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Nova series infinita maxime convergens perimetrum ellipsis exprimens

Leonhard Euler

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NOVA SERIES INFINITA
 MAXIME CONVERGENS
 PERIMETRV M ELLIPSIS
 EXPRIMENS.

Auctore

L. E V L E R O.

§. 1.

Postquam olim multum fuisset occupatus, ut plures series infinitas, quibus cuiusque ellipsis perimenter exprimeretur, inuestigarem, vix eram suspicatus adhuc simpliciores atque ad calculum magis accommodatas huiusmodi series erui posse, quam passim dedi, siue in Comment. Petrop. siue in Actis Berolin.

§. 2. Nunc autem cum forte cogitationes meae in idem argumentum inciderent, alia ac, ni fallor, multo simplicior et commodior series se mihi obtulit, cuius inuestigationem ita animo institui.

Considero scilicet quadrantem ellipticum ACB, cuius semiaxes sint CA = a; CB = b, quibus coordinatae parallelae vocentur CP = x; et PM = y, ita, ut ex natura ellipsis habeatur ista aequatio

$$bbx^2 + aay^2 = aa.bb$$

siue $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Tab. I.
 Fig. 1.

ex qua singulari modo definitio longitudinem totius arcus A M B siue quartae partis perimetri.

§. 3. Cum igitur esse debeat

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

nouam variabilem z in calculum introduco, statuen-
do $\frac{x^2}{a^2} = \frac{1+z}{2}$; vt fiat $\frac{y^2}{b^2} = \frac{1-z}{2}$; vnde prodit
 $x = a \sqrt{\frac{1+z}{2}}$ et $y = b \sqrt{\frac{1-z}{2}}$, hincque differen-
tando

$$dx = \frac{a dz}{2\sqrt{2(1+z)}}; \text{ et } dy = \frac{-b dz}{2\sqrt{2(1-z)}}$$

ex quo si vocemus arcum B M = s , statim colli-
gimus

$$ds^2 = dx^2 + dy^2 = \frac{a^2 dz^2}{2(1+z)} + \frac{b^2 dz^2}{2(1-z)}$$

siue $ds^2 = \frac{dz^2}{2} \left(\frac{a^2}{1+z} + \frac{b^2}{1-z} \right)$
 $= \frac{dz^2 (a^2 + b^2 - (a^2 - b^2)z)}{2(1-z^2)}$

hincque integrando

$$s = \frac{1}{2\sqrt{2}} \int dz \sqrt{\frac{a^2 + b^2 - (a^2 - b^2)z}{1-z^2}}$$

integrali ita sumto, vt euanescat posito $x = 0$ siue
 $z = -1$ tum vero integrale extendatur vsque ad
terminum $x = a$, vbi fit $z = +1$ sicque obtinebi-
tur quaesitus quadrans ellipticus A M B.

§. 4. Quo hanc formulam tractabiliorem red-
damus, ponamus breuitatis gratia

$$a^2 + b^2 = c^2; \text{ et } \frac{a^2 - b^2}{a^2 + b^2} = n.$$

Hoc enim modo consequimur

$$s = \frac{c}{2\sqrt{2}} \int dz \sqrt{\frac{(1-nz)}{(1-z^2)}}$$

vbi

vbi superius radicale more solito in seriem convertamus :

$$\sqrt{1-nz} = 1 - \frac{1}{2}nz - \frac{1 \cdot 1}{2 \cdot 4} n^2 z^2 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} n^3 z^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} n^4 z^4 - \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} n^5 z^5 \text{ etc.}$$

qui singuli termini nos ad singulares integrationes perducunt; ac bini quidem priores secundum legem datam integrati, vt scilicet euanescent sumto $z = -1$ dabunt

$$\int \frac{dz}{\sqrt{1-z^2}} = A. \text{fin. } z - A. \text{fin. } (-1) = A. \text{fin. } z + \frac{1}{2}\pi$$

$$\int \frac{z dz}{\sqrt{1-z^2}} = -\sqrt{1-z^2} + 0$$

hinc ergo si sumamus $z = +1$ prodibit

$$\int \frac{dz}{\sqrt{1-z^2}} = \pi \text{ et } \int \frac{z dz}{\sqrt{1-z^2}} = 0.$$

