



1773

Exercitationes analyticae

Leonhard Euler

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EXERCITATIONES ANALYTICAE.

Auctore

L. EVLERO.

I.

Non parum notatu digna videtur ratio, quam inter summas harum serierum diuergentium

$$1 - 2^m + 3^m - 4^m + 5^m - \text{etc.}$$

et istarum conuergentium

$$1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \text{etc.}$$

intercedere olim obseruauit, et quae ita se habet:

$$1 - 2^0 + 3^0 - 4^0 + \text{etc.} = \frac{1}{2}$$

$$1 - 2^1 + 3^1 - 4^1 + \text{etc.} = \frac{1}{4} = + \frac{2 \cdot 1}{\pi^2} (1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.})$$

$$1 - 2^2 + 3^2 - 4^2 + \text{etc.} = \frac{0}{8}$$

$$1 - 2^3 + 3^3 - 4^3 + \text{etc.} = -\frac{2}{18} = -\frac{2 \cdot 1 \cdot 2 \cdot 3}{\pi^4} (1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.})$$

$$1 - 2^4 + 3^4 - 4^4 + \text{etc.} = \frac{0}{32}$$

$$1 - 2^5 + 3^5 - 4^5 + \text{etc.} = \frac{16}{64} = + \frac{2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{\pi^6} (1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \text{etc.})$$

$$1 - 2^6 + 3^6 - 4^6 + \text{etc.} = \frac{0}{128}$$

$$1 - 2^7 + 3^7 - 4^7 + \text{etc.} = -\frac{272}{256} = -\frac{2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}{\pi^8} (1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \text{etc.})$$

$$1 - 2^8 + 3^8 - 4^8 + \text{etc.} = \frac{0}{512}$$

$$1 - 2^9 + 3^9 - 4^9 + \text{etc.} = \frac{720}{1024} = + \frac{2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9}{\pi^{10}} (1 + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \frac{1}{7^{10}} + \text{etc.})$$

Y 3

vbi

vbi π denotat peripheriam circuli, cuius diameter $= 1$.

2. Hinc concludere licet, in genere inter has series infinitas

$$1 - 2^{n-1} + 3^{n-1} - 4^{n-1} + \text{etc. et } 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \text{etc.}$$

huiusmodi relationem locum habere, vt fit

$$1 - 2^{n-1} + 3^{n-1} - 4^{n-1} + \text{etc.}$$

$$= \frac{2 \cdot 1 \cdot 2 \cdot 3 \dots (n-1)}{\pi^n} N \left(1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \text{etc.} \right)$$

vbi quidem nouimus, quoties n fuerit numerus impar excepto casu $n = 1$, fore $N = 0$; quoties autem sit n numerus par, esse N vel $+1$ vel -1 . Scilicet si sit n numerus impariter par formae $4m + 2$, erit $N = +1$, si autem n numerus pariter par formae $4m$, erit $N = -1$. Vnde cuiusmodi functio N sit ipsius n haud difficulter conijicere licebit; cum

si $n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13$ etc.
sit $N = +1, 0, -1, 0, +1, 0, -1, 0, +1, 0, -1, 0$ etc.

3. Neque etiam, si rem attentius perpendamus, casus $n = 1$ huic legi aduersatur, qua fieri debet $N = 0$; nihil enim impedit, quo minus hanc aequalitatem admittamus:

$$1 - 2^0 + 3^0 - 4^0 + \text{etc.} = \frac{2}{\pi} 0 \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \text{etc.} \right)$$

quandoquidem seriei

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \text{etc.}$$

summa

summa est infinita, unde utique fieri potest $\frac{2}{\pi} \cdot \infty = 1$
seu summae seriei

$$1 - 1 + 1 - 1 + \text{etc.}$$

Quamobrem sine vlla exceptione

si fuerit $n = 1, 2, 3, 4, 5, 6, 7, 8, 9$ etc.

erit $N = 0, +1, 0, -1, 0, +1, 0, -1, 0$ etc.

cui quidem legi innumerae formulae pro N assumendae satisfacere possunt. Verum dubitare non licet, quin simplicissima maximeque naturalis hic locum inueniat, quae est $N = \cos. \frac{n-2}{2} \pi$, denotante hic π angulum duobus rectis aequalem, quoniam sinus totus $= 1$ assumitur, ut π sit semicircumferentia circuli.

4. Admissa ergo hac coniectura, habebimus in genere:

$$1 - 2^{n-1} + 3^{n-1} - 4^{n-1} + \text{etc.} = 2 \cos. \frac{n-2}{2} \pi \cdot \frac{1 \cdot 2 \cdot 3 \dots (n-1)}{\pi^n}$$

$$\left(1 + \frac{1}{3^n} + \frac{1}{5^n} + \text{etc.}\right)$$

sive conuertendo:

$$1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \text{etc.} = \frac{1}{2 \cos. \frac{n-2}{2} \pi \cdot 1 \cdot 2 \cdot 3 \dots (n-1)}$$

$$\left(1 - 2^{n-1} + 3^{n-1} - 4^{n-1} + \text{etc.}\right)$$

atque ex praecedentibus manifestum est, hanc aequalitatem reuera consistere quoties fuerit n numerus par, neque etiam a veritate discedere casibus, quibus n est numerus impar. Quare si ea vera sit pro
cas-

casibus, quibus n est numerus fractus, formulae
 $1. 2. 3 \dots (n-1)$ valores per interpolationem as-
 signari debent, qui quidem pro semilibus ita se
 habent:

$$\text{si } n-1 = \frac{1}{2}, \quad \frac{3}{2}, \quad \frac{5}{2}, \quad \frac{7}{2}, \quad \frac{9}{2}, \quad \text{etc.}$$

$$\text{sunt } \frac{1}{2}\sqrt{\pi}; \frac{1 \cdot 3}{2 \cdot 2}\sqrt{\pi}; \frac{1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 2}\sqrt{\pi}; \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 2 \cdot 2}\sqrt{\pi}; \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}\sqrt{\pi} \text{ etc.}$$

$$\text{et } \cos \frac{(n-2)\pi}{2} = \frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}; -\frac{1}{\sqrt{2}}; -\frac{1}{\sqrt{2}}; +\frac{1}{\sqrt{2}}.$$

5. Pro his ergo casibus habebimus:

$$1 + \frac{1}{3\sqrt{3}} + \frac{1}{5\sqrt{5}} + \frac{1}{7\sqrt{7}} + \text{etc.} = + \frac{\sqrt{2}}{1} \pi (1 - \sqrt{2} + \sqrt{3} - \sqrt{4} + \sqrt{5} - \text{etc.})$$

$$1 + \frac{1}{3^2\sqrt{3}} + \frac{1}{5^2\sqrt{5}} + \frac{1}{7^2\sqrt{7}} + \text{etc.} = + \frac{2\sqrt{2}}{1 \cdot 3} \pi^2 (1 - 2\sqrt{2} + 3\sqrt{3} - 4\sqrt{4} + \text{etc.})$$

$$1 + \frac{1}{3^3\sqrt{3}} + \frac{1}{5^3\sqrt{5}} + \frac{1}{7^3\sqrt{7}} + \text{etc.} = - \frac{4\sqrt{2}}{1 \cdot 3 \cdot 5} \pi^3 (1 - 2^2\sqrt{2} + 3^2\sqrt{3} - 4^2\sqrt{4} + \text{etc.})$$

$$1 + \frac{1}{3^4\sqrt{3}} + \frac{1}{5^4\sqrt{5}} + \frac{1}{7^4\sqrt{7}} + \text{etc.} = - \frac{8\sqrt{2}}{1 \cdot 3 \cdot 5 \cdot 7} \pi^4 (1 - 2^3\sqrt{2} + 3^3\sqrt{3} - 4^3\sqrt{4} + \text{etc.})$$

$$1 + \frac{1}{3^5\sqrt{3}} + \frac{1}{5^5\sqrt{5}} + \frac{1}{7^5\sqrt{7}} + \text{etc.} = + \frac{16\sqrt{2}}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9} \pi^5 (1 - 2^4\sqrt{2} + 3^4\sqrt{3} - 4^4\sqrt{4} + \text{etc.})$$

etc.

quae aequalitates, an absolute sint verae, pertinaci-
 ter asseuerare non ausim; scrutari ergo conuenit,
 num seriebus per approximationem summatis satis-
 fiat; ac pro prima quidem colligimus

$$1 - \sqrt{2} + \sqrt{3} - \sqrt{4} + \sqrt{5} - \text{etc.} = 0, 380317 \text{ proxime}$$

qui numerus per $\pi\sqrt{2}$ multiplicatur dat 1, 689665,
 cui summa seriei

$$1 + \frac{1}{3\sqrt{3}} + \frac{1}{5\sqrt{5}} + \frac{1}{7\sqrt{7}} \text{ etc.}$$

proxime aequalisprehenditur.

6. Quoniam autem numeris imparibus pro n
 accipiendis hinc nihil concludi posse videtur, propte-
 rea

rea quod alterum membrum nostrae aequationis abit in ∞ ; vt hos valores inuestigemus, pro n ponamus numerum infinite parum excedentem numerum integrum, seu scribamus $n + \omega$ loco n , denotante ω fractionem infinite paruum; atque habebimus:

$$1 + \frac{1}{3^{n+\omega}} + \frac{1}{5^{n+\omega}} + \frac{1}{7^{n+\omega}} + \text{etc.} = \frac{1}{2 \operatorname{cof.} \frac{n-2+\omega}{2} \pi \cdot 1 \cdot 2 \cdot \dots \cdot (n-1+\omega)} \frac{\pi^{n+\omega}}{(1 - 2^{n-1+\omega} + 3^{n-1+\omega} - 4^{n-1+\omega} + \text{etc.})}$$

Hic igitur primo obseruo esse

$$\frac{1}{a^{n+\omega}} = a^{-n-\omega} = a^{-n} (1 - \omega \log a)$$

vbi logarithmi naturales seu hyperbolici sunt intelligendi, ita vt fit

$$\frac{1}{a^{n+\omega}} = \frac{1}{a^n} - \frac{\omega \log a}{a^n}$$

Simili modo erit

$$a^{n-1+\omega} = a^{n-1} + a^{n-1} \omega \log a, \text{ et } \pi^{n+\omega} = \pi^n (1 + \omega \log \pi):$$

tum vero fit

$$\operatorname{cof.} \frac{n-2+\omega}{2} \pi = \operatorname{cof.} \frac{n-2}{2} \pi - \frac{1}{2} \omega \log \pi \operatorname{fin.} \frac{n-2}{2} \pi.$$

Denique cum ostenderim olim formulae $1 \cdot 2 \cdot \dots \cdot (n-1+\omega)$ valorem casu $n=1$ esse $= 1 - 0,57721566 \omega$, si scribamus breuitatis gratia $\lambda = 0,577216649015325$

sumendo $n=1; 2; 3; 4; 5$ etc.

fit $1 \cdot 2 \cdot \dots \cdot (n-1+\omega) = 1 - \lambda \omega; 1 + (1-\lambda)\omega; 2 + (3-2\lambda)\omega; 6 + (11-6\lambda)\omega; 24 + (50-24\lambda)\omega$ etc.

