



1773

Consideratio aequationis differentio-differentialis  
 $(a+bx)ddz + (c+ex)(dx dz/x) + (f+gx)(z dx^2/xx) = 0$

Leonhard Euler

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CONSIDERATIO  
 AEQVATIONIS DIFFERENTIO-  
 DIFFERENTIALIS

$$(a+bx)ddz+(c+ex)\frac{dx dz}{x}+(f+gx)\frac{x dx^2}{x^2}=0.$$

Auctore  
 L. E V L E R O.

I.

Primo haec aequatio ad formam differentialem simplicem reuocari potest ponendo  $lz = f v dx$ , vt fit

$$\frac{dz}{z} = v dx \quad \text{et} \quad \frac{ddz}{z} - \frac{dz^2}{z^2} = dx dv,$$

ideoque

$$\frac{ddz}{z} = dx dv + v v dx^2.$$

Diuisa enim illa aequatione per  $z dx$  hinc orietur:  
 $(a+bx)dv+(a+bx)v v dx+(c+ex)\frac{v dx}{x}+(f+gx)\frac{dx}{x^2}=0$   
 cuius integratio si pateret, foret pro proposita  
 $lz = f v dx$ .

II. Hinc duplici modo terminus simplici quantitate  $v$  affectus elidi potest. Pro altero ponamus  $v = u + X$  denotante  $X$  functionem ipsius  $x$  mox determinandam; et facta substitutione fiet

$$\left. \begin{aligned} (a+bx)du+(a+bx)u dx+(c+ex)\frac{u dx}{x}+(f+gx)\frac{dx}{x^2} \\ +2(a+bx)u X dx+(a+bx)dX \\ + (a+bx)XX dx \\ + (c+ex)\frac{X dx}{x} \end{aligned} \right\} = 0.$$

Iam

Q 3

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Iam statuatur  $X = \frac{-(c+ex)}{2x(a+bx)}$ , vt fit

$$dX = \frac{(ac+2bcx+bcxx)dx}{2xx(a+bx)^2}, \text{ prodibitque}$$

$$(a+bx)du + (a+bx)uudx + (f+gx)\frac{dx}{xx} + \left. \begin{aligned} &+ \frac{(ac+2bcx+bcxx)dx}{2xx(a+bx)} \\ &- \frac{(cc+2cex+eexx)dx}{4xx(a+bx)} \end{aligned} \right\} = 0$$

feu hoc modo :

$$du + uudx + \frac{(a+bx)(f+gx) + 2(ac+2bcx+bcxx) - (c+ex)^2}{4xx(a+bx)^2} dx = 0.$$

III. Per alteram methodum cum priore con-  
iunctam ponatur  $v = Pu + X$ , existentibus P et X  
functionibus ipsius x, et substitutione facta obtinebitur:

$$(a+bx)Pdu + (a+bx)PPuudx + (c+ex)\frac{Pu dx}{xx} + \left. \begin{aligned} &+ \frac{(f+gx)dx}{xx} \\ &+ (a+bx)udP + (a+bx)dX \\ &+ 2(a+bx)PXudx + (a+bx)XXdx \\ &+ (c+ex)\frac{X dx}{xx} \end{aligned} \right\} = 0$$

vnde fieri debet

$$\frac{(c+ex)P dx}{xx} + (a+bx)dP + 2(a+bx)PX dx = 0.$$

Introduxi autem hic binas functiones P et X, quo  
inuestigatio latius pateat; vulgo enim hac altera me-  
thodo vtentes ponere solemus  $v = Pu$ , vt fit  $X = 0$ ,  
quo casu erit

$$\frac{dP}{P} + \frac{(c+ex)dx}{xx(a+bx)} = 0 \text{ seu } \frac{dP}{P} + \frac{c}{a} \frac{dx}{x} + \frac{ae-bc}{a} \frac{dx}{a+bx} = 0$$

vnde integrando colligitur:

$$Px^{\frac{c}{a}}(a+bx)^{\frac{ae-bc}{ab}} = C \text{ et } P = Cx^{-\frac{c}{a}}(a+bx)^{\frac{c}{a}-\frac{e}{b}}$$

et aequatio nostra differentialis primi gradus erit

Cx

$$Cx^{-\frac{c}{a}}(a+bx)^{\frac{c}{a}-\frac{e}{b}+1} du + CCx^{-\frac{2c}{a}}(a+bx)^{\frac{2c}{a}-\frac{2e}{b}+1} u u dx + \frac{(f+gx) dx}{xx} = 0$$

sive hoc modo

$$Cdu + CCx^{-\frac{c}{a}}(a+bx)^{\frac{c}{a}-\frac{e}{b}} u u dx + x^{\frac{c}{a}}(a+bx)^{\frac{e}{b}-\frac{c}{a}-1} (f+gx) \frac{dx}{xx} = 0.$$

IV. Sin autem in genere ponatur  $v = Pu + X$  statuaturque

$$\frac{dP}{P} + \frac{(c+ex) dx}{x(a+bx)} + 2X dx = 0,$$

aequatio nostra differentialis hanc induit formam:

$$Pdu + PPu dx + dX + XX dx + \frac{(c+ex)X dx}{x(a+bx)} + \frac{(f+gx) dx}{xx(a+bx)} = 0$$

in qua vel P vel X pro lubitu accipi potest, unde altera definietur. Veluti si capiatur  $P = \alpha x^n$ , fiet

$$X = \frac{-n}{2x} - \frac{c+ex}{2x(a+bx)} = \frac{-na-c-(nb+e)x}{2x(a+bx)}$$

Ex his formis casus, quibus aequatio fit integrabilis, elicere licet, quos autem facilius ex ipsa aequatione proposita cognoscere poterimus.

V. Quodsi quaerere velimus casus, quibus aequatio proposita integrationem admittit, in quo quidem omnis opera collocanda videtur, quamdiu integrationem in genere instituere non licet, primum quidem statim se offert forma  $z = Ax^m(a+bx)^n$  quae ut satisficiat, definiendum oportet relationem quantitatum constantium  $a, b, c, e, f, g$ . Cum igitur fit

$$\frac{dz}{z} = \frac{m dx}{x} + \frac{nb dx}{a+bx} \text{ et } \frac{d dz}{z} - \frac{dz^2}{z^2} = \frac{-m dx^2}{xx} - \frac{nb dx^2}{(a+bx)^2},$$

erit

erit

$$\frac{d d x}{x} = \frac{m(m-1) d x^2}{x x} + \frac{2 m n b d x^2}{x(a+b x)} + \frac{n(n-1) b b d x^2}{(a+b x)^2},$$

hincque nascitur facta diuisione per  $d x^2$  haec aequatio

$$\left. \begin{aligned} \frac{m(m-1)(a+b x)}{x x} + \frac{2 m n b}{x} + \frac{n(n-1) b b}{a+b x} \\ + \frac{f+g x}{x x} + \frac{m(c+e x)}{x x} + \frac{n b(c+e x)}{x(a+b x)} \end{aligned} \right\} = 0$$

quae vt subsistere possit, bini postremi termini collecti

$$\frac{n b((n-1) b x + c + e x)}{x(a+b x)}$$

denominatorem  $a+b x$  amittere debent ex quo fit

$$a : c = b : (n-1) b + e, \text{ seu } n-1 = \frac{c}{a} - \frac{e}{b},$$

$$\text{et } n = 1 + \frac{c}{a} - \frac{e}{b},$$

ita vt habeatur haec aequatio

$$\frac{m(m-1)a + mc + f}{x x} + \frac{m(m-1)b + 2mnb + me + g + nbc : a}{x} = 0$$

unde haec duae nascuntur aequationes loco  $n$  valorem inuentum scribendo :

$$m(m-1)a + mc + f = 0 \text{ et}$$

$$m(m-1)b + \frac{(2m+1)bc}{a} - me - \frac{cc}{a} + \frac{bcc}{aa} + g = 0.$$

Multiplicetur haec per  $a$  et illa per  $-b$  fiet summa :

$$2mab + mbc - mae + bc - ce + \frac{bcc}{a} + ag - bf = 0$$

$$\text{hincque } m = \frac{abf - aag - abc + ace - bcc}{2abb + abc - aae} \text{ et}$$

$$(m-1)a + c = \frac{abf - aag - 2aab + aae}{2ab + bc - ae}$$

qui valores in prima substituti dant

$$\begin{aligned} (bf-ag)^2 + f(2abb + 3bcc - 3abe - bce + aae) + c(2b-e)(ab-ae+bc) = 0 \\ + g(2aab - aae + abc - ace + bcc) \end{aligned} \text{ quae}$$

quae resoluta praebet,

$$ag = bf - \frac{1}{2}(2ab - ae + bc) + \frac{c}{2a}(ae - bc) \pm (2ab - ae + bc) \sqrt{\frac{(a-c)^2}{4aa} - \frac{f}{a}}$$

siue

$$g = \frac{c(ae - bc) + 2abf - (2ab - ae + bc)(a \pm \sqrt{(a-c)^2 - 4af})}{2aa}$$