§. 5. Pro reliquis terminis consideremus reductionem consuetam generalem

$$\int \frac{z^{\lambda+2} dz}{\sqrt{1-z^2}} = A. \int \frac{z^{\lambda} dz}{\sqrt{1-z^2}} + B. z^{\lambda+1} \sqrt{1-z^2}$$

vbi esse oportet

$$A = + \frac{\lambda+1}{\lambda+2}; \text{ et } B = \frac{-1}{\lambda+2}$$

ita, vt fit

$$\int \frac{z^{\lambda+2} dz}{\sqrt{1-z^2}} = \frac{\lambda+1}{\lambda+2} \int \frac{z^{\lambda} dz}{\sqrt{1-z^2}} - \frac{1}{\lambda+2} z^{\lambda+1} \sqrt{1-z^2}$$

vbi constantem non adiciamus, quia haec formula iam euanescit sumto $z = -1$; vnde si iam ponatur $z = +1$ obtinebitur

$$\int \frac{z^{\lambda+2} dz}{\sqrt{1-z^2}} = \frac{\lambda+1}{\lambda+2} \int \frac{z^{\lambda} dz}{\sqrt{1-z^2}}$$

§. 6. Ex hac reductione statim liquet, omnia integralia ex potestatibus imparibus ipsius z oriunda per se evanescere; pro potestatibus autem paribus ad scopum nostrum adipiscimur

$$\int \frac{dz}{\sqrt{(1-z^2)}} = \pi; \int \frac{z^2 dz}{\sqrt{(1-z^2)}} = \frac{1}{2} \pi;$$

$$\int \frac{z^4 dz}{\sqrt{(1-z^2)}} = \frac{1 \cdot 3}{2 \cdot 4} \pi; \int \frac{z^6 dz}{\sqrt{(1-z^2)}} = \frac{1 \cdot 3 \cdot 5}{3 \cdot 4 \cdot 6} \pi$$

etc.

§. 7. His igitur valoribus substitutis longitudo quadrantis elliptici colligitur fore

$$AMB = \frac{c\pi}{2\sqrt{2}} \left\{ 1 - \frac{1 \cdot 1}{2 \cdot 4} n^2 \cdot \frac{1}{2} - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} n^4 \cdot \frac{1 \cdot 5}{2 \cdot 4} \right.$$

$$\left. - \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12} n^6 \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} - \text{etc.} \right.$$

Pro hac autem forma scribamus tanisper breuitatis gratia

$$AMB = \frac{c\pi}{2\sqrt{2}} [1 - \alpha n^2 - \beta n^4 - \gamma n^6 - \delta n^8 - \epsilon n^{10} \text{ etc.}]$$

qui coefficientas sequenti modo succinctius exprimi poterunt

$$\alpha = \frac{1 \cdot 1}{2 \cdot 4} \cdot \frac{1}{2} = \frac{1 \cdot 1}{4 \cdot 4}$$

$$\frac{\beta}{2} = \frac{3 \cdot 5}{2 \cdot 4}; \frac{\gamma}{\beta} = \frac{7 \cdot 9}{12 \cdot 12}; \frac{\delta}{\gamma} = \frac{11 \cdot 13}{16 \cdot 16}$$

§. 8. Cum igitur inuenti coefficientes tam simplicem et egregiam constituent seriem, haec expressio, quam eruimus, utique maxime videtur attentione digna, cum termini vehementer conuergant idque pro omnibus plane ellipsis, propterea quod semper $\frac{a^2 - b^2}{a^2 + b^2} = n$ fractio est unitate minor. Habebimus scilicet

