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De tautochrona in medio rarissimo, quod resistit in ratione multiplicata quacunque celeritatis

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DE TAVTOCHRONA IN MEDIO RARISSIMO, QVOD RESISTIT IN RATIONE MVLTIPLICATA QVACVNQVE CELERITATIS.

Auctore
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Quamquam huius quaestionis iam in Mechanicae meae Volumine secundo plenam dedi solutionem, nemo tamen eorum, qui hoc argumentum deinceps pertractarunt ac penitus exhausisse videntur, eandem inuestigationem est aggressus, etiamsi hinc via tutissima parari videatur ad eiusmodi media, quorum resistentia potestati cuicunque celeritatis fuerit proportionalis. Quare ne haec materia obliuioni tradatur, eam hic denuo retractare visum est, idque eo magis, quod nunc methodum faciliorem sum exhibiturus.

1. Sit igitur ASC huiusmodi Tautochrona siue ascensus, siue descensus, namque modo ostensum est eundem calculum ad vtrumque casum accommodari posse, sitque A punctum infimum, et C terminus siue ascensus, siue descensus, cuius eleuatio super terminum imum A in axe verticali sit $Ak=k$, ita vt totum tempus per arcum AC requiratur constantis quantitatis pro omnibus valoribus litterae k. Tab. IV. Fig. 6.

 2. Tum

2. Tum vero pro puncto indefinito S, vocetur abſciſſa $AX = x$ et arcus $AS = s$, celeritas porro in S vocetur $= u$ et tempus, quo arcus AS abſoluitur $= t$, ita vt ſit $t = \int \frac{ds}{u}$. Porro denotet u^{2n} eam celeritatis poteſtatem, cui reſiſtentia ſit proportionalis, atque pro motus determinatione habebitur huiusmodi aequatio:

$$2u\,du = -a\,dx + \lambda u^{2n}\,ds$$

denotante λ eum coefficientem minimum, quo actio reſiſtentiae determinatur, huiusque litterae λ valor poſitiuus ad deſcenſum, negatiuus autem ad aſcenſum referetur.

3. Iam cum neglecta reſiſtentia hinc prodiret

$$uu = a(k-x),$$

ſiquidem poſito $x = k$, celeritas in puncto C euaneſcere debet, pro caſu reſiſtentiae fingamus hanc formulam:

$$uu = a(k-x) + \lambda P + \lambda^2 Q + \lambda^3 R \text{ etc.}$$

quae ſeries vtique maxime erit conuergens ob λ quantitatem quam minimam, atque adeo ſufficere poſſet ſolam litteram P in calculum introduxiſſe, interim tamen hic operae pretium erit, noſtram inveſtigationem etiam ad maiorem praeciſionis gradum proſequi. Hinc autem porro colligitur

$$u^{2n} = a^n(k-x)^n + \lambda n a^{n-1}(k-x)^{n-1}P + \lambda^2 n a^{n-1}(k-x)^{n-1}Q + \lambda^3 n a^{n-1}(k-x)^{n-1}R$$
$$+ \frac{\lambda^2 (n)(n-1)}{1 \cdot 2} a^{n-2}(k-x)^{n-2}P^2 + \lambda^3 n(n-1) a^{n-2}(k-x)^{n-2}PQ$$
$$+ \frac{\lambda^3 n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}(k-x)^{n-2}P^3.$$

His

His igitur valoribus in aequatione nostra substitutis, fit

$$+\lambda dP + \lambda^2 dQ + \lambda^3 dR \text{ etc.} = \lambda a^n (k-x)^n ds + \lambda^2 n a^{n-1} (k-x)^{n-1} P ds$$
$$+ \lambda^3 n a^{n-1} (k-x)^{n-1} Q ds$$
$$+ \frac{\lambda^3 n(n-1)}{1 \cdot 2} a^{n-2} (k-x)^{n-2} P^2 ds$$

ideoque

$$P = a^n \int (k-x)^n ds; \quad Q = n . a^{2n-1} \int (k-x)^{n-1} ds \int (k-x)^n ds$$

vel $Q = n . a^{n-1} \int (k-x)^{n-1} P dy$; $R = n . a^{n-1} \int (k-x)^{n-1} Q ds$
$$+ \frac{n(n-1)}{1 \cdot 2} a^{n-2} \int (k-x)^{n-2} P^2 ds.$$

