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De motu gravium citissimo super curvis specie datis

Leonhard Euler

DE MOTV GRAVIVM CITISSIMO SVPER CVRVIS SPECIE DATIS.

Auctore

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Problema I.

I.

Tab. VI. Fig. 5. Datis in plano horizontali punctis A et B, inuenire eum arcum circularem APB super quo corpus ex A descendendo citissime in B perueniat.

Solutio.

Bisecta distantia AB in C sit $AC = BC = a$, ac per C ducta verticali PO sit O centrum arcus quaesiti, eiusque radius

$OA = OP = r$ erit $PC = r - \sqrt{(rr - aa)} = p$,

vt fiat $r = \frac{aa + pp}{2p}$, sumantur coordinatae orthogonales $PX = x$, $XY = y$, erit $y = \sqrt{(2rx - xx)} = \sqrt{(\frac{aa + pp}{p}x - xx)}$, et elementum curuae $= \frac{r\,dx}{\sqrt{(2rx - xx)}}$, quare cum celeritas in Y sit $= \sqrt{(p - x)}$ erit elementum temporis

$$dt = \frac{r\,dx}{\sqrt{(p - x)(2rx - xx)}},$$

ita repraesentandum

$$dt = \frac{r\,dx}{\sqrt{(px - xx)(2r - x)}},$$

cuius

cuius integrale ita ſumtum vt euaneſcat poſito $x=0$, ſi ſtatuatur $x=p$, dabit tempus deſcenſus per arcum AP, cuius duplum erit tempus motus ab A ad B, quod minimum eſſe oportet. Cum igitur ſit

$$dt=\frac{dx\sqrt{\frac{1}{2}r}}{\sqrt{(px-xx)}}\left(1-\frac{x}{2r}\right)^{-\frac{1}{2}}$$ erit per ſeriem infinitam

$$dt=\frac{dx\sqrt{\frac{1}{2}r}}{\sqrt{(px-xx)}}\left(1+\frac{1}{4}\cdot\frac{x}{r}+\frac{1\cdot 3}{4\cdot 8}\cdot\frac{xx}{rr}+\frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}\cdot\frac{x^3}{r^3}+\text{etc.}\right).$$

At poſito poſt integrationem $x=p$, fit $\int\frac{dx}{\sqrt{(px-xx)}}=\pi$ denotante π peripheriam circuli cuius diameter $=1$; tum vero eſt in genere

$$\int\frac{x^n\,dx}{\sqrt{(px-xx)}}=\frac{2n-1}{2n}p\int\frac{x^{n-1}\,dx}{\sqrt{(px-xx)}}-\frac{1}{n}x^{n-1}\sqrt{(px-xx)}$$

quod poſtremum membrum facto $x=p$ euaneſcit.

Quare cum ſit $\int\frac{dx}{\sqrt{(px-xx)}}=\pi$

erit
$$\int\frac{x\,dx}{\sqrt{(px-xx)}}=\frac{1}{2}\pi p$$
$$\int\frac{x^2\,dx}{\sqrt{(px-xx)}}=\frac{1\cdot 3}{2\cdot 4}\pi p^2$$
$$\int\frac{x^3\,dx}{\sqrt{(px-xx)}}=\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\pi p^3$$
$$\int\frac{x^4\,dx}{\sqrt{(px-xx)}}=\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\pi p^4$$
etc.

Quibus valoribus ſubſtitutis obtinebimus tempus per AP

$$=\frac{\pi\sqrt{r}}{\sqrt{2}}\left(1+\frac{1\cdot 1}{2\cdot 4}\cdot\frac{p}{r}+\frac{1\cdot 1\cdot 3\cdot 3}{2\cdot 4\cdot 4\cdot 8}\cdot\frac{pp}{rr}+\frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 5}{2\cdot 4\cdot 4\cdot 8\cdot 6\cdot 12}\cdot\frac{p^3}{r^3}+\text{etc.}\right.$$
$$=\frac{\pi\sqrt{r}}{\sqrt{2}}\left(1+\frac{1^2}{2^2}\cdot\frac{p}{2r}+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}\cdot\left(\frac{p}{2r}\right)^2+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}\left(\frac{p}{2r}\right)^3+\text{etc.}\right.$$