Quare si littera  $g$  hunc habeat valorem, aequatio nostra integrale habebit  $z = Ax^m(a + bx)^n$  existente

$$m = \frac{a-c \pm \sqrt{(a-c)^2 - 4af}}{2a} \text{ et } n = 1 + \frac{c}{a} - \frac{e}{b}.$$

VI. Alia via casus integrabiles reperiuntur, si valor ipsius  $z$  in seriem conuertatur, quae si alicubi abrumpatur, expressionem finitam pro  $z$  exhibet. Fingatur ergo:

$$z = Ax^n + Bx^{n+1} + Cx^{n+2} + Dx^{n+3} + Ex^{n+4} + \text{etc.}$$

et facta substitutione consequemur:

$$\begin{array}{l} n(n-1)Aax^{n-2} + (n+1)nBax^{n-1} + (n+2)(n+1)Cax^n + (n+3)(n+2)Dax^{n+1} \text{ etc.} \\ + n(n-1)Ab + (n+1)nBb + (n+2)(n+1)Cb \\ nAc + (n+1)Bc + (n+2)Cc + (n+3)Dc \\ + nAe + (n+1)Be + (n+2)Ce \\ Af + Bf + Cf + Df \\ + Ag + Bg + Dg \end{array}$$

quos singulos terminos ad nihilum reduci oportet.

Primo ergo erit  $n(n-1)a + nc + f = 0$  hincque

$$n = \frac{a-c \pm \sqrt{(a-c)^2 - 4af}}{2a} \text{ porro vero}$$

$$B = \frac{-n(n-1)b - ne - g}{(n+1)na + (n+1)c + f} \quad A = \frac{-n(n-1)b - ne - g}{2na + c} \quad A$$

$$C = \frac{-(n+1)nb - (n+1)e - g}{(n+2)(n+1)a + (n+2)c + f} \quad B = \frac{-(n+1)nb - (n+1)e - g}{2((2n+1)a + c)} \quad B$$

$$D = \frac{-(n+2)(n+1)b - (n+2)e - g}{(n+3)(n+2)a + (n+3)c + f} \quad C = \frac{-(n+2)(n+1)b - (n+2)e - g}{2((2n+2)a + c)} \quad C$$

etc.

Tom. XVII. Nou. Comm.

R

Haec

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Haec ergo series alicubi abrumpitur, si sumto pro  $i$  numero quocunque integro positivo, quo etiam cyphra referatur, fuerit

$$g = -(n+i)(n+i-1)b - (n+i)e.$$

Cum autem sit  $n+i = \frac{(2i+1)a-c \pm \sqrt{(a-c)^2 - 4af}}{2a}$

et  $n+i-1 = \frac{(2i-1)a-c \pm \sqrt{(a-c)^2 - 4af}}{2a}$  erit

$$g = \frac{-((2i+1)a-c \pm \sqrt{(a-c)^2 - 4af})(2i-1)ab - bc + 2ae + b \sqrt{(a-c)^2 - 4af}}{4a^2}$$

et euoluendo

$$g = \frac{2abf + c(ae-bc) - a^2(2iab + (2i+1)(ae-bc)) \mp (2iab + ae-bc) \sqrt{(a-c)^2 - 4af}}{2a^2}$$

si ergo effet  $i = -1$ , quod autem hic sumere non licet, casus praecedens emergeret. Hinc igitur innumerabiles alii casus similes eruuntur.

VII. Possimus etiam seriem, in qua exponentes ipsius  $x$  decrescant, assumere, hoc modo.

$$z = Ax^n + Bx^{n-1} + Cx^{n-2} + Dx^{n-3} + Ex^{n-4} + \text{etc.}$$

qua substituta nostra aequatio fit

$$\begin{aligned} &+n(n-1)Ax^{n-1} + n(n-1)Aax^{n-2} + (n-1)(n-2)Bax^{n-2} + (n-2)(n-3)Cax^{n-3} + \text{etc.} \\ &+ (n-1)(n-2)Bb + (n-2)(n-3)Cb + (n-3)(n-4)Db \\ &+ nAc + (n-1)Bc + (n-2)Cc \\ &+ nAe + (n-1)Be + (n-2)Ce + (n-3)De \\ &+ Ag + Bg + Cg + Dg \end{aligned}$$

hincque esse debet  $n(n-1)b + ne + g = 0$  seu

$$n = \frac{b-e \pm \sqrt{(b-e)^2 - 4bg}}{2b}, \text{ vel } g = -nbn + nb - ne.$$

Praeterea vero:

$$B =$$

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$$B = \frac{+n(n-1)a + nc + f}{(2n-2)b + e} A; C = \frac{(n-1)(n-2)a + (n-1)c + f}{2((2n-3)b + e)} B$$

$$D = \frac{(n-2)(n-3)a + (n-2)c + f}{2((2n-4)b + e)} C; D = \frac{(n-3)(n-4)a + (n-3)c + f}{4((2n-5)b + e)} C$$

etc.

Sit ut ante  $i$  numerus integer positivus, cyphra non exclusa, et integrale finitum obtinebitur, quoties fuerit

$$(n-1)(n-i-1)a + (n-i)c + f = 0 \text{ vnde fit}$$

$$n = \frac{+(2i+1)a - c + \sqrt{((a-c)^2 - 4af)}}{2a}$$

ut invenimus;  $g = -n((n-1)b + e)$ , ideoque

$$g = \frac{-(2i+1)a - c + \sqrt{((a-c)^2 - 4af)}((2i-1)ab - bc + 2ae + b\sqrt{((a-c)^2 - 4af)})}{4aa}$$

quae evoluta praebet ut ante

$$g = \frac{2abf + c(ae - bc) - a(2iab + (2i+1)(ae - bc)) - (2iab + ae - bc)\sqrt{((a-c)^2 - 4af)}}{2aa}$$

ita ut hinc iidem casus ac ante prodeant, atque adeo eadem integralia ordine retrogrado scripta obtineantur.

VIII. Verum ante quam integrale per seriem inuestigemus, nostra aequatio transformari potest in aliam eiusdem formae ponendo

$$z = (a + bx)^m v, \text{ vnde fit } \frac{dz}{z} = \frac{dv}{v} + \frac{mb dx}{a + bx}$$

$$\text{et } \frac{ddz}{z} = \frac{ddv}{v} - \frac{mb^2 dx^2}{(a + bx)^2} + \frac{2mb dx dv}{v(a + bx)} + \frac{mmb dx^2}{(a + bx)^2}$$

factaque substitutione

$$\begin{aligned} \frac{(a + bx) ddv}{v} + \frac{2mb dx dv}{v} + \frac{m(m-1)bb dx^2}{a + bx} &= 0 \\ + \frac{(c + ex) dx dv}{xv} + \frac{mb(c + ex) dx^2}{x(a + bx)} \\ + \frac{(f + gx) dx^2}{xx} \end{aligned}$$

R 2

fiat



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fiat  $(m-1)bx + c + ex$  diuisibile per  $a + bx$ , eritque

$$(m-1)b + e = \frac{bc}{a} \quad \text{et} \quad m = 1 + \frac{c}{a} - \frac{e}{b},$$

nostraque aequatio.

$$(a+bx)ddv + (c + (\frac{2bc}{a} + 2b - ex)\frac{dx dv}{x}) + (f + (g + \frac{bc}{a} + \frac{bcc}{aa} - \frac{ex}{a})x)\frac{v dx^2}{xx} = 0.$$

Ponatur breuitatis gratia  $\frac{2bc}{a} + 2b - e = \varepsilon$  et

$$g + \frac{bc}{a} + \frac{bcc}{aa} - \frac{ex}{a} = \eta.$$

vt habeatur forma propositae similis.

$$(a+bx)ddv + (c + \varepsilon x)\frac{dx dv}{x} + (f + \eta x)\frac{v dx^2}{xx} = 0.$$

quae ergo est integrabilis, si fuerit.

$$\eta = \frac{2abf + c(ac - bc) - a(2iab + (2i+1)(ae - bc)) + (iab + ae - bc)\sqrt{(a-c)^2 - 4af}}{2aa}$$

at est  $ae - bc = 2ab - ae + bc$ , vnde habetur

$$\eta = \frac{2abf + c(ac - bc) - a(2i+1)^2 ab - (2i+1)(ae - bc) + (2i+1)ab - ae + bc\sqrt{(a-c)^2 - 4af}}{2aa}$$

$$= g + \frac{c(ab - ae + bc)}{aa}, \quad \text{ideoque}$$

$$g = \frac{2abf + c(ac - bc) - a(2i+1)^2 ab - (2i+1)(ae - bc) + (2i+1)ab - ae + bc\sqrt{(a-c)^2 - 4af}}{2aa}$$

quae expressio congruit cum praecedente, si ibi loco  $i$  ponatur  $i - 1$ . Quare hic iam pro  $i$  omnes numeros integros tam positivos quam negatiuos sumere licet.