AMB

$$A M B = \frac{\pi a}{2\sqrt{2}} \left\{ \begin{aligned} &1 - \frac{1 \cdot 1}{4 \cdot 4} n^2 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{4 \cdot 4 \cdot 8 \cdot 8} n^4 \\ &- \frac{1 \cdot 3 \cdot 5 \cdot 7}{4 \cdot 4 \cdot 8 \cdot 8 \cdot 12 \cdot 12} n^6 \\ &- \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13}{4 \cdot 4 \cdot 8 \cdot 8 \cdot 12 \cdot 12 \cdot 16 \cdot 16} n^8 \\ &\text{etc.} \end{aligned} \right.$$

§. 9. Contemplemur hinc casum, quo ellipsis nostra fit circulus radii = a; tum enim erit b = a hinc c = a√2 et n = 0, ex quo quadrans circularis prodit, uti quidem notissimum est, = $\frac{1}{2} \pi a$.

§. 10. Deinde vero etiam casus occurrit maxime notatu dignus, quo semiaxis CB = b = 0; tum enim quadrans ellipticus PMB ipsi semiaxi CA = a fit aequalis; at pro nostra formula erit c = a et n = 1 quibus valoribus substitutis nanciscimur sequentem aequationem

$$a = \frac{\pi a}{2\sqrt{2}} \left\{ \begin{aligned} &1 - \frac{1 \cdot 1}{4 \cdot 4} - \frac{1 \cdot 1 \cdot 3 \cdot 5}{4 \cdot 4 \cdot 8 \cdot 8} \\ &- \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{4 \cdot 4 \cdot 8 \cdot 8 \cdot 12 \cdot 12} \text{ etc.} \end{aligned} \right.$$

qui praecise ipse ille casus est, quo series nostra quam minime est conuergens, et qui propterea nostram attentionem eo magis meretur, quod huius seriei summa adcurate assignari potest, cum fit

$$1 - \frac{1 \cdot 1}{4 \cdot 4} - \frac{1 \cdot 1 \cdot 3 \cdot 5}{4 \cdot 4 \cdot 8 \cdot 8} \text{ etc. in infin.} = \frac{2\sqrt{2}}{\pi}$$

§. 10. Si cui lubuerit super hac serie calculos numericos instituire, subiungamus hic valores litterarum a, β, γ etc. in fractionibus decimalibus, qui ita se habent

K 2

a = 0,

A M B

$$\alpha = 0,0625000$$

$$\beta = 0,0146484$$

$$\gamma = 0,0064087$$

$$\delta = 0,0035798$$

$$\varepsilon = 0,0022821$$

$$\zeta = 0,0015808$$

etc.

serie autem hucusque tantum continuata prodit $1 - \alpha - \beta - \gamma - \delta - \varepsilon - \zeta = 0,9090002$; iam vero reperitur $\frac{2\sqrt{2}}{\pi} = 0,9003200$; vnde videmus, sequentium litterarum $\eta, \vartheta, \iota, \kappa$ etc. omnium summam efficere debere $0,0086802$.

§. 11. Ceterum pro calculo numerico non parum notasse iuuabit, nostros coefficients etiam sequenti modo concinnius exprimi posse

$$\alpha = \frac{1}{16}$$

$$\beta = \frac{1}{64} \cdot \frac{15}{16}$$

$$\gamma = \frac{1}{144} \cdot \frac{15}{16} \cdot \frac{63}{64}$$

$$\delta = \frac{1}{256} \cdot \frac{15}{16} \cdot \frac{63}{64} \cdot \frac{143}{144}$$

$$\varepsilon = \frac{1}{4096} \cdot \frac{15}{16} \cdot \frac{63}{64} \cdot \frac{143}{144} \cdot \frac{255}{256}$$

etc.