 Cum

Cum igitur ſit

$$\frac{p}{2r} = \frac{pp}{aa+pp}, \text{ ob } r = \frac{aa+pp}{2p},$$

ſi ſtatuamus

$$\frac{p}{2r} = nn; \text{ erit } pp = \frac{nnaa}{1-nn}, \text{ et } 2r = \frac{a}{n\sqrt{(1-nn)}};$$

hincque

$$\frac{\sqrt{r}}{\sqrt{2}} = \frac{\sqrt{a}}{2\sqrt{n}\sqrt{(1-nn)}}.$$

Vel poſito potius $n = \frac{1}{m}$, vt ſit

$$p = \frac{a}{\sqrt{(mm-1)}} \text{ et } r = \frac{mma}{2\sqrt{(mm-1)}},$$

erit tempus per

$$AP = \frac{\pi m \sqrt{a}}{2\sqrt[4]{(mm-1)}}\left(1 + \frac{1^2}{2^2}\cdot\frac{1}{m^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}\cdot\frac{1}{m^4} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}\cdot\frac{1}{m^6} + \text{etc.}\right)$$

vbi m ita definiri debet, vt haec expreſſio minimum valorem conſequatur. Quare ob

$$d.\frac{m}{\sqrt[4]{(mm-1)}} = \frac{dm\left(\frac{1}{2}mm-1\right)}{(mm-1)\sqrt[4]{(mm-1)}}$$

habebitur per $\frac{\pi\sqrt{a}}{2}$ diuidendo

$$\frac{\frac{1}{2}mm-1}{(mm-1)\sqrt[4]{(mm-1)}}\left(1 + \frac{1^2}{2^2}\cdot\frac{1}{m^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}\cdot\frac{1}{m^4} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}\cdot\frac{1}{m^6} + \text{etc.}\right)$$

$$-\frac{1}{\sqrt[4]{(mm-1)}}\left(\frac{1^2}{2}\cdot\frac{1}{m^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4}\cdot\frac{1}{m^4} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6}\cdot\frac{1}{m^6} + \text{etc.}\right) = 0$$

quae multiplicata per $(mm-1)\sqrt[4]{(mm-1)}$ praebet.

$\frac{1}{2}mm$

$$\begin{aligned}
&\tfrac{1}{2}mm+\tfrac{1^2}{2^2}\cdot\tfrac{1}{2}+\tfrac{1^2\cdot 3^2}{2^2\cdot 4^2}\cdot\tfrac{1}{2}\cdot\tfrac{1}{m^2}+\tfrac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}\cdot\tfrac{1}{2}\cdot\tfrac{1}{m^4}+\tfrac{1^2\cdot 3^2\cdot 5^2\cdot 7^2}{2^2\cdot 4^2\cdot 6^2\cdot 8^2}\cdot\tfrac{1}{2}\cdot\tfrac{1}{m^6}\\
&-1-\tfrac{1^2}{2^2}\cdot\tfrac{1}{m^2}\quad-\tfrac{1^2\cdot 3^2}{2^2\cdot 4^2}\cdot\tfrac{1}{m^4}\quad-\tfrac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}\cdot\tfrac{1}{m^6}\\
&-\tfrac{1^2}{2}-\tfrac{1^2\cdot 3^2}{2^2\cdot 4}\cdot\tfrac{1}{m^2}\quad-\tfrac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6}\cdot\tfrac{1}{m^4}\quad-\tfrac{1^2\cdot 3^2\cdot 5^2\cdot 7^2}{2^2\cdot 4^2\cdot 6^2\cdot 8}\cdot\tfrac{1}{m^6}\\
&-\tfrac{1^2}{2}\cdot\tfrac{1}{m^2}\quad+\tfrac{1^2\cdot 3^2}{2^2\cdot 4}\cdot\tfrac{1}{m^4}\quad+\tfrac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6}\cdot\tfrac{1}{m^6}
\end{aligned}$$

nihilo aequandum; vnde ſinguli termini collecti dabunt:

$$\tfrac{1}{2}mm-\tfrac{11}{2^2\cdot 2}-\tfrac{1^2\cdot 31}{2^2\cdot 4^2\cdot 2}\cdot\tfrac{1}{m^2}-\tfrac{1^2\cdot 3^2\cdot 59}{2^2\cdot 4^2\cdot 6^2\cdot 2}-\tfrac{1^2\cdot 3^2\cdot 5^2\cdot 95}{2^2\cdot 4^2\cdot 6^2\cdot 8^2\cdot 2}\cdot\tfrac{1}{m^6}-\text{etc.}=0$$