IX. Fieri autem potest, vt casus, qui per priorem seriem sunt integrabiles, iidem quoque per posteriorem integrari sicque pro eadem aequatione gemina integralia exhiberi queant. Ponamus enim numerum  $i$  pro hac posteriori forma superare numerum integrum  $i$  praecedentis formae excessu  $\alpha - 1$ ,  
ita

ita ut hic pro  $i$  scribamus  $i + \alpha - 1$ . Quo factio  
ut ambo valores ipsius  $g$  congruant fieri necesse est

$$2(i+\alpha)^2 ab - (2i+2\alpha-1)(ae-bc) = 2i^2 ab + (2i+1)(ae-bc)$$

$$\text{et } 2(i+\alpha)ab - ae + bc = 2iab + ae - bc$$

ex qua sequitur  $aab = ae - bc$ . In priori autem  
scribendo  $aab$  loco  $ae - bc$ , prodit per  $ab$  dividendo

$$2(i+\alpha)^2 - 2\alpha i - 2\alpha\alpha + \alpha = 2i^2 + 2\alpha i + \alpha$$

quae cum sit identica pro omnibus valoribus ipsius  $i$ ,  
habebimus  $\alpha = \frac{ae-bc}{ab}$ ; quae expressio debet esse  
numerus integer.

X. Quoniam igitur infinitos valores pro litte-  
ra  $g$  eruimus, quibus aequatio proposita integratio-  
nem admittit, atque adeo formulam algebraicam pro  $z$   
satisfaciens assignari potest; operae pretium est, ut  
hos casus accuratius perpendamus. Denotante ergo  
 $i$  numerum quemcunque integrum siue positivum  
siue negativum, evolutio prior §. 7. facta has duas  
conditiones postulat:

$$n(n-1)b + ne + g = \alpha \quad \text{et}$$

$$(n-i)(n-i-1)a + (n-i)c + f = 0$$

ex quibus deducitur:

$$n = \frac{b-e + \sqrt{(b-e)^2 - 4bg}}{2b} \quad \text{et}$$

$$n-i = \frac{a-c + \sqrt{(a-c)^2 - 4af}}{2a} \quad \text{unde fit}$$

$$i = \frac{bc - ae + a\sqrt{(b-e)^2 - 4bg} - b\sqrt{(a-c)^2 - 4af}}{2ab}$$

Quoties ergo haec formulam

$$\frac{ac - ae + a\sqrt{(b-e)^2 - 4bg} - b\sqrt{(a-c)^2 - 4af}}{2ab}$$

R 3.

vbi

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vbi partes irrationales tam positivae quam negativae accipi possunt, aequatur numero integro siue positivo siue negativo, toties proposita aequatio integrationem admittit.

XI. Si coefficientes  $a, b, c, e, f, g$  sint rationales, ut hoc fieri possit vel vtrumque signum radicale fieri rationale debet, vel se mutuo destruere. Pro hoc casu fit

$$aa(b-e)^2 - 4aabg = bb(a-c)^2 - 4abbf \text{ seu}$$

$$4ab(ag-bf) = (ae-bc)(ae+bc-2ab)$$

tum vero fit necesse est  $i = \frac{bc-ae}{2ab}$ .

Pro illo vero casu si statuamus:

$$\sqrt{(a-c)^2 - 4af} = b \text{ et } \sqrt{(b-e)^2 - 4bg} = k \text{ erit}$$

$$f = \frac{(a+c)^2 - bb}{4a} \text{ et } g = \frac{(b-e)^2 - kk}{4b}$$

Tales igitur valores si habeant litterae  $f$  et  $g$ , dispiciatur an haec expressio  $\frac{bc-ae+ak-bb}{2ab}$  sit numerus integer? Tum enim si sit numerus integer positivus, valor ipsius  $x$  per seriem priorem, sin autem negativus, per posteriorem exhiberi poterit. Ac si insuper  $\frac{ae-ae}{ab}$  fuerit numerus integer, vtroque modo integratio absolui poterit, vnde integrale completum algebraicum obtinebitur.

XII. Casus etiam integrabiles inuestigari possunt quaerendo factorem, per quem aequatio multiplicata fiat integrabilis. In hunc finem consideremus aequationem huius formae:

$$ddz + Q dx dz + R z dx^2 = 0$$

ita

ita vt fit

$$Q = \frac{c + ex}{x(a + bx)} \quad \text{et} \quad R = \frac{f + gx}{xx(a + bx)}$$

Sitque multiplicator  $z p dz + q z dx$ , ideoque aequatio integrabilis:

$$z p dz ddz + q z dx ddz + 2p Q dx dz^2 + Q q z dx^2 dz + 2p R z dx^2 dz + q R z z dx^2 = 0.$$

Statuatur aequatio integralis:

$$p dz^2 + q z dx dz + z z dx^2 \int R q dx = C dx^2$$

cuius differentiali inde ablato fieri debet:

$$\left. \begin{aligned} &+ 2Q p dx dz^2 - d p dz^2 - q dx dz^2 \\ &+ Q q z dx^2 dz + 2R p z dx^2 dz - z dx d q dz - z z dx^2 dz \int R q dx \end{aligned} \right\} = 0$$

vnde hae duae aequationes existunt:

$$d p + q dz = 2Q p dx \quad \text{seu} \quad \frac{d p}{p} + \frac{q dz}{p} = 2Q dx$$

$$\text{et} \quad Q q dz + 2R p dx - d q - 2 dx \int R q dx = 0.$$

$$\text{Ponatur} \int R q dx = S, \quad \text{erit} \quad R dx = \frac{d S}{q}, \quad \text{et}$$

$$Q q dx + \frac{2p d S}{q} - d q - 2 S dx = 0 \quad \text{seu} \quad d S - \frac{S q dx}{p} = \frac{q dq}{2p} - \frac{Q q q dx}{2p}$$

et pro  $Q dx$  scripto superiori valore

$$d S - \frac{S q dx}{p} = \frac{q dq}{2p} - \frac{q q d p}{4 p p} - \frac{q^2 dx}{4 p p}$$

Sit  $x$  numerus cuius logarithmus =  $z$  et integrando, eruitur

$$x \frac{-\int q dx}{p} S = \int x \frac{-\int q dx}{p} \left( \frac{q dq}{2p} - \frac{q q d p}{4 p p} - \frac{q^2 dx}{4 p p} \right) \text{ seu}$$

$$x \frac{-\int q dx}{p} S = \frac{1}{2} C + x \frac{-\int q dx}{p} \frac{q q}{4 p}, \quad \text{vnde fit}$$

$$S = \frac{1}{2}$$

$$S = \frac{1}{2} C x^{\frac{\int q dx}{p}} + \frac{q}{4p} = \int R q dx, \text{ hincque porro}$$

$$R = \frac{C x^{\frac{\int q dx}{p}}}{4p} + \frac{2pdq - qdp}{2pp dx} \text{ et } Q = \frac{dp}{2p dx} + \frac{q}{2p}$$

Ex quibus colligimus, proposita hac aequatione

$$ddz + \frac{(dp + qdx)dz}{2p} + (C x^{\frac{\int q dx}{p}} p dx + 2pdq - qdp) \frac{z dz}{4pp} = 0$$

si ea ducatur in  $2p dz + qz dx$  fore integrale

$$p dz^2 + qz dx dz + (C x^{\frac{\int q dx}{p}} p + q) \frac{z dz^2}{4p} = A dx^2.$$

XIII. His ad propositum accommodatis primo obtinemus:

$$\frac{(c+ex)dx}{x(a+bx)} = \frac{dp}{2p} + \frac{q dx}{2p} \text{ vnde colligimus}$$

$$\frac{\int q dx}{p} = -\ln p + \frac{2c}{a} \ln x + \frac{2(ae-bc)}{ab} \ln(a+bx), \text{ ideoque}$$

$$\frac{\int q dx}{x^{\frac{2c}{a}} (a+bx)^{\frac{2(ae-bc)}{ab}}} = \frac{p}{4pp dx} \text{ hincque}$$

$$\frac{f+gx}{xx(a+bx)} = \frac{C x^{\frac{2c}{a}} (a+bx)^{\frac{2(ae-bc)}{ab}} dx + 2pdq - qdp}{4pp dx}$$

At est  $q = \frac{2p(c+ex)}{x(a+bx)} - \frac{dp}{dx}$  indeque

$$dq = \frac{2(c+ex)dp}{x(a+bx)} - \frac{2pdx(ac+2bcx+bcxx)}{xx(a+bx)^2} - \frac{dp}{dx}$$

quibus substitutis aequatio resoluenda erit

$$\frac{4(f+gx)pp dx}{xx(a+bx)} = C x^{\frac{2c}{a}} (a+bx)^{\frac{2(ae-bc)}{ab}} dx + \frac{2(c+ex)dp}{x(a+bx)} - \frac{4pp dx(ac+2bcx+bcxx)}{xx(a+bx)^2} - \frac{2pd dp}{dx} + \frac{dp^2}{dx}.$$

XIV.

