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De motu vibratorio chordarum crassitie utcunque variabili praeditarum

Leonhard Euler

DE MOTV VIBRATORIO CHORDARVM CRASSITIE VTCVNQVE VARIABILI PRAEDITARVM.

Auctore

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I.

Tab. VI. Fig. 4. Sit huiusmodi chorda in terminis A et B fixa et tensa a vi seu pondere K, ponatur eius longitudo $AB = a$ et abscissa portione quacunque $AX = x$, crassities in x ita sit comparata, vt si haberetur chorda vniformis eiusdem crassitiei, cuius longitudo $= k$, eius pondus futurum esset $= X$ denotante X functionem quamcunque ipsius x, quippe qua ratio crassitiei variabilis continetur, hinc ergo elementi chordae dx massula seu pondusculum erit $= \frac{X\,dx}{k}$, ideoque pondus portionis $AX = \frac{1}{k}\int X\,dx$ in quo integrali si fiat $x = a$, prodibit pondus totius chordae AB.

II. Ponamus nunc elapso t min. sec. punctum chordae X peruenisse in Y, vbi quidem semper assumimus hoc interuallum $XY = y$ esse quasi infinite paruum, ita vt omnes tangentes in his punctis Y infinite parum ab axe declinent, ac pro puncto Y haec declinatio aequetur angulo minimo $= u$. Haec

Haec igitur applicata $XY = y$ erit functio binarum variabilium t et x, vnde celeritas puncti Y a X digredientis erit $(\frac{dy}{dt})$ eiusque incrementum seu acceleratio $= \frac{1}{2b}(\frac{ddy}{dt^2})$, denotante b altitudinem lapsus pro vno minuto secundo, quae ducta in massam elementi dx dabit ipsam vim, qua elementum in Y secundum directionem XY sollicitatur $= \frac{X dx}{2bk}(\frac{ddy}{dt^2})$ quia enim excursiones sunt infinite paruae, punctum y aliter moueri nequit nisi in directione XY.

III. Nunc igitur etiam vires, quibus punctum y reuera sollicitatur, expendi oportet, quae aliae non sunt nisi tensiones, quibus punctum Y cum versus A, tum in plagam oppositam versus B vrgetur; sit igitur Θ tensio punctum Y versus A trahens, hincque orietur vis in directione

abscissae $XA = \Theta \operatorname{cos.} \omega$

at in directione

applicatae $YX = \Theta \operatorname{sin.} \omega$;

at pro elemento sequente, quum sit eius inclinatio ad axem $= \omega + d\omega$ et tensio $= \Theta + d\Theta$; hinc nascetur vis

secundum $XB = (\Theta + d.\Theta) \operatorname{cos.} (\omega + d\omega)$

et vis

in directione $XY = (\Theta + d.\Theta) \operatorname{sin.} (\omega + d.\omega)$.

Iam vero vires in directione axis agentes se mutuo destruere debent, quia alioquin punctum Y non secundum applicatam moueretur, necesse igitur est, fiat

$\Theta \operatorname{cos.} \omega = (\Theta + d\Theta) \operatorname{cos.} (\omega + d\omega)$

 ideoque

ideoque

$$d\,\Theta\,\text{coſ.}\,\omega - \Theta\,d\,\omega\,\text{ſin.}\,\omega = 0,$$

ideoque

$$d\,\Theta = \Theta\,d\,\omega.\,\frac{\text{ſin.}\,\omega}{\text{coſ.}\,\omega},$$

quia autem angulus ω eſt infinite paruus, vtique hinc ſequitur $d\,\Theta = 0$, ita vt tenſio Θ ſit conſtans ſeu per totam cordae longitudinem eadem; quare quum in termino A tenſio manifeſto aequetur ponderi K, quo chorda tenditur, erit vtique $\Theta = K$.

