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De variis integrabilitatis generibus

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DE VARIIS INTEGRABILITATIS GENERIBVS.

Auctore

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1.

Si quantitas variabilis p abſolute ſpectetur et quaeratur, quomodo quantitatem V comparatam eſſe oporteat, vt formula $V\,dp$ fiat integrabilis; tunc nullum eſt dubium, quin iſta quantitas V debeat eſſe functio quaepiam ipſius p. Vocabulum enim integrabilitatis ita hic ſenſu latiſſimo accipio, vt quaecunque functio ipſius p fuerit V, formulam $V\,dp$, ſemper integrabilem eſſe dicam, nihilque interſit, ſiue eius integrale algebraice, ſiue per logarithmos, ſiue per arcus circulares, ſiue per quascunque altiores quantitates tranſcendentes, exprimatur, quandoquidem formulam integralem $\int V\,dp$, ſemper per quadraturam cuiuspiam curuae exhibere licet.

2. Longe aliter autem ſe res habet, quando quantitas p certa quadam ratione ad alias quantitates variabiles refertur, tum enim praeter functiones ipſius p quantitati illi V, etiam alios valores tribuere licet, quibus formula $V\,dp$ integrabilis redditur. Veluti ſi p ita ad binas coordinatas x et y refe-

referatur, vt fit $dy = p\,dx$ fiue $p = \frac{dy}{dx}$, tum loco V etiam fumi poterit x, quoniam formula $x\,dp$ reuera eft integrabilis, quum enim fit

$$\int x\,dp = px - \int p\,dx, \quad \text{ob } \int p\,dx = y$$

erit vtique

$$\int x\,dp = px - y.$$

3. Quin etiam idem locum habet in ipfis differentialibus primitiuis dx et dy; vt enim formula $V\,dx$ fit integrabilis, hoc non folum vfu venit, fi fuerit V functio quaecunque ipfius x; fed etiam cafu, quo $V = p$, quum fit $\int p\,dx = y$, fimili modo, vt formula $\int V\,dy$ fiat integrabilis, loco V non folum functio quaecunque ipfius y accipi poterit, fed etiam cafu $V = \frac{1}{p}$, fit $\int \frac{dy}{p} = x$.

4. Quo haec generalius profequamur, vocemus quantitatem V multiplicatorem, per quem quaepiam formula differentialis reddatur integrabilis, vnde ex praemiffis patet, fi formula differentialis fuerit vel dp vel dx vel dy, tum multiplicatorem effe vel $V = x$, vel $V = p$, vel $V = \frac{1}{p}$.

5. Quantumuis haec facilia et obuia videantur; tamen faepenumero inueftigatio huiusmodi multiplicatorum maxime ardua deprehenditur, et quod eius vfum maxime commendat, faepius hoc modo aequationes differentiales, tam fecundi, quam altiorum graduum, fatis commode refoluere licet, ad quas absque his fubfidiis vix alius aditus patere videatur. Saepius enim iam notaui, praecipuum nego-

negotium in aequationibus differentialibus integrandis ad inuentionem idoneorum multiplicatorum reduci, ex quo inuestigatio huiusmodi multiplicatorum sine dubio maximi momenti censeri debet.

6. Haec accuratius perscrutandi occasionem mihi dedit haec quaestio, qua linea curua inter binas coordinatas x et y contenta quaeritur, cuius radius osculi aequalis sit futurus lineae rectae

$$\sqrt{(xx+yy)},$$

quum enim posito $dy = p\,dx$ sit radius osculi

$$= \frac{dx}{dp}(1+pp)^{\frac{3}{2}}$$

haec habetur aequatio resoluenda

$$\frac{dp}{(1+pp)^{\frac{3}{2}}} = \frac{dx}{\sqrt{(xx+yy)}},$$

cuius vtrumque membrum non sine admiratione integrabile fieri deprehendi, ope multiplicatoris $x+py$, tum enim, pro posteriori membro fit

$$\frac{dx(x+py)}{\sqrt{(xx+yy)}} = \frac{x\,dx+y\,dy}{\sqrt{(xx+yy)}}$$

cuius integrale est

$$\sqrt{(xx+yy)}$$

pro priori vero membro, res non adeo est manifesta, posito autem

$$y = px+v,$$

vt fiat

$$dy = p\,dx = p\,dx + x\,dp + dv, \text{ ideoque } dv = -x\,dp$$

habebimus

$$dp(x+py) = x\,dp + yp\,dp = x(1+pp)dp + vp\,dp = -dv(1+pp) + vp\,dp$$

ita

ita vt prius membrum fiat

$$-\frac{dv(1+pp)+vpdp}{(1+pp)^{\frac{3}{2}}}$$

cuius integrale manifesto est

$$=-\frac{v}{\sqrt{(1+pp)}}=\frac{px-y}{\sqrt{(1+pp)}}$$

sicque tota aequatio integralis erit

$$\sqrt{(xx+yy)}+C=\frac{px-y}{\sqrt{(1+pp)}}.$$

7. Haec igitur perpendens, non amplius dubitaui, quin omnes huiusmodi formulae differentiales duplicem admittant multiplicatorem, ita vt, si talis formula iam per se sit integrabilis, quae in genere sit dv, praeter multiplicatores naturales, qui sunt functiones ipsius v, etiam dentur multiplicatores alius indolis, qui non sint functiones ipsius v, quemadmodum in exemplis allatis fieri vidimus.