7. Consideremus hinc potissimum casum $n=3$, quandoquidem haec series

$$1 + \frac{1}{5^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.}$$

Tom. XVII. Nou. Comm.

Z

ita

ita est comparata, vt omnes adhuc labores ad eius summam inuestigandam frustra suscepti sint. Cum igitur fit $\cos. \frac{n-2}{2} \pi = 0$ et $\sin. \frac{n-2}{2} \pi = 1$, nostra aequatio hanc induet formam:

$$\left. \begin{aligned} &1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} \\ &-\omega(l1 + \frac{l3}{3^3} + \frac{l5}{5^3} + \frac{l7}{7^3} + \text{etc.}) \end{aligned} \right\} = \frac{-1 \pi^2 (1 + \omega / \pi)}{\pi \omega 2 + (3 - 2\lambda)\omega} \left. \begin{aligned} &1 - 2^2 + 3^2 - 4^2 + \text{etc.} \\ &-\omega(2^2 l2 - 3^2 l3 + 4^2 l4 - \text{etc.}) \end{aligned} \right\}$$

Verum quia

$$1 - 2^2 + 3^2 - 4^2 + \text{etc.} = 0,$$

posito $\omega = 0$, habebimus

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} = \frac{1}{3} \pi^2 (2^2 l2 - 3^2 l3 + 4^2 l4 - 5^2 l5 + \text{etc.})$$

ficque ad scopum pertingeremus, si huius seriei logarithmicae

$$2^2 l2 - 3^2 l3 + 4^2 l4 - 5^2 l5 + \text{etc.}$$

summam assignare liceret. Simili autem modo pro reliquis potestatibus reperitur

$$1 + \frac{1}{3^5} + \frac{1}{5^5} + \frac{1}{7^5} + \text{etc.} = \frac{-\pi^4}{1 \cdot 2 \cdot 3 \cdot 4} (2^4 l2 - 3^4 l3 + 4^4 l4 - 5^4 l5 + \text{etc.})$$

$$1 + \frac{1}{3^7} + \frac{1}{5^7} + \frac{1}{7^7} + \text{etc.} = \frac{-\pi^6}{1 \cdot 2 \cdot \dots \cdot 6} (2^6 l2 - 3^6 l3 + 4^6 l4 - 5^6 l5 + \text{etc.})$$

$$1 + \frac{1}{3^9} + \frac{1}{5^9} + \frac{1}{7^9} + \text{etc.} = \frac{-\pi^8}{1 \cdot 2 \cdot \dots \cdot 8} (2^8 l2 - 3^8 l3 + 4^8 l4 - 5^8 l5 + \text{etc.})$$

8. Proposita ergo nobis fit haec series infinita:

$$2^2 l2 - 3^2 l3 + 4^2 l4 - 5^2 l5 + 6^2 l6 - 7^2 l7 + \text{etc.} = Z$$

$$\text{vt fiat } 1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} = \frac{1}{3} \pi \pi Z,$$

et quo minus de eius summa desperemus, notetur esse

$$l2 - l3 + l4 - l5 + l6 - \text{etc.} = \frac{\pi}{2} l \frac{\pi}{2}.$$

Illa

Illa autem series Z in plures formas transmutari potest, veluti

$$Z = 12 - 3l_{\frac{2}{2}}^{\frac{2}{2}} + 6l_{\frac{2}{2}}^{\frac{4}{2}} - 10l_{\frac{2}{2}}^{\frac{6}{2}} + 15l_{\frac{2}{2}}^{\frac{8}{2}} - 21l_{\frac{2}{2}}^{\frac{10}{2}} + \text{etc. et}$$

$$Z = l_{\frac{1.2}{1.2}}^{\frac{2.2}{1.2}} + 4l_{\frac{2.4}{2.4}}^{\frac{4.4}{2.4}} + 9l_{\frac{3.6}{3.6}}^{\frac{6.6}{3.6}} + 16l_{\frac{4.8}{4.8}}^{\frac{8.8}{4.8}} + 25l_{\frac{5.10}{5.10}}^{\frac{10.10}{5.10}} \text{ etc.}$$

$$- 2l_{\frac{2.4}{2.4}}^{\frac{3.5}{2.4}} - 6l_{\frac{4.6}{4.6}}^{\frac{5.5}{4.6}} - 12l_{\frac{6.8}{6.8}}^{\frac{7.7}{6.8}} - 20l_{\frac{8.10}{8.10}}^{\frac{9.9}{8.10}} - \text{etc.}$$

Si enim in genere ponamus

$$Z = \alpha l_{\frac{1.2}{1.2}}^{\frac{2.2}{1.2}} - \beta l_{\frac{2.4}{2.4}}^{\frac{3.3}{2.4}} + \gamma l_{\frac{3.6}{3.6}}^{\frac{4.4}{3.6}} - \delta l_{\frac{4.8}{4.8}}^{\frac{5.5}{4.8}} + \varepsilon l_{\frac{5.10}{5.10}}^{\frac{6.6}{5.10}} - \zeta l_{\frac{6.8}{6.8}}^{\frac{7.7}{6.8}} \text{ etc.}$$

esse debet $2\alpha + \beta = 4$ hincque $\beta = 4 - 2\alpha$ $\zeta = 18 - 6\alpha$

$$\alpha + 2\beta + \gamma = 9 \quad \gamma = 1 + 3\alpha \quad \eta = 9 + 7\alpha$$

$$\beta + 2\gamma + \delta = 16 \quad \delta = 10 - 4\alpha \quad \theta = 28 - 8\alpha$$

$$\gamma + 2\delta + \varepsilon = 25 \quad \varepsilon = 4 + 5\alpha \quad \varepsilon = 15 + 9\alpha$$

hic vero sumimus $\alpha = 1$, ut progressio maxime fiat regularis.

9. Haec posterior forma maxime ad institutum nostrum videtur accommodata, quoniam logarithmi in series convergentes resolvuntur. Hunc in finem pro terminis positivis hac resolutione utar; cum quilibet hac forma contineatur

$$x x l_{\frac{4 x x}{4 x x - 1}}^{\frac{4 x x}{4 x x - 1}} = - x x l \left(1 - \frac{1}{4 x x} \right)$$

inde nascitur haec series infinita:

$$x x \left(\frac{1}{4 x x} + \frac{1}{2 \cdot 2^2 x^4} + \frac{1}{3 \cdot 2^6 x^6} + \frac{1}{4 \cdot 2^8 x^8} + \text{etc.} \right) \text{ seu haec}$$

$$\frac{1}{2^2} + \frac{1}{2 \cdot 2^4} \cdot \frac{1}{x x} + \frac{1}{3 \cdot 2^6} \cdot \frac{1}{x^4} + \frac{1}{4 \cdot 2^8} \cdot \frac{1}{x^6} + \frac{7}{5 \cdot 2^{10}} \cdot \frac{1}{x^8} + \text{etc.}$$

Pro terminis autem negativis forma generalis est

$$- x(x+1) l_{\frac{(2x+1)^2}{4x(x+1)}}^{\frac{(2x+1)^2}{4x(x+1)}} = - x(x+1) l \left(1 + \frac{1}{4x(x+1)} \right),$$

Z 2

quae

quae resoluitur in hanc seriem :

$$-\frac{1}{2^2} + \frac{1}{2 \cdot 2^4} \cdot \frac{1}{x(x+1)} - \frac{1}{3 \cdot 2^6} \cdot \frac{1}{x^2(x+1)^2} + \frac{1}{4 \cdot 2^8} \cdot \frac{1}{x^3(x+1)^3} + \text{etc.}$$

vnde valor ipsius Z in has series transformatur :

$$\begin{aligned} Z = & \frac{1}{2^2} (1 - 1 + 1 - 1 + \text{etc.}) + \frac{1}{2 \cdot 2^4} (1 + \frac{1}{1 \cdot 2} + \frac{1}{2^2} + \frac{1}{2 \cdot 3} + \frac{1}{3^2} + \frac{1}{3 \cdot 4} + \text{etc.}) \\ & + \frac{1}{3 \cdot 2^6} (1 - \frac{1}{1^2 \cdot 2^2} + \frac{1}{2^4} - \frac{1}{2^2 \cdot 3^2} + \text{etc.}) + \frac{1}{4 \cdot 2^8} (1 + \frac{1}{1^3 \cdot 2^3} + \frac{1}{2^6} + \frac{1}{2^3 \cdot 3^3} + \frac{1}{3^6} + \text{etc.}) \\ & + \frac{1}{5 \cdot 2^{10}} (1 - \frac{1}{1^4 \cdot 2^4} + \frac{1}{2^8} - \frac{1}{2^4 \cdot 3^4} + \text{etc.}) + \frac{1}{6 \cdot 2^{12}} (1 + \frac{1}{1^5 \cdot 2^5} + \frac{1}{2^{10}} + \frac{1}{2^5 \cdot 3^5} + \frac{1}{3^{10}} + \text{etc.}) \end{aligned}$$

10. Quodsi nunc breuitatis gratia ponamus :

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \text{etc.} = \alpha \pi^2 ;$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{etc.} = \beta \pi^4 ;$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \text{etc.} = \gamma \pi^6$$

$$1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \text{etc.} = \delta \pi^8$$

vbi quidem numeri α , β , γ , δ etc. sunt cogniti ; et quia

$$1 - 1 + 1 - 1 \text{ etc.} = \frac{1}{2} \text{ erit}$$

$$\begin{aligned} Z = & \frac{1}{2^2} \cdot \frac{1}{2} + \frac{1}{2 \cdot 2^4} (\alpha \pi^2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \text{etc.}) \\ & + \frac{1}{3 \cdot 2^6} (\beta \pi^4 - \frac{1}{2^2} - \frac{1}{6^2} - \frac{1}{12^2} - \text{etc.}) + \frac{1}{4 \cdot 2^8} (\gamma \pi^6 + \frac{1}{2^3} + \frac{1}{6^3} + \frac{1}{12^3} + \text{etc.}) \\ & + \frac{1}{5 \cdot 2^{10}} (\delta \pi^8 - \frac{1}{2^4} - \frac{1}{6^4} - \frac{1}{12^4} - \text{etc.}) + \frac{1}{6 \cdot 2^{12}} (\epsilon \pi^{10} + \frac{1}{2^5} + \frac{1}{6^5} + \frac{1}{12^5} + \text{etc.}) \end{aligned}$$

vbi iam totum negotium ad summationem harum serierum

$$\frac{1}{2^n} + \frac{1}{6^n} + \frac{1}{12^n} + \frac{1}{20^n} + \text{etc.}$$

est reductum, quarum potestatum radices 2, 6, 12, 20 etc. sunt numeri pronic.