IV. Contemplemur nunc vires in directione XY agentes, et quia ob

$$\text{ſin.}\,\omega = \left(\frac{d\,y}{d\,x}\right) \text{ prior vis in directione YX eſt } K\left(\frac{d\,y}{d\,x}\right)$$

altera autem huic contraria ipſo ſui differentiali augetur eritque idcirco

$$= K\left(\frac{d\,y}{d\,x}\right) + K\left(\frac{d\,d\,y}{d\,x^2}\right)$$

ſicque punctum Y reuera in directione XY ſollicitabitur

$$\text{vi} = K\,d\,x\left(\frac{d\,d\,y}{d\,x^2}\right)$$

quae ergo neceſſario aequalis eſſe debet vi illi acceleratrici ſupra inuentae, vnde exoritur iſta aequatio:

$$\frac{X\,d\,x}{2\,b\,k}\left(\frac{d\,d\,y}{d\,t^2}\right) = K\,d\,x\left(\frac{d\,d\,y}{d\,x^2}\right);$$

quae per $K\,d\,x$ diuiſa producit hanc aequationem finalem:

$$\frac{X}{2\,b\,k.\,K}\left(\frac{d\,d\,y}{d\,t^2}\right) = \left(\frac{d\,d\,y}{d\,x^2}\right).$$

Ponamus breuitatis gratia

$$\frac{K}{2\,b\,k.\,K} = w,$$

quae

quae ergo littera defignat functionem ipfius x, a tempore t immunem, ita vt habeamus

$$w\left(\frac{ddy}{dt^2}\right) = \left(\frac{ddy}{dx^2}\right),$$

quam autem aequationem adhuc nullo modo in genere refoluere licuit, interim tamen per feriem infinitam eius integrale adeo completum exhiberi poteft. Denotante enim Θ functionem quamcunque temporis t, litteris vero P, Q, R, S etc. functiones tantum alterius variabilis x; fingatur

$$y = P\Theta + Q\left(\frac{dd\Theta}{dt^2}\right) + R\left(\frac{d^4\Theta}{dt^4}\right) + S\left(\frac{d^6\Theta}{dt^6}\right) \text{ etc.}$$

et habebimus

$$\left(\frac{ddy}{dt^2}\right) = P\left(\frac{dd\Theta}{dt^2}\right) + Q\left(\frac{d^4\Theta}{dt^4}\right) + R\left(\frac{d^6\Theta}{dt^6}\right) + S\left(\frac{d^8\Theta}{dt^8}\right)$$

$$\text{et } \left(\frac{ddy}{dx^2}\right) = \Theta\left(\frac{ddP}{dx^2}\right) + \left(\frac{dd\Theta}{dx^2}\right)\left(\frac{ddQ}{dx^2}\right) + \left(\frac{d^4\Theta}{dt^4}\right)\left(\frac{ddR}{dx^2}\right) \text{ etc.}$$

quae feries vt praecedenti in w ductae aequetur, confequimur has determinationes:

I°. $\left(\frac{ddP}{dx^2}\right) = 0$ ideoque $P = \alpha x + \beta$

II°. $\left(\frac{ddQ}{dx^2}\right) = wP$; fiue $Q = \int dx \int wP\,dx$

III°. $\left(\frac{ddR}{dx^2}\right) = wQ$; fiue $R = \int dx \int wQ\,dx$

IV°. $\left(\frac{ddS}{dx^2}\right) = wR$; fiue $S = \int dx \int wR\,dx$ etc.

Verum hinc parum lucri impetramus, atque hanc ob rem conueniet cafus aliquot particulares euoluere, id quod in Problematibus fequentibus expediemus.

Problema I.

Inuenire cafus, quibus aequationi differentiali inventae:

$$w\left(\frac{ddy}{dt^2}\right) = \left(\frac{ddy}{dx^2}\right),$$

ſatisfacit haec forma integralis:

$$y = p\,\Phi(t + \int z\,dx),$$

vbi p, z ſint functiones tantum ipſius x.

Solutio.