8. Statim autem ac vnus quispiam multiplicator fuerit cognitus, ex eo mox infinitos alios multiplicatores concludere licet, ita quum formulae dx multiplicator sit p, et $\int p\,dx=y$, tum functio quaecunque ipsius y, quae sit Y, per p multiplicata, dabit etiam multiplicatorem idoneum, nam dx multiplicatum in Yp dat $Y\,dy$, quod manifesto est integrabile. Deinde si X denotet functionem quamcunque ipsius x, formula dy multiplicatorem habebit $\frac{X}{p}$, tum enim prodit $\frac{X\,dy}{p}=X\,dx$. Simili modo quum sit

$$\int x\,dp=px-y,$$

si V denotet functionem quamcunque formulae $px-y$; tum Vx erit multiplicator ipsius dp, si quidem erit

$$V x dp = V. d(px - y),$$

huiusmodi autem multiplicatores, qui hoc modo ex vno quodam multiplicatore cognito concluduntur, omnes eiusdem generis sunt censendi, vnde simplicissimum eorum in quouis genere primitiuum appellabo, quippe quo cognito, reliqui omnes innotescunt.

9. Ita si in genere proposita sit huiusmodi formula differentialis,

$$P dp + Q dx + R dy$$

vbi P, Q, R sint functiones quaecunque ipsarum x, y et p, quae integrabilis reddatur ope multiplicatoris M, sequenti modo omnes reliqui multiplicatores eiusdem generis inueniri poterunt. Ponatur

$$M(P dp + Q dx + R dy) = dv,$$

ita vt dv sit verum differentiale, ac denotet V functionem quamcunque ipsius v, manifestumque erit, multiplicatorem quoque fore V M, si quidem tum habebitur

$$V M(P dp + Q dx + R dy) = V dv,$$

quae formula per hypothesin est integrabilis.

10. Simili modo, si pro eadem formula proposita

$$P dp + Q dx + R dy$$

adhuc

adhuc alius multiplicator primitiuus N fuerit repertus; tum ex eo etiam infiniti alii eiusdem generis erui poterunt, ita vt hoc modo duae inueniantur formulae generales, pro multiplicatoribus formulae differentialis propositae. Hinc ergo ista quaestio maximi momenti nascitur, quam seorsim proponi operae pretium erit.

Problema.

Si formula

$$P\,dp + Q\,dx + R\,dy$$

integrabilis fiat tam per multiplicatorem M, quam per alium diuersae naturae N, inuenire expressionem generalem, quae omnes plane multiplicatores possibiles eiusdem formulae in se complectatur.

Solutio.

Quum M et N sint multiplicatores, ponamus

$$M\,(P\,dp + Q\,dx + R\,dy) = dv$$

et $$N\,(P\,dp + Q\,dx + R\,dy) = du$$

eruntque quantitates v et u cognitae functiones, iam denotet z functionem quamcunque binarum harum variabilium v et u, cuius differentiale propterea huiusmodi formam habebit:

$$dz = S\,dv + T\,du$$

vbi functiones S et T ex functione z erunt cognitae, hac forma iam inuenta, dico expressionem ge-

 neralem,

neralem, omnes plane multiplicatores in se complectentem, fore:

$$= S M + T N,$$

tum enim habebitur:

$$(S M + T N)(P dp + Q dx + R dy) = S dv + T du = dz,$$

cuius integrale per hypothesin est z, vbi pro z functio quaecunque binarum variabilium z et u pro lubitu sumi potest.

11. Vt hoc exemplo illustremus, sit proposita formula dp, cuius duo multiplicatores constant

$M = 1$ et $N = x$, hinc ergo fit

$dp = dv$ et $x dp = du$, ideoque

$v = p$ et $u = px - y$,

quare si z denotet functionem quamcunque harum duarum variabilium v et u, sitque

$$dz = S dv + T du,$$

multiplicator vniuersalis erit $S + T x$.

12. Circa hanc formulam obseruandum est, non absolute necessarium esse, vt valores litterarum S et T ex certa quadam functione z deriuentur. Dummodo enim pro litteris S et T eiusmodi functiones ipsarum v et u capiantur, vt sit $\left(\frac{dS}{du}\right) = \left(\frac{dT}{dv}\right)$, tum enim semper formula $S M + T N$ erit multiplicator idoneus formulae differentialis

$$P dp + Q dx + R dy,$$

et producti integrale erit ipsa functio illa z, quam ex litteris S et T facile inuenire licet.

13.

13. Quod autem ista formula $SM + TN$ omnes plane multiplicatores formulae differentialis propositae in se complectatur, ratio in eo est sita, quod semper duo tantum eiusmodi multiplicatores primitiui M et N exhiberi queant, qui a se inuicem non pendeant; si enim plures eiusmodi multiplicatores locum haberent; tum forma ista vtique non foret generalis, sed alia multo generalior exhiberi posset; ratio autem, cur duo tantum huiusmodi multiplicatores locum inueniant, in eo est quaerenda, quod inter ternas nostras variabiles, x, y et p vnica detur relatio, scilicet $p = \frac{dy}{dx}$, si enim vlterius progredi et insuper litteram q introducere velimus, vt sit

$$dp = q\,dx, \text{ siue } q = \frac{dp}{dx}$$

quaelibet formula differentialis tres adeo multiplicatores admitteret, quemadmodum ex forma simplicissima dq manifestum est, quae primum ipsa est integrabilis, seu multiplicator $= 1$, secundus multiplicator est y, quoniam

$$\int y\,dq = qy - \int q\,dy \text{ at } \int q\,dy = \int pq\,dx = \int p\,dp = \frac{pp}{2}$$

vnde fit

$$\int y\,dq = qy - \frac{pp}{2},$$

tertius vero multiplicator est x, quum sit

$$\int x\,dq = xq - p,$$

ex quo satis patet his casibus tria integrabilitatis genera locum habere.