11. Huius autem seriei singuli termini, quorum forma est $\frac{1}{x^n(x+1)^n}$ in partes simplicium potestatem resolui possunt, quæ ita se habent:

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

$$\frac{1}{x^2(x+1)^2} = \frac{1}{x^2} + \frac{1}{(x+1)^2} - 2\left(\frac{1}{x} - \frac{1}{x+1}\right)$$

$$\frac{1}{x^3(x+1)^3} = \frac{1}{x^3} - \frac{1}{(x+1)^3} - 3\left(\frac{1}{x^2} + \frac{1}{(x+1)^2}\right) + \frac{6}{1 \cdot 2}\left(\frac{1}{x} - \frac{1}{x+1}\right)$$

$$\frac{1}{x^4(x+1)^4} = \frac{1}{x^4} + \frac{1}{(x+1)^4} - 4\left(\frac{1}{x^3} + \frac{1}{(x+1)^3}\right) + \frac{6 \cdot 5}{1 \cdot 2}\left(\frac{1}{x^2} + \frac{1}{(x+1)^2}\right) - \frac{4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3}\left(\frac{1}{x} - \frac{1}{x+1}\right)$$

Cum iam summas præfixo signo \int indicando sit

$$\int \frac{1}{(x+\frac{1}{2})^n} = \int \frac{1}{x^n} - 1, \text{ erit}$$

$$\int \frac{1}{x(x+1)} = 1,$$

$$\int \frac{1}{x^2(x+1)^2} = 2 \int \frac{1}{x^2} - 1 - 2$$

$$\int \frac{1}{x^3(x+1)^3} = 1 - 3\left(2 \int \frac{1}{x^2} - 1\right) + \frac{6 \cdot 5}{1 \cdot 2}$$

$$\int \frac{1}{x^4(x+1)^4} = 2 \int \frac{1}{x^4} - 1 - 4 + \frac{4 \cdot 5}{1 \cdot 2}\left(2 \int \frac{1}{x^2} - 1\right) - \frac{4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3}$$

12. In singulis his expressionibus numeros absolutos commode in vnum colligere licet, et cum deinde sit

$$\int \frac{1}{x^2} = a \pi^2; \int \frac{1}{x^3} = \beta \pi^3; \int \frac{1}{x^4} = \gamma \pi^4; \int \frac{1}{x^5} = \delta \pi^5 \text{ etc.}$$

habebimus

$$\int \frac{1}{x(x+1)} = 1$$

$$\int \frac{1}{x^2(x+1)^2} = 2\alpha\pi^2 - \frac{2 \cdot 3}{1 \cdot 2}$$

$$\int \frac{1}{x^3(x+1)^3} = -3 \cdot 2\alpha\pi^2 + \frac{3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3}$$

$$\int \frac{1}{x^4(x+1)^4} = 2\beta\pi^4 + \frac{4 \cdot 5}{1 \cdot 2} 2\alpha\pi^2 - \frac{4 \cdot 5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4}$$

$$\int \frac{1}{x^5(x+1)^5} = -5 \cdot 2\beta\pi^4 - \frac{5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3} 2\alpha\pi^2 + \frac{5 \cdot 6 \cdot 7 \cdot 8 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

$$\int \frac{1}{x^6(x+1)^6} = 2\gamma\pi^6 + \frac{6 \cdot 7}{1 \cdot 2} 2\beta\pi^4 + \frac{6 \cdot 7 \cdot 8 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} 2\alpha\pi^2 - \frac{6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$$

etc.

vbi haec reductio notatu digna est obseruanda :

$$1 + \frac{n}{1} + \frac{n(n+1)}{1 \cdot 2} + \dots + \frac{n(n+1)\dots(2n-2)}{1 \cdot 2 \dots (n-1)} = \frac{n(n+1)(n+2)\dots(2n-1)}{1 \cdot 2 \cdot 3 \dots n}$$

fit enim summa ex lege nota

$$= \frac{(n+1)(n+2)(n+3)\dots(2n-1)}{1 \cdot 2 \cdot 3 \dots (n-1)}$$

13. His igitur valoribus substitutis obtinebimus :

$$\begin{aligned} Z = & \frac{1}{2^2} \cdot \frac{1}{2} + \frac{1}{2 \cdot 2^2} (\alpha\pi^2 + 1) + \frac{1}{3 \cdot 2^3} (\beta\pi^4 + \frac{2 \cdot 3}{1 \cdot 2} - 2\alpha\pi^2) + \frac{1}{4 \cdot 2^4} (\gamma\pi^6 + \frac{3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3} - \frac{3}{1} \cdot 2\alpha\pi^2) \\ & + \frac{1}{5 \cdot 2^5} (\delta\pi^8 + \frac{4 \cdot 5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{4 \cdot 5}{1 \cdot 2} 2\alpha\pi^2 - 2\beta\pi^4) + \frac{1}{6 \cdot 2^6} (\epsilon\pi^{10} + \frac{5 \cdot 6 \cdot 7 \cdot 8 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3} 2\alpha\pi^2 - \frac{1}{3} \cdot 2\beta\pi^4) \\ & + \frac{1}{7 \cdot 2^7} (\zeta\pi^{12} + \frac{6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \frac{6 \cdot 7 \cdot 8 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} 2\alpha\pi^2 - \frac{6 \cdot 7}{1 \cdot 2} 2\beta\pi^4 - 2\gamma\pi^6) + \text{etc.} \end{aligned}$$

quae expressio in has series resoluitur :

$$\begin{aligned} Z = & \frac{1}{2^2 \cdot 2} + \frac{1}{2 \cdot 2^4} - \frac{2 \cdot 3}{2 \cdot 3 \cdot 2^6} + \frac{3 \cdot 4 \cdot 5}{2 \cdot 3 \cdot 4 \cdot 2^8} + \frac{4 \cdot 5 \cdot 6 \cdot 7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 2^{10}} + \frac{5 \cdot 6 \cdot 7 \cdot 8 \cdot 9}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 2^{12}} + \text{etc.} \\ & + \frac{\alpha\pi^2}{2 \cdot 2^4} + \frac{\beta\pi^4}{3 \cdot 2^6} + \frac{\gamma\pi^6}{4 \cdot 2^8} + \frac{\delta\pi^8}{5 \cdot 2^{10}} + \frac{\epsilon\pi^{10}}{6 \cdot 2^{12}} + \text{etc.} \\ & - 2\alpha\pi^2 \left(\frac{1}{3 \cdot 2^6} + \frac{3}{1 \cdot 4 \cdot 2^8} + \frac{4 \cdot 5}{1 \cdot 2 \cdot 5 \cdot 2^{10}} + \frac{5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 6 \cdot 2^{12}} + \frac{6 \cdot 7 \cdot 8 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 7 \cdot 2^{14}} \right) \\ & - 2\beta\pi^4 \left(\frac{1}{5 \cdot 2^{10}} + \frac{5}{1 \cdot 6 \cdot 2^{12}} + \frac{6 \cdot 7}{1 \cdot 2 \cdot 7 \cdot 2^{14}} + \frac{7 \cdot 8 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 6 \cdot 2^{16}} + \frac{8 \cdot 9 \cdot 10 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 9 \cdot 2^{18}} \right) \text{ etc.)} \\ & - 2\gamma\pi^6 \left(\frac{1}{7 \cdot 2^{14}} + \frac{7}{1 \cdot 8 \cdot 2^{16}} + \frac{8 \cdot 9}{1 \cdot 2 \cdot 9 \cdot 2^{18}} + \frac{9 \cdot 10 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 10 \cdot 2^{20}} + \frac{10 \cdot 11 \cdot 12 \cdot 13}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 11 \cdot 2^{22}} + \text{etc.)} \right) \end{aligned}$$

etc.

14.

14. Hinc deducimur ad istam feriem infinitam generalem, quae illas series numericas omnes in se complectitur :

$$\frac{1}{n \cdot 2^{2n}} + \frac{n}{(n+1) 2^{2n+2}} + \frac{(n+1)(n+2)}{2(n+2) 2^{2n+4}} + \frac{(n+2)(n+3)(n+4)}{2 \cdot 3 \cdot (n+3) 2^{2n+6}} \\ + \frac{(n+3)(n+4)(n+5)(n+6)}{2 \cdot 3 \cdot 4 (n+4) 2^{2n+8}} + \text{etc.}$$

cuius igitur summam inuestigari oportet. Quodsi enim huius seriei summam in genere hoc signo $S(n)$ indicemus, habebimus

$$Z = -\frac{1}{2} + S(1) + 2\alpha \pi^2 \left(\frac{1}{4 \cdot 2^4} - S(3)\right) + 2\beta \pi^4 \left(\frac{1}{6 \cdot 2^6} - S(5)\right) \\ + 2\gamma \pi^6 \left(\frac{1}{8 \cdot 2^8} - S(7)\right) + 2\delta \pi^8 \left(\frac{1}{10 \cdot 2^{10}} - S(9)\right) + \text{etc.}$$