V. Character Φ hic denotat functionem quamcunque quantitatis ſuffixae $t + \int z\,dx$, nequidem functionibus discontinuis exceptis, cuius differentialia more ſolito, per Φ', Φ'', Φ''' etc. deſignabimus, ex hac autem forma ſtatim colligimus:

$\left(\frac{ddy}{dt^2}\right) = p\,\Phi''(t + \int z\,dx)$, tum vero reperimus

$\left(\frac{dy}{dx}\right) = \left(\frac{dp}{dx}\right)\Phi(t + \int z\,dx) + pz\,\Phi'(t + \int z\,dx)$, hincque

$$\left(\frac{ddy}{dx^2}\right) = \frac{ddp}{dx^2}\Phi(t + \int z\,dx) + \frac{2z\,dp + p\,dz}{dx}\Phi'(t + \int z\,dx)$$
$$+ pzz.\,\Phi''(t + \int z\,dx),$$

cui illa forma in w ducta aequari debet, vnde ſequitur:

I°. $\frac{ddp}{dx^2} = 0$; ideoque $p = \alpha x + \beta$

II°. $2z\,dp + p\,dz = 0$; ſiue $zp^2 = C$ ideoque $z = \frac{C}{pp} = \frac{C}{(\alpha x + \beta)^2}$

III°. tandem eſſe debet $w = zz = \frac{C^2}{(\alpha x + \beta)^4}$.

Conſequenter ſi craſſities chordae ita fuerit comparata, vt fiat

$$w = \frac{X}{2bk.K} = \frac{CC}{(\alpha x + \beta)^4},$$

tum motum chordae actu definire licebit.

Euolutio caſus quo craſſities chordae ita variatur, vt fiat:

VI.

$$\frac{X}{2bk.K} = \frac{cc}{(nx + f)^4}.$$

VI. In hac expreſſione litterae c, et f perinde ac b, k et x denotant quantitates lineares, dum litterae t, n aeque ac pondera K, X in meris numeris dantur, ita vt conditio praeſcripta homogeneitate gaudeat. Hac forma cum ſupra inuenta collata erit primo $C = c$; $\alpha = w$; et $\beta = f$ ideoque

$$p = nx + f \text{ et } z = \frac{c}{(nx+f)^2}$$

hincque

$$\int z\,dx = \int \frac{c\,dx}{(nx+f)^2} = \frac{c}{n}\left(\frac{1}{x} - \frac{1}{nx+f}\right) = \frac{cx}{f(nx+f)}.$$

Quum igitur pro eadem chorda, cuius craſſitiem quadratum cc inuoluit, loco c etiam $-c$ ſcribere liceat; haec quoque forma integralis locum habebit:

$$y = (nx+f)\,\psi\left(t - \frac{cx}{f(nx+f)}\right),$$

quocirca introducendo binas functiones arbitrarias ϕ et ψ nanciſcimur verum integrale completum pro noſtro caſu, ſcilicet:

$$y = (nx+f)\,\phi\left(t + \frac{cx}{f(nx+f)}\right) + (nx+f)\,\psi\left(t - \frac{cx}{f(nx+f)}\right).$$

VII. Nihil igitur aliud ſupereſt, niſi vt hoc integrale completum ad ſtatum noſtrae quaeſtionis adcommodemus. Ac primo quidem quum poſito $x = 0$, eſſe debeat $y = 0$; euidens eſt, functionem ψ alteri ϕ ſub ſigno contrario aequari debere, ita vt iam habeamus

$$y = (nx+f)\,\phi\left(t + \frac{cx}{f(nx+f)}\right) - (nx+f)\,\phi\left(t - \frac{cx}{f(nx+f)}\right).$$

Quo igitur etiam in altero termino ſumto $x = a$ denuo fiat $y = 0$, neceſſe eſt vt ſit

$$\phi\left(t + \frac{ca}{f(na+f)}\right) = \phi\left(t - \frac{ca}{f(na+f)}\right),$$

vnde intelligimus, functionem Φ siue lineam curuam hac functione designatam ita comparatam esse debere, vt dum abscissae interuallo $= \frac{2ca}{f(na+f)}$ augentur, applicatae continuo eaedem recurrant, quemadmodum hanc conditionem in Dissertatione superiori de chordis vniformibus fuse explicauimus. Hinc porro sequitur, quum initio fuisset $t = 0$; elapso tempore $t = \frac{2ca}{f(na+f)}$ min. sec., chordam perfecte in pristinum statum restitui, duplici vibratione peracta, ita vt tempus vnius vibrationis futurum sit $= \frac{ca}{f(na+f)}$ sec. siue quod eodem redit haec chorda singulis minutis secundis $\frac{f(na+f)}{ca}$ absoluet vibrationes. Hocque adeo casu coniectura, quam in superiore dissertatione innuimus, egregie confirmatur, quod enim hic vocauimus X, ibi erat V, tempus vibrationis autem asseruimus ibi esse $= \int \frac{dx\sqrt{V}}{\sqrt{2bk.K}}$, dum integrale per totam chordae longitudinem $x = a$ extenditur, hic autem tempus vibrationis prodiit $= \int z\,dx$, integrali etiam per totam chordae longitudinem extenso existente

$$z = \sqrt{w} = \frac{\sqrt{X}}{\sqrt{2bk.K}},$$

quae est prorsus eadem forma.