quae ad hanc reducitur:

$$1-\tfrac{1}{2^2}\cdot\tfrac{11}{m^2}-\tfrac{1^2\cdot 31}{2^2\cdot 4^2\cdot m^4}-\tfrac{1^2\cdot 3^2\cdot 59}{2^2\cdot 4^2\cdot 6^2\cdot m^6}-\tfrac{1^2\cdot 3^2\cdot 5^2\cdot 95}{2^2\cdot 4^2\cdot 6^2\cdot 8^2\cdot m^8}-\text{etc.}=0$$

vbi numeri 11, 31, 59, 95 ſunt quadrati pares quinario minuti. Quo hinc valorem ipſius m eruamus, hanc aequationem ita repraeſentemus:

$$1=\frac{A}{m^2}+\frac{B}{m^4}+\frac{C}{m^6}+\frac{D}{m^8}+\frac{E}{m^{10}}+\text{etc.}$$

eruntque harum litterarum valores tam ipſi quam eorum logarithmi:

A = 2, 7500000	lA = 0, 4393327
B = 0, 4843750	lB = 9, 6851817
C = 0, 2304687	lC = 9, 3626120
D = 0, 1449584	lD = 9, 1612436
E = 0, 1039755	lE = 9, 0169309
F = 0, 0803659	lF = 8, 9050720
G = 0, 0651991	lG = 8, 8142416
H = 0, 0547023	lH = 8, 7380053
I = 0, 0470380	lI = 8, 6724492
K = 0, 0412122	lK = 8, 6150254.

Ex primo termino ſtatim patet, eſſe $mm > 2\frac{3}{4}$, verum tamen valor $mm = 3$ nimis magnus deprehenditur; ex quo mm intra limites 2, 75 et 3, 00 continetur.

Tribuamus ergo ipſi mm quosdam valores a veritate parum diſcrepantes, et omnes terminos ſeriei colligamus, vti hinc videre licet:

	$mm = 2, 94$	$mm = 2, 95$
	0, 9353742	0, 9322034
	560386	556593
	90692	89773
	19402	19140
	4734	4654
	1245	1219
	343	335
	98	96
	29	28
	8	8
ſumma	1, 0030682	0, 9993883
error	+ 0, 0030682	− 0, 0006117

ex quibus binis erroribus concluditur valor vero proximus

$mm = 2, 94833$, hinc $\sqrt{(mm - 1)} = 1, 39581$, et

$CP = p = \frac{a}{1, 39581} = 0, 71643\, a$, atque

$PO = r = \frac{2, 94833}{2 \cdot 1, 39581}\, a = 1, 056136, a$

vnde arcus AP continebit $71^{\circ}, 14', 7\frac{3}{17}''$.

Coroll.

Coroll. 1.

2. Arcus ergo quaesitus A P C ita commodissime describitur vt per rectae A B punctum medium C ducta verticali, capiatur $CP = 0,71643 . AC$ seu $CO = 0,33970 . AC$, eritque O centrum circuli describendi eiusque radius $OA = 1,056136 . AC$.

Coroll. 2.

3. Cum angulus B O P sit $71^\circ, 14', 7\frac{3}{13}''$, si ducatur corda B P erit angulus $CBP = 35^\circ, 37', 3\frac{8}{13}''$ vnde colligitur ipsa corda $BP = 1,2301315 . AC$. Nulla autem harum rationum rationalis esse videtur.

Scholion.

4. Tabulae logarithmorum, quibus in superiori calculo sum vsus, vix sufficiunt, vt valorem ipsius mm accuratius definiamus. Interim tamen, cum is intra limites 2,948 et 2,949 contineatur, faciamus pro vtroque calculos qui ita se habent:

	$mm = 2,948$	$mm = 2,949$
	0,9328358	0,9325194
	557249	556971
	89956	89864
	19193	19167
	4670	4662
	1224	1222
	337	336
	96	96
	28	28
	8	8
	3	3
	1,0001222	0,9997551
Err. =	+ 1222	− 2449

 vnde

vnde colligitur $mm = 2,94833288$, ita vt valor ſupra inuentus tam prope ad veritatem accedat, vt hic vix certior aeſtimari queat; diſcrimen enim facile ab errore vltimarum notarum oriri potuit. Hinc foret $mm = \frac{1769}{600}$ cuius quippe valor eſt $2,948\frac{1}{3}$ neque propius veritatem aſſequi licet, niſi quis velit maioribus logarithmorum tabulis vti.