Series autem nostra generalis ita commodius exhiberi potest

$$n(n+1) S(n) = \frac{n+1}{2^{2n}} + \frac{nn}{2^{2n+2}} + \frac{n(n+1)(n+1)}{2 \cdot 2^{2n+4}} + \frac{n(n+1)(n+2)(n+4)}{2 \cdot 3 \cdot 2^{2n+6}} \\ + \frac{n(n+1)(n+3)(n+5)(n+6)}{2 \cdot 3 \cdot 4 \cdot 2^{2n+8}} + \frac{n(n+1)(n+4)(n+6)(n+7)(n+8)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 2^{2n+10}} + \text{etc.}$$

vbi denominatores numero n carent. Possunt etiam singuli termini ita per factores repraesentari, vt statuatur :

$$S(n) = A + AB + ABC + ABCD + ABCDE + \text{etc.}$$

eritque

$$A = \frac{1}{n \cdot 2^{2n}}; B = \frac{nn}{4(n+1)}; C = \frac{(n+1)(n+1)}{4 \cdot 2n}; D = \frac{(n+2)(n+4)}{4 \cdot 3(n+1)}; \\ E = \frac{(n+3)(n+5)(n+6)}{4 \cdot 4(n+2)(n+4)}; F = \frac{(n+4)(n+7)(n+8)}{4 \cdot 5(n+3)(n+5)} \text{ etc.}$$

vbi

vbi factor in genere hanc habet formam

$$\frac{(n+\lambda-1)(n+2\lambda-3)(n+2\lambda-2)}{4\lambda(n+\lambda-2)(n+\lambda)}$$

15. Incipiamus a casu simplicissimo $n=1$,
et quia factor in genere est

$$= \frac{\lambda(2\lambda-2)(2\lambda-1)}{4\lambda(\lambda-1)(\lambda+1)} = \frac{2\lambda-1}{2\lambda+2}; \text{ erit}$$

$$A = \frac{1}{4}; B = \frac{1}{8}; C = \frac{5}{8}; D = \frac{7}{8}; E = \frac{9}{16}; F = \frac{9}{16} \text{ etc.}$$

unde fit

$$S(1) = \frac{1}{4} + \frac{1}{8} \left(1 + \frac{5}{8} + \frac{3 \cdot 5}{6 \cdot 7} + \frac{3 \cdot 5 \cdot 7}{6 \cdot 8 \cdot 10} + \frac{3 \cdot 5 \cdot 7 \cdot 9}{6 \cdot 8 \cdot 10 \cdot 12} + \text{etc.} \right)$$

Cum autem fit

$$V(1-1) = 1 - \frac{1}{2} - \frac{1 \cdot 1}{2 \cdot 4} - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} - \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} - \text{etc.} = 0$$

$$\text{erit } 1 + \frac{5}{8} + \frac{3 \cdot 5}{6 \cdot 8} + \frac{3 \cdot 5 \cdot 7}{6 \cdot 8 \cdot 10} + \text{etc.} = \frac{2 \cdot 4}{1 \cdot 1} \left(1 - \frac{1}{4} \right) = 4$$

$$\text{ideoque } S(1) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8} \text{ et } -\frac{1}{8} + S(1) = \frac{1}{4}.$$

16. Quo autem summas reliquarum serierum
facilius definire queamus, loco $\frac{1}{2}$ scribamus x , ut
fit $x = \frac{1}{4}$, et cum habeamus:

$$S(n) = \frac{1}{n} x^n + \frac{n}{(n+1)} x^{n+1} + \frac{(n+1)(n+2)}{2(n+2)} x^{n+2} + \frac{(n+2)(n+3)(n+4)}{2 \cdot 3(n+3)} x^{n+3} + \text{etc.}$$

quae casu $n=1$ abit in

$$S(1) = x + \frac{1}{2} x x + \frac{3}{2 \cdot 3} x^3 + \frac{3 \cdot 4 \cdot 5}{2 \cdot 3 \cdot 4} x^4 + \frac{4 \cdot 5 \cdot 6 \cdot 7}{2 \cdot 3 \cdot 4 \cdot 5} x^5 + \text{etc.}$$

$$\text{seu } S(1) = x + \frac{1}{2} x x + x^3 + \frac{5}{2} x^4 + \frac{6 \cdot 7}{2 \cdot 3} x^5 + \frac{7 \cdot 8 \cdot 9}{2 \cdot 3 \cdot 4} x^6 + \frac{3 \cdot 9 \cdot 10 \cdot 11}{2 \cdot 3 \cdot 4 \cdot 5} x^7 + \text{etc.}$$

$$\text{vel } S(1) = x + \frac{1}{2} x x (1 + \frac{2}{1} x + \frac{2 \cdot 5}{1 \cdot 2} x x + \frac{2 \cdot 5 \cdot 14}{1 \cdot 2 \cdot 3} x^3 + \frac{2 \cdot 5 \cdot 14}{2 \cdot 3 \cdot 4 \cdot 5} x^4 + \text{etc.})$$

$$\text{vel } S(1) = x + \frac{2 \cdot 2}{2} x x (1 + \frac{3}{2} 4 x + \frac{3 \cdot 5}{6 \cdot 8} 4^2 x^2 + \frac{3 \cdot 5 \cdot 7}{6 \cdot 8 \cdot 10} 4^3 x^3 + \frac{3 \cdot 5 \cdot 7 \cdot 9}{6 \cdot 8 \cdot 10 \cdot 12} 4^4 x^4 + \text{etc.})$$

At est

$$V(1-4x) = 1 - \frac{1}{2} 4x - \frac{1 \cdot 1}{2 \cdot 4} 4^2 x^2 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} 4^3 x^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} 4^4 x^4 - \text{etc.}$$

$$\text{unde } \frac{1 \cdot 1}{2 \cdot 4} 4^2 x^2 (1 + \frac{5}{2} 4 x + \frac{3 \cdot 5}{6 \cdot 8} 4^2 x^2 + \text{etc.}) = 1 - 2x - V(1-4x)$$

ergo

$$\text{ergo } S(1) = x + \frac{1-2x-\sqrt{1-4x}}{4} = \frac{1+2x-\sqrt{1-4x}}{4}$$

ficqueposito $x = \frac{1}{4}$ fit $S(1) = \frac{1}{4}(1 + \frac{1}{2}) = \frac{3}{8}$ vt supra.

17. Ponamus nunc $n = 3$, fitque $S(3) = Q$,
existente $S(1) =$

$$P = \frac{1+2x-\sqrt{1-4x}}{4}, \text{ ita vt fit}$$

$$P = x + \frac{1}{2}xx + \frac{2 \cdot 3}{2 \cdot 3}x^3 + \frac{3 \cdot 4 \cdot 5}{2 \cdot 3 \cdot 4}x^4 + \frac{4 \cdot 5 \cdot 6 \cdot 7}{2 \cdot 3 \cdot 4 \cdot 5}x^5 + \text{etc.}$$

$$Q = \frac{1}{5}x^5 + \frac{2}{4}x^4 + \frac{4 \cdot 5}{2 \cdot 3}x^3 + \frac{5 \cdot 6 \cdot 7}{2 \cdot 3 \cdot 4}x^2 + \frac{6 \cdot 7 \cdot 8 \cdot 9}{2 \cdot 3 \cdot 4 \cdot 7}x^7 + \text{etc.}$$

Ex his colligitur:

$$Pxx - Q = \frac{2}{3}x^3 - \frac{1}{4}x^4 - \frac{2 \cdot 3 \cdot 5}{2 \cdot 3 \cdot 5}x^5 - \frac{3 \cdot 4 \cdot 5 \cdot 8}{2 \cdot 3 \cdot 4 \cdot 5}x^6 - \frac{4 \cdot 5 \cdot 6 \cdot 7 \cdot 11}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 7}x^7 - \frac{5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 14}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 8}x^8 - \text{etc.}$$

hincque differentians

$$2Px + \frac{xxdP}{dx} - \frac{dQ}{dx} = 2xx - x^3 - \frac{2 \cdot 3}{2 \cdot 3}5x^4 - \frac{3 \cdot 4 \cdot 5}{2 \cdot 3 \cdot 4}8x^5 - \frac{4 \cdot 5 \cdot 6 \cdot 7}{2 \cdot 3 \cdot 4 \cdot 5}11x^6 - \frac{5 \cdot 6 \cdot 7 \cdot 8 \cdot 9}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}14x^7$$

$$\text{at } \frac{xxdP}{dx} = xx + x^3 + \frac{2 \cdot 3}{2 \cdot 3}3x^4 + \frac{3 \cdot 4 \cdot 5}{2 \cdot 3 \cdot 4}4x^5 + \frac{4 \cdot 5 \cdot 6 \cdot 7}{2 \cdot 3 \cdot 4 \cdot 5}5x^6 + \frac{5 \cdot 6 \cdot 7 \cdot 8 \cdot 9}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}6x^7$$

cuius triplum ad priorem additum praebet

$$2Px + \frac{xxdP}{dx} - \frac{dQ}{dx} = 5xx + 2x^3 + 4 \cdot \frac{2 \cdot 3}{2 \cdot 3}x^4 + 4 \cdot \frac{3 \cdot 4 \cdot 5}{2 \cdot 3 \cdot 4}x^5 + 4 \cdot \frac{4 \cdot 5 \cdot 6 \cdot 7}{2 \cdot 3 \cdot 4 \cdot 5}x^6 + \text{etc.}$$

$$\text{et } 4Px = 4xx + 2x^2 + 4 \cdot \frac{2 \cdot 3}{2 \cdot 3}x^4 + 4 \cdot \frac{3 \cdot 4 \cdot 5}{2 \cdot 3 \cdot 4}x^5 + 4 \cdot \frac{4 \cdot 5 \cdot 6 \cdot 7}{2 \cdot 3 \cdot 4 \cdot 5}x^6 + \text{etc.}$$

$$\text{hinc } -2Px + \frac{xxdP}{dx} - \frac{dQ}{dx} = xx \text{ et } dQ = 4xxdP - 2Pxdx - xx dx$$

$$\text{vnde ob } dP = \frac{1}{2}dx + \frac{dx}{2\sqrt{1-4x}} \text{ colligitur}$$

$$dQ = -\frac{1}{2}xx dx + \frac{1}{2}xx dx \sqrt{1-4x} + \frac{2xx dx}{\sqrt{1-4x}} = -\frac{1}{2}xx dx + \frac{xx dx}{2\sqrt{1-4x}}$$

et integrando

$$Q = -\frac{1}{2}xx - \frac{(1+2x)\sqrt{1-4x}}{24} + \frac{1}{24}$$

Ponatur $x = \frac{1}{4}$; quo casu fit $P = \frac{3}{8}$, erit $Q = \frac{5}{192}$
ita vt fit

$$S(1) = \frac{3}{8}; S(3) = \frac{5}{192} \text{ et } -\frac{1}{2} + S(1) = \frac{3}{8}; \frac{1}{4,2^4} - S(3) = -\frac{1}{96}$$

18. Ponamus nunc in genere $S(n) = P$, et sequentem summam $S(n+2) = Q$ erit

$$P = \frac{1}{n} x^n + \frac{n}{n+1} x^{n+1} + \frac{n+1}{2} x^{n+2} + \frac{(n+2)(n+3)}{2 \cdot 3} x^{n+3} + \frac{(n+3)(n+4)(n+5)}{2 \cdot 3 \cdot 4} x^{n+4} \text{ etc.}$$

$$Q = \frac{1}{n+2} x^{n+2} + \frac{n+2}{n+3} x^{n+3} + \frac{n+3}{2} x^{n+4} + \frac{(n+4)(n+5)}{2 \cdot 3} x^{n+5} + \frac{(n+5)(n+6)(n+7)}{2 \cdot 3 \cdot 4} x^{n+6}$$

ex quibus colligitur fore

$$Q = Px - \frac{1}{2}(n+1) \int P dx - \frac{1}{2}(n-1) x x \int \frac{P dx}{x^2}$$

Hinc ergo ex valore $S(n)$ definiri potest valor $S(n+2)$ excepto quidem casu $n=1$, quia tum in $\int \frac{P dx}{x^2}$ occurret $\int \frac{dx}{x}$.