14. Contemplemur autem hic tantum ternas variabiles x, y et p, exiſtente $p = \frac{dy}{dx}$, et quo clarius appareat, ſemper duos multiplicatores primitivos locum habere, varios caſus ſimpliciores in medium afferamus, quibus hos duos multiplicatores, ſiue diuinando, ſiue quocunque alio modo, reperire licuit, quos caſus ſequenti modo adiungamus.

I. $\alpha x\, dp + \beta p\, dx$.

15. Haec formula primo per ſe eſt integrabilis, quum eius integrale ſit

$$\alpha(px - y) + \beta y$$

ita vt multiplicator primus ſit $= 1$. Alter multiplicator erit $p^{\alpha - 1} x^{\beta - 1}$; tum enim integrale ſit $p^\alpha x^\beta$. Pro multiplicatore igitur vniuerſali inueniendo erit ex §. 10.

$$M = 1 \quad \text{et } N = p^{\alpha - 1} x^{\beta - 1} \text{ hinc}$$

$$v = \alpha(px - y) + \beta y = \alpha p x + (\beta - \alpha) y$$

et $u = p^\alpha x^\beta$,

quare ſi fuerit

$$dZ = S\, dv + T\, du,$$

multiplicator generalis erit

$$S.\, 1 + T p^{\alpha - 1} x^{\beta - 1}.$$

16. Vnico autem caſu, quo $\alpha = 1$ et $\beta = 1$, haec ſolutio ſit incongrua, quippe quo ambo multiplicatores non amplius erunt diuerſi, vterque enim fieret $= 1$, hocque incommodum etiam vſu venit, quoties $\beta = \alpha$, tum enim prius integrale eſt $\alpha p x$ et

et alter multiplicator $p^{\alpha-1}x^{\beta-1}$ eius foret poteſtas, neque propterea a priori multiplicatore differret, quod quidem per ſe euidens eſt, quum totum negotium a ratione inter α et β pendeat, quare hic noua quaeſtio oritur, num caſu $\beta=\alpha$ exhiberi queat alius multiplicator, et quomodo futurus ſit expreſſus, quem caſum ſeorſim euoluamus.

II. $x\,dp+p\,dx$.

17. Circa priorem multiplicatorem $=1$ hic nulla eſt difficultas, quum integrale ſit $p\,x$, alter vero multiplicator, non tam facile ſe offert, re autem diligentius perpenſa, multiplicator ſe obtulit, $=\mathrm{L}\,x$, erit enim

$$\int(x\,dp+p\,dx)\,\mathrm{L}\,x=p\,x\,\mathrm{L}\,x-y,$$

ſimili autem modo alius colligitur multiplicator $\frac{y}{pp+xy}$, fiet enim integrale

$$=\frac{-y}{px}+\int\frac{dy}{px}=\frac{-y}{px}+\mathrm{L}\,x,$$

neque vero hic multiplicator tertius a duobus prioribus diſcrepat, quum enim ex prioribus ſit

$$\mathrm{M}=1,\ v=px \text{ et } \mathrm{N}=\mathrm{L}x \text{ et } u=px\,\mathrm{L}x-y,$$

manifeſtum eſt, tertium integrale eſſe functionem ipſius u et v, quum ſit

$$\frac{u}{v}=\mathrm{L}\,x-\frac{y}{px}.$$

Ex hoc igitur exemplo intelligitur, ſaepenumero plures multiplicatores diuerſos videri poſſe, quum tamen ad duos reduci queant, ad quod diiudicandum tantum ex binis multiplicatoribus eliciantur litterae

v et u

v et u, ex quibus ſemper reliqua integralia, quotcunque fuerint inuenta, componi reperientur.

III. $\alpha y\, dp + \beta p\, dy$.

18. Hic iterum vnus multiplicator ſponte ſe offert, ſcilicet $p^{\alpha-1} y^{\beta-1}$, cui reſpondet integrale $p^{\alpha} y^{\beta}$ ſiue quod eodem redit, ſumto multiplicatore $\frac{1}{py}$, erit integrale

$$= \alpha\, lp + \beta\, ly = \mathrm{L}\, p^{\alpha} y^{\beta},$$

quod quum ſit illius logarithmus, etiam a priore differre non eſt cenſendum; alter multiplicator re probe perpenſa colligitur $x p y^{\frac{\beta}{\alpha}}$, integrale enim erit

$$x p y^{\frac{\beta}{\alpha}} - \left(\frac{\alpha}{\beta+\alpha}\right) y^{\frac{\beta+\alpha}{\alpha}},$$

tum etiam quaſi ſponte ſe prodit multiplicator $\frac{1}{pp}$, erit enim

$$\int \frac{\alpha y\, dp}{pp} = -\frac{dy}{p} + \int \frac{\alpha\, dy}{p} = -\frac{\alpha y}{p} + \alpha x \text{ et } \int \frac{p\, dy}{pp} = \int \frac{dy}{p} = x,$$

vnde totum integrale erit

$$= \frac{-\alpha y}{p} + (\alpha + \beta)\, x,$$

hoc autem iam in duobus praecedentibus continetur, erit enim

$$\mathrm{M} = p^{\alpha-1} y^{\beta-1} \text{ et } v = p^{\alpha} y^{\beta},\ \mathrm{N} = x p y^{\frac{\beta}{\alpha}}$$

$$\text{et } u = x p y^{\frac{\beta}{\alpha}} - \left(\frac{\alpha}{\beta+\alpha}\right) y^{\frac{\beta+\alpha}{\alpha}},$$

vnde diuidendo u per

$$\frac{\mathrm{V}^{\frac{1}{\alpha}}}{\alpha+\beta} = \frac{p y^{\frac{\beta}{\alpha}}}{\alpha+\beta}, \text{ oritur integrale } (\alpha+\beta)x - \frac{\alpha y}{p}.$$

Haec

Haec autem reductio non succedit, casu quo $\beta = -\alpha$, quem casum seorsim euoluamus.