Scholion 2.

5. Forſitan iuuabit ex inuento hoc valore mm ipſum tempus deſcenſus per arcum AP definiuiſſe. Quia igitur erat

$$\frac{mma}{\sqrt{(mm-1)}} = 2r = 2,112272\,a, \text{ erit}$$

$$\frac{m\sqrt{a}}{\sqrt[4]{(mm-1)}} = 1,453365\sqrt{a};$$

et ob tempus per

$$AP = \frac{\pi m \sqrt{a}}{\sqrt[4]{(nm-1)}}\left(1 + \frac{\alpha}{m^2} + \frac{\beta}{m^4} + \frac{\gamma}{m^6} + \frac{\delta}{m^8} + \text{etc.}\right)$$

ſingulos terminos per logarithmos euoluendo habebimus:

$la =$

	$1 = 1,00000000$
$l\alpha = 9,3979400$	$\frac{\alpha}{m^2} = 0,08479368$
$l\beta = 9,1480625$	$\frac{\beta}{m^4} = 0,01617743$
$l\gamma = 8,9897000$	$\frac{\gamma}{m^6} = 0,00381040$
$l\delta = 8,8737161$	$\frac{\delta}{m^8} = 0,00098949$
$l\varepsilon = 8,7824011$	$\frac{\varepsilon}{m^{10}} = 0,00027197$
$l\zeta = 8,7068240$	$\frac{\zeta}{m^{12}} = 0,00007751$
$l\eta = 8,6424546$	$\frac{\eta}{m^{14}} = 0,00002267$
$l\theta = 8,5863971$	$\frac{\theta}{m^{16}} = 0,00000676$
$l\iota = 8,5367499$	$\frac{\iota}{m^{18}} = 0,00000205$
$l\varkappa = 8,4921971$	$\frac{\varkappa}{m^{20}} = 0,00000063$
	reliquae 26
	summa $= 1,10615285$

vnde colligitur tempus per A P $= 0,803822.\pi\sqrt{a}$
et pro π fubftito valore eft $= 2,525280.\sqrt{a}$
ideoque tempus per A P B $= 5,050560.\sqrt{a}$.

Scholion 3.

6. Problema hoc ideo notatu dignum videtur, quod folutio ex aequatione infinita, cuius radix inveftigari debet, fit petenda. Cum enim quaeftio, in genere qua inter omnes omnino curuas ab A ad B ducendas ea quaeritur, fuper qua motus citiffime abfoluatur, methodo maximorum et minimorum expedite conficiatur, videri poterat, fi eadem quaeftio tantum ad arcus circulares reftringatur, folutionem vix difficiliorem effe futuram; quod tamen multo fecus euenit. Quamobrem in doctrina maximorum et

et minimorum etiamnum methodus desideratur, inter omnes tantum curuas, quae ad certam quandam speciem referantur, eam determinandi, quae certa quapiam maximi minimiue proprietate sit praedita. In hoc quidem genere alia methodus adhuc non patet, nisi qua hic sum vsus, qua ea quantitas, quae maxima minimaue fieri debet, per seriem exprimitur, indeque more solito valor maximo minimoue conueniens eruitur. Quodsi solutio problematis propositi ad eiusmodi numeros perduxisset, quorum certa quaedam proprietas agnosci potuisset, inde fortasse aliam methodum magis directam coniectura assequi licuisset; verum numeri inuenti ita ab omni proportione cognita abhorrent, vt nullum vestigium aliter eo perueniendi pateat.

Problema 2.

7. Datis in recta horizontali binis punctis A et B inter omnes semi-ellipses super axe A B describendas eam definire A P B, super qua graue in A descendens citissime ad B perueniat.

Solutio.

Bisecta A B in C ponatur $CA = CB = a$, qui erit alter semi-axis ellipsis datus, quaesitus autem ponatur $CP = p$. Vocatis coordinatis $CX = x$, $XY = y$, erit $aayy + ppxx = aapp$; vnde

$$x = \frac{a}{p}\surd(pp - yy), \text{ et } dx = -\frac{aydy}{p\surd(pp-yy)}.$$

Ergo

Ergo elementum curuae in

$$Y = \frac{dy \sqrt{(p^4 + (aa - pp)yy)}}{p\sqrt{(pp - yy)}}.$$

Quare cum celeritas in Y sit $= \sqrt{y}$, erit temporis quo arcus A Y conficitur, elementum

$$\frac{dy \sqrt{(p^4 + (aa - pp)yy)}}{p\sqrt{y(pp - yy)}}$$

quod ita integrari debet, vt euanescat facto $y = 0$ tum vero posito $y = p$, habebitur tempus descensus per A P, quod minimum esse oportet.

