Summatio progressionum sin(ÏÎ») + sin(2ÏÎ») + sin(3ÏÎ») + ... + sin(nÏÎ»); cos(ÏÎ») + cos(2ÏÎ») + cos(3ÏÎ») + ... + cos(nÏÎ»)

Leonhard Euler

SVMMATIO PROGRESSIONVM

$\sin.\Phi^\lambda+\sin.2\Phi^\lambda+\sin.3\Phi^\lambda+\ldots\ldots+\sin.n\Phi^\lambda$
$\cos.\Phi^\lambda+\cos.2\Phi^\lambda+\cos.3\Phi^\lambda+\ldots\ldots+\cos.n\Phi^\lambda.$

Auctore

L. EVLERO.

§. 1.

Posito $\cos.\Phi+\sqrt{-1}\sin.\Phi=p$ et $\cos.\Phi-\sqrt{-1}\sin.\Phi=q$ notum est fore

$$\cos.n\Phi=\frac{p^n+q^n}{2} \text{ et } \sin.n\Phi=\frac{p^n-q^n}{2\sqrt{-1}}$$

tum vero etiam esse $pq=1$. His positis euidens est, summationem harum serierum semper reduci posse ad has duas series vel progressiones geometricas

$$p^\alpha+p^{2\alpha}+p^{3\alpha}+\ldots\ldots+p^{n\alpha}=\frac{p^{(n+1)\alpha}-p^\alpha}{p^\alpha-1}=\frac{p^\alpha(1-p^{n\alpha})}{1-p^\alpha}$$

$$q^\alpha+q^{2\alpha}+q^{3\alpha}+\ldots\ldots+q^{n\alpha}=\frac{q^{(n+1)\alpha}-q^\alpha}{q^\alpha-1}=\frac{q^\alpha(1-q^{n\alpha})}{1-q^\alpha}$$

§ 2. Quod si iam hae duae progressiones invicem addantur, vt prodeat ista

$$\begin{matrix} p^\alpha+p^{2\alpha}+p^{3\alpha}+\ldots\ldots+p^{n\alpha} \\ +q^\alpha+q^{2\alpha}+q^{3\alpha}+\ldots\ldots+q^{n\alpha} \end{matrix} \text{ eius summa erit } \begin{matrix} \dfrac{p^\alpha-p^{(n+1)\alpha}}{1-p^\alpha} \\ +\dfrac{q^\alpha-q^{(n+1)\alpha}}{1-q^\alpha} \end{matrix}$$

=

$$= \frac{p^\alpha - p^{(n+1)\alpha} - p^\alpha q^\alpha + p^{(n+1)\alpha} q^\alpha + q^\alpha - q^{(n+1)\alpha} - p^\alpha q^\alpha + p^\alpha q^{(n+1)\alpha}}{1 - p^\alpha - q^\alpha + p^\alpha q^\alpha}$$

quae expressio ob $p\,q = 1$ transformatur in hanc

$$\frac{p^\alpha - p^{(n+1)\alpha} - 1 + p^{n\alpha} + q^\alpha - q^{(n+1)\alpha} - 1 + q^{n\alpha}}{2 - p^\alpha - q^\alpha}$$

quae porro ob

$$p^\alpha + q^\alpha = 2\cos.\alpha\Phi \text{ et } p^{(n+1)\alpha} + q^{(n+1)\alpha} = 2\cos.(n+1)\alpha\Phi$$
$$\text{et } p^{n\alpha} + q^{n\alpha} = 2\cos.n\alpha\Phi$$

reducitur ad hanc formam:

$$\frac{\cos.\alpha\Phi - \cos.(n+1)\alpha\Phi - 1 + \cos.n\alpha\Phi}{1 - \cos.\alpha\Phi} = -1 + \frac{\cos.(n\alpha\Phi) - \cos.(n+1)\alpha\Phi}{1 - \cos.\alpha\Phi}$$

quae ergo est summa seriei propositae.

