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Observationes circa aequationem differentialem $y dy + My dx + N dx = 0$

Leonhard Euler

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OBSERVATIONES
CIRCA AEQVATIONEM
DIFFERENTIALIEM

$$y dy + My dx + N dx = 0.$$

Auctore

L. E V L E R O.

I.

In hac aequatione quantitates M et N vt functiones quaecunqve variabilis x spectantur; et cum haec aequatio ita sit comparata, vt in genere nullo modo integrari queat, methodis saltem etiamnunc cognitissimis; praecipua quaestio in ea indole binarum functionum M et N inuestiganda versatur, qua integratio absolui queat; vnde quidem casus per se obuios, veluti quando vel altera euanescit, vel ambas in ratione constante consistunt, excludi conuenit.

II. Cum forma huius aequationis satis sit simplex, vt mirari liceat, quod eius integratio vires Analyseos adhuc eluserit, eius certe consideratio eo maiori attentione digna videtur; idque potissimum quod in forma generaliori, a Comite Riccati olim tractata

$$dz + Pz^m dx + Q dx = 0,$$

continetur, ad quam adeo duplici modo reuocari potest. Primo enim per y diuisa praebet

Tom. XVII. Nou. Comm.

0

dy

$$dy + N y^{-1} dx + M dx = 0,$$

ita vt $n = -1$, tum vero posito $yy = z$, prodit

$$dz + 2 M z^{\frac{1}{2}} dx + 2 N dx = 0,$$

ita vt sit $n = \frac{1}{2}$. Vbi in genere obseruasse iuuabit, posito $z = y^{\frac{2}{n-1}}$ formam generalem in hanc mutari

$$dy + (1-n) Q y^{\frac{n}{n-1}} + (1-n) P dx = 0,$$

ita vt in hoc negotio exponentes n et $\frac{n}{n-1}$ pro aequivalentibus sint habendi.

III. Patet ergo, quod tantum in transitu monuerim, casum $n = 2$, quem Ricatus olim imprimis est contemplatus, hac proprietate prae reliquis esse praeditum, vt hac reductione ad se ipsum reuocetur, dum $\frac{n}{n-1}$, iterum dat binarium. Tum vero casus $n = 1$, quo aequatio

$$dz + P z dx + Q dx = 0$$

generatim est integrabilis, perducitur ad alterum $\frac{n}{n-1} = \infty$; vnde cum potestati y^{∞} aequiualet quasi forma exponentialis e^y , etiam haec aequatio

$$dy + P e^y dx + Q dx = 0$$

pro integrabili est habenda, cuius integratio ponendo

$$e^y = \frac{1}{v}, \text{ vt fiat } y = -lv \text{ et } dy = -\frac{dv}{v},$$

hincque prodeat

$$dv - Q v dx - P dx = 0,$$

per se est manifesta.

IV. Vt autem ad ipsam aequationem propositam reuertar, ante omnia obseruo me ad similem formam, cum olim aequationem differentialem tertii gradus hanc:

$$d^3 v + A dt d d v + B dt^2 d v + C v dt^3 = 0$$

tractassem, vbi quidem A, B, C et dt sunt constantes, esse perductum. Posito scilicet $v = e^{f x dt}$ obtinueram

$$d d x + 3 x dt d x + A dt d x + (x^3 + A x x + B x + C) dt^2 = 0$$

seu rationem elementi constantis dt exuendo:

$$d. \frac{d x}{d t} + (3 x + A) dx + (x^3 + A x x + B x + C) dt = 0.$$

Nunc posueram $\frac{d x}{d t} = y$ hincque $dt = \frac{d x}{y}$, ex quo nata erat haec aequatio

$$d y + (3 x + A) dx + C x^3 + A x x + B x + C) \frac{d x}{y} = 0$$

sive $y dy + (3 x + A) y dx + (x^3 + A x x + B x + C) dx = 0$ quae vtrique in forma proposita comprehenditur.

V. Cum igitur aequatio differentialis tertii ordinis, vnde hanc deriuauimus, sit integrabilis, ac posito

$$x^3 + A x x + B x + C = (x - a)(x - b)(x - \gamma)$$

eius integrale completum fit

$$v = F e^{\alpha t} + G e^{\beta t} + H e^{\gamma t}$$

hinc sequitur, etiam nostrae aequationis

$$y dy + (3 x + A) y dx + (x^3 + A x x + B x + C) dx = 0$$

integrale in genere exhiberi posse; quod quidem adeo algebraice ita expressum inde elici

$$(y + (x - \alpha)(x - \beta))^{\beta - \alpha} (y + (x - \beta)(x - \gamma))^{\gamma - \beta} (y + (x - \gamma)(x - \alpha))^{\alpha - \gamma} = E.$$

vbi notandum est, esse

$$A = -\alpha - \beta - \gamma; \quad B = \alpha\beta + \alpha\gamma + \beta\gamma \quad \text{et} \quad C = -\alpha\beta\gamma.$$