4. Valores autem harum litterarum P, Q, R per integrationem ita determinari oportet, vt euanescant posito $x = k$, id quod ideo est necesse, vt celeritas in termino C euanescat. Manifestum igitur est, istas litteras factorem esse habituras $(k - x)$ vel adeo eius quandam potestatem altiorem, quam sequente ratiocinio cognoscere licebit. Sit $(k - x)^e$ maxima potestas, quam littera P inuoluit, ita vt fractio $\frac{P}{(k-x)^e}$ non amplius per $k - x$ sit diuisibilis, seu quod eodem redit, quae posito $x = k$ non amplius euanescat. Quoniam autem istius fractionis, tam numerator P, quam denominator $(k - x)^e$ euanescit facto $x = k$, secundum regulam notissimam eius valor pro hoc casu eruetur, si tam numerator, quam denominator seorsim differentientur, sicque casu $x = k$ haec fractio erit

$$= \frac{dP}{-e dx (k-x)^{e-1}} = \frac{a^n . (k-x)^n ds}{-e dx (k-x)^{e-1}} = -\frac{1}{e} a^n (k-x)^{n-e+1} \frac{ds}{dx}$$

quae

quae fractio vt facto $x = k$ fiat finita, necesse est, sit $n = e - 1$, ideoque $e = n + 1$. Nam quia ratio $\frac{ds}{dx}$ non inuoluit k, hinc nostra conclusio non turbatur, quocirca efficimus quantitatem P inuoluere factorem $(k - x)^{n+1}$.

5. Eodem modo ratiocinium circa litteram Q instituamus, cuius maximus factor itidem sit $(k-x)^e$, et casu $x = k$ quoque erit

$$\frac{Q}{(k-x)^e} = \frac{dQ}{-e\,dx(k-x)^{e-1}} = \frac{a^{n-1}(k-x)^{n-1}P\,ds}{-e\,dx(k-x)^{e-1}},$$

quae forma ob

$$P = (k-x)^{n+1}P' \text{ abit in } \frac{Q}{(k-x)^e} = \frac{n\,a^{n-1}(k-x)^{2n}P'ds}{-e\,dx(k-x)^{e-1}}$$

quae ergo vt facto $x = k$ prodeat finita necesse est, sit $e = 2n + 1$, ita vt littera Q certe factorem sit habitura $(k - x)^{2n+1}$. Eodem modo iudicium circa litteram R instituetur, erit enim

$$\frac{R}{(k-x)^e} = \frac{dR}{-e\,dx(k-x)^{e-1}} = \frac{n.\,a^{n-1}(k-x)^{n-1}Q\,dx}{-e\,dx(k-x)^{e-1}} + \frac{n(n-1)}{1.\;2}\,\frac{a^{n-2}(k-x)^{n-2}P^2\,dx}{-e\,dx(k-x)^{e-1}},$$

cuius expressionis pars prior ob

$$Q = (k-x)^{2n+1}Q',$$

praebet $e - 1 = 3n$, siue $e = 3n + 1$, quem eundem valorem quoque pars posterior ob $P^2 = (k-x)^{2n+2}$ praebet sicque lex manifesta est, si quis vlterius progredi voluerit.

6. In-

6. Inuentis autem his litteris P, Q, R quum sit

$$uu = a(k-x) + \lambda P + \lambda^2 Q + \lambda^3 R \text{ etc.}$$

huius potestas exponentis $= -\frac{1}{2}$, praebebit

$$\frac{1}{u} = \frac{1}{\sqrt{a(k-x)}} - \frac{1}{2}\frac{\lambda P}{a^{3:2}(k-x)^{3:2}} - \frac{1}{2}\frac{\lambda^2 Q}{a^{\frac{3}{2}}(k-x)^{\frac{3}{2}}} - \frac{1}{2}\frac{\lambda^3 R}{a^{\frac{3}{2}}(k-x)^{\frac{3}{2}}}$$
$$+ \frac{3}{8}\frac{\lambda^2 P^2}{a^{\frac{5}{2}}(k-x)^{\frac{5}{2}}} + \frac{3}{4}\frac{\lambda^3 PQ}{a^{\frac{5}{2}}(k-x)^{\frac{5}{2}}}$$
$$- \frac{5}{16}\frac{\lambda^3 P^3}{a^{\frac{7}{2}}(k-x)^{\frac{7}{2}}}$$

ergo quum sit $dt = \frac{ds}{u}$, pro elemento temporis sequentem nanciscimur formulam:

$$dt = \frac{ds}{\sqrt{a(k-x)}} - \frac{1}{2}\frac{\lambda P ds}{a^{\frac{3}{2}}(k-x)^{\frac{3}{2}}} - \frac{1}{2}\frac{\lambda^2 Q ds}{a^{\frac{3}{2}}(k-x)^{\frac{3}{2}}} - \frac{1}{2}\frac{\lambda^3 R ds}{a^{\frac{3}{2}}(k-x)^{\frac{3}{2}}}$$
$$+ \frac{3}{8}\frac{\lambda^2 P^2 ds}{a^{\frac{5}{2}}(k-x)^{\frac{5}{2}}} + \frac{3}{4}\frac{\lambda^3 PQ ds}{a^{\frac{5}{2}}(k-x)^{\frac{5}{2}}}$$
$$- \frac{5}{16}\frac{\lambda^3 P^3 ds}{a^{\frac{7}{2}}(k-x)^{\frac{7}{2}}}$$

cuius integrale quum exprimat tempus per $AS = s$, ita debet capi, vt euanescat posito $s = 0$, vel $x = 0$, quo facto si statuatur $x = k$, obtinebitur tempus totius siue descensus siue ascensus, quod ob tautochronismi indolem ita debet esse comparatum, vt quan-

 titas

titas k inde prorſus exulat atque adeo termini, vbi occurreret, ſe mutuo deſtruant.

7. Neglecta autem reſiſtentia, notum eſt, huic indoli ſatisfieri, ſumendo

$$ds = dx \sqrt{\frac{b}{x}},$$

tum enim primum membrum prodit

$$dt = \sqrt{\frac{b}{a}} \frac{dx}{\sqrt{(kx - xx)}},$$

cuius integrale eſt

$$\frac{2\sqrt{b}}{\sqrt{a}} \text{ Arc. ſin. } \frac{\sqrt{x}}{\sqrt{k}}$$

poſito ergo $x = k$, colligitur totum tempus

$$\frac{2\sqrt{b}}{\sqrt{a}} \frac{\pi}{2} = \frac{\pi . \sqrt{b}}{\sqrt{a}},$$

in quam expreſſionem littera k non amplius ingreditur. Quodſi ergo admiſſa reſiſtentia effici potuerit, vt totum tempus eidem formulae aequetur, ac ſequentia membra ſe mutuo deſtruant, negotium penitus erit confectum.

8. Quum ſublata reſiſtentia inuenimus

$$ds = dx \sqrt{\frac{b}{x}},$$

quem valorem etiam in medio reſiſtente propemodum valere certum eſt, reuera ſumamus eſſe

$$ds = dx\sqrt{\frac{b}{x}} + \lambda p dx + \lambda^2 q dx + \lambda^3 r dx + \text{etc.}$$

quo valore ſubſtituto, elementum temporis ſequentibus membris exprimetur

$dt =$

$$dt = \sqrt{\tfrac{b}{a}}\,\frac{dx}{\sqrt{(kx-xx)}} - \tfrac{1}{2}\,\frac{\lambda P dx \sqrt{\tfrac{b}{x}}}{a^{\frac{3}{2}}(k-x)^{\frac{3}{2}}} - \tfrac{1}{2}\,\frac{\lambda^2 Q dx \sqrt{\tfrac{b}{x}}}{a^{\frac{3}{2}}(k-x)^{\frac{3}{2}}} - \tfrac{1}{2}\,\frac{\lambda^3 R dx \sqrt{\tfrac{b}{x}}}{a^{\frac{3}{2}}(k-x)^{\frac{3}{2}}}$$

$$+ \frac{\lambda p dx}{\sqrt{a(k-x)}} + \tfrac{3}{8}\,\frac{\lambda^2 P^2 dx \sqrt{\tfrac{b}{x}}}{a^{\frac{5}{2}}(k-x)^{\frac{5}{2}}} + \tfrac{3}{4}\,\frac{\lambda^3 PQ dx \sqrt{\tfrac{b}{x}}}{a^{\frac{5}{2}}(k-x)^{\frac{5}{2}}}$$

$$- \tfrac{1}{2}\,\frac{\lambda^2 P p dx}{a^{\frac{3}{2}}(k-x)^{\frac{3}{2}}} - \tfrac{5}{16}\,\frac{\lambda^3 P^3 dx \sqrt{\tfrac{b}{x}}}{a^{\frac{3}{2}}(k-x)^{\frac{3}{2}}}$$

$$+ \frac{\lambda^2 q dx}{\sqrt{a(k-x)}} - \tfrac{1}{2}\,\frac{\lambda^3 Q p dx}{a^{\frac{3}{2}}(k-x)^{\frac{3}{2}}}$$