§. 3. Sin autem altera nostrarum progressionum ab altera subtrahatur, vt habeatur ista

$$\begin{matrix} +p^\alpha + p^{2\alpha} + p^{3\alpha} + \dots + p^{n\alpha} \\ -q^\alpha - q^{2\alpha} - q^{3\alpha} + \dots - q^{n\alpha} \end{matrix} \quad \text{eius summa erit} \quad \begin{matrix} \dfrac{p^\alpha - p^{(n+1)\alpha}}{1 - p^\alpha} \\ \dfrac{-q^\alpha + q^{(n+1)\alpha}}{1 - q^\alpha} \end{matrix}$$

quae partes ad eundem denominatorem reductae producent

$$\left\{\begin{matrix} +p^\alpha - p^{(n+1)\alpha} - p^\alpha q^\alpha + p^{(n+1)\alpha} q^\alpha \\ -q^\alpha + q^{(n+1)\alpha} + p^\alpha q^\alpha - q^{(n+1)\alpha} p^\alpha \end{matrix}\right\} : 1 - p^\alpha - q^\alpha + p^\alpha q^\alpha$$

ob $p\,q = 1$ vero haec expressio ad hanc reducitur

$$\begin{matrix} p^\alpha - p^{(n+1)\alpha} - 1 + p^{n\alpha} \\ -q^\alpha + q^{(n+1)\alpha} + 1 - q^{n\alpha} \end{matrix} : 2 - p^\alpha - q^\alpha$$

 et

et porro ob

$$p^{\alpha}-q^{\alpha}=2\sqrt{-1}\,\text{fin.}\,\alpha\Phi \text{ et } p^{(n+1)\alpha}-q^{(n+1)\alpha}=2\sqrt{-1}\,\text{fin.}\,(n+1)\alpha\Phi$$
$$\text{et } p^{n\alpha}-q^{n\alpha}=2\sqrt{-1}\,\text{fin.}\,n\alpha\Phi$$

in hanc transformatur expreffionem

$$\frac{(\text{fin.}\,\alpha\Phi-\text{fin.}\,(n+1)\alpha\Phi)+\text{fin.}\,(n\alpha\Phi)}{1-\text{cof.}\,\alpha\Phi}\sqrt{-1}.$$

§. 4. Defignemus breuitatis gratia fummas harum ferierum, vltimo termino feu generali praefigendo fignum fummationis $\int$, ita vt bini cafus euoluti praebeant fequentes fummationes

$$\int(p^{n\alpha}+q^{n\alpha})=-1+\frac{\text{cof.}\,(n\alpha\Phi)-\text{cof.}\,(n+1)\alpha\Phi}{1-\text{cof.}\,\alpha\Phi}, \text{ et}$$
$$\int(p^{n\alpha}-q^{n\alpha})=\frac{(\text{fin.}\,\alpha\Phi+\text{fin.}\,(n\alpha\Phi)-\text{fin.}\,(n+1)\alpha\Phi)}{1-\text{cof.}\,\alpha\Phi}(\sqrt{-1})$$

ex quibus formulis facile erit omnes cafus propofitos deducere.

§. 5. Sit igitur primo $\lambda=1$, vt habeantur hae duae feries fummandae:

$$s=\text{fin.}\,\Phi+\text{fin.}\,2\Phi+\text{fin.}\,3\Phi+\ldots\ldots+\text{fin.}\,n\Phi \text{ fiue } s=\int\text{fin.}\,n\Phi \text{ et}$$
$$t=\text{cof.}\,\Phi+\text{cof.}\,2\Phi+\text{cof.}\,3\Phi+\ldots\ldots+\text{cof.}\,n\Phi \text{ fiue } t=\int\text{cof.}\,n\Phi$$

cum igitur fit

$$\text{fin.}\,n\Phi=\frac{p^n-q^n}{2\sqrt{-1}} \text{ et } \text{cof.}\,n\Phi=\frac{p^n+q^n}{2}$$

habebimus

$$2s\sqrt{-1}=\int(p^n-q^n) \text{ et } 2t=\int(p^n+q^n)$$

vnde ex paragrapho praecedente ftatim nancifcimur ob $\alpha=1$

$$2s\sqrt{-1}=\frac{(\text{fin.}\,\Phi+\text{fin.}\,n\Phi-\text{fin.}\,(n+1)\Phi}{1-\text{cof.}\,\Phi}\sqrt{-1} \text{ et}$$
$$2t=-1+\frac{\text{cof.}\,n\Phi-\text{cof.}\,(n+1)\Phi}{1-\text{cof.}\,\Phi} \text{ ideoque}$$

$$s = \frac{\text{fin.}\,\Phi + \text{fin.}\,n\Phi - \text{fin.}\,(n+1)\Phi}{2(1-\text{cof.}\,\Phi)} \quad \text{et} \quad t = -\tfrac{1}{2} + \frac{\text{cof.}\,n\Phi - \text{cof.}\,(n+1)\Phi}{2(1-\text{cof.}\,\Phi)}.$$