IV. $y\,dp - p\,dy$.

19. Cum hoc casu sit $\alpha = 1$ et $\beta = -1$, erit prior multiplicator $\frac{1}{yy}$, praeterea vero colligitur multiplicator $\frac{x}{yy}$, tum enim erit

$$\int \frac{x(y\,dp - p\,dy)}{yy} = \frac{px}{y} - \mathrm{L}y$$

haec enim formula differentiata praebet

$$\frac{x(y\,dp - p\,dy)}{yy} = \frac{p\,dx + x\,dp}{y} - \frac{dy\,px}{yy} - \frac{dy}{y} \text{ ob } dy = p\,dx$$

quum nunc habeamus duos multiplicatores, alterum $M = \frac{1}{yy}$, et alterum $N = \frac{x}{yy}$, vnde fit

$$v = \frac{p}{y} \text{ et } u = \frac{px}{p} - \mathrm{L}y$$

si Z denotet functionem quamcunque binarum quantitatum v et u, erit multiplicator generalis

$$= M\left(\frac{dZ}{dv}\right) + N\left(\frac{dZ}{du}\right) = \frac{1}{yy}\left(\frac{dZ}{dv}\right) + \frac{x}{yy}\left(\frac{dZ}{du}\right).$$

V. $p\,dp + x\,dx$.

20. Primo haec formula ipsa per se est integrabilis, ita vt sit

$$M = 1 \text{ et } v = \tfrac{1}{2}(pp + xx),$$

tum vero alius multiplicator deprehenditur arcus cuius tangens est $\frac{x}{p} = N$, tum enim erit

$$\int (p\,dp + x\,dx)\,\mathrm{A.\,tang.}\frac{x}{p} = \tfrac{1}{2}(pp+xx)\,\mathrm{A.\,tang.}\frac{x}{p}$$
$$-\int \tfrac{1}{2}(pp + xx)\,d.\,\mathrm{A.\,tang.}\frac{x}{p} = \tfrac{1}{2}(pp+xx)\,\mathrm{A.tang.}\frac{x}{p}$$
$$-\int \tfrac{1}{2}(p\,dx - x\,dp)$$

 at

at eſt

$$\int \tfrac{1}{2}(p\,dx - x\,dp) = y - \tfrac{px}{2},$$

vnde integrale quaeſitum erit

$$\tfrac{1}{2}(pp + xx)\,\text{A. tang.}\,\tfrac{x}{p} - y + \tfrac{px}{2}.$$

21. Ex his exemplis abunde patet, inuentionem huiusmodi multiplicatorum neutiquam eſſe obuiam, ſed ſaepenumero admodum eſſe abſconditam, quin etiam euenire poteſt, vt vires analyſeos plane ſuperet. Interim tamen methodum quandam hic aperiam, ad hoc inſtitutum accommodatam, cuius ope plurimis caſibus tales multiplicatores inuenire licet.

22. Quum ratio duplicis multiplicatoris in eo lateat, quod huiusmodi formulae differentiales ſint ſecundi gradus, vnde fit, vt vterque multiplicator vnam tantum quaſi integrationem inuoluat, ideoque duplex integratio etiam duplicem multiplicatorem requirat, hinc viciſſim, ambos multiplicatores reperire licebit, ſi vtramque integrationem abſoluamus. Quemadmodum igitur hac methodo vti oporteat, in ſequentibus exemplis docebimus.

Exemplum 1.

23. Propoſita formula differentiali, $x\,dp + p\,dx$, eius vtrumque multiplicatorem inuenire. Quum haec formula per ſe ſit integrabilis, ideoque $M = 1$, ponatur

$$x\,dp + p\,dx = dv \text{ eritque } px = v,$$

ſicque

ſicque vna integratio eſt abſoluta, pro altera vero quum ſit $p = \frac{v}{x}$, per dx multiplicando ob $pdx = dy$ habebimus $dy = \frac{v\,dx}{x}$, vnde integrando elicimus

$$y = v\,\mathrm{L}\,x - \int dv.\,\mathrm{L}\,x, \text{ idcoque}$$

$$\int dv\,\mathrm{L}\,x = v\,\mathrm{L}\,x - y = p\,x\,\mathrm{L}\,x - y,$$

vnde intelligimus, formulam $dv\,\mathrm{L}\,x$ eſſe integrabilem, ſiquidem eius integrale eſt $p\,x\,\mathrm{L}\,x - y$, quare quum dv denotet ipſam formulam noſtram propoſitam, $x\,dp + p\,dx$, patet eius multiplicatorem fore $\mathrm{L}\,x$.