At quia iam constat casus

$$S(3) = \frac{1-6xx - (1+2x)\sqrt{1-4x}}{24}$$

si hic pro P sumatur, fiet

$$S(5) = Px - 2 \int P dx - x x \int \frac{P dx}{x^2},$$

quae per integrationem euoluta dat

$$S(5) = \frac{1}{60}(1-15xx+10x^3 - (1+2x-9xx)\sqrt{1-4x}).$$

Posito ergo $x = \frac{1}{4}$ fit $S(5) = \frac{7}{2250} = \frac{7}{1920}$, ideoque

$$\frac{1}{6 \cdot 2^6} - S(5) = -\frac{1}{960}.$$

19. Sit porro $n=5$ et

$$P = \frac{1}{60}(1-15xx+10x^3 - (1+2x-9xx)\sqrt{1-4x})$$

$$\text{erit } S(7) = Px - 3 \int P dx - 2 x x \int \frac{P dx}{x^2},$$

quibus integralibus euolutis tandem reperitur :

$$S(7)$$

$$S(7) = \frac{1}{112} (1 - 28xx + 56x^3 - 14x^4 - (1 - 2x - 22xx + 20x^2) \sqrt{1-4x})$$

Posito ergo $x = \frac{1}{4}$ fit

$$S(7) = \frac{1}{112} (1 - \frac{7}{4} + \frac{7}{8} - \frac{7}{128}) = \frac{1}{2^{11} \cdot 7}$$

$$\text{hincque } \frac{1}{8 \cdot 2^8} - S(7) = -\frac{1}{2^{10} \cdot 7}$$

Quare si, quae hactenus inuenimus, colligamus, etiam sequentes valores coniectura facile obtinebimus:

$$S(1) - \frac{1}{2 \cdot 2^2} = \frac{1}{4} = \frac{1}{1 \cdot 2 \cdot 2^1}$$

$$S(3) - \frac{1}{4 \cdot 2^4} = \frac{1}{96} = \frac{1}{2 \cdot 4 \cdot 2^2}$$

$$S(5) - \frac{1}{6 \cdot 2^6} = \frac{1}{960} = \frac{1}{5 \cdot 6 \cdot 2^5}$$

$$S(7) - \frac{1}{8 \cdot 2^8} = \frac{1}{2^{10} \cdot 7} = \frac{1}{7 \cdot 8 \cdot 2^7}$$

20. Hinc igitur tandem pro Z sequentem valorem adipiscimur:

$$Z = \frac{1}{4} - \frac{\alpha \pi^2}{3 \cdot 4 \cdot 2^2} - \frac{\beta \pi^4}{5 \cdot 6 \cdot 2^4} - \frac{\gamma \pi^6}{7 \cdot 8 \cdot 2^6} - \frac{\delta \pi^8}{9 \cdot 10 \cdot 2^8} - \frac{\epsilon \pi^{10}}{11 \cdot 12 \cdot 2^{10}} - \text{etc.}$$

ita vt fit

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} = \frac{1}{2} \pi \pi Z, \text{ seu}$$

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} = \frac{1}{8} \pi \pi - \frac{2 \alpha \pi^4}{3 \cdot 4 \cdot 2^4} - \frac{2 \beta \pi^6}{5 \cdot 6 \cdot 2^6} - \frac{2 \gamma \pi^8}{7 \cdot 8 \cdot 2^8} - \text{etc.}$$

Cuius seriei vt summam inuestigemus, consideremus π tanquam quantitatem variabilem, factoque $\frac{\pi}{2} = \Phi$ ponamus

$$\frac{\alpha \Phi^4}{3 \cdot 4} + \frac{\beta \Phi^6}{5 \cdot 6} + \frac{\gamma \Phi^8}{7 \cdot 8} + \frac{\delta \Phi^{10}}{9 \cdot 10} + \text{etc.} = s$$

$$\text{erit } \frac{d s}{d \Phi^2} = \alpha \Phi^2 + \beta \Phi^4 + \gamma \Phi^6 + \delta \Phi^8 + \text{etc.} = z$$

vnde formemus:

$$2sz = 2\alpha\alpha\Phi^4 + 4\alpha\beta\Phi^6 + 4\alpha\gamma\Phi^8 + 4\alpha\delta\Phi^{10} + \text{etc.} \\ + 2\beta\beta\Phi^6 + 4\beta\gamma\Phi^8$$

A a 2

Iam

Iam quia $\xi = \frac{2\alpha\alpha}{5}$; $\gamma = \frac{4\alpha\alpha}{7}$; $\delta = \frac{4\alpha\gamma + 2\xi\xi}{9}$ etc.
erit

$$2 \int z z d\Phi = \xi \Phi^5 + \gamma \Phi^7 + \delta \Phi^9 + \text{etc.} = z\Phi - \alpha\Phi^3$$

hincque

$$2 z z d\Phi = z d\Phi + \Phi dz - 3\alpha\Phi d\Phi.$$

21. Cum nunc sit $\alpha = \frac{1}{2}$, reperitur per integrationem

$$z = \frac{1}{2} - \frac{\Phi}{2 \tan \Phi}, \text{ vti tentanti patebit.}$$

Hinc cum $dd s = z d\Phi^2$, colligitur

$$\frac{ds}{d\Phi} = \int z d\Phi = \frac{1}{2} \Phi - \frac{1}{2} \int \frac{\Phi d\Phi}{\tan \Phi}$$

et $s = \frac{1}{4} \Phi^2 - \frac{1}{2} \int d\Phi \int \frac{\Phi d\Phi}{\tan \Phi}$, ideoque

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.} = \frac{1}{8} \pi \pi - \frac{1}{2} \Phi \Phi + \int d\Phi \int \frac{\Phi d\Phi}{\tan \Phi}$$

et ob $\Phi = \frac{\pi}{2}$ feu $\pi = 2\Phi$ erit

$$1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \text{etc.} = \int d\Phi \int \frac{\Phi d\Phi}{\tan \Phi} = \frac{\pi}{2} \int \frac{\Phi d\Phi}{\tan \Phi} - \int \frac{\Phi \Phi d\Phi}{\tan \Phi} \\ = 2 \int \Phi d\Phi / \sin \Phi - \frac{\pi}{2} \int d\Phi / \sin \Phi$$

at est $\int d\Phi / \sin \Phi = -\frac{\pi}{2}$

ergo

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.} = \frac{\pi \pi}{4} + 2 \int \Phi d\Phi / \sin \Phi$$

vbi integralibus ita sumtis, vt euanescant, posito $\Phi = 0$, statui deinceps debet $\Phi = \frac{\pi}{2}$, vt obtineatur summa seriei propositae. Et si autem integratio institui nequit, tamen per quadraturas eius valor

valor definiti potest. At vero ipsa series ante inventa per Z maxime est idonea ad summam proximè determinandam.

22. Hinc ansam arripio accuratius istam seriem perpendendi :

$$P = \frac{1}{n} x^n + \frac{n}{n+1} x^{n+1} + \frac{(n+1)}{2} x^{n+2} + \frac{(n+2)(n+1)}{2 \cdot 3} x^{n+3} \\ + \frac{(n+3)(n+2)(n+1)}{2 \cdot 3 \cdot 4} x^{n+4} + \text{etc.}$$

cuius valores pro casibus, quibus n est numerus integer impar, methodo singulari determinauimus, qui ita se habent :

$$\text{si } n=1; P = \frac{1}{4}(1+2x-V(1-4x))$$

$$n=3; P = \frac{1}{24}(1-6xx-(1+2x)V(1-4x))$$

$$n=5; P = \frac{1}{240}(1-15xx+10x^3-(1+2x-9xx)V(1-4x))$$

$$n=7; P = \frac{1}{112}(1-28xx+56x^3-14x^5-(1+2x-22xx+20x^3)V(1-4x)).$$

Quare cum summatio in genere ad aequationem differentialem reduci queat, operae pretium videtur examinare, quomodo his casibus isti valores satisficiant. Conueniet autem potius differentialia considerari, quae sunt

$$\text{si } n=1; \frac{dP}{dx} = \frac{1}{2} \left(1 + \frac{x}{\sqrt{1-4x}} \right)$$

$$n=3; \frac{dP}{dx} = \frac{1}{2} \left(-x + \frac{x}{\sqrt{1-4x}} \right)$$

$$n=5; \frac{dP}{dx} = \frac{1}{2} \left(-x + xx + \frac{x-3xx}{\sqrt{1-4x}} \right)$$

$$n=7; \frac{dP}{dx} = \frac{1}{2} \left(-x + 3xx - x^3 + \frac{x-5xx+5x^3}{\sqrt{1-4x}} \right).$$

23. In genere autem est differentiando :

$$\frac{dP}{dx} = x^{n-1} + nx^n + \frac{(n+1)(n+2)}{1 \cdot 2} x^{n+1} + \frac{(n+2)(n+3)(n+4)}{1 \cdot 2 \cdot 3} x^{n+2} + \text{etc.}$$

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Ponamus $x = y y$ et $\frac{dP}{dx} = \frac{dP}{2y dy} = s$, vt habeamus

$$s = y^{2n-2} + ny^{2n} + \frac{(n+1)(n+2)}{1 \cdot 2} y^{2n+2} + \frac{(n+2)(n+3)(n+4)}{1 \cdot 2 \cdot 3} y^{2n+4} + \text{etc.}$$

unde fit

$$y^{2-n}s = y^n + ny^{n+2} + \frac{(n+1)(n+2)}{1 \cdot 2} y^{n+4} + \frac{(n+2)(n+3)(n+4)}{1 \cdot 2 \cdot 3} y^{n+6}$$

hincque perro:

$$\frac{dd. y^{2-n} s}{dy^2} = n(n-1)y^{n-2} + \frac{n(n+1)(n+2)}{1} y^n + \frac{(n+1)(n+2)(n+3)(n+4)}{1 \cdot 2} y^{n+2} + \text{etc.}$$