§. 6. Sit nunc $\lambda = 2$ et ſtatuamus iterum

$s = \text{fin.}\,\Phi^2 + \text{fin.}\,2\Phi^2 + \ldots\ldots + \text{fin.}\,n\Phi^2$ ſiue $s = \int \text{fin.}\,n\Phi^2$ et

$t = \text{cof.}\,\Phi^2 + \text{cof.}\,2\Phi^2 + \ldots\ldots + \text{cof.}\,n\Phi^2$ ſiue $t = \int \text{cof.}\,n\Phi^2$

cum nunc ſit

$$\text{fin.}\,n\Phi^2 = \frac{p^{2n} - 2p^n q^n + q^{2n}}{-4} = \tfrac{1}{2} - \frac{p^{2n} - q^{2n}}{4} \text{ et}$$

$$\text{cof.}\,n\Phi^2 = \frac{p^{2n} + 2p^n q^n + q^{2n}}{4} = +\tfrac{1}{2} + \frac{p^{2n} + q^{2n}}{4}$$

habebimus has formulas

$$4s = 2\int 1 - \int (p^{2n} + q^{2n}) \quad \text{et} \quad 4t = 2\int 1 + \int (p^{2n} + q^{2n})$$

vbi cum numerus terminorum ſit n, manifeſtum eſt fore $\int 1 = n$; hinc quia $\alpha = 2$, ex ſuperioribus erit

$$\int (p^{2n} + q^{2n}) = -1 + \frac{\text{cof.}\,2n\Phi - \text{cof.}\,2(n+1)\Phi}{1 - \text{cof.}\,2\Phi}$$

quibus valoribus ſubſtitutis, facta diuiſione per 4 obtinebimus

$$s = \tfrac{n}{2} + \tfrac{1}{4} - \frac{\text{cof.}\,2n\Phi + \text{cof.}\,2(n+1)\Phi}{4(1-\text{cof.}\,2\Phi)} \quad \text{et} \quad t = \tfrac{n}{2} - \tfrac{1}{4} + \frac{\text{cof.}\,2n\Phi - \text{cof.}\,2(n+1)\Phi}{4(1-\text{cof.}\,2\Phi)}$$

atque hinc ſtatim liquet fore

$s + t = n$, prorſus vti rei natura poſtulat.

§. 7. Statuamus nunc $\lambda = 3$ et ſeries ſummandas ita repraeſentemus

$s = \text{fin.}\,\Phi^3 + \text{fin.}\,2\Phi^3 + \ldots\ldots + \text{fin.}\,n\Phi^3$ ſiue $s = \int \text{fin.}\,n\Phi^3$ et

$t = \text{cof.}\,\Phi^3 + \text{cof.}\,2\Phi^3 + \ldots\ldots + \text{cof.}\,n\Phi^3$ ſiue $t = \int \text{cof.}\,n\Phi^3$.

 Cum

Cum nunc fit

$$\sin. n\Phi^3 = \frac{p^{3n} - 3p^{2n}q^n + 3p^nq^{2n} - q^{3n}}{-8\sqrt{-1}} \text{ et}$$

$$\cos. n\Phi^3 = \frac{p^{3n} + 3p^{2n}q^n + 3p^nq^{2n} + q^{3n}}{8}$$

vnde ob $pq = 1$ consequimur

$$s = \frac{-1}{8\sqrt{-1}}\int(p^{3n} - q^{3n})\frac{-3p^n + 3q^n}{-8\sqrt{-1}} = \frac{-1}{-8\sqrt{-1}}\int(p^{3n} - q^{3n}) + \frac{3}{8\sqrt{-1}}\int(p^n - q^n)$$

tum vero

$$t = + \tfrac{1}{8}\int(p^{3n} + q^{3n}) + \tfrac{3}{8}\int(p^n + q^n)$$

quod si nunc valores supra inuentos hic substituamus, ambae summae quaesitae ita prodibunt expressae.