VI. Cum enim ob $v = e^{\int x dt}$ fit $x = \frac{dv}{v dt}$, fiet.

$$x = \frac{\alpha F e^{\alpha t} + \beta G e^{\beta t} + \gamma H e^{\gamma t}}{v} \quad \text{et} \quad y = \frac{dx}{dt}.$$

$$y = \frac{(\beta - \alpha)^2 F G e^{(\alpha + \beta)t} + (\gamma - \alpha)^2 F H e^{(\alpha + \gamma)t} + (\gamma - \beta)^2 G H e^{(\beta + \gamma)t}}{v v}$$

Inde porro colligitur:

$$x - \alpha = \frac{(\beta - \alpha) G e^{\beta t} + (\gamma - \alpha) H e^{\gamma t}}{v} \quad \text{et} \quad x - \beta = \frac{(\alpha - \beta) F e^{\alpha t} + (\gamma - \beta) H e^{\gamma t}}{v}$$

sicque conficitur:

$$y + (x - \alpha)(x - \beta) = \frac{(\gamma - \alpha)(\gamma - \beta) H e^{\gamma t}}{v}$$

similique modo

$$y + (x - \beta)(x - \gamma) = \frac{(\alpha - \beta)(\alpha - \gamma) F e^{\alpha t}}{v}$$

$$\text{et} \quad y + (x - \gamma)(x - \alpha) = \frac{(\beta - \gamma)(\beta - \alpha) G e^{\beta t}}{v}$$

vnde veritas integralis exhibiti fit manifesta, et quia id ineluit quantitatem constantem E per arbitrias F, G, H definitam, pro completo erit habendum.

VII. Haec consideratio viam nobis aperit, a priori ad aequationes formae propositae perueniendi.

Sumtis

Sumtis enim tribus functionibus ipsius x , quae sint P, Q, R statuatur aequatio integralis :

$$(y + P)^\lambda (y + Q)^\mu (y + R)^\nu = \text{Const.}$$

unde haec nascitur aequatio differentialis :

$$\frac{\lambda dy + \lambda dP}{y + P} + \frac{\mu dy + \mu dQ}{y + Q} + \frac{\nu dy + \nu dR}{y + R} = 0.$$

quae a fractionibus liberata hanc induit formam :

$$\begin{aligned} &(\lambda + \mu + \nu) y y dy + y dy (\lambda(Q + R) + \mu(P + R) + \nu(P + Q)) \\ &+ dy (\lambda QR + \mu PR + \nu PQ) + y y (\lambda dP + \mu dQ + \nu dR) \\ &+ y (\lambda(Q + R) dP + \mu(P + R) dQ + \nu(P + Q) dR) \\ &+ \lambda QR dP + \mu PR dQ + \nu PQ dR = 0 \end{aligned}$$

ex qua forma proposita resultat statuendo :

1°. $\lambda + \mu + \nu = 0$

2°. $\lambda QR + \mu PR + \nu PQ = 0$

3°. $\lambda dP + \mu dQ + \nu dR = 0$ seu $\lambda P + \mu Q + \nu R = \text{Const.}$

VIII. Si hic ponamus $\lambda P + \mu Q + \nu R = a$ et $P + Q + R = S$, ut singulas litteras P, Q, R hinc definire valeamus, ratione habita primae conditionis, qua esse oportet $\lambda + \mu + \nu = 0$, reperiemus hos valores :

$$P = \frac{\lambda \mu \nu S + (\lambda \lambda + 2 \mu \nu) a + (\mu - \nu) \sqrt{(\lambda \lambda - \mu \nu) a a - 3 \lambda \mu \nu a S}}{3 \lambda \mu \nu}$$

$$Q = \frac{\lambda \mu \nu S + (\mu \mu + 2 \lambda \nu) a + (\nu - \lambda) \sqrt{(\mu \mu - \lambda \nu) a a - 3 \lambda \mu \nu a S}}{3 \lambda \mu \nu}$$

$$R = \frac{\lambda \mu \nu S + (\nu \nu + 2 \lambda \mu) a + (\lambda - \mu) \sqrt{(\nu \nu - \lambda \mu) a a - 3 \lambda \mu \nu a S}}{3 \lambda \mu \nu}$$

vbi signa radicalia ob $\lambda \lambda - \mu \nu = \mu \mu - \nu \nu = \lambda \mu$ inter se conueniunt.