Est vero etiam:

$$\frac{1}{2} \frac{d}{dy} d. \frac{s}{yy} = (n-2)y^{2n-3} + \frac{n(n+1)(n+2)}{1 \cdot 2} y^{2n-1} + \frac{(n+1)(n+2)(n+3)(n+4)}{1 \cdot 2 \cdot 3} y^{2n+1}$$

quae multiplicata per y^{5-2n} et denuo differentiata producit

$$\frac{1}{4} \frac{d}{dy^2} d. (y^{5-2n} d. \frac{s}{yy}) = n(n-1)y + \frac{n(n+1)(n+2)}{1} y^3 + \frac{(n+1)(n+2)(n+3)(n+4)}{1 \cdot 2} y^5 + \text{etc.}$$

quae series per superiorem etiam est

$$= \frac{y^{5-n} dd. (y^{2-n} s)}{dy^2},$$

ita vt inter s et y hanc habeamus aequationem

$$d(y^{5-2n} d. \frac{s}{yy}) = 4y^{5-n} dd. (y^{2-n} s).$$

24. Sumto elemento dy constante haec aequatio euoluta dat :

$$y^{2-2n} dds + (1-2n)y^{2-2n} dy ds - 4(1-n)y^{1-2n} s dy^2 = 4y^{5-2n} dds + 8(2-n)y^{4-2n} dy ds + 4(2-n)(1-n)y^{3-2n} s dy^2$$

quae per y^{2n-1} multiplicata abit in hanc :

$$yy(1-4yy)dds + (1-2n)ydyds - 4(1-n)sdy^2 - 8(2-n)y^2 dy ds - 4(2-n)(1-n)yysdy^2 = 0$$

quae posito $yy = x$ sumtoque dx constante transformatur in hanc

$$xx(1-4x)dds + (1-n)xdxds - (1-n)sdx^2 - 2(5-2n)xxdxds - (2-n)(1-n)sxdx^2 = 0$$

vbi est $s = \frac{dP}{dx}$ feu $P = \int s dx$. Integrationes autem hac lege institui debent, vt existente x infinite parvo fiat,

$$\frac{ds}{dx} = (n-1)x^{n-2}; s = x^{n-1} \text{ et } P = \frac{1}{n}x^n.$$

25. Si haec aequatio per seriem infinitam integretur, incipiendo a termino x^{n-1} , ipsa proposita series reproducitur, sed etiam initium fieri potest a termino constante x^0 , vnde quoque integrale obtinetur, quod autem nostris conditionibus non satis facit; verum praeterea aliud integrale elici potest, quod cum illo coniunctim negotium conficit. Fingatur ergo

$$s = O + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + \text{etc.}$$

$$\text{erit } \frac{ds}{dx} = A + 2Bx + 3Cxx + 4Dx^3 + 5Ex^4 + 6Fx^5 + \text{etc.}$$

$$\text{et } \frac{d ds}{dx^2} = 2B + 6Cx + 12Dxx + 20Ex^3 + 38Fx^4 + 42Gx^5 + \text{etc.}$$

quibus

quibus seriebuis substitutis fieri oportet:

$$\begin{array}{r}
 -(1-n)O - (2-n)(1-n)Ox - (2-n)(1-n)Ax^2 - (2-n)(1-n)Bx^3 - (2-n)(1-n)Cx^4 - (2-n)(1-n)Dx^5 \\
 + (1-n)A + 2(1-n)B + 3(1-n)C + 4(1-n)D + 5(1-n)E \\
 - (1-n)A - 2(5-2n)A - 4(5-2n)B - 6(5-2n)C - 8(5-2n)D \text{ etc.} \\
 + 2B + 6C + (1-n)D + 12D + 20E \\
 - 3B - 9B - 12C - 24C - 48D
 \end{array} \} = 0$$

quae aequatio reducitur ad hanc formam:

$$\begin{array}{r}
 -(1-n)O - (2-n)(1-n)Ox + (s-n)Bx^2 + 2(4-n)Cx^3 + 3(5-n)Dx^4 + 4(6-n)Ex^5 \\
 - (s-n)(1-n)A - (s-n)(6-n)B - (7-n)(s-n)C - (8-n)(10-n)D \text{ etc.}
 \end{array} \} = 0.$$

26. Singulis ergo terminis ad nihilum perductis, fieri debet $O = 0$ nisi sit $n = 1$, at pro reliquis coefficientibus habebitur

$$\begin{aligned}
 B &= \frac{(s-n)(4-n)}{1(s-n)} A & A &= \frac{4-n}{1} A \\
 C &= \frac{(s-n)(5-n)}{2(4-n)} B & B &= \frac{(4-n)(6-n)}{1 \cdot 2} A \\
 D &= \frac{(7-n)(8-n)}{3(5-n)} C & C &= \frac{(6-n)(7-n)(8-n)}{1 \cdot 2 \cdot 3} A \\
 E &= \frac{(9-n)(10-n)}{4(6-n)} D & D &= \frac{(7-n)(8-n)(9-n)(10-n)}{1 \cdot 2 \cdot 3 \cdot 4} A
 \end{aligned}$$

etc.

vnde lex progressionis est manifesta. Casu autem $n = 1$ quantitas O manet indefinita, tum autem aequationi satisficit, reliquis coefficientibus omnibus annihilatis, ita ut sit $s = 0$, etiamsi ex his determinationibus valores quoque finiti pro iis assumi possent veluti

$$B = \frac{3}{2} A; C = \frac{4 \cdot 5}{3 \cdot 2} A; D = \frac{5 \cdot 6 \cdot 7}{4 \cdot 3 \cdot 2} A \text{ etc.}$$

vnde integrale completum foret:

$$s = 0 + A + (x + \frac{s}{2}x^2 + \frac{4 \cdot 5}{1 \cdot 2}x^3 + \frac{5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3}x^4 + \frac{6 \cdot 7 \cdot 8 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4}x^5 + \text{etc.}$$

27. Simili modo pro reliquis casibus, quibus n est numerus integer, fit quidem $O = 0$, sed A numerus arbitrarius, sed praeterea alius quidam coeffi-

coefficientis etiam non definitur, quem propterea pro lubitu assumere licet. Quare si $is = 0$ ponatur, habebitur integrale terminis finitis contentum, quod ita se habebit:

$$\text{si } n=3; O=0; A \text{ indef. } B=0; C=0 \text{ etc.}$$

$$\text{si } n=4; O=0; A \text{ indef. } B=a; C=0 \text{ etc.}$$

$$\text{si } n=5; O=0; A \text{ indef. } B=-A; C=0; D=0 \text{ etc.}$$

$$\text{si } n=6; O=0; A \text{ indef. } B=-2A; C=0; D=0 \text{ etc.}$$

$$\text{si } n=7; O=0; A \text{ indef. } B=-3A; C=A; D=0; \text{ etc.}$$

$$\text{si } n=8; O=0; A \text{ indef. } B=-4A; C=\frac{2 \cdot 3}{1 \cdot 2} A; D=0; E=0 \text{ etc.}$$

$$\text{si } n=9; O=0; A \text{ indef. } B=-5A; C=\frac{3 \cdot 4}{1 \cdot 2} A; D=-\frac{1 \cdot 2 \cdot 3}{1 \cdot 2 \cdot 3} A; E=0$$

$$\text{si } n=10; O=0; A \text{ indef. } B=-6A; C=\frac{4 \cdot 5}{1 \cdot 2} A; D=-\frac{2 \cdot 1 \cdot 4}{1 \cdot 2 \cdot 3} A; E=0$$

$$\text{si } n=11; O=0; A \text{ indef. } B=-7A; C=\frac{5 \cdot 6}{1 \cdot 2} A; D=-\frac{2 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3} A; E=\frac{1 \cdot 3 \cdot 3 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} A.$$

28. Ecce ergo pro omnibus casibus, quibus n est numerus integer positivus praeter $n=2$ integralia particularia, unde partes rationales formularum supra pro $\frac{d^p}{dx}$ inuentarum colligi possunt:

$$\text{si } n=1; s=0$$

$$\text{si } n=3; s=Ax$$

$$\text{si } n=4; s=Ax^2$$

$$\text{si } n=5; s=A(x-xx)$$

$$\text{si } n=6; s=A(x-2xx)$$

$$\text{si } n=7; s=A(x-3xx+x^3)$$

$$\text{si } n=8; s=A(x-4xx+3x^3)$$

$$\text{si } n=9; s=A(x-5xx+6x^3-x^5)$$

$$\text{si } n=10; s=A(x-6xx+10x^3-4x^5)$$

$$\text{si } n=11; s=A(x-7xx+15x^3-10x^5+x^7)$$

$$\text{si } n=12; s=A(x-8xx+21x^3-20x^5+5x^7).$$

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29. Pro numero ergo quocunque n integrale hoc particulare est

$$s = A \left\{ x - \frac{(n-4)}{1} x x + \frac{(n-5)(n-6)}{1 \cdot 2} x^3 - \frac{(n-6)(n-7)(n-8)}{1 \cdot 2 \cdot 3} x^4 \right. \\ \left. + \frac{(n-7)(n-8)(n-9)(n-10)}{1 \cdot 2 \cdot 3 \cdot 4} x^5 - \text{etc.} \right\}$$

quae series etsi in infinitum continuata satisfacit, tamen cum quispiam terminus euanuerit, sequentes omnes omittere licet, quippe qui seorsim sumti aliud integrale particulare praeberent.

Ceterum hinc evidens est, quamlibet harum formularum ex binis praecedentibus ita definiri, vt si pro casibus $n = \nu$, $n = \nu + 1$, $n = \nu + 2$, valor ipsius s ponantur s , s' , s'' futurum sit

$$s'' = s' - s x$$

liquidem constans A in omnibus eundem valorem retineat. Atque vi huius legis pro casu $n = 2$ statui oportet $s = 0$. Verum vti iam monui, haec integralia particularia nostris conditionibus non satisfaciunt, verumtamen partes racionales suppeditant vti mox videbimus.