Problema II.

Inuenire casus, quibus aequationi nostrae differentiali talis satisfacit forma integralis:

$$y = p\,\Phi\,(t + \int z\,dx) + q\,\Phi'\,(t + \int z\,dx).$$

Solutio.

Solutio.

VIII. Euolutione huius formae facta, reperiemus vt sequitur:

$$\begin{array}{r|l|l|l|l}
\left(\frac{ddy}{dx^2}\right) = & \Phi(t+\int z\,dx) & \Phi'(t+\int z\,dx) & \Phi''(t+\int z\,dx) & \Phi'''(t+\int z\,dx) \\
 & +\frac{ddp}{dx^2} & +\frac{2z\,dp+p\,dz}{dx} & p\,z\,z & +q\,z\,z \\
 & & +\frac{ddq}{dx^2} & +\frac{2z\,dq+q\,dz}{dx} & \\
w\left(\frac{ddy}{dt^2}\right) = & & & p\,w & +q\,w
\end{array}$$

quocirca sequentes conditiones adimpleri oportet

I°. $\frac{ddp}{dx} = 0$; siue $p = \alpha x + \beta$;

II°. $2z\,dp + p\,dz + \frac{ddq}{dx} = 0$; siue $\frac{ddq}{dx} = -(2z\,dp + p\,dz)$

III°. $\frac{2z\,dq + q\,dz}{dx} + p\,z\,z = p\,w$;

IV°. $q\,z\,z = q\,w$, ideoque $z\,z = w$.

Qui postremus valor $w = z\,z$ in penultima substitutus praebet

$$2z\,dq + q\,dz = 0,$$

quae in q ducta et integrata praebet $z\,q\,q = C$. Hinc igitur habemus

$$z = \frac{C}{q\,q} \text{ et } dz = -\frac{2C\,dq}{q^3},$$

vnde nostra secunda aequatio fit

$$\frac{ddq}{dx} + \frac{2C\,dp}{q\,q} - \frac{2C\,p\,dq}{q^3} = 0:$$

verum ob

$$p = \alpha x + \beta \text{ erit } dx = \frac{dp}{\alpha};$$

et per $2C$ diuidendo fiet

$$\frac{\alpha\,ddq}{2C\,dp} + \frac{q\,dp - p\,dq}{q^3} = 0,$$

vbi

vbi elementum dp ſumtum eſt conſtans. Verum ſecunda aequatio ſponte fit integrabilis multiplicata per p, quum enim ſit $dp = \frac{dx}{\alpha}$, ea transformatur in hanc:

$$\frac{\alpha p\, ddq}{dp} + 2zp\,dp + pp\,dz = 0,$$

cuius integrale eſt

$$\frac{\alpha(p\,dq - q\,dp)}{dp} + 2zpp = D.$$

Modo ante autem habebamus $z = \frac{C}{qq}$, ita vt nunc habeamus

$$\frac{\alpha(p\,dq - q\,dp)}{dp} + \frac{Cpp}{qq} = D$$

quae poſito $q = up$ abit in hanc:

$$\alpha pp\,du + \frac{C\,dp}{uu} = D\,dp$$

vnde colligitur

$$\frac{dp}{pp} = \frac{\alpha uu\,du}{Duu - C} = \frac{4}{D} \cdot du + \frac{\alpha C}{D} \cdot \frac{du}{Duu - C}$$

vnde u per p vel viciſſim facile definitur. Quo inuento erit

$$q = pu \text{ et } z = \frac{C}{qq} \text{ et } w = zz = \frac{CC}{q^4}.$$

IX. Quia autem hinc quantitas u non per p, ſed viciſſim p per u definitur, excepto ſolo caſu $D = 0$, quandoquidem quantitatem C nihilo aequare non licet; caſus $D = 0$ vtique ſingularem euolutionem meretur, pro quo ſtatim habemus