24. Eodem modo etiam alios multiplicatores reperire licet, quum enim ſit

$$dy = \frac{v\,dx}{x} \text{ erit etiam } \frac{dy}{v} = \frac{dx}{x},$$

hinc integrando

$$\frac{y}{v} + \int \frac{y\,dv}{vv} = \mathrm{L}\,x, \text{ ergo } \int \frac{y\,dv}{vv} = \mathrm{L}\,x - \frac{y}{v} = \mathrm{L}\,x - \frac{y}{p\,x},$$

integrabilis ergo eſt formula

$$\frac{y\,dv}{vv} \text{ ſeu } dv.\frac{y}{vv},$$

ſicque multiplicator erit

$$\frac{y}{vv} = \frac{y}{ppxx},$$

quem ergo loco N aſſumere licet, vnde, quum ſit

$$v = p\,x \quad \text{et} \quad u = \mathrm{L}\,x - \frac{y}{p\,x},$$

ſi Z denotet functionem quamcunque ipſarum v et u erit multiplicator generalis

$$\left(\frac{dZ}{dv}\right) + \frac{y}{ppxx}\left(\frac{dZ}{du}\right),$$

 veluti

veluti si fuerit $Z = v\,u$ erit

$$\left(\frac{d\,Z}{d\,v}\right) = u \quad \text{et} \quad \left(\frac{d\,Z}{d\,u}\right) = v\,,$$

vnde oritur hic multiplicator:

$$u + \frac{y}{p\,p\,x\,x}\,v = \mathrm{L}\,x - \frac{y}{p\,x} + \frac{y}{p\,x} = \mathrm{L}\,x\,,$$

qui est multiplicator priori loco inuentus.

Exemplum 2.

25. Proposita formula differentiali $\alpha\,x\,d\,p + \beta\,p\,d\,x$, eius ambos multiplicatores inuenire. Quum haec formula iam per se sit integrabilis, erit $M = 1$ et posito:

$$\alpha\,x\,d\,p + \beta\,p\,d\,x = d\,v \quad \text{erit} \quad v = \alpha\,p\,x + (\beta - \alpha)\,y\,,$$

vnde colligitur

$$p = \frac{v}{\alpha\,x} + \frac{(\alpha - \beta)\,y}{\alpha\,x}\,,$$

quae per $d\,x$ multiplicata praebet,

$$d\,y = p\,d\,x = \frac{v\,d\,x}{\alpha\,x} + \frac{\alpha - \beta}{\alpha} \cdot \frac{y\,d\,x}{x}\,,$$

hincque

$$\frac{d\,x}{x} = \frac{\alpha\,d\,y}{v + (\alpha - \beta)\,y}\,,$$

integrando igitur obtinebimus

$$\mathrm{L}\,x = \frac{\alpha}{\alpha - \beta}\,\mathrm{L}\,v + (\alpha - \beta)\,y - \int \frac{\alpha}{\alpha - \beta} \cdot \frac{d\,v}{v + (\alpha - \beta)\,y}$$

sicque erit

$$\alpha \int \frac{d\,v}{v + (\alpha - \beta)\,y} = \alpha\,\mathrm{L}\,v + (\alpha + \beta)\,y - (\alpha - \beta)\,\mathrm{L}\,x,$$

vnde patet, formulae nostrae $d\,v$ multiplicatorem fore

$$\frac{\alpha}{v + (\alpha - \beta)\,y} = \frac{1}{p\,x}\,,$$

vti

vti per se est manifestum, tum enim integrale erit $\alpha L p + \beta L x$.

Exemplum 3.

26. Proposita formula $p\,dp + x\,dx$, eius ambos multiplicatores inuenire. Hic iterum primus multiplicator est $M = 1$ et posito

$$p\,dp + x\,dx = dv \text{ erit } pp + xx = 2v, \text{ hinc}$$

$$p = \sqrt{(2v - xx)}$$

et per dx multiplicando

$$dy = p\,dx = dx\sqrt{(2v - xx)},$$

ponatur tantisper $2v = ss$, fietque

$$y = \tfrac{1}{2}x.\sqrt{(ss - xx)} + \tfrac{ss}{2}.\text{ Arc. sin. }\tfrac{x}{s} - \int\tfrac{xsds}{2\sqrt{(ss-xx)}}$$

$$+ s\,ds.\text{ Arc. sin.}\tfrac{x}{s} - \tfrac{sxds}{2\sqrt{(ss-xx)}} = \tfrac{1}{2}x\sqrt{(ss - xx)}$$

$$+ \tfrac{ss}{2}\text{ Arc. sin. }\tfrac{x}{s} - \int s\,ds.\text{ Arc. sin. }\tfrac{x}{s},$$

ideoque

$$\int s\,ds.\text{ A. sin.}\tfrac{x}{s} = \tfrac{1}{2}x\sqrt{(ss - xx)} + \tfrac{ss}{2}\text{ A. sin. }\tfrac{x}{s} - y = \tfrac{px}{2}$$

$$+ \tfrac{1}{2}(pp + xx)\text{ A. sin. }\tfrac{x}{\sqrt{(pp+xx)}} - y,$$

at est

$$s\,ds = p\,dp + x\,dx,$$

vnde patet, nostrae formulae multiplicatorem esse

$$\text{Ar. sin. }\tfrac{x}{\sqrt{(pp+xx)}},\text{ siue Arc. tang. }\tfrac{x}{p}.$$