$$+ \tfrac{3}{8}\,\frac{\lambda^3 P^2 p dx}{a^{\frac{5}{2}}(k-x)^{\frac{5}{2}}}$$

$$- \tfrac{1}{2}\,\frac{\lambda^3 P q dx}{a^{\frac{3}{2}}(k-x)^{\frac{3}{2}}}$$

$$+ \frac{\lambda^3 r dx}{a^{\frac{3}{2}}(k-x)^{\frac{3}{2}}}.$$

9. Hic praeter terminum primum occurrunt membra ſiue littera λ ſimpliciter, ſiue eius quadrato, ſiue eius cubo affecta, totumque negotium nunc huc redit, vt ſingulorum horum membrorum integralia facto $x = k$, ſe mutuo tollant pro quolibet ordine, quod quomodo praeſtari queat, in ordine primo oſtendamus, vbi efficiendum eſt, vt

posito $x = k$, fiat $-\frac{1}{2}\int \frac{P\,dx\sqrt{\frac{b}{x}}}{a^{\frac{3}{2}}(k-x)^{\frac{3}{2}}} + \int \frac{p\,dx}{\sqrt{a(k-x)}} =$

$$-\frac{\sqrt{b}}{2a^{\frac{3}{2}}}\int \frac{P\,dx}{(k-x)^{\frac{3}{2}}\sqrt{x}} + \int \frac{p\,dx}{\sqrt{a(k-x)}} = 0.$$

Quum autem sit

$$P = a^n \int (k-x)^n\,ds,$$

loco ds valore principali substituto $dx\sqrt{\frac{b}{x}}$, caeteris enim partibus haec formula ad ordines sequentes deuoluitur, pro hoc ordine erit

$$P = a^n \sqrt{b} \int \frac{(k-x)^n\,dx}{\sqrt{x}}.$$

Quo autem istae formulae pleniores reddantur, statuamus $k = cc$ et $x = zz$, fietque

$$P = 2a^n \sqrt{b} \int dz\,(cc - zz)^n,$$

quem valorem nouimus inuoluere factorem

$$(cc - zz)^{n+1} \text{ siue } (c - z)^{n+1}.$$

Quum nunc efficiendum sit, vt

$$\int \frac{p\,dx}{\sqrt{(k-x)}} = \frac{\sqrt{b}}{2a}\int \frac{P\,dx}{(k-x)^{\frac{3}{2}}\sqrt{x}} = \frac{\sqrt{b}}{a}\int \frac{P\,dz}{(cc-zz)^{\frac{3}{2}}},$$

hoc postremum integrale ita reducatur

$$\int \frac{P\,dz}{(cc-zz)^{\frac{3}{2}}} = \frac{zP}{cc\sqrt{(cc-zz)}} - \int \frac{z\,dP}{cc\sqrt{(cc-zz)}},$$

cuius membrum primum sponte euanescit facto $z = 0$, at si in posteriori ponatur $x = k$, siue $z = c$, quoniam

niam P factorem habet $(c-z)^{n+1}$, id facto $z=c$ necessario euanescit, dummodo fuerit $n+1>\frac{1}{2}$ siue $n>-\frac{1}{2}$, semper autem assumi conuenit $n+1>\frac{1}{2}$, quia aliae resistentiae hypotheses forent maxime absurdae. Hanc ob rem nobis superest haec aequatio

$$\int\frac{p\,dx}{\sqrt{(k-x)}}=-\frac{\sqrt{b}}{acc}\int\frac{z\,dP}{\sqrt{(cc-zz)}}=-\frac{a^{n-1}b}{cc}\int\frac{zzdz(cc-zz)^n}{\sqrt{(cc-zz)}}$$

ideoque integrando

$$\int\frac{p\,dx}{\sqrt{(k-x)}}=\frac{1}{n+\frac{1}{2}}\cdot\frac{a^{n-1}b}{cc}(cc-zz)^{n+\frac{1}{2}}-\frac{1}{n+\frac{1}{2}}a^{n-1}b.c^{2n-1}.$$