$$-\frac{\alpha uu\,du}{C} = \frac{dp}{pp},$$

cuius integrale eſt

$$\frac{1}{p} + E = \frac{\alpha u^3}{3C}$$

ideo-

ideoque

$$u = \sqrt[3]{\left(\frac{3C}{\alpha p} \cdot (1 + Ep)\right)}; \text{ hincque porro}$$

$$q = p\sqrt[3]{\frac{3C}{\alpha p}(1 + Ep)} = \sqrt[3]{\frac{3Cpp}{\alpha}(1 + Ep)}$$

$$z = \frac{C}{pp\sqrt[3]{\frac{9CC}{\alpha^2 p^2}(1+Ep)^2}} = \frac{C}{p\sqrt[3]{\frac{9CCp}{\alpha\alpha}(1+Ep)^2}} = \frac{\sqrt[3]{\frac{\alpha\alpha C}{9}}}{p\sqrt[3]{p(1+Ep)^2}}$$

$$w = \frac{CC}{pp\sqrt[3]{\frac{81 \cdot C^4 pp}{\alpha}(1+Ep)^4}} = \frac{CC}{\frac{3C}{\alpha}pp(1+Ep)\sqrt[3]{\frac{3Cpp}{\alpha}(1+Ep)}}$$

$$= \frac{\sqrt[3]{\alpha^4 CC}}{\sqrt[3]{81 \cdot p^8 (1 + Ep)^4}}.$$

Hinc autem consequitur formula integralis

$$\int z\,dx = -\sqrt[3]{\frac{3C(1+Ep)}{\alpha p}}.$$

X. Quo has formulas concinniores reddamus et simul homogeneitati consulamus, statuamus

$$p = nx + f; \text{ vt sit } \alpha = n,$$

tum vero sumamus

$$C = \tfrac{1}{3}c \text{ et } 1 + Ep = \frac{nx + f + g}{g}$$

et habebimus

$$\int z\,dx = -\sqrt[3]{\frac{c(nx + f + g)}{ng(nx + f)}}$$

hinc

$$z = \frac{\sqrt[3]{nnggc}}{3\sqrt[3]{(nx+f)^4(nx+f+g)^2}}; w = \frac{\sqrt[3]{n^4 g^4 cc}}{9\sqrt[3]{(nx+f)^8(nx+f+g)^4}}$$

ac denique

$$q = \sqrt[3]{\frac{c(nx+f)^2(nx+f+g)}{ng}}.$$

Caeterum quia integrale $\int z\,dx$ ita capi conuenit, vt euanescat posito $x = 0$, hoc obseruato erit

$$\int z\,dx = \sqrt[3]{\frac{c(f+g)}{nfg}} - \sqrt[3]{\frac{c(nx+f+g)}{ng(nx+f)}}.$$

XI. Quoniam $w = zz$, patet, pro eodem valore ipsius w litteram z tam positiue, quam negatiue accipi posse, at si eam negatiue capiamus, quod fit, dum loco c scribimus $-c$, etiam littera q valorem sortitur negatiuum, vnde pro eadem aequatione differentiali etiam sequens forma integralis valebit

$$y = p\,\psi(t - \int z\,dx) - q\,\psi'(t - \int z\,dx)$$

quae forma si ad ante inuentam addatur, habebitur integrale completum nostrae aequationis, quippe quod erit

$$y = p\,\Phi(t + \int z\,dx) + p\,\psi(t - \int z\,dx)$$
$$+ q\,\Phi'(t + \int z\,dx) - q\,\psi'(t - \int z\,dx).$$

Haec autem forma difficulter ad motum chordarum adcommodari potest, quandoquidem etiamsi sumamus $\psi = -\Phi$ pro casu $x = 0$, non fit $y = 0$, multo minus pro casu $x = a$ quaestioni satisfieri poterit.

Problema III.