$$s = -\frac{\sin.3\Phi - \sin.3n\Phi + \sin.(3(n+1)\Phi)}{8(1 - \cos.3\Phi)} + \frac{3\sin.\Phi + 3\sin.n\Phi - 3\sin.(n+1)\Phi}{8(1 - \cos.\Phi)}$$

$$t = -\tfrac{1}{2} + \frac{\cos.3n\Phi - \cos.3(n+1)\Phi}{8(1 - \cos.3\Phi)} + \frac{3\cos.n\Phi - 3\cos.(n+1)\Phi}{8(1 - \cos.\Phi)}.$$

§. 8. Sit nunc $\lambda = 4$ ita vt quaerantur hae summae

$s = \sin.\Phi^4 + \sin.2\Phi^4 + \ldots\ldots + \sin.n\Phi^4$ siue $s = \int\sin.n\Phi^4$ et
$t = \cos.\Phi^4 + \cos.2\Phi^4 + \ldots\ldots + \cos.n\Phi^4$ siue $t = \int\cos.n\Phi^4$.

Cum igitur sit

$$\sin.n\Phi^4 = \frac{p^{4n} - 4p^{3n}q^n + 6p^{2n}q^{2n} - 4p^nq^{3n} + q^{4n}}{+16}$$

et $$\cos.n\Phi^4 = \frac{p^{4n} + 4p^{3n}q^n + 6p^{2n}q^{2n} + 4p^nq^{3n} + q^{4n}}{16}$$

ob $pq = 1$ sequuntur hi valores

$$s = \tfrac{1}{16}\int(p^{4n} + q^{4n}) - \tfrac{1}{4}\int(p^{2n} + q^{2n}) + \tfrac{3}{8}\int 1 \text{ et}$$

$$t = \tfrac{1}{16}\int(p^{4n} + q^{4n}) + \tfrac{1}{4}\int(p^{2n} + q^{2n}) + \tfrac{3}{8}\int 1$$

valo-

valoribus igitur quos supra dedimus substitutis erit:

$$S = \frac{3n}{8} + \frac{5}{16} + \frac{\cos.4n\Phi - \cos.4(n+1)\Phi}{16(1-\cos.4\Phi)} - \frac{\cos.2n\Phi + \cos.2(n+1)\Phi}{4(1-\cos.2\Phi)} \text{ et}$$

$$s = \frac{3n}{8} - \frac{5}{16} + \frac{\cos.4n\Phi - \cos.4(n+1)\Phi}{16(1-\cos.2\Phi)} + \frac{\cos.2n\Phi - \cos.2(n+1)\Phi}{4(1-\cos.2\Phi)}$$

sicque facile erit etiam maiores valores exponentis λ euoluere.

§. 9. Quod si iam quaeratur, cuiusmodi summae hinc sint prodíturae, si istae series in infinitum continuentur, non exigua circumspectione erit vtendum. Primo enim si exponens λ fuerit numerus par, euidens est, sumto pro n numero infinito, summas harum serierum etiam fore infinite magnas; verum si λ fuerit numerus impar, tum nihil est, quod has summas in infinitum augere possit, tota autem quaestio huc redigitur, vt valores formularum sinus $n\alpha\Phi$ et cos. $n\alpha\Phi$ assignentur quando pro n numeri infinite magni accipiuntur, perspicuum autem est hos valores hoc casu aeque a termino vsque ad terminum -1 variari posse, ac si n esset numerus finitus, vnde si res in se spectetur, nihil certi de his summis affirmari licet, cum quaecunque summa proferretur, si insuper vnus pluresue termini adderentur, prorsus alia summa esset prodítura; Interim tamen ab illustri Auctore praecedentis dissertationis summae hoc casu per rationes metaphysicas perquam ingeniose assignantur, quibus in analysi perfecte acquiescere queamus.