Facto igitur $z=c$, pro toto tempore prodibit

$$\int\frac{p\,dx}{\sqrt{(k-x)}}=-\frac{2}{2n+1}a^{n-1}.c^{2n-1}.b.$$

10. Quantitatem igitur p ita definiri oportet, vt formula $\int\frac{p\,dx}{\sqrt{(k-x)}}$ integrata positoque $x=k$, proveniat eadem quantitas

$$-\frac{2}{2n+1}.a^{n-1}c^{2n-1}b,$$

atque facile perspicitur, pro p accipi debere quandam potestatem ipsius x, quandoquidem k inesse nequit. Hunc in finem statuatur
$p=\theta x^m$ et adhibita substitutione $k=cc$ et $x=zz$ producitur

$$\int\frac{\theta x^m dx}{\sqrt{(k-x)}}=2\theta\int\frac{z^{2m+1}dz}{\sqrt{(cc-zz)}}=-\frac{2}{2n+1}a^{n-1}c^{2n-1}.b$$

fiat nunc $z=cv$, ita vt post integrationem poni debeat $v=1$ et adipiscimur

$$2\,\theta\,c^{2m+1}\int\frac{v^{2m+1}\,dv}{\sqrt{(1-vv)}}=-\frac{2}{2n+1}\,a^{n-1}\,c^{2n+1}\,b.$$

Hinc autem loco k penitus exulare debet littera c, quod euenit ſumendo $m=n-1$, eritque nunc

$$\theta\int\frac{v^{2n-1}\,dv}{\sqrt{(1-vv)}}=-\frac{1}{2n+1}\,a^{n-1}\,b.$$

Iam vero formula

$$\int\frac{v^{2n-1}\,dv}{\sqrt{(1-vv)}}$$

poſito poſt integrationem $v=1$, praebet certum numerum abſolutum, quem littera N indicemus, ita vt nunc etiam coefficientem conſequamur

$$\theta=-\frac{a^{n-1}\,b}{(2n+1)\,\mathrm{N}}.$$

11. Inuentis igitur litteris θ et m, prodit littera

$$p=-\frac{a^{n-1}.\,b}{(2n+1)\,\mathrm{N}}\,x^{n-1},$$

ſicque pro primo ordine approximationis littera λ ſimpliciter affecto adepti ſumus pro curua tautochrona hanc aequationem

$$ds=dx\sqrt{\tfrac{b}{x}}-\frac{\lambda\,a^{n-1}\,b.\,x^{n-1}\,dx}{(2n+1)\,\mathrm{N}}$$

qua primus gradus noſtrae approximationis continetur et quia medium rariſſimum eſſe aſſumitur, in hoc gradu acquieſcere poterimus. Interim tamen hinc ſatis intelligitur, quomodo ſequentem approximatiomationis gradum, quadrato $\lambda\lambda$ affectum, expediri oporteat.

12. Hinc si resistentia ipsi celeritati fuerit proportionalis ideoque exponens $2n = 1$ et $n = \frac{1}{2}$, aequatio pro Tautochrona hinc reperitur

$$ds = dx \sqrt{\frac{b}{x}} - \frac{\lambda b}{(2n+1) N a^{\frac{1}{2}}} \cdot \frac{dx}{\sqrt{x}}$$

qua aequatione manifesto cyclois exprimitur, id quod egregie conuenit. Sin autem resistentia sequatur quadratum celeritatis erit $n = 1$ et pro curua Tautochrona oritur

$$ds = dx \sqrt{\frac{b}{x}} - \frac{\lambda b dx}{3N},$$

id quod etiam conuenit cum Tautochrona pro hoc medio resistente inuenta.

13. Totum ergo negotium hic redit ad inventionem numeri N ex integratione formulae

$$\int \frac{v^{2n-1} dv}{\sqrt{(1-vv)}}$$

deriuandi, postquam scilicet positum fuerit $v = 1$, quare a casibus simplicissimis inchoemus. Ac *primo* quidem si fuerit $2n = 1$, prodit

$$N = \int \frac{dv}{\sqrt{(1-vv)}} = \frac{\pi}{2}.$$

Tum vero pro casu $2n = 2$ fit

$$N = \int \frac{v\,dv}{\sqrt{(1-vv)}} = 1.$$

Pro aliis casibus in subsidium vocetur haec reductio generalis:

$$\int \frac{v^{\nu+2} dv}{\sqrt{(1-vv)}} = -\frac{1}{\nu+2} v^{\nu+1} \sqrt{(1-vv)} + \frac{\nu+1}{\nu+2} \int \frac{v^{\nu} dv}{\sqrt{(1-vv)}},$$

quae

quae casu $v = 1$, quo opus habemus, praebet

$$\int \frac{v^{\nu+2}\, dv}{\sqrt{(1 - vv)}} = + \frac{\nu + 1}{\nu + 2} \int \frac{v^{\nu}\, dv}{\sqrt{(1 - vv)}},$$

vnde sequentem geminam tabellam valorum ipsius N deducimus;