§. 12. Occasione huius seriei, quam inuenimus, operae pretium erit, in eius summam a posteriore inquirere, id quod duplici modo fieri potest; prior modus, quem iam olim proposui ac deinceps saepissime ad usum accommodaui, nos deducit ad aequationem differentialem, cuius integrale per ipsam seriem propositam exprimatur. Quo nunc

haec

haec methodus facilius adhiberi queat, ponamus $n = 2v$, ut series summanda fiat

$$s = 1 - \frac{1 \cdot 1}{2 \cdot 2} v^2 - \frac{1 \cdot 1}{2 \cdot 2} \frac{3 \cdot 5}{4 \cdot 4} v^4 - \frac{1 \cdot 1}{2 \cdot 2} \frac{3 \cdot 5}{4 \cdot 4} \frac{7 \cdot 9}{6 \cdot 6} v^6 \text{ etc.}$$

§. 13. Differentiemus hanc seriem, tum vero iterum per v multiplicemus, ut prodeat

$$\frac{v ds}{dv} = -\frac{1 \cdot 1}{2} v^2 - \frac{1 \cdot 1}{2 \cdot 2} \frac{3 \cdot 5}{4} v^4 - \frac{1 \cdot 1}{2 \cdot 2} \frac{3 \cdot 5}{4 \cdot 4} \frac{7 \cdot 9}{6} v^6 \text{ etc.}$$

quae denuo differentiata praebet

$$\frac{d \cdot v ds}{dv^2} = -1 \cdot 1 v - \frac{1 \cdot 1}{2 \cdot 2} 3 \cdot 5 v^3 - \frac{1 \cdot 1}{2 \cdot 2} \frac{3 \cdot 5}{4 \cdot 4} 7 \cdot 9 v^5 \text{ etc.}$$

hoc scilicet modo ex singulis denominatoribus duos factores sustulimus.

§. 14. Nunc vero denuo ope differentiationis numeratores binis novis factoribus augeamus; hunc in finem primam aequationem in \sqrt{v} ductam differentiemus, prodibitque

$$\begin{aligned} \frac{d \cdot s \sqrt{v}}{dv} &= + v^{-\frac{1}{2}} - \frac{1 \cdot 1}{2 \cdot 2} 5 \cdot v^{\frac{3}{2}} \\ &\quad - \frac{1 \cdot 1}{2 \cdot 2} \frac{3 \cdot 5}{4 \cdot 4} 9 \cdot v^{\frac{5}{2}} \\ &\quad - \frac{1 \cdot 1}{2 \cdot 2} \frac{3 \cdot 5}{4 \cdot 4} \frac{7 \cdot 9}{6 \cdot 6} 13 \cdot v^{\frac{7}{2}} \\ &\quad \text{etc.} \end{aligned}$$

haec denuo differentietur et per v iterum multiplicando fit

$$\begin{aligned} \frac{d \cdot d \cdot s \sqrt{v}}{dv^2} &= -v^{-\frac{3}{2}} - \frac{1 \cdot 1}{2 \cdot 2} 3 \cdot 5 \cdot v^{\frac{1}{2}} \\ &\quad - \frac{1 \cdot 1}{2 \cdot 2} \frac{3 \cdot 5}{4 \cdot 4} 7 \cdot 9 \cdot v^{\frac{3}{2}} \\ &\quad \text{etc.} \end{aligned}$$

K 3

quæ

quae per $v^{\frac{5}{2}}$ multiplicata producit

$$\frac{4 \cdot v^{\frac{5}{2}} d d . s \sqrt{v}}{d v^2} = -v - \frac{1 \cdot 1}{2 \cdot 2} 3 \cdot 5 \cdot v^{\frac{3}{2}} \\ - \frac{1 \cdot 1}{2 \cdot 2} \frac{3 \cdot 5}{4 \cdot 4} 7 \cdot 9 \cdot v^{\frac{1}{2}} \text{ etc.}$$

supra vero iam inuenimus

$$\frac{d \cdot v d s}{d v^2} = -v - \frac{1 \cdot 1}{2 \cdot 2} 3 \cdot 5 \cdot v^{\frac{3}{2}} - \frac{1 \cdot 1}{2 \cdot 2} \frac{3 \cdot 5}{4 \cdot 4} 7 \cdot 9 \cdot v^{\frac{1}{2}} \text{ etc.}$$

quae series cum sint aequales, inde deducimus hanc aequationem

$$4 \cdot v^{\frac{5}{2}} d d . s \sqrt{v} = d \cdot v d s$$

quae aequatio continet relationem summae quaesitae s ad variabilem v .