§. 10. Cum autem in his seriebus aeque ac in omnibus aliis non conuergentibus, notio summae

 pro-

proprie locum inuenire nequeat, quandoquidem quotcunque etiam termini actu addantur tamen nunquam ad summam determinatam peruenitur; Iam pridem validissimis rationibus innixus admonui his casibus voci summae alium significatum ad analysin magis accommodatum tribui debere, quam nouam notionem ita constitui debere censeo, vt summa cuiusque seriei infinitae, siue fuerit conuergens siue divergens, vocetur ea formula analytica, ex cuius euolutione eae series nascantur, hacque admissa definitione omnia dubia circa huiusmodi summationes sponte euanescunt.

§. 11. Quod quo clarius appareat, consideremus primam seriem supra exhibitam

$$s = \text{sin.}\,\Phi + \text{sin.}\,2\Phi + \text{sin.}\,3\Phi + \ldots\ldots + \text{sin.}\,n\Phi$$

pro qua inuenimus

$$\frac{\text{sin.}\,\Phi + \text{sin.}\,n\Phi - \text{sin.}\,(n+1)\Phi}{2(1-\text{cos.}\,\Phi)}$$

in quam expressionem formulae $\text{sin.}\,n\Phi$ et $\text{sin.}\,(n+1)\Phi$ propter vltimum terminum ingrediuntur, quod si ergo series reuera in infinitum continuetur; ob nullum terminum vltimum etiam hae formulae sponte excedunt, ita vt hoc casu fiat $s = \frac{\text{sin.}\,\Phi}{2(1-\text{cos.}\,\Phi)}$, quae etiam ea ipsa est formula, ex cuius euolutione ista series elicitur, vnde vi meae definitionis haec formula in analysi recte pro summa istius seriei haberi potest, quod idem de altera serie

$$t = \text{cos.}\,\Phi + \text{cos.}\,2\Phi + 3\Phi + \ldots\ldots + \text{cos.}\,n\Phi$$

est

est tenendum, pro qua inuenimus

$$t = -\tfrac{1}{2} + \frac{\text{cos.}\, n\,\Phi - \text{cos.}\,(n+1)\,\Phi}{2(1 - \text{cos.}\,\Phi)}$$

omisso enim postremo membro vtpote a termino vltimo pendente, summa per meam definitionem vtique erit $t = -\tfrac{1}{2}$ quod cum non tam facile pateat, notandum est, hunc valorem natum esse ex formula $t = \frac{\text{cos.}\,\Phi - 1}{2(1 - \text{cos.}\,\Phi)}$, quem valorem seriei propositae esse aequalem ita ostendi potest: Multiplicetur vtrinque per $2 - 2\,\text{cof.}\,\Phi$ fierique debebit

$$\text{cof.}\,\Phi - 1 = \begin{cases} 2\,\text{cof.}\,\Phi + 2\,\text{cof.}\,2\,\Phi + 2\,\text{cof.}\,3\,\Phi + 2\,\text{cof.}\,4\,\Phi \\ -2\,\text{cof.}\,\Phi^2 - 2\,\text{cof.}\,\Phi\,\text{cof.}\,2\,\Phi - 2\,\text{cof.}\,\Phi\,\text{cof.}\,3\,\Phi \end{cases}$$

cum nunc sit in genere

$$2\,\text{cof.}\,a\,\text{cof.}\,b = \text{cof.}\,(a-b) + \text{cof.}\,(a+b) \text{ erit}$$

$$2\,\text{cof.}\,\Phi^2 = 1 + \text{cof.}\,2\,\Phi \qquad 2\,\text{cof.}\,\Phi\,\text{cof.}\,4\,\Phi = \text{cof.}\,3\,\Phi + \text{cof.}\,5\,\Phi$$
$$2\,\text{cof.}\,\Phi\,\text{cof.}\,2\,\Phi = \text{cof.}\,\Phi + \text{cof.}\,3\,\Phi \qquad 2\,\text{cof.}\,\Phi\,\text{cof.}\,5\,\Phi = \text{cof.}\,4\,\Phi + \text{cof.}\,6\,\Phi$$
$$2\,\text{cof.}\,\Phi\,\text{cof.}\,3\,\Phi = \text{cof.}\,2\,\Phi + \text{cof.}\,4\,\Phi \qquad 2\,\text{cof.}\,\Phi\,\text{cof.}\,6\,\Phi = \text{cof.}\,5\,\Phi + \text{cof.}\,7\,\Phi$$
$$\text{etc.}$$