IX. Irrationalitate harum formularum sublata ad eandem aequationem peruenitur, cuius integrale supra exhibui (§. V.) vnde hanc evolutionem ulterius non prosequor. Interim tamen maximi momenti esse arbitror, obseruasse aequationem differentialem generalem §. VII. expositam per se reddi integrabilem, si ea diuidatur per

$$(y + P)(y + Q)(y + R),$$

quod si ad aequationem superiorem

$$y dy + (3x + A)y dx (x^3 + Axx + Bx + C) dx = 0,$$

attendamus; reperiemus, eam per se integrabilem reddi, si diuidatur per hanc formam:

$$y^3 + yy(3xx + 2Ax + B) + y(3x + A)(x^3 + Axx + Bx + C) + (x^3 + Axx + Bx + C)^2$$

etiamsi hinc minus pateat, integrale adeo algebraice exhiberi posse. Quae obseruatio me deducit ad methodum illam generalem iam dudum a me expositam, qua ostendi, omnium aequationum differentialium integrationes commodissime per multiplicatores absolui posse.

X. Cum igitur hic multiplicator seu diuisor ita comparatus esse debeat, vt formula per se fiat integrabilis, vtique necesse est, vt criteria huiusmodi formularum perspecta habeamus, quae integrabilitatem certo indicent, etiamsi forte ipsa integratio difficulter ac nonnisi per quadraturas satis complicatas confici queat. Omnium autem formularum integrabilium, cuiuscunque gradus differentialia impli-

cent,

cent, hanc esse proprietatem nuper demonstraui, vt positis.

$$dy = p dx, dp = q dx, dq = r dx \text{ etc.}$$

quo pacto eae semper ad talem formam $V dx$ reducuntur, in qua littera V vtcunque quantitates x, y, p, q, r etc. implicabit, tam futurum fit

$$\left(\frac{dV}{dy}\right) - \frac{1}{dx} d\left(\frac{dV}{dp}\right) + \frac{1}{dx^2} dd\left(\frac{dV}{dq}\right) - \frac{1}{dx^3} d^3\left(\frac{dV}{dr}\right) + \text{etc} = 0$$

ac vicissim quoties haec conditio locum habeat, toties quoque formulam $V dx$ esse integrabilem.

XI. Hanc igitur quaestionem nunc euoluendam suscipio.

Inuestigare eiusmodi functionem Z binarum variabilium x et y, per quam aequatio nostra

$$y dy + M y dx + N dx = 0$$

diuisa fiat integrabilis !!

Hoc ergo casu erit $V = \frac{y^2 + My + N}{Z}$ vnde colligitur

$$\left(\frac{dV}{dy}\right) = \frac{p}{Z} - \frac{y p}{Z Z} \left(\frac{dZ}{dy}\right) + \frac{M}{Z} - \frac{(My + N)}{Z Z} \left(\frac{dZ}{dy}\right)$$

tum vero ob $\left(\frac{dV}{dp}\right) = \frac{y}{Z}$ porro differentiando reperitur

$$\frac{1}{dx} d\left(\frac{dV}{dp}\right) = \frac{p}{Z} - \frac{y p}{Z Z} \left(\frac{dZ}{dy}\right) - \frac{y}{Z Z} \left(\frac{dZ}{dx}\right) \text{ ob } \frac{dZ}{dx} = \left(\frac{dZ}{dx}\right) + p \left(\frac{dZ}{dy}\right)$$

Cum itaque fieri oporteat

$$\left(\frac{dV}{dy}\right) - \frac{1}{dx} d\left(\frac{dV}{dp}\right) = 0$$

habebimus hanc aequationem pro definienda functione Z .

$$\frac{M}{Z} - \frac{(My + N)}{Z Z} \left(\frac{dZ}{dy}\right) + \frac{y^2}{Z Z} \left(\frac{dZ}{dx}\right) = 0$$

$$\text{feu } M Z - M y \left(\frac{dZ}{dy}\right) - N \left(\frac{dZ}{dy}\right) + y \left(\frac{dZ}{dx}\right) = 0.$$

XII. Si loco diuisoris Z sumatur potestas quaecunque Z^n , vt integrabilis reddi debeat haec forma:

$$\frac{y dy + M y dx + N dx}{Z^n}$$

functionem Z ex hac aequatione definiiri oportebit

$$M Z - n(M y + N) \left(\frac{dZ}{dy}\right) + n y \left(\frac{dZ}{dx}\right) = 0$$

vnde vicissim inuestigatio ita institui poterit, vt sumta pro lubitu forma functionis Z , inde indoles quantitatum M et N , quae per solam variabilem x determinantur, quaeratur, vt aequatio proposita hoc modo integrabilis reddi queat. Quamobrem his vestigiis insistens sequentes casus euoluam, vbi quidem litteris P , Q , R etc. functiones solius variabilis x indicari moneo.