§. 15. Haec ergo aequatio euoluta fit differentiale 2^{di} gradus; sumto enim elemento $d v$ constante ob

$$d \cdot s \sqrt{v} = d s \sqrt{v} + \frac{s d v}{2 \sqrt{v}}; \text{ erit } d d . s \sqrt{v} = d d s \cdot \sqrt{v} \\ + \frac{d v d s}{\sqrt{v}} - \frac{s d v^2}{4 v \sqrt{v}}; \text{ ergo}$$

$$4 v^{\frac{5}{2}} d d . s \sqrt{v} = 4 v^{\frac{5}{2}} d d s + 4 v^{\frac{3}{2}} d v d s - v d v^2$$

tum vero ob $d \cdot v d s = v d d s + d v d s$, habebitur haec aequatio

$$v d d s (1 - 4 v^2) + d v d s (1 - 4 v^2) + s v d v^2 = 0$$

sive

$$v d d s + d v d s + \frac{s v d v^2}{1 - 4 v^2} = 0.$$

§. 16. Huius igitur aequationis differentialis secundi gradus constructio in nostra est potestate; fiat

fiat enim ellipsis, cuius semiaxes sint a et b ; eiusque peripheriae quarta pars $= q = A M B$; tum vero capiatur $c = \sqrt{a^2 + b^2}$ et $\frac{a^2 - b^2}{a^2 + b^2} = n = 2v$; unde cum sit

$$q = \frac{\pi c}{2\sqrt{2}} s; \text{ fiet } s = \frac{2q\sqrt{2}}{\pi c}.$$

Iam ob

$$a^2 + b^2 = c^2 \text{ et } a^2 - b^2 = 2c^2 v$$

$$\text{crit } a^2 = \frac{c^2(1+2v)}{2}; \text{ et } b^2 = \frac{c^2(1-2v)}{2}.$$

Quocirca nostra constructio ita erit comparata, sumtis ellipsis semiaxibus

$$a = c \sqrt{\frac{1+2v}{2}} \text{ et } b = c \sqrt{\frac{1-2v}{2}}$$

fit q quarta pars perimetri huius ellipsis eritque pro resolutione nostrae aequationis $s = \frac{2q\sqrt{2}}{\pi c}$.

Haec aequatio si ponamus $s = \frac{z}{\sqrt{v}}$, induet hanc formam simpliciore

$$d d z + \frac{z d v^2}{4 v^2 (1 - 4 v^2)} = 0$$

pro qua erit $z = \frac{2q\sqrt{2}v}{\pi c}$.

§. 17. Haec porro aequatio ad differentialem primi gradus reducetur, ponendo $z = e^{t d v}$ tum enim resultabit

$$d t + t^2 d v + \frac{d v}{4 v^2 (1 - 4 v^2)} = 0$$

unde, si liceret t per v definire, ita, vt innotesceret integrale $\int t d v$, foret $z = e^{t d v}$.

§. 18. Hic erat primus modus ex proposita serie infinita in eius summam inquirendi, vbi scilicet loco numeri constantis n quantitatem variabilem v in-

ut introduximus; altero autem modo idem praestandi, cuius plurima specimina iam passim occurrunt, quantitas constans n talis relinquitur, ponamus autem $n = 2m$; ita, ut nostra series summanda sit

$$1 - \frac{1 \cdot 1}{2 \cdot 2} m^2 - \frac{1 \cdot 1}{2 \cdot 2} \cdot \frac{2 \cdot 5}{4 \cdot 4} m^4 \\ - \frac{1 \cdot 1}{2 \cdot 2} \cdot \frac{2 \cdot 5}{4 \cdot 4} \cdot \frac{2 \cdot 9}{6 \cdot 6} m^6 \text{ etc.}$$

§. 19. Nunc fingamus esse

$$s = f. dz \sqrt{1 - 2m^2 p}$$

postquam scilicet absoluta integratione quantitati variabili z certus valor determinatus fuerit tributus, litteram vero p etiam ut variabilem spectemus, quae cuiusmodi functio ipsius z capi debeat, ut haec integratio ad nostram seriem infinitam perducatur, sequenti modo inuestigabimus.