27. Haec autem operatio nimis est molesta, quam vt ea, in formulis magis complicatis, vti

 quea-

queamus, quare eam ſequenti modo faciliorem reddere conemur. Quum inuenerimus:

$$dy = dx\,\sqrt{(ss - xx)},$$

ſtatuamus hic $x = sz$ fietque

$$dy = ss\,dz\,\sqrt{(1 - zz)} + zs\,ds\,\sqrt{(1 - zz)},$$

quae per ss diuiſa dat

$$\frac{dy}{ss} = dz\,\sqrt{(1 - zz)} + \frac{z\,ds}{s}\sqrt{(1 - zz)}$$

et integrando

$$\frac{y}{ss} + 2\int\frac{y\,ds}{s^3} = \int dz.\,\sqrt{(1 - zz)} + \int\frac{z\,ds}{s}\sqrt{(1 - zz)}$$

vbi membrum penultimum

$$\int dz\,\sqrt{(1 - zz)}$$

abſolute datur, quum ſit certa quaedam functio ipſius $z = \frac{x}{s}$, poſtremum autem membrum reſtituto pro z valore $\frac{x}{s}$, abit in

$$\int\frac{x\,ds}{s^3}\sqrt{(ss - xx)},$$

vnde binis integralibus per $\frac{ds}{s^3}$ affectis coniungendis, conſequimur

$$+\int\frac{ds}{s^3}(2y - x\sqrt{(ss - xx)}) = \int dz\sqrt{(1 - zz)} - \frac{y}{ss}$$

ex quo patet, formulae noſtrae

$$p\,dp + x\,dx = s\,ds$$

multiplicatorem eſſe

$$\frac{1}{s^4}(2y - x\sqrt{(ss - xx)})$$

qui ob

$$ss = pp + xx,$$

trans-

transmutatur in hanc formam $\frac{2y-px}{(pp-xx)^2}$, qui si ponatur $= M$, erit

$$u = \int dz \sqrt{(1-zz)} - \frac{y}{pp+xx} \text{ existente } z = \frac{x}{\sqrt{(pp+xx)}}$$

atque hinc multiplicatorem generalem facile elicere licet.

28. Hinc patet, si formula proposita sit $\alpha p\,dp + \beta x\,dx$, quo casu iterum est $M = 1$ et

$$v = \frac{\alpha pp}{2} + \frac{\beta}{2} \cdot xx,$$

alterum multiplicatorem repertum iri

$$N = \frac{2y-xp}{(\alpha pp+\beta xx)^2}$$

tum enim integrale hinc natum erit

$$u = \int \frac{s\,ds}{s^4}(2y-px) = \int dz \sqrt{\frac{(1-\beta zz)}{\alpha}} - \frac{y}{\alpha pp+\beta xx},$$

existente

$$z = \frac{x}{s} = \frac{x}{\sqrt{(\alpha pp+\beta xx)}}$$

Exemplum 4.

29. Proposita formula differentiali:

$$p^{n-1}\,dp + \beta x^{n-1}\,dx,$$

eius multiplicatores inuenire. Quum alter multiplicator iterum sit cognitus $= 1$, posita nostra formula $= dv$, erit

$$p^n + \beta x^n = nv$$

vnde fit

$$p = (nv - \beta x^n)^{\frac{1}{n}},$$

hincque

$$dy = p\,dx = dx(nv - \beta x^n)^{\frac{1}{n}},$$

statua-

statuatur nunc

$$n v = s^n, \text{ sitque } x = s z,$$

habebimus

$$d y = (s d z + z d s) s (1 - \beta z^n)^{\frac{1}{n}},$$

quae per $s s$ diuisa, dat

$$\frac{d y}{s s} = d z (1 - \beta z^n)^{\frac{1}{n}} + \frac{z d s}{s} (1 - \beta z^n)^{\frac{1}{n}},$$

hinc integrando

$$\frac{y}{s s} + \int \frac{2 y d s}{s^3} = \int d z (1 - \beta z^n)^{\frac{1}{n}} + \int \frac{z d s}{s} (1 - \beta z^n)^{\frac{1}{n}},$$

vbi membrum penultimum est determinatum, quippe certa quaedam functio ipsius

$$z = \frac{x}{s} = \frac{x}{(n v)^{\frac{1}{n}}},$$

vltimum vero membrum, si in eo restituatur $z = \frac{x}{s}$ abit in

$$\int \frac{x d s}{s^3} (s^n - \beta x^n)^{\frac{1}{n}},$$

atque ob

$$(s^n - \beta x^n)^{\frac{1}{n}} = p$$

erit

$$\int \frac{x d s}{s^3} (s^n - \beta x^n)^{\frac{1}{n}} = \int \frac{p x d s}{s^3},$$

quibus substitutis fiet

$$\int \frac{d s}{s^3} (2 y - p x) = \int d z (1 - \beta z^n)^{\frac{1}{n}} - \frac{y}{s s}.$$

Quum autem sit $s = (n v)^{\frac{1}{n}}$, erit

$$d s = (n v)^{\frac{1}{n} - 1} d v \text{ et } \frac{d s}{s^3} = \frac{d v}{(n v)^{\frac{2+n}{n}}}$$

ex

ex quo primum membrum fiet

$$\int \frac{dv}{(nv)^{\frac{2+n}{n}}}(2y-px),$$

cuius integrale quum iam fit inuentum, patet formulae propofitae dv multiplicatorem effe

$$=\frac{2y-px}{(nv)^{\frac{2+n}{n}}}=\frac{2y-px}{(p^n+\beta x^n)^{\frac{2+n}{n}}}.$$

Exemplum 5.