quibus valoribus substitutis aequalitas manifesto in oculos incurrit, prodibit enim

$$\begin{aligned} \text{cof.}\,\Phi - 1 = {} & 2\,\text{cof.}\,\Phi + 2\,\text{cof.}\,2\,\Phi + 2\,\text{cof.}\,3\,\Phi + 2\,\text{cof.}\,4\,\Phi \\ & -1 - \text{cof.}\,\Phi - \text{cof.}\,2\,\Phi - \text{cof.}\,3\,\Phi - \text{cof.}\,4\,\Phi \text{ etc.} \\ & \qquad - \text{cof.}\,2\,\Phi - \text{cof.}\,3\,\Phi - \text{cof.}\,4\,\Phi \end{aligned}$$

§. 12. Iisdem obseruatis cautelis etiam pro casu $\lambda = 3$ quo posueramus

$$s = \text{fin.}\,\Phi^3 + \text{fin.}\,2\,\Phi^3 + \text{fin.}\,3\,\Phi^3 + \text{etc. in infinitum}$$

et $t = \text{cof.}\,\Phi^3 + \text{cof.}\,2\,\Phi^3 + \text{cof.}\,3\,\Phi^3 + \text{cof.}\,4\,\Phi^3 + \text{etc. in infinitum}$

fum-ſumma harum ſerierum infinitarum ita erunt expreſſae

$$s = -\frac{\sin. 3\Phi}{2(1-\cos. 3\Phi)} + \frac{3\sin.\Phi}{2(1-\cos.\Phi)} \text{ et } t = -\tfrac{1}{2}.$$

hic quidem non tam facile apparet, harum formularum euolutionem ad ipſas has ſeries perducere, nihilo vero minus certum eſt, perfectam aequalitatem locum habere, id quod, qui in hoc calculi genere ſatis ſunt exercitati, perſpicient. Interim tamen veritatem poſterioris ſummationis hoc modo oſtendiſſe iuuabit, cum ſit

$$\cos. a^3 = \tfrac{3}{4}\cos. a + \tfrac{1}{4}\cos. 3a$$

ſeries haec in duas ſequentes reſoluitur

$$t = \tfrac{3}{4}(\cos.\ \Phi + \cos. 2\Phi + \cos. 4\Phi \text{ etc.})$$
$$+ \tfrac{1}{4}(\cos. 3\Phi + \cos. 6\Phi + \cos. 9\Phi \text{ etc.})$$

prioris autem ſeriei ſumma ex precedentibus eſt $\frac{3}{4}\cdot -\frac{1}{2} = -\frac{3}{8}$ poſterioris vero ob eandem rationem ſumma eſt $\frac{1}{4}\cdot -\frac{1}{2} = -\frac{1}{8}$ vnde ambae coniunctim faciunt ſummam $-\frac{1}{2}$.

SVMMATIO GENERALIS

INFINITARVM ALIARVM PROGRESSIONVM AD HOC GENVS REFERENDARVM.

Theorema.

Si cognita fuerit ſummatio huius progreſſionis

$$Az + Bz^2 + Cz^3 + Dz^4 \ldots\ldots + Nz^n$$

tum

tum ſemper etiam has progreſſiones ſummare licebit

$$S = A x \,\text{ſin.}\, \Phi + B x^2 \,\text{ſin.}\, 2\Phi + C x^3 \,\text{ſin.}\, 3\Phi + \ldots\ldots N x^n \,\text{ſin.}\, n\Phi$$

et $T = A x \,\text{coſ.}\, \Phi + B x^2 \,\text{coſ.}\, 2\Phi + C x^3 \,\text{coſ.}\, 3\Phi + \ldots\ldots N x^n \,\text{coſ.}\, n\Phi.$

Demonſtratio.