§. 20. Euoluta autem formula irrationali

$(1 - 2m^2 p)^{\frac{1}{2}}$ in hanc seriem infinitam

$$1 - \frac{1}{2} m^2 p - \frac{1 \cdot 3}{2 \cdot 4} m^4 p^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} m^6 p^3$$

quantitas s sequenti serie formularum integralium definietur

$$s = z - \frac{1}{2} m^2 \int p dz - \frac{1 \cdot 3}{2 \cdot 4} m^4 \int p^2 dz \\ - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} m^6 \int p^3 dz \text{ etc.}$$

Nunc vero statuamus, si post singulas integrationes variabili z certus valor determinatus tribuatur, tum fore

$$\int p dz = \frac{1}{2} z; \int p^2 dz = \frac{2}{3} \int p dz \\ \int p^3 dz = \frac{3}{2} \int p^2 dz; \int p^4 dz = \frac{12}{5} \int p^3 dz \\ \text{etc.}$$

sic

sic enim fiet

$$s = z \left(1 - \frac{1 \cdot 1}{2 \cdot 2} m^2 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4} m^4 - \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} m^6 \text{ etc.} \right)$$

quae est ipsa nostra series proposita.

§. 21. Nunc igitur tota quaestio huc redit, cuiusmodi functionem ipsius z pro p sumi oporteat, ut stabilita illa ratio integralium, dum scilicet variabili z certus valor tribuitur, obtineatur, ista autem relatio generatim ita exprimitur,

$$\int p^\lambda dz = \frac{4\lambda - 3}{2\lambda} \int p^{\lambda-1} dz$$

ponamus igitur integralibus adhuc indefinite sumtis fore

$$\int p^\lambda dz = \frac{4\lambda - 3}{2\lambda} \int p^{\lambda-1} dz + \frac{p^\lambda Q}{2\lambda}$$

facta ergo differentiatione prodibit

$$p^\lambda dz = \frac{4\lambda - 3}{2\lambda} p^{\lambda-1} dz + \frac{1}{2} p^{\lambda-1} Q dp + \frac{p^\lambda}{2\lambda} dQ$$

quae per $p^{\lambda-1}$ diuisa et per 2λ multiplicata praebet

$$2\lambda p dz = (4\lambda - 3) dz + \lambda Q dp + p dQ$$

et cum haec aequatio subsistere debeat, quicquid sit λ , suppeditat nobis has duas aequationes

$$2p dz - 3 dz - Q dp = 0$$

$$-3 dz + p dQ = 0$$

ex quibus utramque functionem p et Q definire licebit.

§. 22. Perinde autem hic est, siue p et Q sint functiones ipsius z siue z et Q ipsius p , dummodo

modo earum relatio inter se stabiliatur; ex posteriore autem statim habemus

$$dz = \frac{1}{2} p dQ$$

qui valor in priore substitutus praebet

$$\frac{1}{2}(p-2)p dQ - Q dp = 0$$

ex qua fit

$$\frac{dQ}{Q} = \frac{p dp}{2p(p-2)} = -\frac{1}{4} \frac{dp}{p} + \frac{1}{4} \frac{dp}{p-2}$$

unde integrando oritur

$$\begin{aligned} \log. Q &= -\frac{1}{4} l. p + \frac{1}{4} l. (p-2) \\ &= +\frac{1}{4} l. \frac{p-2}{p} \end{aligned}$$

unde fit $Q = 2 \left(\frac{p-2}{p}\right)^{\frac{1}{2}}$

tum vero quia ex prima aequatione est $dz = \frac{Q dp}{2(p-2)}$; hinc fit

$$dz = \frac{dp}{p^{\frac{3}{2}}(p-2)^{\frac{3}{2}}} = \frac{dp}{\sqrt{p^3(p-2)}}$$

