac{4635}{289}.$$

Omnes hi valores per 289 multiplicati per 9 deprimantur, et habebitur ista solutio

$a = 491$; $b = 807$; $c = 466$

$f = 1223$; $g = 515$; $h = 1252$

quae eadem resultat ex altero casu invento $q = \frac{13}{23}$, unde est $aa + bb + cc = 2.7.19.43.97$.

Casus 2. $p = 2$.

18. Pro hoc ergo casu primo habemus:

$a+b-c = 2-(1-q)(3+q)$; $b-c+f = \frac{b+c-a}{2}$

$b+c-a = 4(1+q)(3-2q)$;

$a+c-b = -6q(3-q)$; $a-c+g = \frac{a+c-b}{q}$

unde fit

$$hh = (q-2)^2(7q+3)^2 - 4(1-q)(3+q)(3-5q)$$

quae evoluta praebet hanc formam:

$$hh = 49q^4 - 174q^3 + 99qq + 216q$$

haecque facto $q = 3r$ transit in hanc simpliciorem

$$\tfrac{1}{81} hh = 49r^4 - 58r^3 + rr + 8r$$

casus autem excludendi sunt $q = \pm 1$; $q = -3$; $q = \frac{5}{2}$; $q = 3$ et $q = \frac{3}{5}$.

19. Cum hic sit $A = 7$, $B = -29$, $C = 1$, $D = 4$, $E = 0$ erit ex solutione prima sumto $\frac{1}{9} h = 7rr - \frac{29}{7} r$

$r - \frac{14.28}{841-9} = \frac{49}{99}$ et $q = \frac{49}{33}$, hincque $h = -\frac{7.470}{11.98}$

tum vero porro

$a + b - c = \frac{2.16.148}{33.33}$; $b - c + f = \frac{4.41}{33.33}$

$b + c - a = \frac{4.82.1}{33.33}$; $a - c + g = -\frac{6.50}{33}$

$a + c - b = -\frac{6.49.50}{33.33}$

multiplicentur hi valores omnes per $\frac{33.33}{4}$ erit

$a+b-c = 1184$; $2c = -3593$; $2f = -4777$

$b+c-a = 82$; $2a = -2491$; $2g = -6092$

$a+c-b = -3675$; $2b = +1266$; $2h = -1645$

$a+b+c = -2409$.

Duplicatis ergo valoribus prodit haec solutio

$a = 2491$; $b = 1266$; $c = 3593$

$f = 4777$; $g = 6092$; $h = 1645$

hinc vero est $aa + bb + cc = 2.19.31.43.409$.

20. Transformemus aequationem nostram ponendo $r = \frac{y+1}{y-1}$ orieturque

$$\tfrac{1}{324} hh(y-1)^4 = 25 + 82y + 73yy + 16y^3$$

statuatur $\frac{1}{18}h(y-1)^2 = 5+\frac{41}{5}y$, fitque $y = -\frac{8}{25}$ hinc $r = -\frac{8}{17}$ et $q = -\frac{24}{17}$; ideoque $h = \frac{45.128}{289}$; Porro

$a+b-c = -\frac{2.41.27}{289}$; $b-c+f = -\frac{2.7.99}{289}$

$b+c-a = -\frac{4.7.99}{289}$; $a-c+g = -\frac{6.17.75}{289}$

$a+c-b = \frac{6.24.75}{289}$.

Multiplicentur omnes hi valores per $\frac{289}{9}$ et habebitur

$a+b-c = -246$; $2c = 892$; $h = 640$

$b+c-a = -308$; $2a = 954$; $b-c+f = -154$; $f = 569$

$a+c-b = 1200$; $2b = 554$; $a-c+g = -850$; $g = 881$

$a+b+c = 1646$

unde colligitur haec solutio:

$a = 477$; $b = 277$; $c = 446$

$f = 569$; $g = 881$; $h = 640$.