Cum ſumma progreſſionis

$$A z + B z^2 + C z^3 \ldots\ldots + N z^n$$

ſit certa quaedam functio quantitatis variabilis z, deſignetur ea hac formula $\Delta : z$; tum vero ponendo vt ante

$$p = \text{coſ.}\,\Phi + \sqrt{-1}\,\text{ſin.}\,\Phi \quad \text{et} \quad q = \text{coſ.}\,\Phi - \sqrt{-1}\,\text{ſin.}\,\Phi,$$

vt fiat

$$\text{ſin.}\, n\Phi = \frac{1}{2\sqrt{-1}}(p^n - q^n) \quad \text{et} \quad \text{coſ.}\, n\Phi = \tfrac{1}{2}(p^n + q^n);$$

ſi hae formulae in propoſitis ſeriebus ſubſtituantur, earum ſummae ita expreſſae obtinebuntur

$$2 S \sqrt{-1} = \Delta : p x - \Delta : q x \quad \text{et} \quad 2 T = \Delta : p x + \Delta : q x$$

vbi notandum, in vtraque formula quantitates imaginarias litteris p et q inuolutas ſponte ſe deſtruere, ita, vt pro ſummis S et T valores reales ſint produturi; atque haec ſummatio perinde ſuccedet, ſiue propoſitae ſeries in infinitum progrediantur, ſiue alicubi terminentur.

Exempl. I.

Sint omnes coëfficientes A, B, C .. = 1 et ſeries in infinitum continuetur; eritque $\Delta : z = \frac{z}{1-z}$; hinc ergo pro ſerie priore

$S = x \,\text{ſin.}\,\Phi + x^2 \,\text{ſin.}\, 2\Phi + x^3 \,\text{ſin.}\, 3\Phi + x^4 \,\text{ſin.}\, 4\Phi + \ldots$ in infi-

 habe-

habebitur

$$2 S \sqrt{-1} = \frac{px}{1-px} - \frac{qx}{1-qx} = \frac{(p-q)x}{1-(p+q)x+pqx^2}$$

quae expressio ob $p-q = 2\sqrt{-1}\,\text{fin.}\,\Phi$ et $p+q = 2\,\text{cof.}\,\Phi$ et $pq = 1$ praebebit

$$S = \frac{x\,\text{fin.}\,\Phi}{1-2x\,\text{cof.}\,\Phi+x^2}.$$

Pro altera vero ferie

$$T = x\,\text{cof.}\,\Phi + x^2\,\text{cof.}\,2\,\Phi + x^3\,\text{cof.}\,3\,\Phi + \ldots\ldots \text{ in infinit.}$$

fiet $2\,T = \frac{px}{1-px} + \frac{qx}{1-qx} = \frac{(p+q)x - 2pq.x^2}{1-(p+q)x+pq.x^2}$

fiue $T = \frac{x\,\text{cof.}\,\Phi - x^2}{1-2x\,\text{cof.}\,\Phi+x^2}$.

Coroll. 1.

Hinc igitur fi $x = 1$, oriuntur fummationes fupra datae, fcilicet

$$S = \frac{\text{fin.}\,\Phi}{2(1-\text{cof.}\,\Phi)} = \tfrac{1}{2}\,\text{cotang.}\,\tfrac{1}{2}\,\Phi$$

et $T = -\frac{1}{2}$

qui cafus eo magis eft notatu dignus, quod finguli termini funt quantitates variabiles, cum tamen fumma fit quantitas conftans.

Coroll. 2.

Semper autem in genere quantitatem x ita affumere licebit, vt fumma feriei datae quantitati a fiat aequalis; pro priore ferie finuum autem erit

$$\frac{x\,\text{fin.}\,\Phi}{1-2x\,\text{cof.}\,\Phi+x^2} = a;$$

ac fi hinc littera x determinetur, certo erit

$$a = x\,\text{fin.}\,\Phi + x^2\,\text{fin.}\,2\,\Phi + x^3\,\text{fin.}\,3\,\Phi + \ldots\ldots;$$

fimisimilique modo si statuatur

$$\frac{x\,\text{cof.}\,\Phi - x^2}{1 - 2x\,\text{cof.}\,\Phi + x^2} = a$$

ex eaque valor litterae x eruatur; etiam erit

$$a = x\,\text{cof.}\,\Phi + x^2\,\text{cof.}\,2\Phi + x^3\,\text{cof.}\,3\Phi + \dots\dots$$

Exempl. 2.