XIV. Verum hoc modo haud minoribus difficultatibus implicamur, quam si ipsam aequationem propositam resolvere vellemus. Aliam ergo viam magis particularem ingrediamur, et quaeramus conditiones coefficientium A, B, C vt haec aequatio:

$$Ax^\lambda ddz + Bx^{\lambda-1} dx dz + Cx^{\lambda-2} z dx^2 = 0$$

si multiplicetur per  $2x dz + az dx$ , fiat integrabilis. Cum igitur productum sit

$$+ 2Ax^{\lambda+1} dz ddz + 2Bx^\lambda dx dz^2 + aBx^{\lambda-1} z dx^2 dz + aCx^{\lambda-2} z z dx^3 = 0$$

$$+ aAx^\lambda z dx ddz + 2Cx^{\lambda-1} z dx^2 dz$$

integrale fit necesse est

$$Ax^{\lambda+1} dz^2 + aAx^\lambda z dx dz + \frac{a}{\lambda-1} Cx^{\lambda-1} z z dx^2 = E dx^2$$

cuius differentiale si inde auferatur, prodibit haec aequatio:

$$\left. \begin{aligned} &+ x^\lambda dx dz^2 (2B - (\lambda+1)A - aA) \\ &+ x^{\lambda-1} z dx^2 dz (aB + 2C - a\lambda A - \frac{2aC}{\lambda-1}) \end{aligned} \right\} = 0.$$

Vnde utroque membro seorsim annihilato fit primo

$$B = \frac{a+\lambda+1}{2} A \text{ hincque}$$

$$\frac{a(a-\lambda+1)}{2} A = \frac{2(a-\lambda+1)}{\lambda-1} C$$

ex qua duplici modo eruitur:

$$\text{vel } \lambda = a + 1 \quad \text{vel } C = \frac{a(\lambda-1)}{4} A.$$

Duae ergo aequationes oriuntur

$$\text{altera } Ax^{\lambda+1} ddz + (a+1)Ax^\lambda dx dz + Cx^{\lambda-1} z dx^2 = 0$$

$$\text{altera } Ax^\lambda ddz + \frac{1}{2}(a+\lambda+1)Ax^{\lambda-1} dx dz + \frac{1}{2}a(\lambda-1)Ax^{\lambda-2} z dx^2 = 0$$

quarum vtraque per  $2xz + \alpha z dx$  multiplicata fit integrabilis; illius enim integrale erit:

$$Ax^{\alpha+2} dz^2 + \alpha Ax^{\alpha+1} z dx dz + Cx^{\alpha} z z dx^2 = E dx^2$$

huius vero

$$Ax^{\lambda+1} dz^2 + \alpha Ax^{\lambda} z dx dz + \frac{1}{2} \alpha \alpha Ax^{\lambda-1} z z dx^2 = F dx^2$$

XV. Summa ergo harum duarum aequationum eodem factore integrabilis reddetur. Scilicet haec aequatio:

$$(Ax^{\alpha+1} + Dx^{\lambda}) dz^2 + ((\alpha+1)Ax^{\alpha+1} + (\alpha+\lambda+1)Dx^{\lambda-1}) dx dz + (Cx^{\alpha-1} + \frac{1}{2} \alpha (\lambda-1) Dx^{\lambda-2}) z dx^2 = 0$$

multiplicata per  $2xz + \alpha z dx$  integrale praebet:

$$(Ax^{\alpha+2} + Dx^{\lambda+1}) dz^2 + \alpha (Ax^{\alpha+1} + Dx^{\lambda}) z dx dz + (Cx^{\alpha} + \frac{1}{2} \alpha \alpha Dx^{\lambda-1}) z z dx^2 = E dx^2$$

quod isto modo representari potest:

$$(Ax^{\alpha} + Dx^{\lambda-1})(xdz + \frac{1}{2} \alpha z dx)^2 = dx^2 (C\alpha\alpha A - C)x^{\alpha} z z + E)$$

ita ut fit

$$xdz + \frac{1}{2} \alpha z dx = \frac{1}{2} dx \sqrt{\frac{4E + (\alpha\alpha A - 4C)x^{\alpha} z z}{Ax^{\alpha} + Dx^{\lambda-1}}}$$

Ponatur

$$x^{\alpha} z z = v \text{ erit } x^{\alpha-1} z (2xdz + \alpha z dx) = 2v dv,$$

ideoque

$$2xdz + \alpha z dx = \frac{dv}{x^{\alpha-1} z} = \frac{2dv}{x^{\frac{1}{2}\alpha - 1}}; \text{ vnde fit}$$

$$2x^{\frac{1}{2}\alpha - 1} dv = dx \sqrt{\frac{4E + (\alpha\alpha A - 4C)v v}{Ax^{\alpha} + Dx^{\lambda-1}}} \text{ seu}$$

$z dx$

$$\frac{2 d v}{\sqrt{(4 E+(a a A-4 C) v v)}} = \frac{x^{\frac{1}{2} \alpha-1} d x}{\sqrt{(A x^{\alpha}+D x^{\alpha-1})}}$$

XVI. Quo nunc hanc aequationem ad nostram formam perducamus, quod duplici modo fieri potest, ponamus primo  $\lambda = a$ , ut facta diuisione per  $x^{\alpha}$  habeatur haec aequatio:

$$(D+A x) d d z + \left( (a + \frac{1}{2}) D + (a + 1) A x \right) \frac{d z d x}{x} + \left( \frac{1}{2} a (a - 1) D + C x \right) \frac{z d x^2}{x x} = 0$$

quae multiplicata per  $x^{\alpha}$  ( $2 x d z + a z d x$ ) fit integrabilis, existente integrali posito

$$x^{\alpha} z z = v v \quad \text{seu} \quad z = x^{-\frac{1}{2} \alpha} v$$

$$\frac{2 d v}{\sqrt{(4 E+(a a A-4 C) v v)}} = \frac{x^{\frac{1}{2} \alpha-1} d x}{\sqrt{x^{\alpha-1} (D+A x)}} = \frac{d x}{\sqrt{x (D+A x)}}$$

Sit iam  $D = a$ ;  $A = b$ ;  $(a + \frac{1}{2}) a = c$ , seu  $\alpha = \frac{c}{a} - \frac{1}{2}$ , et  $C = g$  et prodibit haec aequatio:

$$(a + b x) d d z + \left( c + \frac{b(a + 2c)x}{2a} \right) \frac{d z d x}{x} + \left( \frac{(2c - a)(2c - 3a)}{16a} + g z \right) \frac{z d x^2}{x x} = 0$$

ita vt pro forma proposita fit

$$e = \frac{b(a + 2c)}{2a} \quad \text{et} \quad f = \frac{(2c - a)(2c - 3a)}{16a}$$

huiusque aequationis posito  $z = x^{\frac{1}{2} - \frac{c}{a}} v$  integrale erit:

$$\frac{2 d v}{\sqrt{(4 E + \frac{b(2c - a)^2}{4a^2} - 4g) v v}} = \frac{d x}{\sqrt{x(a + b x)}}$$

XVII. Statuatur nunc secundo  $\lambda = a + 2$ , vt facta diuisione per  $x^{\alpha+1}$  oriatur haec aequatio:

$$S \quad 2 \quad (A + D)$$



$$(A + Dx) ddz + ((\alpha + 1)A + (\alpha + \frac{1}{2})Dx) \frac{dx dz}{x} + (C + \frac{1}{2}\alpha(\alpha + 1)Dx) \frac{dx^2}{xx} = 0$$

quae multiplicata per  $x^{\alpha+1}(2xdz + azdx)$ , posito  $x^{\alpha}zz = vv$  seu  $z = x^{-\frac{1}{2}\alpha}v$  habebit integrale:

$$\frac{2dv}{\sqrt{(4E + (\alpha\alpha A - 4C)vv)}} = \frac{x^{\frac{1}{2}\alpha-1}dx}{\sqrt{x^{\alpha}(A + Dx)}} = \frac{dx}{\sqrt{x(A + Dx)}}$$