Cum autem hinc series concinna elici nequeat, alteram variabilem x in calculum introducamus; et quia

$$y = \frac{p}{a}\sqrt{(aa - xx)}; \quad dy = -\frac{pxdx}{a\sqrt{(aa - xx)}};$$

hincque elementum curuae

$$= \frac{dx\sqrt{(a^4 - (aa - pp)xx)}}{a\sqrt{(aa - xx)}},$$

erit temporis elementum

$$= \frac{1}{\sqrt{ap}} \cdot \frac{dx\sqrt{(a^4 - (aa - pp)xx)}}{(aa - xx)^{\frac{3}{4}}};$$

quod ita integratum vt posito $x = 0$ euanescat, facto $x = a$ dabit tempus descensus per A P. Hoc autem integrale haud difficulter in seriem infinitam conuertimus, posito enim breuitatis gratia

$$\frac{aa - pp}{aa} = n; \text{ erit } \sqrt{(a^4 - (aa - pp)xx)}$$

$$= aa\left(1 - \frac{nxx}{aa}\right)^{\frac{1}{2}} = aa\left(1 - \frac{1 \cdot nxx}{2aa} - \frac{1 \cdot 1 \cdot n^2x^4}{2 \cdot 4a^4} - \frac{1 \cdot 1 \cdot 3n^3x^6}{2 \cdot 4 \cdot 6a^6} - \text{etc.}\right)$$

 vnde

vnde fit elementum temporis

$$\frac{a\sqrt{a}}{\sqrt{p}} \cdot \frac{dx}{(aa-xx)^{\frac{3}{4}}} \left(1 - \frac{1.\,nxx}{2aa} - \frac{1.\,1\,n^2 x^4}{2.\,4\,a^4} - \frac{1.\,1.\,3\,n^3 x^6}{2.\,4.\,6\,a^6} - \text{etc.}\right).$$

Spectemus integrale $\int \frac{dx}{(aa-xx)^{\frac{3}{4}}}$ vt datum, fitque eius valor $= \frac{\alpha}{\sqrt{a}}$ casu $x = a$, et cum in genere fit

$$\int \frac{x^{\lambda+2}\,dx}{(aa-xx)^{\frac{3}{4}}} = \frac{2(\lambda+1)}{2\lambda+3} a^2 \int \frac{x^{\lambda}\,dx}{(aa-xx)^{\frac{3}{4}}} - \frac{2}{2\lambda+3} x^{\lambda+1} \sqrt[4]{(aa-xx)}$$

erit casu $x = a$ vt sequitur

$$\int \frac{dx}{(aa-xx)^{\frac{3}{4}}} = \frac{\alpha}{\sqrt{a}}$$

$$\int \frac{xx\,dx}{(aa-xx)^{\frac{3}{4}}} = \frac{2}{3}\,\alpha\, a \sqrt{a}$$

$$\int \frac{x^4\,dx}{(aa-xx)^{\frac{3}{4}}} = \frac{2.\,6}{3.\,7}\,\alpha\, a^3 \sqrt{a}$$

$$\int \frac{x^6\,dx}{(aa-xx)^{\frac{3}{4}}} = \frac{2.\,6.\,10}{3.\,7.\,11}\,\alpha\, a^5 \sqrt{a}$$

Hisque substitutis colligitur tempus per **A P**

$$= \frac{\alpha\, a}{\sqrt{p}}\left(1 - \frac{1}{2}.\,\frac{2}{3} n - \frac{1.\,1}{2.\,4}.\,\frac{2.\,6}{3.\,7} nn - \frac{1.\,1.\,3}{2.\,4.\,6}.\,\frac{2.\,6.\,10}{3.\,7.\,11} n^3 - \text{etc.}\right)$$

$$= \frac{\alpha\, a}{\sqrt{p}}\left(1 - \frac{1}{1}.\,\frac{1}{3} n - \frac{1.\,1}{1.\,2}.\,\frac{1.\,3}{3.\,7} nn - \frac{1.\,1.\,3}{1.\,2.\,3}.\,\frac{1.\,3.\,5}{3.\,7.\,11} n^3 - \frac{1.\,1.\,3.\,5}{1.\,2.\,3.\,4}.\,\frac{1.\,3.\,5.\,7}{3.\,7.\,11.\,15} n^4\right) \text{etc.}$$

vbi ob $\frac{aa-pp}{aa} = n$, eft $p = a\sqrt{(1-n)}$ et $-2p\,dp = aa\,dn$.