Casus I

Quo integrabilis reddi debet haec forma:

$$\frac{y dy + M y dx + N dx}{(y + P)^n}$$

XIII. Cum ergo sit $Z = y + P$ erit

$$\left(\frac{dZ}{dy}\right) = 1 \text{ et } \left(\frac{dZ}{dx}\right) = \frac{dP}{dx}$$

vnde §. praec. hanc suppeditat aequationem:

$$0 = M y + M P - n M y - n N + \frac{n y dP}{dx}$$

quoniam igitur M , N et P sunt functiones solius x , seorsim esse debet:

$$1^\circ. n dP = (n-1) M dx \text{ et } 2^\circ. n N dx = M P dx.$$

Quare

Quare pro lubitu sumta functione P habebimus

$$M dx = \frac{n}{n-1} dP \quad \text{et} \quad N dx = \frac{1}{n-1} P dP$$

unde discimus hanc aequationem:

$$y dy + \frac{n}{n-1} y dP + \frac{1}{n-1} P dP = 0$$

integrabilem reddi, si diuidatur per formam $(y+P)^n$.

XIV. Haec autem aequatio nullam plane habet difficultatem, quoniam est homogenea, atque adeo per hanc formam $(n-1)yy + nyP + PP$ diuisa integrabilis euadat, ex quo diuisore, quia constat factoribus $(y+P)((n-1)y+P)$ deducitur aequatio:

$$\frac{1}{n-2} \cdot \frac{dy+dP}{y+P} - \frac{1}{(n-1)(n-2)} \cdot \frac{(n-1)dy+dP}{(n-1)y+P} = 0$$

cuius integralis manifesto est

$$(y+P)^{n-1} = A((n-1)y+P)$$

quae etiam ex illo diuisore concluditur. Tantum obseruo casu $n=2$, quo haec forma fit incongrua, integrale fore

$$I(y+P) - \frac{y}{y+P} = \text{Const.}$$

quippe cuius differentiatio praebet

$$\frac{y dy + 2y dP + P dP}{(y+P)^2} = 0.$$

XV. Singularis hic se obtulit casus, quo aequatio:

$$(y+P)^{n-1} = A((n-1)y+P),$$

quae ob constantem arbitrariam A est indefinita, sumto $n=2$ hac indole penitus priuatur. Vt autem tum eius vera forma eliciatur, statuatur more soli-

to $n = 2 + \omega$, denotante ω fractionem infinite parvam, vt fit

$$(y + P)^\omega = 1 + \omega l(y + P);$$

fic illa aequatio hanc induet formam:

$$y + P + \omega(y + P)l(y + P) = A(y + P) + A\omega y$$

fit nunc $A = 1 + B\omega$ ac prodibit

$$(y + P)l(y + P) = y + B(y + P)$$

ficque loco constantis A alia arbitraria B est introducta.

XVI. Vt aequationem inuentam elegantiore reddamus, ponamus $P = x^{n-1}$, vt prodeat

$$y dy + n x^{n-2} y dx + x^{2n-2} dx = 0,$$

quam ergo integrabilem fieri nouimus, si diuidatur per $(y + x^{n-1})^n$, et integrale eius completum erit

$$(y + x^{n-1})^{n-1} = A((n-1)y + x^{n-1})$$

dum obseruetur casu $n = 2$ integrale esse

$$l(y + x^{n-1}) - \frac{y}{y + x^{n-1}} = \text{Const.} = l(y+x) - \frac{y}{y+x}$$

Hic vero nouus casus singularis occurrit $n = 1$, aequationem praebens

$$y dy + \frac{y dx}{x} + \frac{dx}{x} = 0,$$

quae per $y + 1$ diuisa integrale dat $y + l \frac{x}{y+1} = \text{Const.}$

Quod vt ex forma illa generali eruatur, posito $n = 1 + \omega$ erit

$$(y + 1 + \omega l x)^\omega = 1 + \omega l(y + 1) = A(\omega y + 1 + \omega l x)$$

fit

fit ergo $A = 1 + B \omega$ prodibitque

$$l(y+1) = y + lx + B \text{ seu } y + l \frac{x}{y+1} = \text{Const. vt ante.}$$

Casus II.

Quo integrabilis reddenda est haec forma

$$\frac{y dy + M y dx + N dx}{(yy + Py + Q)^n}$$

XVII. Quia hic est $Z = yy + Py + Q$ pro §. 12. habebimus:

$$\left(\frac{dZ}{dy}\right) = 2y + P \text{ et } \left(\frac{dZ}{dx}\right) = y \frac{dP}{dx} + \frac{dQ}{dx}$$

hincque istam aequationem resoluendam

$$\begin{aligned} 0 &= M y y + M P y + M Q \\ &- 2 n M P \quad - n M P \quad - n N P \\ &+ \frac{n d P}{d x} \quad + 2 n N \\ &\quad + \frac{n d Q}{d x}, \end{aligned}$$