30. Vt autem completa integralia eruamus, alia integralia particularia inuestigemus, quae partes irrationales praebeant. In hunc finem ponamus

$$s = \frac{t}{\sqrt{(1-4x)}} \text{ erit } ds = \frac{dt}{\sqrt{(1-4x)}} + \frac{2t dx}{(1-4x)^{\frac{3}{2}}}$$

$$\text{et } dds = \frac{d dt}{\sqrt{(1-4x)}} + \frac{4 dx dt}{(1-4x)^{\frac{3}{2}}} + \frac{12 t dx^2}{(1-4x)^{\frac{5}{2}}}$$

quibus

quibus substitutis nostra aequatio differentio-differentialis abibit in hanc formam:

$$xx(1-4x)ddt - (n-1)x dt dx + (n-1)t dx^2 - 2(2n-3)xxdt dx - n(n-1)t dx^2 = 0.$$

Ponatur hic

$$t = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6 + \text{etc.}$$

factaque substitutione peruenitur ad hanc aequationem:

$$0 = (n-1)A - n(n-1)A - (n-3)Cxx - 2(n-4)Dx^3 - 3(n-5)Ex^4 - (n-2)(n-3)B - (n-4)(n-5)C - (n-6)(n-7)D \text{ etc.}$$

nisi ergo $n = 1$ debet esse $A = 0$, et pro reliquis fit

$$C = -\frac{(n-2)(n-3)}{1(n-3)} B = -\frac{(n-2)}{1} B$$

$$D = -\frac{(n-4)(n-5)}{2(n-4)} C = +\frac{(n-2)(n-5)}{1 \cdot 2} B$$

$$E = -\frac{(n-6)(n-7)}{3(n-5)} D = -\frac{(n-2)(n-6)(n-7)}{1 \cdot 2 \cdot 3} B.$$

31. Ex his ergo valores finiti ipsius t pro singulis numeris integris n ita se habebunt:

$$\text{fi } n = 1; t = A$$

$$\text{fi } n = 2; t = Bx$$

$$\text{fi } n = 3; t = Bx$$

$$\text{fi } n = 4; t = B(x - 2xx)$$

$$\text{fi } n = 5; t = B(x - 3xx)$$

$$\text{fi } n = 6; t = B(x - 4xx + 2x^3)$$

$$\text{fi } n = 7; t = B(x - 5xx + 5x^3)$$

$$\text{fi } n = 8; t = B(x - 6xx + 9x^3 - 2x^4)$$

$$\text{fi } n = 9; t = B(x - 7xx + 14x^3 - 7x^4)$$

$$\text{fi } n = 10; t = B(x - 8xx + 20x^3 - 16x^4 + 2x^5)$$

et in genere

$$B b 2$$

$$t = B$$

$$t = B \left\{ x - \frac{(n-2)}{1} x x + \frac{(n-2)(n-5)}{1 \cdot 2} x^3 - \frac{(n-2)(n-6)(n-7)}{1 \cdot 2 \cdot 3} x^4 \right. \\ \left. + \frac{(n-2)(n-7)(n-8)(n-9)}{1 \cdot 2 \cdot 3 \cdot 4} x^5 - \text{etc} \right\}$$

vbi iterum vt ante $t'' = t' - t x$.

32. Quemadmodum supra seriem P designauimus per $S(n)$, si seriem $s = \frac{d}{d x} P$ per $\Sigma(n)$ designemus, erit generatim

$$\Sigma(1) = -A + \frac{B}{\sqrt{1-4x}} \quad \text{hic } A = -\frac{1}{2} \text{ et } B = \frac{1}{2}$$

$$\Sigma(2) = 0 + \frac{2 B x}{\sqrt{1-4x}} \quad \text{vt hae formae ad seriem pro-$$

$$\Sigma(3) = A x + \frac{B x}{\sqrt{1-4x}} \quad \text{positam accommodentur.}$$

$$\Sigma(4) = A x + \frac{B(x-2xx)}{\sqrt{1-4x}}$$

$$\Sigma(5) = A(x-xx) + \frac{B(x-3xx)}{\sqrt{1-4x}}$$

$$\Sigma(6) = A(x-2xx) + \frac{B(x-4xx+2xx^2)}{\sqrt{1-4x}}$$

$$\Sigma(7) = A(x-3xx+x^3) + \frac{B(x-5xx+5xx^2)}{\sqrt{1-4x}}$$

$$\Sigma(8) = A(x-4xx+3x^3) + \frac{B(x-6xx+9xx^2-2xx^3)}{\sqrt{1-4x}}$$

$$\Sigma(9) = A(x-5xx+6x^3-x^4) + \frac{B(x-7xx+14xx^2-7xx^3)}{\sqrt{1-4x}}$$

$$\Sigma(10) = A(x-6xx+10x^3-4x^4) + \frac{B(x-8xx+20xx^2-16xx^3+2xx^4)}{\sqrt{1-4x}}$$

etc.

33. Quoniam hi valores seriem recurrentem constituunt, cum quisque aequetur praecedentium ultimo, demto penultimo per x multiplicato, terminus generalis seu valor $\Sigma(n)$ finite exprimi poterit, erit enim ex proprietate serierum recurrentium:

$$\Sigma(n) = M \frac{(1 + \sqrt{1-4x})^n}{2} + N \frac{(1 - \sqrt{1-4x})^n}{2}$$

vbi

vbi ex binis primis coefficientes M et N ita definiuntur vt fit

$$M = \frac{(A+B)(1-2x-\sqrt{1-4x})}{2x\sqrt{1-4x}} = \frac{A+B}{x\sqrt{1-4x}} \left(\frac{1-\sqrt{1-4x}}{2} \right)^2$$

$$N = \frac{(B-A)(1-2x+\sqrt{1-4x})}{2x\sqrt{1-4x}} = \frac{B-A}{x\sqrt{1-4x}} \left(\frac{1+\sqrt{1-4x}}{2} \right)^2$$

Cum autem pro nostro casu fit $A = -\frac{1}{2}$ et $B = \frac{1}{2}$ erit

$$\Sigma(n) = \frac{1}{x\sqrt{1-4x}} \left(\frac{1+\sqrt{1-4x}}{2} \right)^2 \left(\frac{1-\sqrt{1-4x}}{2} \right)^n$$

$$\text{feu } \Sigma(n) = \frac{x}{\sqrt{1-4x}} \left(\frac{1-\sqrt{1-4x}}{2} \right)^{n-2} = \frac{dP}{dx}$$

existente

$$P = \frac{1}{n} x^n + \frac{n}{n+1} x^{n+1} + \frac{(n+1)}{2} x^{n+2} + \frac{(n+2)(n+1)}{2 \cdot 3} x^{n+3} \\ + \frac{(n+3)(n+5)(n+5)}{2 \cdot 3 \cdot 4} x^{n+4} + \text{etc.}$$

34. Huius ergo seriei valor, quem ponimus P est

$$P = \int \frac{x dx}{\sqrt{1-4x}} \left(\frac{1-\sqrt{1-4x}}{2} \right)^{n-2}$$

ad quod integrale inueniendum ponatur $\frac{1-\sqrt{1-4x}}{2} = y$ erit

$$dy = \frac{dx}{\sqrt{1-4x}} \text{ et } x = y - yy, \text{ vnde fit}$$

$$P = \int dy (y - yy) y^{n-2} = \frac{y^n}{n} - \frac{y^{n+1}}{n+1} \text{ ideoque}$$

$$P = \frac{n+1-ny}{n(n+1)} y^n \text{ feu}$$

$$P = \frac{n+2+n\sqrt{1-4x}}{2n(n+1)} \left(\frac{1-\sqrt{1-4x}}{2} \right)^n = S(n) \text{ posito } x = \frac{1}{2}$$

B b 3

vnde

vnde pro formulis supra (14) expositis colligitur

$$S(n) = \frac{n+2}{2^{n+1}n(n+1)}$$

hincque vti formulae ibi se habent:

$$\frac{1}{(n+1)2^{n+1}} - S(n) = -\frac{1}{2^n(n+1)n}$$

quae expressio prorsus congruit cum iis, quas supra §. 19. sola inductione nixi dedimus, ita vt nunc quidem nullum amplius dubium superesse possit.

35. Deinde memorabile est, huius aequationis differentio-differentialis

$$\left. \begin{aligned} xx(1-4x)dds - (n-1)x dx ds + (n-1)s dx^2 \\ + 2(2n-5)xx dx ds - (n-1)(n-2)s dx^2 \end{aligned} \right\} = 0$$

integrale completum et quidem algebraicum assignari posse, quod ex praecedentibus ita se habet:

$$s = \frac{Cx}{\sqrt{1-4x}} \left(\frac{1+\sqrt{1-4x}}{2} \right)^{n-2} + \frac{Dx}{\sqrt{1-4x}} \left(\frac{1-\sqrt{1-4x}}{2} \right)^{n-2}$$

quod quomodo per integrationem inde erui queat, non tam facile patet. Hinc tamen statim intelligitur, substitutionem $s = \frac{x^u}{\sqrt{1-4x}}$ plurimum esse futuram, posito enim in §. 30; $t = ux$ haec aequatio nascitur

$$\begin{aligned} xx(1-4x)ddu - (n-3)xdxdu - (n-2)(n-3)xudx^2 = 0 \text{ seu} \\ + 2(2n-7)xx dx du \\ x(1-4x)ddu - (n-3)dx du - (n-2)(n-3)udx^2 = 0 \\ + 2(2n-7)xx dx du \end{aligned}$$

cuius ergo integrale est

$$u = C \left[\frac{1+\sqrt{1-4x}}{2} \right]^{n-2} + D \left[\frac{1-\sqrt{1-4x}}{2} \right]^{n-2}$$

36. Si in hac aequatione ponatur $\sqrt{(1-4x)}=y$, et elementum dy vt constans introducatur, prodibit ista aequatio simplicior

$$(1-yy)d^2u + 2(n-3)ydydu - (n-2)(n-3)udy^2 = 0$$

cuius integrale iam esse constat:

$$u = C \left(\frac{1+y}{2} \right)^{n-2} + D \left(\frac{1-y}{2} \right)^{n-2}.$$