Sit iam  $A = a$ ;  $D = b$ ;  $(\alpha + 1)a = c$ , seu  $\alpha = \frac{c}{a} - 1$  et  $C = f$  vt obtineatur haec aequatio:

$$(a + bx) ddz + (c + \frac{b(a+2c)}{2}x) \frac{dx dz}{x} + (f + \frac{bc(c-a)}{4aa}x) \frac{dx^2}{xx} = 0$$

et pro forma proposita fit:

$$e = \frac{b(a+2c)}{2a} \quad \text{et} \quad g = \frac{bc(c-a)}{4aa}$$

cuius posito  $z = x^{\frac{1}{2}} - \frac{c}{2a}v$  integrale est:

$$\frac{2dv}{\sqrt{(4E + (\frac{c-a}{a})^2 - 4f)vv)}} = \frac{dx}{\sqrt{x(a + bx)}}$$

XVIII. Non solum autem quoties ipsa aequatio proposita:

$$(a + bx) ddz + (c + ex) \frac{dx dz}{x} + (f + gx) \frac{dx^2}{xx} = 0$$

in altera harum formarum est contenta, quod euenit:

$$\text{si fuerit vel } e = \frac{b(a+2c)}{2a} \quad \text{et} \quad f = \frac{(2c-a)(2c-3a)}{16a}$$

$$\text{vel } e = \frac{b(a+2c)}{2a} \quad \text{et} \quad g = \frac{bc(c-a)}{4aa}$$

integrationem admittit, sed etiam quoties eadem transformata in alterutra continetur. Transformatio autem vt supra §. 8. vidimus fit substitutione

$z =$

$$z = (a + bx)^{\frac{1}{2}} + \frac{c}{a} - \frac{e}{b} v,$$

vnde oritur

$$(a + bx)ddv + (c + \epsilon x)\frac{d^2 x dv}{x} + (f + \eta x)\frac{v dx^2}{x^2} = 0$$

existente

$$\epsilon = \frac{2b(a+c)}{a} - e \text{ et } \eta = g - \frac{ce}{a} + \frac{bc(a+c)}{aa}.$$

Haec autem ponendo  $v = x^n s$ , ob

$$\frac{dv}{v} = \frac{n dx}{x} + \frac{ds}{s} \text{ et } \frac{d^2 v}{v} = \frac{n(n-1)dx^2}{x^2} + \frac{2n dx ds}{xs} - \frac{d ds}{s},$$

transformatur in hanc:

$$\left. \begin{aligned} (a + bx)\frac{d ds}{s} + 2n(a + bx)\frac{d^2 x ds}{x^2} + n(n-1)(a + bx)\frac{d^2 x^2}{x^2} \\ + (c + \epsilon x)\frac{d^2 x ds}{x^2} + n(c + \epsilon x)\frac{d^2 x^2}{x^2} \\ + (f + \eta x)\frac{d^2 x^2}{x^2} \end{aligned} \right\} = 0$$

vnde hi bini casus integrabiles eruuntur.

Primus. si  $D = a$ ;  $A = b$ ;  $(\alpha + \frac{1}{2})a = 2na + c$

$$(\alpha + 1)b = 2nb + \frac{2b(a+c)}{a} - e$$

$$\frac{1}{2}\alpha(\alpha - 1)a = n(n-1)a + nc + f$$

$$C = n(n-1)b + \frac{2nb(a+c)}{a} - ne + g - \frac{ce}{a} + \frac{bc(a+c)}{aa}$$

hincque  $\alpha = 2n - \frac{1}{2} + \frac{c}{2a}$ , et  $e = \frac{3}{2}b + \frac{bc}{a} = \frac{b(3a+2c)}{2a}$

atque  $(n - \frac{1}{2} + \frac{c}{2a})(n - \frac{3}{4} + \frac{c}{2a})a = n(n-1)a + nc + f$

vnde  $f = \frac{c^2 c}{4a^2} - \frac{c^2}{2a} + \frac{3ca}{16} = \frac{(2c-a)(2c-3a)}{16a}$

Alter casus his conditionibus continetur:

$A = a$ ;  $D = b$ ;  $(\alpha + 1)a = 2na + c$ ;  $(\alpha + \frac{3}{2})b = 2nb + \frac{2b(a+c)}{a} - e$

$$C = n(n-1)a + nc + f;$$

$$\frac{1}{2}\alpha(\alpha + 1)b = n(n-1)b + \frac{2nb(a+c)}{a} - ne + g - \frac{ce}{a} + \frac{b^2 a(a+c)}{aa}$$

vnde fit  $\alpha = 2n - 1 + \frac{c}{a}$ ;  $e = \frac{b(2a+2c)}{2a}$  atque

$$g = \frac{bc}{4a} + \frac{bcc}{4aa} = \frac{bc(a+c)}{4aa}$$

vbi constat numerum  $n$  nihil conferre.

XIX. Quatuor ergo hinc nacti sumus casus integrabiles, qui sunt:

$$1^{\circ}. e = \frac{b(a+2c)}{2a}; \quad f = \frac{(2c-a)(2c-3a)}{16a}$$

$$2^{\circ}. e = \frac{b(a+2c)}{2a}; \quad g = \frac{bc(c-a)}{4aa}$$

$$3^{\circ}. e = \frac{b(2a+2c)}{2a}; \quad f = \frac{(2c-a)(2c-3a)}{4aa}$$

$$4^{\circ}. e = \frac{b(2a+2c)}{2a}; \quad g = \frac{bc(a+c)}{4a}$$

quibus adeo integrale completum exhibuimus. Videamus ergo quomodo hi casus se habeant ad conditionem, quam supra ex serie deduximus, vtrum in ea contineantur nec ne?

Pro primo igitur habemus

$$bc - ae = \frac{ab}{2}; \quad b - e = \frac{b(a-2c)}{2a}$$

et  $\sqrt{(a-c)^2 - 4af} = \pm \frac{a}{2}$ , vnde haec formula

$$-\frac{1}{4} \pm \frac{1}{4} + \frac{\sqrt{(b^2(a-2c)^2 - 16a^2bg)}}{4aa}$$

numerus integer esse deberet.

Pro secundo est

$$bc - ae = \frac{ab}{2}; \quad \text{et } \sqrt{(b-e)^2 - 4bg} = \pm \frac{b}{2},$$

ergo haec formula

$$-\frac{1}{4} \pm \frac{1}{4} - \frac{\sqrt{(a-c)^2 - 4af}}{2a}$$

numerus integer esse deberet.

Pro

Pro tertio est

$$bc - ae = \frac{-3ab}{2}; \quad b - e = \frac{-b(a+2c)}{2a}, \quad \text{et}$$

$$\sqrt{(a-c)^2 - 4af} = \pm \frac{a}{2};$$

vnde haec formula

$$-\frac{3}{4} \mp \frac{1}{4} + \frac{\sqrt{(b(a+2c))^2 - 16a^2bg}}{4ab}$$

numerus integer esse deberet.

Pro quarto est

$$bc - ae = \frac{-3ab}{2} \quad \text{et} \quad \sqrt{(b-e)^2 - 4bg} = \pm \frac{b}{2}$$

vnde haec formula

$$-\frac{3}{4} \mp \frac{1}{4} - \frac{\sqrt{(a-c)^2 - 4af}}{2a}$$

numerus integer esse deberet.

Vnde perspicitur hos quatuor casus in superiori conditione non contineri, ideoque hinc omnino novos casus integrabilitatis erui.

XX. Cum igitur hi casus, quibus integrale completum eruimus, omnino discrepent ab iis, quibus supra integrale particulare exhibuimus, iuuabit ostendisse, quomodo etiam his casibus integrale completum obtineri possit; quod sequenti modo facillime praestari videtur.