$2n = 1$	$N = \frac{\pi}{2}$	$2n = 2$	$N = 1$
$= 3$	$N = \frac{1}{2} \cdot \frac{\pi}{2}$	$= 4$	$N = \frac{2}{3}$
$= 5$	$N = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{\pi}{2}$	$= 6$	$N = \frac{2}{3} \cdot \frac{4}{5}$
$= 7$	$N = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{\pi}{2}$	$= 8$	$N = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7}$
$= 9$	$N = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{\pi}{2}$	$= 10$	$N = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{8}{9}$

At si $2n$ non fuerit numerus integer, numerus N aliter definiri nequit, nisi per quadraturas curuarum altiores.

14. Ita ergo pro medio quocunque rarissimo, cuius resistentia rationem quamcunque multiplicatam celeritatis sequitur, Tautochronae tam pro descensu quam pro ascensu satis expedite sunt assignatae, quatenus scilicet prima approximatione sumus contenti. Verum haec adeo multo latius patent et Tautochronas inuenire licebit, si resistentia huiusmodi formula exprimatur

$$\lambda u^{2n} + \lambda' u^{2n'} + \lambda'' u^{2n''} \text{ etc.}$$

vbi coefficientes λ, λ', λ'' quam minimi reputentur, si enim exponentibus illis $2n$, $2n'$, $2n''$ etc. quaerantur numeri respondentes N, N', N'' aequatio pro curua Tautochrona descensus erit

$$ds = dx \sqrt{\frac{b}{x}} - \frac{\lambda a^{n-1} b . x^{n-1}\, dx}{(2n + 1) N} - \frac{\lambda' a^{n'-1} b x^{n'-1}\, dx}{(2n' + 1) N'} - \frac{\lambda'' a^{n''-1} b x^{n''-1}\, dx}{(2n'' + 1) N''} \text{ etc.}$$

sumitur

sumtis autem litteris λ, λ', λ'' negatiuis, haec aequatio Tautochronam ascensus declarabit.

15. Circa illas autem Tautochronas pro resistentiae hypothesibus simplicibus notandum est, ex Tautochrona descensus inueniri Tautochronam ascensus, si λ negatiue capiatur, vnde sufficiet pro singulis hypothesibus Tautochronas descensus assignasse, quae dum motus hac aequatione exprimitur

$$2\,u\,d\,u = -\,a\,d\,x + \lambda\,u^{2n}\,d\,s$$

sequenti modo se habebunt

Pro resistentia	
λu^0 vbi $n = 0$	$ds = dx\sqrt{\frac{b}{x}}$
λu^2 vbi $n = 1$	$ds = dx\sqrt{\frac{b}{x} - \frac{\lambda b\,dx}{3}}$
λu^4 vbi $n = 2$	$ds = dx\sqrt{\frac{b}{x} - \frac{3.1}{5.2}\lambda b a x\,dx}$
λu^6 vbi $n = 3$	$ds = dx\sqrt{\frac{b}{x} - \frac{1.3.5}{7.2.4}\lambda a b^2 x^2\,dx}$
λu^8 vbi $n = 4$	$ds = dx\sqrt{\frac{b}{x} - \frac{1.3.5.7}{9.2.4.6}\lambda b a^3 x^3\,dx}$
	etc.

Pro resistentia	
λu vbi $n = \frac{1}{2}$	$ds = dx\sqrt{\frac{b}{x} - \frac{\lambda b\,dx}{\pi\sqrt{ax}}}$
λu^3 vbi $n = \frac{3}{2}$	$ds = dx\sqrt{\frac{b}{x} - \frac{2\lambda b\,dx\sqrt{ax}}{2.\,\pi}}$
λu^5 vbi $n = \frac{5}{2}$	$ds = dx\sqrt{\frac{b}{x} - \frac{2.4}{3.3}\frac{\lambda b a^{3:2} x^{3:2}\,dx}{\pi}}$
λu^7 vbi $n = \frac{7}{2}$	$ds = dx\sqrt{\frac{b}{x} - \frac{2.4.6}{4.3.5}\frac{\lambda b a^{5:2} x^{5:2}}{\pi}}$
λu^9 vbi $n = \frac{9}{2}$	$ds = dx\sqrt{\frac{b}{x} - \frac{2.4.6.8}{5.3.5.7}\frac{\lambda b a^{7:2} x^{7:2}}{\pi}}$
	etc.

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