Proposita aequatione differentiali secundi gradus:

$$zz\left(\frac{ddy}{dt^2}\right) = \left(\frac{ddy}{dx^2}\right);$$

vbi

vbi z denotet quamcunque functionem ipsius x tantum; eius integrale per huiusmodi seriem infinitam exprimere:

$$y = p\Phi(t + \int z\,dx) + q\Phi'(t + \int z\,dx) + r\Phi''(t + \int z\,dx) + s\Phi'''(t + \int z\,dx) \text{ etc.}$$

Solutio.

XII. Si singuli isti termini per differentiationem, vt ante, euoluantur, vbi iam vidimus esse $w = zz$, sequentes conditiones erunt adimplendae:

$$\text{I. } ddp = 0; \text{ II. } \frac{ddq}{dx} + 2z\,dp + p\,dz = 0$$

$$\text{III}^{\circ}. \frac{ddr}{dx} + 2z\,dq + q\,dz = 0; \text{ IV}^{\circ}. \frac{dds}{dx} + 2z\,dr + r\,dz = 0 \text{ etc.}$$

Ex prima statim colligitur $p = nx + f$, secunda ad hanc formam reducta

$$\frac{ddq}{dx} + \frac{1}{p}\,d.ppz = 0,$$

statim praebet

$$\frac{dq}{dx} = -\int \frac{1}{p}\,d.ppz \text{ ideoque } q = -\int dx \int \frac{1}{p}\,d.ppz,$$

quemadmodum autem hic q exprimitur per z et p, ita quoque r exprimetur per z et q ita, vt sit

$$r = -\int dx \int \frac{1}{q}\,d.qqz,$$

eodemque modo porro

$$s = -\int dx \int \frac{1}{r}\,d.rrz \text{ etc.}$$

Alia Solutio.

Solutio adhuc concinnior reddi potest, sumendo $z = \frac{1}{2}vv$, vt aequatio proposita sit

$$\frac{1}{4}v^4\left(\frac{ddy}{dt^2}\right) = \left(\frac{ddy}{dx^2}\right),$$

tum enim manente $p = nx + f$, ſecunda conditio hanc induet formam

$$\frac{ddq}{dx^2} + vv\,dp + pv\,dv = 0;\ \text{ſiue}$$

$$\frac{ddq}{dx} = -v(v\,dp + p\,dv) = -v\,d.pv,$$

vnde fit

$$q = -\int dx \int v.\,d.\,pv,$$

eodemque modo reperitur

$$r = -\int dx \int v d.qv \quad \text{et} \quad s = -\int dx \int v d.rv \text{ etc.}$$

XIII. Haec ergo forma in infinitum eſt continuanda, niſi forte eueniat, vt quaepiam litterarum q, r, s etc. euaneſcat tum enim totum integrale forma finita exprimetur; et quia pro eodem valore $zz = \frac{1}{4} v^4$, quantitatem z ſiue uu etiam negatiue ſumere licet, gemina huiusmodi forma obtinebitur, quae coniunctim integrale completum referet, id quod caſibus, quibus omnes hae quantitates euadunt ſimplices poteſtates ipſius x, clarius elucebit. Quodſi autem ſumto $v = \alpha x^\lambda$ quaecunque litterarum p, q, r, s etc. fuerit $= A x^n$, ſequens erit

$$-\frac{A\alpha^2(n+\lambda)}{(n+2\lambda)(n+2\lambda+1)}.\,x^{n+2\lambda+1}.$$

Quam ob rem quum pro his caſibus ſit vel $p = 1$, vel $p = x$, ſequentes valores ſequenti modo ſe habebunt:

Pro primo caſu, quo $p = 1$, habebimus

$$q = -\frac{\alpha\alpha\lambda}{2\lambda(2\lambda+1)}.\,x^{2\lambda+1};\ r = +\frac{\alpha^4\lambda(3\lambda+1)\,x^{4\lambda+e}}{2\lambda(2\lambda+1)(4\lambda+1)(4\lambda+2)}$$

$$s = -\frac{\alpha^6\lambda(3\lambda+1)(5\lambda+2).\,x^{6\lambda+3}}{2\lambda(2\lambda+1)(4\lambda+1)(4\lambda+2)(6\lambda+2)(6\lambda+3)} \text{ etc.}$$

haec

haec progreſſio igitur abrumpitur, quoties fuerit vel $\lambda = 0$, vel $3\lambda + 1 = 0$ ſeu $\lambda = -\frac{1}{3}$, vel $\lambda = -\frac{2}{5}$, vel $\lambda = -\frac{3}{7}$ ideoque in genere ſi $\lambda = -\frac{i}{2i+1}$, quo caſu fit $v = \alpha x^{\frac{-i}{2i+1}}$ hincque $z = \frac{1}{2}\alpha\alpha x^{\frac{-2i}{2i+1}}$ ergo $zz = \frac{1}{4}\alpha^4 . x^{\frac{-4i}{2i+1}}$.