Si aequationi

$$Pdz + Qdx + Rzdx^2 = 0$$

satisfaciat valor  $z = V$ , vt fit

$$P'dV + Q'dx + R'Vdx^2 = 0,$$

illa aequatio reddetur integrabilis ducta in  $\frac{V}{P(Vdx - x'dV)}$

Posita

Posito enim

$$\int \frac{P V d d z + Q V d x d z + R V z d x^2}{P(V d z - z d V)} = S d x, \text{ erit}$$

$$\begin{aligned} S d x - I P (V d z - z d V) &= \int \frac{Q V d x d z + R V z d x^2 - V d P d z + P z d d V + z d V d P}{P(V d z - z d V)} \\ &= \int \frac{Q d x}{P} - I P + \int \frac{z(P d d V + Q d x d V + R V d x^2)}{P(V d z - z d V)}. \end{aligned}$$

$$\text{At } P d d V + Q d x d V + R V d x^2 = 0,$$

ideoque habetur:

$$S d x = I(V d z - z d V) + \int \frac{Q d x}{P} + \text{Const.} = \text{Const.}$$

unde erit

$$V d z - z d V = C x^{\frac{-\int Q d x}{P}} d x \text{ et}$$

$$z = C V \int \frac{d x}{V} x^{\frac{-\int Q d x}{P}}$$

quod est integrale completum ex particulari  $z = V$  erutum.

XXI. Quoniam utriusque generis casus ex aequatione proposita eliciantur, si ea per formam  $z p d z + q d z d x$  multiplicata integrabilis efficiatur, posito  $p = u u$ , ut sit

$$q = \frac{z u u (c + b x)}{x(a + b x)} - \frac{z u d x}{d x},$$

erit aequatio integralis:

$$u u d z^2 + q z d x d z + (C x^{\frac{\int q d x}{u u}} + q q) \frac{z z d x^2}{u u} = A d x^2$$

vbi est

$$\frac{\int q d x}{x u u} = \frac{z c (a + b x) \frac{z(a c - b c)}{a b}}{u u}$$

ideo-

ideoque

$$\left(udx + \frac{qz dx}{2u}\right)^2 = Adx^2 - \frac{Cx^{\frac{2c}{a}}(a+bx)^{\frac{2(ae-bc)}{ab}} z z dx^2}{4u^2}$$

verum quantitati  $u$  ex hac aequatione elici oportet.

$$\frac{d du}{dx} = \frac{(c+ex) du}{x(a+bx)} + \frac{(f+gx) u dx}{x x(a+bx)} + \frac{(ac + 2bcx + be x x) u dx}{x x(a+bx)^2} - \frac{Cx^{\frac{2c}{a}}(a+bx)^{\frac{2(ae-bc)}{ab}} dx}{4u^2}$$

et prioris quidem generis casus hinc sumta constante  $C = 0$  sunt deducti. Verum haec aequatio posito

$$u = x^{\frac{c}{a}}(a+bx)^{\frac{ae-bc}{ab}} v \text{ abit in hanc}$$

$$\frac{Cx^{\frac{2c}{a}}(a+bx)^{\frac{2(bc-de)}{ab}} dx^2}{4v^2} = ddv + \frac{(c+ex) dx dv}{x(a+bx)} + \frac{(f+gx) v dx^2}{x x(a+bx)}$$

cuius applicatio est facilior, unde si  $C = 0$ , quantitas  $v$  satisfacere debet huic aequationi

$$(a+bx) ddv + \frac{(c+ex) dx dv}{x} + \frac{(f+gx) v dx^2}{x x} = 0$$

ita vt hinc ex valore particulari obtineatur completus. At si ponamus  $u = x^m(a+bx)^n$  erit

$$\frac{1}{2} Cx^{\frac{2c}{a}-4m}(a+bx)^{\frac{2(ae-bc)}{ab}-4n} = \frac{m(m-1)}{x x} + \frac{(f-mc+(g-me+mnb)x)}{x x(a+bx)} + \frac{a(c+(2-n)bx+(n-1)(nbb-be)xx)}{x x(a+bx)^2}$$

ideoque tam exponentes  $m$  et  $n$  cum constante  $C$ , quam ratio coefficientium  $a, b, c, e, f, g$  ex hac aequatione definiri debet

$$\frac{1}{2} Cx^{\frac{2c}{a}-4m+2}(a+bx)^{\frac{2(ae-bc)}{ab}-4n+2} = m(m-1)(a+bx)^2$$

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$$+(a+bx)(f-mc+(g-me+2mnb)x)+ac(2-n)bcx \\ + (n-1)(nb-e)bx^2.$$

XXII. Hinc plures casus resultant, quos evol-  
vamus :

Primus. Si exponens

$$\frac{2(ae-bc)}{ab} - 4n + 2 = 2 \text{ seu } n = \frac{ae-bc}{2ab},$$

quo esse debet

$$\frac{2c}{a} - 4m + 2 = 0 \text{ seu } m = \frac{a+c}{2a},$$

ut habeatur

$$\frac{1}{2}C(a+bx)^2 = \frac{cc-aa}{4aa}(a+bx)^2 + \left(f - \frac{c(a+c)}{2a} + \left(g - \frac{bc(a+c)}{2aa}\right)x\right)(a+bx) \\ - ac + (2-n)bcx + (n-1)(nb-e)bx^2$$

vbi postremum membrum per  $a+bx$  diuisibile  
esse debet, id quod duplici modo fieri potest.

1°. Vel est  $n=1$ ; ideoque  $e = \frac{2ab+bc}{a} = \frac{b(2a+c)}{a}$ ,

sicque erit

$$\frac{1}{2}C(a+bx) = \frac{cc-aa}{4aa}(a+bx) + f - \frac{c(a+c)}{2a} + \left(g - \frac{bc(a+c)}{2aa}\right)x + e$$

vnde fit

$$\frac{1}{2}Ca = \frac{cc-aa}{4a} + f + \frac{c(a+c)}{2a} = f - \frac{(a-c)^2}{4a}$$

$$\text{et } \frac{1}{2}Cb = \frac{b(cc-aa)}{4aa} + g - \frac{bc(a+c)}{2aa} = g - \frac{b(a+c)^2}{4aa}$$

Ergo

$$bf - ag - \frac{b(a+c)^2}{4a} + \frac{b(a+c)^2}{4a} = 0 \text{ seu } g = \frac{bf}{a} + \frac{bc}{a} = \frac{b(c+a)}{a}$$

$$\text{et } \frac{a}{2u} = \frac{u(c+ex)}{x(a+bx)} - \frac{du}{dx} = x^{\frac{c-a}{2a}}(c+ex) - \frac{(a+c)}{2a}x^{\frac{c-a}{2a}}(a+bx) - bx^{\frac{a+c}{2a}}$$

$$\text{seu } \frac{q}{2u} = \frac{a(c-a) + b(a+c)}{2a}x^{\frac{c-a}{2a}}. \text{ Consequenter ae-} \\ \text{quatio integralis}$$

$$\left(\frac{a+c}{2a}\right)$$

$$(x^{\frac{a+c}{2a}}(a+bx)dx + \frac{c(a-b) + b(a+c)x}{2a} x^{\frac{c-a}{2a}} dx)^2 = Adx^2$$

$$- \left( \frac{f}{a} - \frac{(a-c)^2}{4aa} \right) \frac{xx dx^2}{xx}$$

2°. Vel est  $n = \frac{ae-bc}{ab} = \frac{ae-bc}{aab}$ , ideoque  $e = \frac{bc}{a}$

et  $n = 0$ ; vnde fit

$$\frac{1}{4}C(a+bx) = \frac{cc-aa}{4aa}(a+bx) + f - \frac{c(a+c)}{2a} + \left( g - \frac{bc(a+c)}{2aa} \right) x + e + \frac{bc}{a} x$$

ergo  $\frac{1}{4}Ca = \frac{cc-aa}{4a} + f + \frac{c(a-c)}{2a} = f - \frac{(a-c)^2}{4a}$

et  $\frac{1}{4}Cb = \frac{b(cc-aa)}{4aa} + g + \frac{bc(a-c)}{2aa} = g - \frac{b(a-c)^2}{4aa}$

vnde colligitur  $bf = ag$  seu  $g = \frac{bf}{a}$ ; qui est casus, quo aequatio proposita per  $a+bx$  diuisibilis existit, sicque nihil habet difficultatis.