30. Propofita formula differentiali:

$$p^{m-1}\,dp+\beta x^{n-1}\,dx=dv,$$

eius multiplicatorem alterum inuenire. Quum hinc fit

$$v=\tfrac{1}{m}p^m+\tfrac{\beta}{n}x^n;\ \text{erit}$$

$$p=(mv-\tfrac{m\beta}{n}.\,x^n)^{\frac{1}{m}}\ \text{et}\ dy=p\,dx=dx(mv-\tfrac{m\beta}{n}.\,x^n)^{\frac{1}{m}}$$

ponatur hic iterum

$$mv=s^n\ \text{et}\ x=sz,\ \text{erit}$$

$$dy=(s\,dz+z\,ds)\,s^{\frac{n}{m}}(1-\tfrac{m\beta}{n}.\,z^n)^{\frac{1}{m}},$$

quae diuifa per $s^{\frac{m+n}{m}}$, praebet

$$\frac{dy}{s^{\frac{m+n}{m}}}=dz(1-\tfrac{m\beta}{n}.\,z^n)^{\frac{1}{m}}+\tfrac{z\,ds}{s}(1-\tfrac{m\beta}{n}.\,z^n)^{\frac{1}{m}},$$

hinc integrando

$$\frac{y}{s^{\frac{m+n}{m}}}+\int\frac{m+n}{m}.\,\frac{y\,ds}{s^{\frac{2m+n}{m}}}=\int dz(1-\tfrac{m\beta}{n}.\,z^n)^{\frac{1}{m}}$$

$$+\int\tfrac{z\,ds}{s}(1-\tfrac{m\beta}{n}.\,z^n)^{\frac{1}{m}},$$

vbi

vbi membrum penultimum eſt functio cognita ipſius z, vltimum autem ſubſtituto loco z valore $\frac{x}{s}$, abit in

$$\int \frac{x\,d\,s}{s^{\frac{2m+n}{m}}}\left(s^{n}-\tfrac{m\beta}{n}.\,x^{n}\right)^{\frac{1}{m}}=\int \frac{x\,d\,s}{s^{\frac{m+n}{m}}},$$

quare hinc colligimus

$$\int \frac{d\,s}{s^{\frac{2m+n}{m}}}\left(\frac{m+n}{m}y-px\right)=\int dz\left(1-\tfrac{m\beta}{n}.\,z^{n}\right)^{\frac{1}{m}}-\frac{y}{s^{\frac{m+n}{m}}},$$

quum autem ſit

$$s=(m\,v)^{\frac{1}{n}} \text{ erit } d\,s=\tfrac{1}{n}.\,m^{\frac{1}{n}}\,v^{\frac{1}{n}-1}\,d\,v$$

et

$$\frac{d\,s}{s^{\frac{2m+n}{m}}}=\tfrac{1}{n}.\,\frac{d\,v}{m^{\frac{m+n}{mn}}\,v^{\frac{m+n}{mn}}},$$

ex quo colligitur formulae noſtrae propoſitae multiplicatorem eſſe

$$\frac{\frac{m+n}{m}\,y-p\,x}{v^{\frac{m+n}{mn}+1}}=\frac{\frac{m+n}{m}y-p\,x}{\left(\frac{1}{m}\,p^{m}+\frac{\beta}{n}\,x^{n}\right)^{\frac{m+n}{mn}+1}}.$$

31. Hactenus eiusmodi formulas ſumus contemplati, quae cum per ſe ſunt integrabiles, tum vero duas tantum variabiles p et x contineant; ſimili autem modo iſtas formulas tractare licebit, quae tantum has duas variabiles p et y inuoluant, vbi quidem aſſumimus, has formulas per ſe eſſe integrabiles.

Exemplum 6.

Propoſita formula:

$$p^{m-1}\,dp+\beta y^{n-1}\,dp=dv,$$

eius

eius alterum multiplicatorem inueſtigare. Primum integrando colligimus

$$v = \frac{1}{m} p^m + \frac{\beta}{n} y,$$

vnde fit

$$p = (m v - \frac{m\beta}{n} . y^n)^{\frac{1}{m}}$$

et $d y = p\, d x = d x (m v - \frac{m\beta}{n} y^n)^{\frac{1}{m}}$

ideoque

$$d x = \frac{d y}{(m v - \frac{m\beta}{n} . y^n)^{\frac{1}{m}}},$$

ſtatuatur iam hic $m v = s^n$ et $y = s z$, vt obtineamus

$$d x = \frac{s\, d z + z\, d s}{s^{\frac{n}{m}} (1 - \frac{m\beta}{n} . z^n)^{\frac{1}{m}}},$$

quae aequatio diuiſa per $s^{\frac{m-n}{m}}$ dat

$$\frac{d x}{s^{\frac{m-n}{m}}} = \frac{d z}{(1 - \frac{m\beta}{n} . z^n)^{\frac{1}{m}}} + \frac{z\, d s}{s (1 - \frac{m\beta}{n} . z^n)^{\frac{1}{m}}},$$

haec aequatio ſimili modo vt ſupra integratur, et ex penultimo membro iterum naſcitur functio determinata ipſius z; ſi in membro poſtremo loco z ipſius valorem $\frac{y}{s}$ reſtituamus, conſequimur,

$$\frac{x}{s^{\frac{m-n}{m}}} + \int \frac{m-n}{m} \frac{x\, d s}{s^{\frac{2m-n}{m}}} = \int \frac{d z}{(1 - \frac{m\beta}{n} . z^n)^{\frac{1}{m}}} + \int \frac{y\, d s}{s^{\frac{2m-n}{n}}},$$