Pro poſteriore caſu, quo $p = x$, habebimus:

$$q = -\frac{\alpha\alpha(\lambda+1)}{(2\lambda+1)(2\lambda+2)} x^{2\lambda+2}; \quad r = +\frac{\alpha^4(\lambda+1)(3\lambda+2) . x^{4\lambda+3}}{(2\lambda+1)(2\lambda+2)(4\lambda+2)(4\lambda+3)}$$

$$s = -\frac{\alpha^6(\lambda+1)(3\lambda+2)(5\lambda+3) . x^{6\lambda+4}}{(2\lambda+1)(2\lambda+2)(4\lambda+2)(4\lambda+3)(6\lambda+3)(6\lambda+4)} \text{ etc.}$$

quae progreſſio abrumpitur, quoties fuerit vel $\lambda = -1$ vel $\lambda = -\frac{2}{3}$; vel $= -\frac{3}{5}$ et in genere ſi fuerit $\lambda = -\frac{i}{2i-1}$, vnde fit $v = \alpha x^{\frac{-i}{2i-1}}$; $z = \frac{1}{2}\alpha\alpha x^{\frac{-2i}{2i-1}}$ et $zz = \frac{1}{4}\alpha^4 x^{\frac{-4i}{2i-1}}$. Hique caſus manifeſto conueniunt cum iis, qui in famoſa aequatione Riccatiana locum habent.

XIV. At ſuper ſolutione generali §. 12. id imprimis memoratu dignum vſu venit, quod omnes litterae p, q, r, s etc. per geminam integrationem definiantur, ideoque binas conſtantes arbitrarias recipiant, quum tamen iam character Φ omnes poſſibiles functiones in ſe inuoluat. Verum hoc nequaquam mirum videri debet, ſi enim haec forma

$$p\Phi(t + \int z\,dx) + q\Phi' \ldots + r\Phi'' \ldots + s\Phi''' \ldots \text{ etc.}$$

aequationi noſtrae differentiali ſatisfaciat, euidens eſt, etiam ſequentes formas eſſe ſatisfacturas:

$$p\Phi'\ldots+q\Phi''\ldots+r\Phi'''\ldots \text{ etc. item } p\Phi''\ldots+q\Phi'''\ldots \text{ etc.}$$

atque hinc huiusmodi formae coniunctae pariter ſatisfacient, veluti

$$p\Phi\ldots+(q+\alpha p)\Phi'\ldots+(r+\alpha q+\beta p)\Phi''\ldots+(s+\alpha r+\beta q+\gamma p)\Phi'''\ldots\text{etc.}$$

hicque adeo prima quantitas p, quum ſit $=nx+f$, in poſterioribus formis adiectis diuerſos valores induere poterit, vnde ratio huius multiplicitatis in oculos incurrit.

XV. Quum igitur litterae p, q, r, s etc. duplicem determinationem ab arbitrio noſtro pendentem recipere queant, ſingulas ita conſtituere licebit, vt pro vtroque chordae termino, ſcilicet tam ſi $x=0$, quam ſi $x=a$, euaneſcant, quod adeo in prima p fieri poſſet ſumendo $f=0$ et $n=0$, quo quidem caſu chorda in quiete perſiſteret, at quomodocunque litteram p accipere lubuerit, omnes litterae ſequentes q, r, s etc. ſemper ita definiri poſſunt, vt euaneſcant poſito tam $x=0$ quam $x=a$, quae circumſtantia nobis facilem ſuppeditat ſolutionem ſequentis Problematis, quod generaliſſime motum omnium plane chordarum in ſe complectitur, quo omnia quae de hoc argumento deſiderari ſolent, referri poſſunt.