Nunc autem imprimis obseruari oportet, ut pro utroque integrationis termino formula algebraica ibi adiecta $p^\lambda Q = 2 \cdot p^{\lambda - \frac{1}{2}} (p-2)^{\frac{1}{2}}$ euanescat, sicque manifestum est, integrationis terminos statui debere $p = 0$ et $p = 2$.

§. 23. Ecce ergo formulam nostram integram initio introductam hoc modo repraesentatam

$$s = \int \frac{dp \sqrt{(1-2m^2 p)}}{\sqrt{p^3(p-2)}}$$

quare

quare cum fit

$$z = \int \frac{dp}{\sqrt{p^3(p-2)}}$$

ipsa nostra series proposita

$$1 - \frac{1 \cdot 1}{2 \cdot 2} m^2 - \frac{1 \cdot 1}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} m^4 \text{ etc.}$$

æquabitur fractioni $\frac{s}{z}$, postquam scilicet haec integralia ita fuerint sumpta, ut evanescant posito $p=0$ cum vero statuatur $p=2$; quamobrem illas duas formulas integrales ita exprimi contineat

$$s = \int \frac{dp \sqrt{1-2m^2p}}{\sqrt{p^3(2-p)}}$$

$$\text{et } z = \int \frac{dp}{\sqrt{p^3(2-p)}}$$

§. 24. Ex his igitur series nostra supra inventa

$$1 - \frac{1 \cdot 1}{4 \cdot 4} n^2 - \frac{1 \cdot 1}{4 \cdot 4} \cdot \frac{3 \cdot 5}{8 \cdot 8} n^4 \text{ etc.}$$

cuius summam iam vidimus esse $\frac{2q\sqrt{x}}{\pi \cdot c}$, etiam hoc modo per duas formulas integrales repraesentari potest, quae facta levi mutatione $p=2r$ erunt, ea, quae numeratorem constituit

$$s = \int \frac{dr \sqrt{1-nn \cdot r}}{\sqrt{r^3(1-r)}}$$

altera vero, quae constituit denominatorem

$$z = \int \frac{dr}{\sqrt{r^3(1-r)}}$$

L 2

ipsa

ipsa autem fractio nostram seriem exhibebit; nunc autem termini integrationis sunt $t = 0$ et $t = 1$.

§. 25. Adhuc succinctius hae formulae transformari possunt, sumendo $t = t'$; tum enim ambae formulae integrales erunt

$$s = \int \frac{dt \sqrt{(1-n^2 t^2)}}{\sqrt{(1-t^2)}}; \text{ et } z = \int \frac{dt}{\sqrt{(1-t^2)}}$$

terminis integrationis existentibus etiam nunc $t = 0$ et $t = 1$, quo observato fractio $\frac{s}{z}$ aequabitur nostrae seriei, siue erit

$$\frac{s}{z} = \frac{2q\sqrt{2}}{\pi \cdot c}$$

vbi q denotat quartam partem peripheriae ellipsis, cuius semiaxes sunt

$$c \sqrt{\frac{1+n}{2}} \text{ et } c \sqrt{\frac{1-n}{2}}.$$

§. 26. Hinc casu $n = 0$ manifesto fit $\frac{s}{z} = 1$ casu vero $n = 1$ ob $s = t = 1$ fiet

$$\frac{1}{z} = \frac{2\sqrt{2}}{\pi} \text{ siue } z = \int \frac{dt}{\sqrt{(1-t^2)}} = \frac{\pi}{2\sqrt{2}};$$

quod quidem iam aliunde constat.