21. In aequatione per q expressa statuatur $q = \frac{y+1}{y-1}$ ac reperietur

$$\tfrac{1}{4}hh(y-1)^4 = 25y^4 - 146y^3 + 69yy + 244y + 4$$

ubi A = 5, B = −73, C = 69, D = 122, E = 2. Ergo

1°. si $\frac{1}{2}h(y-1)^2 = 5yy - \frac{73}{5}y \pm 2$ fit

$$y = \frac{10(610 \pm 146)}{73^2 - 25(69 \mp 20)} = \frac{5(305 \pm 73)}{901 \pm 125}$$

2°. si $\frac{1}{2}h(y-1)^2 = 5yy + 61y + 2$ fit

$$y = \frac{4.61^2 - 4(69 \mp 20)}{4(-146 \mp 610)} = \frac{1826 \pm 10}{-73 \mp 305}$$

Ex priori dat signum sup. $y = \frac{5.378}{1026} = \frac{35}{19}$ et $q = \frac{27}{8}$ at signum inf. $y = \frac{5.232}{776} = \frac{145}{97}$ et $q = \frac{121}{24}$.

Ex secunda dat signum sup. $y = \frac{1836}{-378} = -\frac{34}{7}$ et $q = \frac{27}{41}$ at signum inf. $y = \frac{1816}{232} = \frac{227}{29}$ et $q = \frac{128}{99}$.

Ex valore $q = \frac{27}{8}$ colligimus hanc solutionem:

$a = 404$; $b = 377$; $c = 619$

$f = 3.314$; $g = 3.325$; $h = 3.159$

ubi fit $aa + bb + cc = 2.3.7.13^2.97$.

Ex valore autem $q = \frac{27}{41}$ nascitur ista solutio

$a = 134$; $b = 823$; $c = 607$

$f = 3.480$; $g = 3.103$; $h = 3.337$

ubi est $aa + bb + cc = 2.3.7.19.31.43$ notandumque est hic bina latera tertio non esse majora.

22. Pluribus casibus involvendis hic non immoror, sed potius animadverto, methodum qua hic sum usus, non satis videri naturalem et ad scopum accommodatam, propterea quod nulla suppeditat criteria solutiones simpliciores distinguendi. Desideratur ergo tam pro hoc problemate, quam pro aliis similibus, quarum solutio ad hujusmodi formam

$$Ax^4 + Bx^3 + Cx^2 + Dx + E$$

ad quadratum reducendam, revocatur. Atque in hoc quidem problemate solutio a quantitate $aa + bb + cc$ inchoanda videtur, quae hujusmodi numero $2(xx + 3yy)$ certe est aequalis; et cum debeat esse

$$4(xx + 3yy) = ff + 3aa = gg + 3bb = hh + 3cc$$

evidens est numerum $xx + 3yy$ factores habere debere, quos constat ejusdem esse formae. Statui igitur poterit

$xx+3yy=(mm+3nn)(pp+3qq)(rr+3ss)$ et $4(xx+3yy)$ octo modis ad formam $AA + 3BB$ referri potest, unde ternas illas eligi oportet. Foret nempe

$$a = 2m(ps+qr)+2n(3qs-pr)$$
$$b = m((3q+p)s \pm (q-p)r) \pm n(3(q-p)s \mp (3q+p)r)$$
$$c = n((3q-p)s \pm (q+q)r) \pm n(3(q+p)s \mp (3q-p)r)$$

et effici restat $aa + bb + cc = 4(xx + 3yy)$. Verum hoc modo calculus fit satis prolixus, nisi forte certis artificiis tractabilior reddi potest.

RESOLUTIO AEQUATIONIS

$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$

PER NUMEROS TAM RATIONALES, QUAM INTEGROS.

Auctore

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I.

Haec forma latissime patens, quae insignem partem Analyseos Diophanteae complectitur, pro varia indole numerorum A, B, C, D, E, F plures in se continet casus, qui vulgo diversis methodis tractari solent. Hic autem singulari modo ejus resolutionem sine radicis extractione ita docebo, ut solutio non solum ad numeros rationales, sed etiam integros accommodari possit.

1. In genere quidem resolutionem hujus aequationis tradere non licet, quia saepe usu venire potest, ut ea sit impossibilis, certissimum autem criterium possibilitatis solutionis sine dubio est, si unicus saltem casus, quo huic aequationi satisfiat, fuerit cognitus. Ponamus igitur hoc contingere casu quo $x = a$ et $y = b$, ita ut revera sit:

$$Aa^2 + 2Bab + Cb^2 + 2Da + 2Eb + F = 0$$

et quemadmodum ex hoc casu cognito, alii sive