XXIII. Secundus casus est quo

$$\frac{2(ae-bc)}{ab} - 4n + 2 = 1 \quad \text{et} \quad \frac{2c}{a} - 4m + 2 = 0$$

ideoque  $m = \frac{a+c}{2a}$  et  $n = \frac{ab+2ae-2bc}{4ab}$ , ita vt habeamus

$$\frac{1}{4}C(a+bx) = \frac{cc-aa}{4aa}(a+bx)^2 + \left( f - \frac{c(a+c)}{2a} + \left( g + \frac{b(a+c)(a-c)}{4aa} \right) x + ac + (2-n)bcx + (n-1)(nb-e)bx \right) x$$

qui casus iterum in duos dispertitur:

1°. Vel est  $n = 1$ , ideoque  $2ae - 2bc = 3ab$

et  $e = \frac{3ab+2bc}{2a} = \frac{b(3a+2c)}{2a}$

vnde fit

$$\frac{1}{4}C = \frac{bc-aa}{4aa}(a+bx) + f + \frac{c(a-c)}{2a} + gx + \frac{b(a+c)(a-c)}{4aa} x$$

hincque

$$\frac{1}{4}C = \frac{cc-aa}{4a} + f + \frac{c(a-c)}{2a} = f - \frac{(a-c)^2}{4aa}$$

T 2

et



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$$\text{et } 0 = \frac{b(cc - aa)}{4aa} + g + \frac{b(a+c)(a-2c)}{4aa} \text{ seu } g = \frac{bc(a+c)}{4aa}$$

qui est casus quartus in §. 19.

$$2^\circ. \text{ Vel est } n = \frac{ae - bc}{ab} = \frac{ab + 2ae - 2bc}{4ab}, \text{ ideoque } e = \frac{b(a+2c)}{2a}$$

et  $n = \frac{1}{2}$ ; vnde aequatio per  $a + bx$  diuisa fit

$$\frac{1}{2}C = \frac{cc - aa}{4aa}(a + bx) + f + \frac{c(a-c)}{2a} + (g + \frac{b(a+c)(a-2c)}{4aa})x + \frac{bc}{2a}x$$

ita vt fit

$$\frac{1}{2}C = \frac{cc - aa}{4aa} + f + \frac{c(a-c)}{2a} = f - \frac{(a-c)^2}{4a}$$

$$\text{et } \frac{b(cc - a)}{4aa} + \frac{bc}{2a} + \frac{b(-c)(a-2c)}{4aa} + g = 0 \text{ seu } g = \frac{bc(c-a)}{4aa}$$

qui erat casus 2°. in §. 19.

XXIV. Tertius casus est quo

$$\frac{2(ae - bc)}{ab} - 4n + 2 = 1 \text{ et } \frac{2c}{a} - 4m + 2 = 1$$

$$\text{ideoque } m = \frac{a+2c}{4a} \text{ et } n = \frac{a^2 - 2ae - 2bc}{4ab}$$

ficque habebimus:

$$\frac{1}{2}C(a+bx) = \frac{(a+2c)(2c-3a)}{16aa}(a+bx)^2 + (f - \frac{c(a+2c)}{4a} + (g + \frac{b(a+2c)(a-c)}{4aa})x)(a+bx) \\ + ac + (2-n)bcx + (n-1)(nb-1)bx^2$$

cuius vltimum membrum duplici modo per  $a + bx$  redditur diuisibile.

$$1^\circ. \text{ Si } n = 1 = \frac{ab + 2ae - 2bc}{4ab}; \text{ ideoque } e = \frac{b(a+2c)}{2a}$$

vnde oritur

$$\frac{1}{2}C = \frac{(a+2c)(2c-3a)}{16aa}(a+bx) + f + \frac{c(2a-2c)}{4a} + gx + \frac{b(a+2c)(a-2c)}{16aa}x$$

ita vt fieri oporteat

$$\frac{(a+2c)(2c-3a)}{16a} + f - \frac{c(2c-3a)}{4a} = 0 \text{ seu } f = \frac{(2c-a)(2c-3a)}{16a}$$

et

$$\text{et } \frac{1}{4} Cx = \frac{b(a+2c)(2c-3a)}{16aa} + g + \frac{b(a+2c)(a-2c)}{8aa} = g - \frac{b(a+2c)^2}{16aa}$$

qui erat casus 3° in §. 19.

$$2^\circ. \text{ Si } n = \frac{ae-bc}{ab} = \frac{ab+2ae-2bc}{4ab} \text{ seu } e = \frac{b(a+2c)}{2a} \text{ et } n = \frac{1}{2}$$

$$\frac{1}{4} Cx = \frac{(a+2c)(2c-3a)}{16aa}(a+bx) + f - \frac{c(a+2c)}{4a} + gx + \frac{b(a+2c)(a-2c)}{8aa} \\ + c + \frac{bc}{2a}x$$

ideoque

$$f + \frac{(a+2c)(2c-3a)}{16a} + \frac{c(3a-2c)}{4a^2} = 0 \text{ seu } f = \frac{(2c-a)(2c-3a)}{16a}$$

$$\text{et } \frac{1}{4} C = \frac{b(a+2c)(2c-3a)}{16aa} + g + \frac{b(a+2c)(a-2c)}{8aa} + \frac{bc}{2a} = g - \frac{b(a-2c)}{16aa}$$

qui erat casus 1° in §. 19.

XXV. *Quartus* casus est quo

$$\frac{2(ae-bc)}{ab} - 4n + 2 = 0 \text{ et } \frac{2c}{a} - 4m + 2 = 0$$

ideoque  $m = \frac{a+c}{2a}$  et  $n = \frac{ab+ae-bc}{2ab}$ , vt habeamus:

$$\frac{1}{4} C = \frac{cc-aa}{4aa}(a+bx)^2 + (a+bx)\left(f - \frac{c(a+c)}{2a}\right) + \left(g + \frac{b(aa-cc)}{2aa}\right)x \\ + ac + \frac{(aa-b-ae+bc)}{2a}cx - \frac{(ab-ae+bc)ab-ae-bc}{4aa}xx$$

unde singulas potestates seorsim tollendo colligimus

$$\frac{bb(3c-aa)}{4aa} + bg + \frac{bb(aa-cc)}{2aa} - \frac{(ab-ae+bc)(ab-ae-bc)}{4aa} = 0$$

$$\frac{b(cc-aa)}{2aa} + bf - \frac{bc(a+c)}{2a} + ag + \frac{b(aa-cc)}{2aa} + \frac{c(ab-ae+bc)}{2a} = 0$$

ex illa fit  $g = \frac{e(e-2b)}{4b}$  ex hac vero  $bf + ag = \frac{c(e-2b)}{2}$

ideoque  $f = \frac{(e-2b)(2bc-ae)}{4bb}$ , quae sunt binae condi-

tiones; tum vero capi debet

$$\frac{1}{4} C = \frac{cc-aa}{4} + af - \frac{c(a+c)}{2} + ac = af - \frac{1}{4}(a-c)^2.$$

## XXVI. Quintus casus quo

$$\frac{2(ae-bc)}{ab} - 4n + 2 = 0; \text{ et } \frac{2c}{a} - 4m + 2 = 1, \text{ ideoque}$$

$$m = \frac{2c+a}{4a} \text{ et } n = \frac{ab+ae-bc}{2ab} \text{ vt habeamus:}$$

$$\frac{1}{4}Cx = \frac{(2c+a)(2c-3a)}{16aa}(a+bx)^2 + (a+bx)\left(f - \frac{c(2c+a)}{2a}\right) + \left(g + \frac{b(a-c)(2c+a)}{4aa}\right)x \\ + ac + \frac{3ab-ae+bc}{2a}cx - \frac{(ab-ae+bc)(ab-ae-bc)}{4aa}xx$$

hincque:

$$\frac{b(2c+a)(2c-3a)}{16aa} + bg + \frac{b(2c+a)(2c+a)}{4aa} - \frac{(ab-ae+bc)(ab-ae-bc)}{4aa} = 0$$

$$\text{seu } g = \frac{(b-2e)(3b-2e)}{16b}$$

$$\frac{(2c+a)(2c-3a)}{16a} + af - \frac{c(2c+a)}{2a} + ac = 0 \text{ seu } f = \frac{(2c-a)(2c-3a)}{16a} \text{ et}$$

$$\frac{1}{4}C = \frac{(ab-ae+bc)^2}{4ab}$$

## XXVII. Sextus casus quo

$$\frac{2(ae-bc)}{ab} - 4n + 2 = 0 \text{ et } \frac{2c}{a} - 4m + 2 = 2 \text{ ideoque}$$

$$m = \frac{c}{2a} \text{ et } n = \frac{ab+ae-bc}{2ab} \text{ vt habeamus:}$$

$$\frac{1}{4}Cxx = \frac{c(c-2a)}{4aa}(a+bx)^2 + (a+bx)\left(f - \frac{c}{2a}\right) + \left(g + \frac{bc(a-c)}{2aa}\right)x \\ + ac + \frac{3ab-ae+bc}{2a}cx - \frac{(ab-ae+bc)(ab-ae-bc)}{4aa}xx$$

vnde fieri oportet:

$$\frac{c(c-2a)}{4a} + af - \frac{c}{2a} + ac = 0 \text{ seu } f = \frac{c(c-2a)}{4a}$$

$$\frac{bc(c-2a)}{2a} + bf - \frac{bc}{2a} + ag + \frac{bc(a-c)}{2a} + \frac{3ab-ae+bc}{2a}c = 0$$

$$\text{seu } g = \frac{-c(3ab-2ae+bc)}{4aa} \text{ atque}$$

$$\frac{1}{4}C = \frac{-bc(3ab-2ae+bc)}{4aa} - \frac{(a-c)^2}{4} = bg - \frac{1}{4}(b-e)^2.$$