vnde resultant hae tres:

$$n d P = (2 n - 1) M d x \text{ seu } M d x = \frac{n d P}{2 n - 1}$$

$$n d Q = (n - 1) M P d x + 2 n N d x$$

$$n N P = M Q, \text{ seu } N d x = \frac{M Q d x}{n P} = \frac{Q d P}{(2 n - 1) P}$$

$$\text{Ergo } n d Q = \frac{n(n-1)P d P}{2n-1} + \frac{2n Q d P}{(2n-1)P}$$

$$\text{seu } d Q - \frac{2 Q d P}{(2n-1)P} = \frac{n-1}{2n-1} P d P$$

quae aequatio per $P^{\frac{-2}{2n-1}}$ multiplicata et integrata dat

$$P^2$$

$$P^{\frac{-2}{2n-1}}$$

$$P^{2n-1} Q = \frac{1}{2} P^{2n-1} + A, \text{ hincque fit}$$

$$Q = \frac{1}{2} P + A P^{2n-1} \text{ et per } P \text{ resolutio erit}$$

$$M dx = \frac{n dP}{2n-1}, \text{ et } N dx = \frac{dP}{2n-1} \left(\frac{1}{2} P + A P^{2n-1} \right).$$

XVIII. Sit $P = 2x^{2n-1}$; vt fiat

$$Q = x^{2n-2} + Bxx; M dx = 2nx^{2n-2} dx \text{ et } N dx = x^{2n-2} dx + Bx dx$$

atque hinc intelligimus hanc aequationem

$$y dy + 2nx^{2n-2} y dx + x^{2n-2} dx + Bx dx = 0$$

integrabilem reddi, si diuidatur per

$$(yy + 2x^{2n-1}y + x^{2n-2} + Bxx)^n.$$

Euidens est hanc aequationem ad simpliciores formam perducī ponendo $y = z - x^{2n-1}$, tum enim prodit

$$z dz + Bx dx + x^{2n-2}(z dx - x dz) = 0$$

quae per $(zz + Bxx)^n$ diuisa vtique fit integrabilis integrali existente:

$$\frac{-1}{2(n-1)(zz + Bxx)^{n-1}} + \int \frac{x^{2n-2}(z dx - x dz)}{(zz + Bxx)^n} = 0$$

cuius posterius membrum posito $z = vx$ abit in

$$-\int \frac{dv}{(vv + B)^n},$$

ita vt nulla supersit difficultas.

Casus

Cafus III.

Quo integrabilis reddenda est haec forma

$$\frac{y dy + M y dx + N dx}{(y^3 + P y^2 + Q y + R)^n}$$

XIX. Ob $Z = y^3 + P y^2 + Q y + R$ erit

$$\left(\frac{dZ}{dy}\right) = 3y^2 + 2Py + Q \text{ et } \left(\frac{dZ}{dx}\right) = \frac{y^2 dP + y dQ + dR}{dx}$$

ex quo fequenti aequationi est facisfaciendum

$$\begin{aligned} 0 &= M y^3 + M P y^2 + M Q y + M R \\ &- 3nM \quad - 2nMP \quad - nMQ \quad - nNQ \\ + \frac{ndP}{dx} &\quad - 3nN \quad - 2nNP \\ &\quad + \frac{ndQ}{dx} \quad + \frac{ndR}{dx} \end{aligned}$$

quae fuppeditat has quatuor determinationes :

- 1°. $ndP = (3n - 1) M dx$
- 2°. $ndQ = (2n - 1) MP dx + 3nN dx$
- 3°. $ndR = (n - 1) MQ dx + 2nNP dx$
- 4°. $nNQ dx = MR dx$

XX. Ex vltima colligimus $M : N = nQ : R$, vnde ex prioribus litteras M et N. elidendo obtinemus :

$$\begin{aligned} (3n - 1)QdQ &= (2n - 1)PQdP + 3RdP \text{ et} \\ (3n - 1)QdR &= (n - 1)QQdP + 2PRdP \end{aligned}$$

hincque $(2n - 1)PQdR + 3RdR = (n - 1)QQdQ + 2PRdQ$. Vnde fi quantitates Q et R. per P definire licuerit, tum erit

$$M dx = \frac{n}{3n - 1} dP \text{ et } N dx = \frac{R dP}{(3n - 1) Q}$$

P 3.

Primum

Primum autem obseruo illis aequationibus satisfieri posse ponendo

$$Q = \alpha P^2 \quad \text{et} \quad R = \xi P^3,$$

hosque coefficientes α et ξ duplicem determinationem sortiri, scilicet

$$\text{vel } \alpha = \frac{1}{3} \quad \text{et} \quad \xi = \frac{n}{27}$$

$$\text{vel } \alpha = \frac{2n-1}{(2n-1)^2} \quad \text{et} \quad \xi = \frac{(n-1)(2n-1)^2}{(2n-1)^3}$$

quo quidem casu aequatio nostra fit homogenea.