Sit nunc

$$\Delta : z = z + \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 + \tfrac{1}{4}z^4 + \dots\dots \text{ in infinit.}$$

$= \text{log.}\,\frac{1}{1-z}$; ita, vt series propositae iam sint

$$S = x\,\text{fin.}\,\Phi + \tfrac{1}{2}x^2\,\text{fin.}\,2\Phi + \tfrac{1}{3}x^3\,\text{fin.}\,3\Phi + \dots\dots$$

et $T = x\,\text{cof.}\,\Phi + \tfrac{1}{2}x^2\,\text{cof.}\,2\Phi + \tfrac{1}{3}x^3\,\text{cof.}\,3\Phi + \dots\dots$

habebimus

$$2S\sqrt{-1} = \text{log.}\,\frac{1}{1-px} - \text{log.}\,\frac{1}{1-qx} = \text{log.}\,\frac{1-qx}{1-px}$$

siue

$$2S\sqrt{-1} = \frac{1 - x\,\text{cof.}\,\Phi + x\sqrt{-1}\,\text{fin.}\,\Phi}{1 - x\,\text{cof.}\,\Phi - x\sqrt{-1}\,\text{fin.}\,\Phi}$$

pro cuius formulae reductione consideretur haec forma

$$\text{log.}\,\frac{f + g\sqrt{-1}}{f - g\sqrt{-1}}$$

de qua constat, si ponatur $\frac{g}{f} = \text{tang.}\,\omega$, hunc logarithmum fore $= 2\omega\sqrt{-1}$; hinc ergo quaeratur angulus ω, vt sit

$$\text{tang.}\,\omega = \frac{x\,\text{fin.}\,\Phi}{1 - x\,\text{cof.}\,\Phi};$$

vnde protenus sequitur $S = \omega$; pro altera progressione cum sit

$$2T = \text{log.}\,\frac{1}{1-px} + \text{log.}\,\frac{1}{1-qx} = -\text{log.}\,(1 - 2x\,\text{cof.}\,\Phi + x^2)$$

habe-

habetur

$$T = -\tfrac{1}{2}\log.(1 - 2x\,\mathrm{cof.}\,\Phi + x^2).$$

Coroll.

Pro priore ergo progreffione cum fit

$$\frac{x\,\mathrm{fin.}\,\Phi}{1 - x\,\mathrm{cof.}\,\Phi} = \mathrm{tang.}\,\omega$$

hinc elicitur

$$x = \frac{\mathrm{tang.}\,\omega}{\mathrm{fin.}\,\Phi + \mathrm{cof.}\,\Phi\,\mathrm{tang.}\,\omega} = \frac{\mathrm{fin.}\,\omega}{\mathrm{fin.}\,(\Phi + \omega)}$$

qua valore fubftituto nancifcimur hanc fummationem attentione maxime dignam

$$\omega = \frac{\mathrm{fin.}\,\omega\,\mathrm{fin.}\,\Phi}{\mathrm{fin.}\,(\Phi+\omega)} + \frac{\mathrm{fin.}\,\omega^2\,\mathrm{fin.}\,2\,\Phi}{2\,\mathrm{fin.}\,(\Phi+\omega)^2} + \frac{\mathrm{fin.}\,\omega^3\,\mathrm{fin.}\,3\,\Phi}{3.\,\mathrm{fin.}\,(\Phi+\omega)^3}$$
$$+ \frac{\mathrm{fin.}\,\omega^4\,\mathrm{fin.}\,4\,\Phi}{4.\,\mathrm{fin.}\,(\Phi+\omega)^4} + \ldots\ldots\ldots$$

vbi fi $\omega = \frac{\pi}{2}$, vt fit $\mathrm{fin.}\,\omega = 1$; et $\mathrm{fin.}(\Phi + \omega) = \mathrm{cof.}\,\Phi$ oritur haec fummatio maxime concinna

$$\frac{\pi}{2} = \frac{\mathrm{fin.}\,\Phi}{1.\,\mathrm{cof.}\,\Phi} + \frac{\mathrm{fin.}\,2\,\Phi}{2.\,\mathrm{cof.}\,\Phi^2} + \frac{\mathrm{fin.}\,3\,\Phi}{3.\,\mathrm{cof.}\,\Phi^3} + \ldots\ldots\ldots$$

quam feriem iam olim in calculo differentiali ex diuerfiffimis principiis fum adeptus; quae eo magis memorabilis erat vifa, quod vtcunque angulus Φ accipiatur, fumma feriei femper maneat eadem $= \frac{\pi}{2}$.

OBSER-