Quo igitur pateat, quomodo haec inde elici queat, ponamus $n = m + 2$ vt habeamus:

$$(1-yy)d^2u + 2(m-1)ydydu - m(m-1)udy^2 = 0$$

vbi commode tentari posse patet hanc positionem

$$u = (\alpha + \xi y)^m$$

vnde fit

$$du = m\xi dy(\alpha + \xi y)^{m-1} \text{ et } d^2u = m(m-1)\xi\xi dy^2(\alpha + \xi y)^{m-2}$$

quo facto erit

$$m(m-1)(\alpha + \xi y)^{m-2}(\xi\xi(1-yy) + 2\xi(\alpha y + \xi yy) - \alpha\alpha - 2\alpha\xi y - \xi\xi yy) = 0$$

ideoque

$$\xi\xi = \alpha\alpha; \text{ ergo } u = C(1 \pm y)^m.$$

Atque ob signum ambiguum obtinebitur integrale completum:

$$u = C(1+y)^m + D(1-y)^m.$$

37. Ceterum notasse iuuabit hanc postremam aequationem

$$(1-yy)d^2u + 2(m-1)ydydu - m(m-1)udy^2 = 0$$

reddi integrabilem, si diuidatur per $(1 \pm y)^m$. Prior vero aequatio:

$$x(1-4x)d^2u - (n-3)xdxdu - (n-2)(n-3)udx^2 = 0$$

+ 2(2n-7)xdxdu in-

integrabilis evadet si multiplicetur per

$$x^{-n+2} du - \frac{(n-2)}{2} x^{-n+2} u dx.$$

In genere autem proposita hac aequatione

$$xx(A+Bx)ddu + \frac{1}{2}(2\alpha+\lambda)Ax dxdu + \frac{1}{2}\alpha(\lambda-2)Audx^2 + \frac{1}{2}(2\alpha+\lambda+1)Bxx dxdu + \frac{1}{2}\alpha(\lambda-1)Bxudx^2 \Big\} = 0$$

si ea multiplicetur per

$$x^{\lambda-2} du + \alpha x^{\lambda-2} u dx,$$

fiet integrabilis, eritque integrale:

$$\frac{1}{2}x^\lambda(A+Bx)du^2 + \alpha x^{\lambda-1}(A+Bx)ududx + \frac{1}{2}\alpha x^{\lambda-1}(A+Bx)u^2 dx^2 = \frac{1}{2}Cdx^2$$

$$\text{feu } x^\lambda du^2 + 2\alpha x^{\lambda-1}ududx + \alpha x^{\lambda-2}u^2 dx^2 = \frac{C dx^2}{A+Bx}$$

$$\text{ergo } x^{\frac{1}{2}\lambda} du + \alpha x^{\frac{1}{2}\lambda-1} u dx = \frac{dx \sqrt{C}}{\sqrt{(A+Bx)}} \text{ hincque}$$

$$u = x^{-\alpha} \int x^{\frac{\alpha-\frac{1}{2}\lambda}{2}} \frac{dx \sqrt{C}}{\sqrt{(A+Bx)}}$$

38. In hac autem aequatione generali nostra superior non continetur: quare conditiones aequationis huius

$$xx(A+Bx)ddu + x(C+Dx)dudx + (E+Fx)udx^2 = 0$$

accuratius inuestigemus, vt per

$$x^{\lambda-2} du + \alpha x^{\lambda-2} u dx$$

multiplicata fiat integrabilis. Ac primo quidem integrale fit

$$\frac{1}{2}x^\lambda(A+Bx)du^2 + \alpha x^{\lambda-1}(A+Bx)ududx + \frac{\alpha E}{\lambda-\frac{1}{2}}x^{\lambda-2}uudx^2 + \frac{\alpha F}{\lambda-\frac{1}{2}}u^2 dx^2 = Gdx^2$$

requiritur autem, vt fit primo

$$C = (\alpha + \frac{1}{2}\lambda)A; \quad D = (\alpha + \frac{1}{2}\lambda + \frac{1}{2})B$$

tum

tum vero triplici modo

I. vel $E = \frac{1}{2}\alpha(\lambda - 2)A$ et $F = \frac{1}{2}\alpha(\lambda - 1)B$ qui est casus superior

II. vel $\lambda = 2\alpha + 2$; $F = \frac{1}{2}\alpha(2\alpha + 1)B$ manente E indefinito

III. vel $\lambda = 2\alpha + 1$; $E = \frac{1}{2}\alpha(2\alpha - 1)A$ manente F indefinito.

39. En ergo duas aequationes differentio-differentiales satis late patentes, quas hac methodo integrale licet:

$$\text{I. } xx(A+Bx)d\dot{u} + (2\alpha+1)Ax dx du + E u dx^2 \\ + (2\alpha+\frac{1}{2})Bxx dx du + \frac{1}{2}\alpha(2\alpha+1)Bx u dx^2 = 0$$

quae per

$$x^{\alpha} du + \alpha x^{\alpha-1} u dx$$

multiplicata integrale dat

$$\frac{1}{2}x^{2\alpha+1}(A+Bx)du^2 + \alpha x^{2\alpha+1}(A+Bx)u du dx + \frac{1}{2}E x^{2\alpha} u u dx^2 \\ + \frac{1}{2}\alpha\alpha Bx^{2\alpha+1} u u dx^2 = G dx^2.$$

Altera vero forma est

$$\text{II. } xx(A+Bx)d\dot{u} + (2\alpha+\frac{1}{2})Ax dx du + \frac{1}{2}\alpha(2\alpha-1)A u dx^2 \\ + (2\alpha+1)Bxx dx du + Fx u dx^2 = 0$$

quae per

$$x^{\alpha-1} du + \alpha x^{\alpha-2} u dx$$

multiplicata istud suppeditat integrale:

$$\frac{1}{2}x^{2\alpha+1}(A+Bx)du^2 + \alpha x^{2\alpha}(A+Bx)u du dx + \frac{1}{2}\alpha\alpha A x^{2\alpha-1} u^2 dx^2 \\ + \frac{1}{2}F x^{2\alpha} u^2 dx^2 = G dx^2.$$

In priori si ponatur $A = 1$, $B = -4$ et $2\alpha + 1 = -n + 3$, atque $E = 0$ prodit aequatio in §. 35 proposita.

40. Verum alia datur via summam progressionis §. 23

$$\frac{dP}{dx} = x^{n-1} + \frac{n}{1}x^n + \frac{(n+1)(n+2)}{1 \cdot 2}x^{n+1} + \frac{(n+2)(n+3)(n+4)}{1 \cdot 2 \cdot 3}x^{n+2} + \text{etc.}$$

inuestigandi, in qua quia x ut constans spectatur, consideremus hanc seriem

$$s = 1 + \frac{n}{2}a^2 + \frac{(n+1)(n+2)}{2 \cdot 4}a^4 + \frac{(n+2)(n+3)(n+4)}{2 \cdot 4 \cdot 6}a^6 + \text{etc.}$$

vbi $aa = 2x$ et $\frac{dP}{dx} = x^{n-1} s$. Iam in subsidium vocetur haec series

$$\frac{(1+ay)^{-n+1} + (1-ay)^{-n+1}}{2} = 1 + \frac{(n-1)n}{1 \cdot 2}aay^2 + \frac{(n-1)n(n+1)(n+2)}{1 \cdot 2 \cdot 3 \cdot 4}a^4y^4 + \text{etc.}$$

pro vna scribamus breuitatis gratia

$$1 + Aa^2y^2 + Ba^4y^4 + Ca^6y^6 + \text{etc.}$$

eritque

$$s = 1 + \frac{1}{n-1}Aa^2 + \frac{1 \cdot 3}{(n-1)n}Ba^4 + \frac{1 \cdot 3 \cdot 5}{(n-1)(n+1)}Ca^6 \text{ etc.}$$

Statuatur nunc

$$s = \frac{1}{2}fdz(1 + Aa^2y^2 + Ba^4y^4 + Ca^6y^6 + \text{etc.})$$

ac fieri oportet

$$\int y y dz = \frac{1}{n-1} \int dz$$

$$\int y^4 dz = \frac{3}{n} \int y y dz$$

$$\int y^6 dz = \frac{5}{n+1} \int y^4 dz$$

ideoque ingenere

$$\int y^{2\lambda} dz = \frac{2\lambda-1}{n+\lambda-2} \int y^{2\lambda-2} dz$$

si post integrationem ipsi y certus valor tribuatur.

41. Ponamus ergo esse ingenere:

$$\int y^{2\lambda} dz = \frac{2\lambda-1}{n+\lambda-2} \int y^{2\lambda-2} dz + \frac{Q y^{2\lambda-1}}{n+\lambda-2}$$

hincque differentiando et per $y^{2\lambda-2}$ diuidendo colligitur

$$(n+\lambda-2)yydz = (2\lambda-1)dz + ydQ + (2\lambda-1)Qdy$$

quae aequatio pro omnibus numeris λ locum habere debet, vnde erit

$$\text{tam } yydz = 2dz + 2Qdy$$

$$\text{quam } (n-2)yydz = -dz + ydQ - Qdy$$

$$\text{ergo } dz = \frac{2Qdy}{yy-2} = \frac{ydQ - Qdy}{(n-2)yy+1} \text{ vnde fit}$$

$$\frac{dQ}{Q} = \frac{-(2n-3)ydy}{2-yy}, \text{ et } Q = (2-yy)^{n-\frac{1}{2}}$$

hincque

$$dz = -2dy(2-yy)^{n-\frac{1}{2}}$$

Quare posito post integrationem

$$y = \sqrt{2} \text{ fit } \int y^{2\lambda} dz = \frac{2\lambda-1}{n+\lambda-2} \int y^{2\lambda-2} dz$$

reperiturque

$$s = \frac{\int dy (2-yy)^{n-\frac{1}{2}} ((1+ay)^{-n+1} + (1-ay)^{-n+1})}{2 \int dy (2-yy)^{n-\frac{1}{2}}}$$

si post integrationem ponatur $y = \sqrt{2}$.

42. Etsi autem haec methodus statim pro summa quaesita formulam integram exhibet, tamen

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verum valorem in expressione algebraica non ostendit. In superioribus autem vidimus esse

$$\frac{dP}{dx} = \frac{x}{\sqrt{(1-4x)}} \left(\frac{1-\sqrt{(1-4x)}}{2} \right)^{n-2},$$

unde concludimus fore hic

$$s = \frac{x^{2-n}}{\sqrt{(1-4x)}} \left(\frac{1-\sqrt{(1-4x)}}{2} \right)^{n-2} = \frac{1}{\sqrt{(1-4x)}} \left(\frac{1-\sqrt{(1-4x)}}{2x} \right)^{n-2}.$$

Quare si statuamus $2x = aa$, et superioris formulae integralis casu $y = \sqrt{2}$ erit valor algebraicus:

$$s = \frac{1}{\sqrt{(1-2aa)}} \left(\frac{1-\sqrt{(1-2aa)}}{aa} \right)^{n-2}$$

quae circumstantia minime contemnenda videtur, cum forte hinc plura alia praeclara in hoc genere deriuare liceat.

DIGRES-