Quare

Quare si ponamus breuitatis gratia hoc tempus:

$$\frac{aa}{\sqrt{p}}(1 - An - Bnn - Cn^3 - Dn^4 - En^5 - \text{etc.})$$

differentiatio praebet hanc aequationem

$$\begin{aligned} 0 = 1 &- An - Bn^2 - Cn^3 - Dn^4 - En^5 \\ &- 4A - 8B - 12C - 16D - 20E - 24F \\ &\quad + 4A + 8B + 12C + 16D + 20E \end{aligned}$$

quae reducitur ad

$$\tfrac{1}{3} = \tfrac{9}{7}An + \tfrac{17}{11}Bnn + \tfrac{25}{15}Cn^3 + \tfrac{33}{19}Dn^4 + \tfrac{41}{23}En^5 \text{ etc.}$$

$$\text{seu } \tfrac{1}{3} = \tfrac{1}{1}\cdot\tfrac{1}{3}\cdot\tfrac{9}{7}\cdot n + \tfrac{1\cdot 1}{1\cdot 2}\cdot\tfrac{1\cdot 3}{3\cdot 7}\cdot\tfrac{17}{11}nn + \tfrac{1\cdot 1\cdot 3}{1\cdot 2\cdot 3}\cdot\tfrac{1\cdot 3\cdot 5}{3\cdot 7\cdot 11}\cdot\tfrac{25}{15}n^3$$
$$+ \tfrac{1\cdot 1\cdot 3\cdot 5}{1\cdot 2\cdot 3\cdot 4}\cdot\tfrac{1\cdot 3\cdot 5\cdot 7}{3\cdot 7\cdot 11\cdot 15}\cdot\tfrac{33}{19}n^4 + \text{etc.}$$

vnde valor numeri n elici debet, quo inuento est $p = a\sqrt{(1-n)}$, in hunc finem, cum sit

$$A = \tfrac{1}{3};\ B = \tfrac{3}{14}A;\ C = \tfrac{15}{33}B;\ D = \tfrac{35}{60}C;\ E = \tfrac{63}{95}D;\ F = \tfrac{99}{138}E$$
$$G = \tfrac{143}{189}F;\ H = \tfrac{195}{248}G;\ I = \tfrac{255}{315}H;\ K = \tfrac{323}{390}I;\ \text{etc.}$$

habentur hi valores in logarithmis:

$lA = 9,5228787;$	$l\frac{9}{7}A = 9,6320232$
$lB = 8,8538720;$	$l\frac{17}{11}B = 9,0429282$
$lC = 8,5114494;$	$l\frac{25}{15}C = 8,7332981$
$lD = 8,2773661;$	$l\frac{33}{19}D = 8,5171264$
$lE = 8,0989830;$	$l\frac{41}{23}E = 8,3500391$
$lF = 7,9547391;$	$l\frac{49}{27}F = 8,2135714$
$lG = 7,8336133;$	$l\frac{57}{31}G = 8,0981265$
$lH = 7,7291962;$	$l\frac{65}{35}H = 7,9980416$
$lI = 7,6374258;$	$l\frac{73}{39}I = 7,9096841$
$lK = 7,5555637;$	$l\frac{81}{43}K = 7,8305802.$

 Primum

Primum autem statim apparet esse $n < \frac{2}{3}$, quin etiam binis sumtis terminis primis $n < \frac{2}{3}$; quare considerentur limites $n = 0,6$ et $n = 0,7$ quibus fit:

	$n = 0,6$	$n = 0,7$
1.	0,25714290	0,30000000
2.	3974026	5409089
3.	1168832	1856060
4.	426316	789802
5.	174099	376295
6.	76292	192379
7.	35090	103232
8.	16720	57389
9.	8185	32777
10.	4094	19123
	0,31597944	0,38836146
cum seq.	4180	38246
summa	0,31602124	0,38874392
	0,33333333	0,33333333
Error =	− 0,01731209	+ 0,05541059

ex quibus erroribus proxime colligitur $n = 0,6238$ Fiant ergo duae nouae hypotheses:

$n = 0,620$	$n = 0,625$
0, 26571430	0, 26785720
4243377	4312093
1292627	1321107
486064	501934
205115	213520
92880	97466
44143	46697
21736	23178
10995	11820
5682	6157
0, 32974150	0, 33319692
8523	9250
0, 32982673	0, 33328942
0, 33333333	0, 33333333
Error − 0, 00350660	− 0, 00004391

vnde patet numerum n adeo maiorem esse quam 0,625 foretque satis exacte $n = 0,625063$, hinc $1 - n$ $= 0,374937$ et $p = a\sqrt{(1-n)} = 0,61232.\ a$ ita vt pro ellipsi quaesita sit

$$AC : CP = 1 : 0,61232$$

quae ratio cum nulla cognita conuenire videtur.

Scholion.

8. Operae pretium videtur valorem ipsius n accuratius inuestigare, quoniam vidimus primis limitibus tantopere esse aberratum; conueniet igitur

praeter valorem $n = 0,625$, alium assumi, qui praebeat errorem fere aequalem at diuersi signi vnde hae duae hypotheses considerentur:

	$n = 0,625$	$n = 0,6252$
	0,26785720	0,26794292
	4312093	4314855
	1321107	1322376
	501934	502579
	213520	213862
	97466	97653
	46697	46801
	23178	23237
	11820	11853
	6157	6177
	0,33319692	0,33333685
Term. seqq.	7453	7474
	0,33327145	0,33341159
	0,33333333	0,33333333
Error	− 0,00006188	+ 0,00007826

Hic etiam necesse est summam sequentium terminorum accuratius colligi, quae fere vti est notata, reperitur. Ex deprehensis ergo erroribus colligitur verus valor $n = 0,6250883$, parum a praecedente discrepans; hincque $p = 0,6123001.a = CP$.

Corollarium I.

9. Ea ergo ellipsis, quae hac minimi proprietate est praedita ita definitur, vt si semiaxis horizonta-

zontalis $CA = BC$ ponatur $= a$, ſit ſemiaxis coniugatus verticalis

$$CP = 0,6123001 . a.$$

Tum vero diſtantia foci F a centro erit

$$CF = \sqrt{(aa - pp)} = a\sqrt{n} = 0,79062\, a,$$

et ſemiparameter

$$= \frac{pp}{a} = (1 - n)\, a = 0,3749117\, a.$$

Coroll. 2.

10. Haec ellipſis ſpecies nullis rationibus cognitis continetur, neque enim ratio elementorum eius rationaliter, neque per indolem circuli exprimi poſſe videtur; ita vt iſta ſpecies omnino ſit ſingularis, neque aliis praeterea proprietatibus praedita exiſtimanda.

Scholion.

11. Ex his exemplis intelligitur, quam inſignis adhuc pars methodi maximorum et minimorum iaceat inculta cum ſi ſpecies curuarum ex quibus electio maximi minimiue fieri debet, proponatur alia via haud pateat, niſi vt radix ex aequatione infinita extrahatur. Atque in his quidem exemplis commode vſu venit, vt termini iſtius aequationis infinitae ſatis promte conuergant, quod ſi in aliis quaeſtionibus ſecus eueniat, multo maiori labore erit opus, quin etiam ſi aequatio plures vel adeo infinitas inuoluat radices reales, reſolutio completa ne expectanda quidem videtur. Quod eo magis mirum

rum videri debet, cum methodus maximorum et minimorum iam ita sit exculta, vt non solum inter omnes omnino curuas sed etiam inter infinitas tantum certa quadam indole praeditas, veluti quae sint eiusdem longitudinis, vel eandem aream includant, ea assignari possit, cui maximi minimiue proprietas quaedam conueniat. Nunc igitur intelligimus plurimum interesse, vtrum curuae infinitae propositae communi quaedam proprietate, veluti eadem longitudine sint praedita, an vero omnes certa quadam curuarum specie contineantur; hoc enim posteriori casu fateri cogimur methodum huiusmodi quaestiones resoluendi etiamnunc penitus latere; quae resolutio enim in casibus hic euolutis successit, in aliis magis complicatis locum omnino non inuenit. Plurimum autem interesse arbitror, quaecunque adhuc in Analysi desiderantur, sollicite annotari.

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