XXI. Consideremus primo casum quo $n = 1$, quippe quem iam supra aliunde eliciimus; eruntque nostrae aequationes:

$$1^\circ. \quad 2QdQ = PQdP + 3RdP \quad \text{et} \quad 2^\circ. \quad QdR = PRdP$$

$$\text{hincque} \quad 3^\circ. \quad PQdR + 3RdR = 2PRdQ.$$

Iam ex 2° . fit $Q = \frac{PRdP}{dR}$, sumtoque dR constante

$$dQ = \frac{PRddP}{dR} + P.dP + \frac{RdP^2}{dR},$$

unde prima abit in hanc:

$$\frac{2PPRRdPddP}{dR^2} + \frac{2PPRdP^2}{dR} + \frac{2PPRdP^3}{dR^2} - \frac{PPRdP^2}{dR} = 3RdP$$

$$\text{seu} \quad \frac{2PPRRdPddP + 2PPRdP^2}{dR} + PPRdP = 3RdR$$

diuidatur per $PR\sqrt{R}$ vt prodeat

$$\frac{2PddP\sqrt{R}}{dR} + \frac{2dP^2\sqrt{R}}{dR} + \frac{PdP}{\sqrt{R}} = \frac{3dR}{P\sqrt{R}}$$

cuius prius membrum integratum dat

$$\frac{2PdP\sqrt{R}}{dR} = 3 \int \frac{dR}{P\sqrt{R}} \quad \text{hincque} \quad dP = \frac{3}{2} \cdot \frac{dR}{P\sqrt{R}} \int \frac{dR}{P\sqrt{R}}$$

quae quidem forma parum lucri attulisse videtur.

XXII.

XXII. Ponamus autem $\int \frac{dR}{P\sqrt{R}} = u$, vt fit
 $P = \frac{dR}{du\sqrt{R}}$ ac postrema aequatio $dP = \frac{3}{2}u du$ dabit
 $P = \frac{3}{2}u \cdot u + A$ vnde fit $\frac{dR}{\sqrt{R}} = \frac{3}{2}u u du + A du$ hinc-
 que integrando

$2\sqrt{R} = \frac{1}{2}u^3 + Au + 2B$ et $R = (\frac{1}{2}u^3 + \frac{1}{2}Au + B)^2$
 et ob $Q = \frac{PRdP}{dR} = \frac{PdP\sqrt{R}}{Pdu} = \frac{3}{2}u\sqrt{R}$ erit

$Q = \frac{3}{2}u(\frac{1}{2}u^3 + \frac{1}{2}Au + B)$ ac denique

$M dx = \frac{1}{2}dP = \frac{3}{2}u du$ et

$N dx = \frac{1}{2}dP \cdot \frac{R}{Q} = \frac{1}{2} \frac{dR}{P} = \frac{1}{2} du \sqrt{R} = \frac{1}{2} du (\frac{1}{2}u^3 + \frac{1}{2}Au + B)$

statuamus nunc $u = 2x + 2f$ vt fiat

$P = 3xx + 6fx + 3ff + A;$

$Q = 3(x+f)(x^3 + 3fxx + 3ffx + f^3 + Ax + Af + B)$

$R = \{x^3 + 3fxx + 3ffx + f^3 + Ax + Af + B\}^2$

$M dx = 3dx(x+f)$ et $N dx = dx(x^3 + 3fxx + 3ffx + f^3 + Ax + Af + B)$

XXIII. Ponamus porro:

$3f = a, 3ff + A = b$ et $f^3 + Af + B = c,$

atque obtinebimus hanc aequationem

$y dy + (3x+a)y dx + (x^3 + axx + bx + c) dx = 0$

quae ergo integrabilis reddetur, si diuidatur per

$y^3 + (3xx + 2ax + b)yy + (3x+a)(x^3 + axx + bx + c)y + (x^3 + axx + bx + c)^2$ sicque

ficque diuisoris forma hoc casu assumta ad eam ipsam integrationem nos deduxit, quam iam supra §. IX. sumus adepti. Circa hunc ergo diuisorem annotasse iuuabit quod supra iam vidimus, si ponatur

$$x^3 + axx + bx + c = (x + \alpha)(x + \beta)(x + \gamma)$$

vt fit

$$a = \alpha + \beta + \gamma; \quad b = \alpha\beta + \alpha\gamma + \beta\gamma; \quad \text{et } c = \alpha\beta\gamma,$$

diuisorem nostrum sic in ternos factores resolutum exhiberi posse

$$(y + (x + \alpha)(x + \beta))(y + (x + \alpha)(x + \gamma))(y + (x + \beta)(x + \gamma))$$

et aequationis nostrae integrale completum fore

$$(y + (x + \alpha)(x + \beta))^{\alpha - \beta} (y + (x + \beta)(x + \gamma))^{\beta - \gamma} (y + (x + \gamma)(x + \alpha))^{\gamma - \alpha} = \text{Const.}$$