XXVIII. Pro his autem casibus omnibus cum fit  $u = x^m (a + bx)^n$

erit

erit

$$\frac{q}{z} = x^{m-1}(a+bx)^{n-1}(c+ex) - mx^{m-1}(a+bx)^n - nbx^m(a+bx)^{n-1}$$

feu  $\frac{q}{z} = x^{m-1}(a+bx)^{n-1}(c-ma + (e-(m+n)b)x)$ .

vnde aequatio integralis colligitur:

$$x^m(a+bx)^{n-1} \left( dz + \frac{c-ma+(e-(m+n)b)x}{x(a+bx)} z dx \right) = A dx^2$$

$$- \frac{1}{4} C x^{\frac{2-c}{a}} - \frac{1}{4} M (a+bx)^{\frac{2(ae-bc)}{ab}} - \frac{1}{4} N z z dx^2$$

vel erit  $dz + \frac{c-ma+(e-(m+n)b)x}{x(a+bx)} z dx =$

$$\frac{dx \sqrt{A - \frac{1}{4} C x^{\frac{2-c}{a}} - \frac{1}{4} M (a+bx)^{\frac{2(ae-bc)}{ab}} - \frac{1}{4} N z z}}{x^m(a+bx)^n}$$

Quare pro casibus inuentis integralia aequationis propositae

$$(a+bx) d dz + (c+ex) \frac{dx dz}{x} + \frac{(f+gx) z dx^2}{xx} = 0$$

sequenti modo se habebunt.

### Casus I.

$$m = \frac{a+c}{2a}; n = 1; e = \frac{b(2a+c)}{a}; g = \frac{b(c+f)}{a} \text{ et } \frac{1}{4} C = \frac{f}{a} - \frac{(a-c)^2}{4a^2}$$

Integrale igitur erit

$$dz + \frac{a(c-a) + (a+c)x}{2ax(a+bx)} z dx =$$

$$\frac{dx}{x^{\frac{a+c}{2a}}(a+bx)} \sqrt{A - \frac{(4af - (a-c)^2)}{4a^2xx}} z z$$

### Casus II.

$$e = \frac{b-c}{a}; g = \frac{bf}{a}; m = \frac{a+c}{2a}; n = 0 \text{ et } \frac{1}{4} C = \frac{4af - (a-c)^2}{4a^2}$$

Inte

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Integrale ergo erit

$$dz + \frac{a(c-a) + b(c-a)x}{2ax(a+bx)} z dx = \frac{dx}{x^{\frac{a+c}{2a}}} \sqrt{A - \frac{(4af - (a-c)^2)}{4aax} z z} \text{ siue}$$

$$dz + \frac{(c-a)z dx}{2ax} = \frac{dx}{x^{\frac{a+c}{2a}}} \sqrt{A + \frac{((a-c)^2 - 4af)}{4aax} z z}.$$

Cafus III.

$$e = \frac{b(a+2c)}{2a}; g = \frac{bc(a+c)}{4aa}; m = \frac{a+c}{2a}; n = 1 \text{ et } \frac{1}{4}C = \frac{4af - (a-c)^2}{4a},$$

vnde integrale est

$$dz + \frac{a(c-a) + bcx}{2ax(a+bx)} z dx = \frac{dx}{x^{\frac{a+c}{2a}}(a+bx)} \sqrt{A + \frac{((a-c)^2 - 4af)z z}{4aax(a+bx)}}$$

Cafus IV.

$$e = \frac{b(a+2c)}{2a}; g = \frac{bc(c-a)}{4aa}; m = \frac{a+c}{2a}; n = \frac{1}{2} \text{ et } \frac{1}{2}C = \frac{4af - (a-c)^2}{4a};$$

vnde integrale est

$$dz + \frac{(c-a)z dx}{2ax} = \frac{dx}{x^{\frac{a+c}{2a}}(a+bx)^{\frac{1}{2}}} \sqrt{A + \frac{((a-c)^2 - 4af)z z}{4aax(a+bx)}}$$

Cafus V.

$$e = \frac{b(3a+2c)}{2a}; g = \frac{(2c-a)(2c-3a)}{16aa}; m = \frac{a+2c}{4a}; n = 1 \text{ et } \frac{1}{4}C = g - \frac{b(a+2c)^2}{16aa};$$

vnde integrale erit

$$dz + \frac{a(2c-a) + b(2c+a)x}{4ax(a+bx)} z dx = \frac{dx}{x^{\frac{a+2c}{4a}}(a+bx)} \sqrt{A + \frac{(b(a+2c)^2 - 16aag)z z}{16aax(a+bx)}}$$

Cafus

## Casus VI.

$$e = \frac{b(a+2c)}{2a}; f = \frac{(2c-a)(2c-a)}{16a}; m = \frac{a+2c}{4a}; n = \frac{1}{2} \text{ et } \frac{1}{2} C = g - \frac{b(a-2c)^2}{16aa},$$

vnde integrale erit

$$dz + \frac{(2c-a)zdx}{4ax} = \frac{dx}{x^{\frac{a+2c}{4a}}(a+bx)^{\frac{1}{2}}} \sqrt{\left(A + \frac{(b(a-2c)^2 - 16aag)zz}{16aax(a+bx)}\right)}$$

## Casus VII.

$$f = \frac{(e-2b)(2bc-ae)}{4bb}; g = \frac{e(e-2b)}{4b}; m = \frac{a+c}{2a}; n = \frac{ab+ae-bc}{2ab}$$

$$\text{et } \frac{1}{2} C = af - \frac{1}{2}(a-c)^2,$$

vnde integrale erit

$$dz + \frac{c-a+(e-2b)x}{2x(a+bx)} z dx = \frac{dx}{x^m(a+bx)^n} \sqrt{\left(A + \frac{((a-c)^2 - 4af)zz}{4xx(a+bx)^2}\right)}$$

## Casus VIII.

$$f = \frac{(2c-a)(2c-3a)}{16a}; g = \frac{(b-2e)(3b-2e)}{16b}; m = \frac{2c+a}{4a}; n = \frac{ab+ae-bc}{2ab}$$

$$\text{et } \frac{1}{2} C = \frac{(ab-ae+bc)^2}{4ab};$$

vnde integrale erit

$$dz + \frac{2c-a+(2e-3b)x}{4x(a+bx)} z dx = \frac{dx}{x^m(a+bx)^n} \sqrt{\left(A - \frac{(ab-ae+bc)^2 zz}{4abx(a+bx)^2}\right)}$$

## Casus IX.

$$f = \frac{c(c-2a)}{4a}; g = \frac{-c(4ab-2ae+bc)}{4aa}; m = \frac{c}{2a}; n = \frac{ab+ae-bc}{2ab}$$

$$\text{et } \frac{1}{2} C = bg - \frac{1}{2}(b-e)^2$$

vnde integrale

$$dz + \frac{c+(e-b)x}{2x(a+bx)} z dx = \frac{dx}{x^m(a+bx)^n} \sqrt{\left(A + \frac{((b-e)^2 - 4bg)zz}{4(a+bx)^2}\right)}$$

Tom. XVII. Nou. Comm.

V

XXIX.

154. CONSID. AEQVAT. DIFFERENTIO-DIFF.

XXIX. Praeter hos vero nouem casus, quibus binæ relationes inter coefficients praescribuntur, initio innumerabiles casus integrabiles duplici modo eruiamus. Altero priori §. VI. integrale algebraicum huius formae:

$$z = Ax^n + Bx^{n+1} + Cx^{n+2} + Dx^{n+3} + \text{etc.}$$

assignari potest, si denotante  $i$  numerum integrum posituum quemcunque fuerit

$$n(n-1)a + nc + f = 0 \quad \text{et} \\ (n+i)(n+i-1)b + (n+i)e + g = 0.$$

Altero vero posteriori §. VIII. Integrale huius est formae

$$z = (a + bx) \frac{ab - ac + bc}{ab} (Ax^n + Bx^{n+1} + Cx^{n+2} + \text{etc.})$$

si fuerit

$$n(n-1)a + nc + f = 0 \quad \text{et} \\ (n+i)(n+i-1)b + (n+i)\left(\frac{2bc}{a} + 2b - e\right) + g + \frac{bc}{a} + \frac{bac}{aa} - \frac{ca}{a} = 0.$$

Quae integralia etsi sunt particularia, tamen ex iis completa facile determinantur.

SOLV.