Problema Generaliſſimum.

Si chorda craſſitiei vtcunque variabilis, in duobus terminis fixa, a vi quacunque fuerit tenſa, definire tempus ſingularum vibrationum, quas edet impulſa.

Solutio.

Chordae craſſities ſe habeat eo modo, vti ſupra §. 1. et vocetur eius longitudo $AB = a$ et vis chordam tendens aequetur ponderi K, tum vero ponatur

$$zz = \frac{X}{2bk.K},$$

et quum motus chordae definiatur hac aequatione generali

$$zz\left(\frac{ddy}{dt^2}\right) = \left(\frac{ddy}{dx^2}\right)$$

modo ante vidimus huic aequationi tali forma:

$$y = p\Phi(t+\int z\,dx) + q\Phi' \ldots r\Phi'' \ldots + s\Phi''' \ldots \text{etc.}$$

ita ſatisfieri poſſe, vt poſt primum ſinguli coefficientes q, r, s etc. pro vtroque chordae termino euaneſcant tam poſito $x = 0$, quam $x = a$. His iam coefficientibus ita determinatis et ſumto $p = 1$, aequatio integralis completa ita exhiberi poterit, vt ſit

$$\begin{aligned} y = {} & \Phi(t+\int z\,dx) && - \psi(t-\int z\,dx) \\ & + q\,\Phi'(t+\int z\,dx) && - q\,\psi'(t-\int z\,dx) \\ & + r\,\Phi''(t+\int z\,dx) && + r\,(\psi''(t-\int z\,dx) \\ & + s\,\Phi'''(t+\int z\,dx) && + s\,(\psi'''(t-\int z\,dx). \\ & && \text{etc.} \end{aligned}$$

Quum enim aequatio inuoluat quadratum zz, eius radicem z tam poſitiue quam negatiue capere licet, et ex ſuperioribus facile colligitur, ſumto z negatiue, coefficientium alternos ſignum contrarium recipere debere; nihil aliud igitur ſupereſt, niſi vt hoc integrale completum ad ſtatum quaeſtionis accommodemodetur, quo requiritur, vt posito siue $x=0$ siue $x=a$, valor ipsius y in nihilum abeat. Ponamus igitur $x=0$ et formulam $\int z\,dx$ ita captum assumamus, vt euanescat posito $x=0$ atque ob euanescentes q, r, s, t etc. habebimus $y=0$, vnde manifesto sequitur, functionem ψ conuenire debere cum functione ϕ ita, vt sit $\psi=\phi$. Nunc ergo pro casu $x=a$, quo litterae q, r, s, t etc. itidem annihilantur, sumamus formulam $\int z\,dx$ accipere valorem $=\mathrm{A}$, et nunc habebitur

$$y=\phi(t+\mathrm{A})-\phi(t-\mathrm{A})=0,$$

vnde discimus, indolem istius functionis ϕ ita comparatam esse debere, vt semper fiat

$$\phi(t+2\,\mathrm{U})=\phi t$$

quicquid fuerit U, id quod etiam in functionibus ϕ', ψ' locum habebit. Quocirca in quocunque statu chorda fuerit pro statu praesenti, post elapsum tempus $\frac{t}{2\,\mathrm{A}}$ in eundem statum reuertetur, hocque temporis interuallo binas absoluisse vibrationes censeri solet, ita vt tempus vnius vibrationis sit $=\mathrm{A}$. Ecce ergo solutionem nostri Problematis, quae ita se habebit: ob $zz=\frac{\mathrm{X}}{2\,b\,k.\,\mathrm{K}}$ sumatur integrale formulae $\int dx\sqrt{\mathrm{X}}$ per totam longitudinem chordae extensum ac tempus singularum vibrationum in minutis secundis expressum erit $\int\frac{dx\sqrt{\mathrm{X}}}{\sqrt{2\,b\,k.\,\mathrm{K}}}$, ex quo vibrationum vno minuto secundo editarum numerus erit $\frac{\sqrt{2\,b\,k.\,\mathrm{K}}}{\int dx\sqrt{\mathrm{X}}}$, omnino vti hactenus coniectura inducti statuimus.

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