XXIV. Circa aequationem differentio-differentialem §. 21. quae per R diuisa est:

$$\frac{2 P P R d d P + 2 P R d P^2 + P P d P d R}{d R} = 3 d R$$

obseruo, eam multo facilius integrari posse, si modo per $\frac{d P}{d R}$ multiplicetur, vt habeatur

$$\sqrt{\frac{2 P P R d P d d P + 2 P R d P^3 + P P d P^2 d R}{d R^2}} = 3 d P$$

cuius integrale statim est

$$\frac{P P R d P^2}{d R^2} = 3 P + A \quad \text{hincque } \frac{d R}{d P} = \frac{P d P}{\sqrt{(3 P + A)}}$$

cuius integrale denuo est

$$2 \sqrt{R} = 2 \left(\frac{P}{9} - \frac{2 A}{27} \right) \sqrt{(3 P + A)} + B.$$

XXV. Pro positione $n = 1$ calculum in genere expedire licuit, pro aliis autem valoribus ipsius n negotium minus succedit, excepto vnico casu $n = \frac{1}{2}$, quo

quo aequatio prima §. 19. statim dat $dP = 0$ ideoque $P = a$; vnde inter Q et R haec prodit aequatio

$$-\frac{1}{3}aQdR + 3RdR = -\frac{2}{3}QQdQ + 2aRdQ$$

feu $9RdR + 2QQdQ = 6aRdQ + aQdR$

quam autem evolvere non licet, nisi sumatur $a = 0$, tum autem oritur

$$\frac{9}{2}RR + \frac{2}{3}Q^3 = \text{Const. seu } R = \sqrt{A - \frac{4}{27}Q^3}.$$

Deinde

$$Ndx = \frac{1}{3}dQ \quad \text{et} \quad Mdx = \frac{1}{3}dQ \cdot \frac{Q}{R}.$$

Statuamus

$$Q = 3x, \text{ vt fiat } R = \sqrt{A - 4x^3}; \quad Ndx = dx \text{ et } Mdx = \frac{x dx}{\sqrt{A - 4x^3}}$$

vnde deducimus hanc aequationem differentialem:

$$y dy + \frac{xy dx}{\sqrt{A - 4x^3}} + dx = 0$$

quam nunc novimus integrabilem reddi, si diuidatur per

$$\sqrt{y^3 + 3xy + \sqrt{A - 4x^3}}.$$

Hinc autem ipsum integrale neutiquam explicite exhiberi potest; quin etiam aequatio ista ita est comparata, vt nulla alia via ad constructionem perduciqueat.

Casus IV.

Quo integrabilis reddenda est haec forma

$$\frac{y dy + My dx + N dx}{(y^4 + Py^3 + Qy^2 + Ry + S)^n}$$

XXVI. Pro hoc casu methodus nostra sequentes suppeditat aequationes :

$$\begin{aligned} 1^\circ. \quad n d P &= (4n-1) M dx \\ 2^\circ. \quad n d Q &= (3n-1) M P dx + 4n N dx \\ 3^\circ. \quad n d R &= (2n-1) M Q dx + 3n N P dx \\ 4^\circ. \quad n d S &= (n-1) M R dx + 2n N Q dx \\ 5^\circ. \quad M S &= n N R \text{ feu } M : N = n R : S \end{aligned}$$

vnde deriuantur istae

$$\begin{aligned} (4n-1) R d Q &= (3n-1) P R d P + 4 S d P \\ (4n-1) R d R &= (2n-1) Q R d P + 3 P S d P \\ (4n-1) R d S &= (n-1) R R d P + 2 Q S d P, \end{aligned}$$

ex binis postremis elidendo Q oritur :

$$2(4n-1) R S d R - (2n-1)(4n-1) R R d S = 6 P S S d P - (n-1)(2n-1) R^2 d P.$$

XXVII. Quoniam hic in genere vix quicquam concludere licet, praeter casum homogeneitatis, quo fieri potest

$$Q = \alpha P^2; \quad R = \beta P^3; \quad \text{et } S = \gamma P^4$$

consideremus casum $n = 1$, vt diuisor fit

$$y^4 + P y^3 + Q y^2 + R y + S.$$

Habebimus ergo $M dx = \frac{1}{3} d P$: $N dx = \frac{S d P}{2 R} = \frac{d S}{2 Q}$ et

$$\begin{aligned} 1^\circ. \quad 3 R d Q &= 2 P R d P + 4 S d P \\ 2^\circ. \quad 3 R d R &= Q R d P + 3 P S d P \\ 3^\circ. \quad 3 R d S &= 2 Q S d P \end{aligned}$$

vnde eliminato Q ex duabus postremis fit

$$2 R S d R - R R d S = 2 P S S d P$$

quae

quae per SS diuisa commode integrationem admittit praebetque

$$\frac{RR}{S} = PP + A \text{ seu } S = \frac{RR}{PP + A}$$

quare ex secunda elicitur

$$Q = \frac{3dR}{dP} - \frac{3PS}{R} = \frac{3dR}{dP} - \frac{3PR}{PP + A}$$

XXVIII. Prima vero dat:

$$dQ = \frac{2}{3}P dP + \frac{4SdP}{3R} = \frac{2}{3}P dP + \frac{4RdP}{3(PP + A)}$$

inde vero sumendo dP constans reperitur:

$$dQ = \frac{3ddR}{dP} - \frac{3PdR}{PP + A} + \frac{3PPRdP - 3ARdP}{(PP + A)^2}$$

ficque oritur haec aequatio differentialis secundi gradus

$$\frac{ddR}{dP} - \frac{PdR}{PP + A} + \frac{3PPRdP - 3ARdP}{9(PP + A)^2} - \frac{2}{9}P dP = 0$$

quam dubito in genere resolui posse.

XXIX. Considerabo ergo casum quo $A = 0$,

ideoque

$$S = \frac{RR}{PP} \text{ et } Q = \frac{3dR}{dP} - \frac{3R}{P};$$

ita vt haec resoluenda fit aequatio:

$$\frac{ddR}{dP} - \frac{dR}{P} + \frac{5RdP}{9PP} - \frac{2}{9}P dP = 0.$$

Statuamus ergo $R = \alpha P^3 + u$ fietque

$$\frac{ddu}{dP} - \frac{du}{P} + \frac{5u dP}{9PP} + 6\alpha P dP - 3\alpha P dP + \frac{5}{9}\alpha P dP - \frac{2}{9}P dP = 0$$

et sumto $\alpha = \frac{1}{16}$ erit

$$\frac{ddu}{dP} - \frac{du}{P} + \frac{5u dP}{9PP} = 0$$

pro qua porro $u = P^\lambda$ et ex aequalitate $\lambda\lambda - 2\lambda + \frac{5}{9} = 0$

colligo $\lambda = 1 \pm \frac{2}{3}$, hincque integrale completum

$$u = \alpha P^{\frac{5}{3}} + \epsilon P^{\frac{1}{3}}$$

Q 2

Quo-

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Quocirca habebimus:

$$R = \frac{1}{16} P^3 + \alpha P^{\frac{5}{2}} + \zeta P^{\frac{1}{2}}$$

$$Q = \frac{2}{3} P^2 + 2\alpha P^{\frac{3}{2}} - 2\zeta P^{-\frac{1}{2}}$$

$$S = \left(\frac{1}{16} P^2 + \alpha P^{\frac{3}{2}} + \zeta P^{-\frac{1}{2}}\right)^2$$

ac tandem $M dx = \frac{1}{3} dP$,

et $N dx = \frac{dP}{3P} \left(\frac{1}{16} P^2 + \alpha P^{\frac{3}{2}} + \zeta P^{-\frac{1}{2}}\right)$.

XXX. Statuamus nunc $P = t^6$, vt sublata irrationalitate fiat:

$$M dx = t dt \text{ et } N dx = \frac{dt}{t} \left(\frac{1}{16} t^2 + \alpha t^6 + \frac{\zeta}{t}\right)$$

et aequatio nostra huius fit formae:

$$y dy + y t dt + \frac{dt}{16} \left(\frac{1}{16} t^2 + \alpha t^6 + \frac{\zeta}{t}\right) = 0$$

quam iam nouimus integrabilem reddi si diuidatur per

$$y^2 + t^2 y^2 + \left(\frac{2}{3} t^6 + 2\alpha t t - \frac{2\zeta}{t}\right) y^2 + \left(\frac{1}{16} t^2 + \alpha t^6 + \zeta t\right) y + \left(\frac{1}{16} t^6 + \alpha t^2 + \frac{\zeta}{t}\right)$$

Hic diuisor duobus constat factoribus:

$$\left(y + \left(\frac{1}{2} t^3 + \frac{2}{t} \sqrt{-\zeta}\right) + \frac{1}{16} t^6 + \alpha t t + \frac{\zeta}{t}\right) \left(y + \left(\frac{1}{2} t^3 \sqrt{-\zeta}\right) + \frac{1}{16} t^6 + \alpha t t + \frac{\zeta}{t}\right)$$

si diuifores magis complicatos adhibere vellemus, vix quicquam ad vsum inde concludere liceret.

CONSI-