ex his igitur colligimus

$$\int \frac{d s}{s^{2-\frac{n}{m}}}\left(\left(1-\frac{n}{m}\right) x-\frac{y}{p}\right)=\int \frac{d z}{\left(1-\frac{m \beta}{n} . z^{n}\right)^{\frac{1}{m}}}-\frac{x}{s^{\frac{m-n}{m}}},$$

quum nunc ſit $s=(m v)^{\frac{1}{n}}$, erit $d s=\frac{1}{n} . m^{\frac{1}{n}} v^{\frac{1}{n}-1} d v$, hinc

$$\frac{d s}{s^{n-\frac{n}{m}}}=\frac{1}{n} d v . \frac{1}{m^{\frac{1}{n}-\frac{1}{m}} v^{\frac{1}{n}-\frac{1}{m}+1}}$$

vnde concludimus formulae noſtrae propoſitae multiplicatorem fore

$$\frac{\left(1-\frac{n}{m}\right) x-\frac{y}{p}}{v^{\frac{1}{n}-\frac{1}{m}+1}}=\frac{\left(1-\frac{n}{m}\right) x-\frac{y}{p}}{\left(\frac{1}{m} p^{m}+\frac{\beta}{n} y^{n}\right)^{\frac{1}{n}-\frac{1}{m}+1}}$$

32. Etſi haec exempla iam ſatis late patere videntur; tamen, ſi res ipſa ſpectetur, ea etiam nunc ſunt vehementer particularia, ſiquidem pro v formula binomialis vtroque caſu prodit, in exemplo penultimo litteras p et x, in vltimo vero p et y inuoluens, atque hinc vix liquet, quomodo operationes inſtitui oporteat, ſi plures termini in valore ipſius v occurrant. Interim tamen ſequenti modo iſta inueſtigatio multo generalior reddi poterit.

Problema.

Si Ω eiusmodi fuerit functio quantitatum p et x, vt ea poſito $p=x^{\lambda} q$, induat hanc formam $x^{n} Q$, ita vt Q tantum ſit functio ipſius q, tum propo-

propofita formula differentiali $d\,\Omega$ eius alternm multiplicatorem inuenire.

Solutio.

Pofito vt ante $d\,\Omega = d\,v$, vt fit $v = \Omega$, ponatur $p = x^\lambda q$ eritque per hypothefin $v = x^n Q$, hinc $Q = \frac{v}{x^n}$, nunc ftatuatur porro $x^n = \frac{v}{z}$, vt fiat $Q = z$, iam quotcunque dimenfiones ipfius q in functione Q contineantur, etiamfi refolutio iftius aequationis vires analyfeos fuperet, tamen certum eft, inde valorem radicis q per certam quandam functionem ipfius z, quae fit Z, expreffum iri, ita vt fit $q = Z$, hinc

$$p = x^\lambda Z \text{ et } dy = p\,dx = x^\lambda Z\,dx,$$

iam cum fit

$$x^n = \frac{v}{z}, \text{ erit } x^{\lambda+1} = \left(\frac{v}{z}\right)^{\frac{\lambda+1}{n}}$$

$$\text{et } x^\lambda dx = \frac{1}{n} \cdot \frac{z\,dv - v\,dz}{zz} \cdot \left(\frac{v}{z}\right)^{\frac{\lambda+1}{n} - 1} = \frac{1}{n}\left(\frac{dv \cdot v^{\frac{\lambda+1}{n} - 1}}{z^{\frac{\lambda+1}{n}}} - \frac{dz \cdot v^{\frac{\lambda+1}{n}}}{z^{\frac{\lambda+1}{n} + 1}}\right)$$

et multiplicando per $\frac{n}{v^{\frac{\lambda+1}{n}}}$, habebimus

$$\frac{n\,dy}{v^{\frac{\lambda+1}{n}}} = \frac{Z\,dv}{v \cdot z^{\frac{\lambda+1}{n}}} - \frac{Z\,dz}{z^{\frac{\lambda+1}{n} + 1}},$$

vbi membrum vltimum variabilem z continens dabit functionem determinatam ipsius z, penultimum vero pro z restituto valore $\frac{v}{x^n}$ ob $Z = \frac{p}{x^\lambda}$, abit in $\frac{p\,x\,d\,v}{v^{\frac{\lambda+1}{n}+1}}$, quare integrando habebimus

$$\frac{n\,y}{v^{\frac{\lambda+1}{n}}} + \int(\lambda+1)\frac{y\,d\,v}{v^{\frac{\lambda+1}{n}+1}} = \int\frac{p\,x\,d\,v}{v^{\frac{\lambda+1}{n}+1}} - \int\frac{Z\,d\,z}{z^{\frac{\lambda+1}{n}+1}},$$

vnde fit

$$\int\frac{d\,v}{v^{\frac{\lambda+1}{n}+1}}(px-(\lambda+1)y) = \int\frac{Z\,d\,z}{z^{\frac{\lambda+1}{n}+1}} + \frac{n\,y}{v^{\frac{\lambda+1}{n}}}.$$

Quocirca concludimus, formulae nostrae $d\,v = d\,\Omega$ multiplicatorem quaesitum esse

$$\frac{p\,x-(\lambda+1)y}{(\Omega)^{\frac{\lambda+1}{n}+1}}.$$

Problema.

33. Si Ω eiusmodi fuerit functio ipsarum p et y, vt posito $p = y^\lambda q$ ea obtineat hanc formam $y^n Q$, existente Q functione ipsius q, tum proposita formula differentiali $d\,\Omega$, eius multiplicatorem invenire.

Solutio.

Posito iterum $d\,\Omega = d\,v$, erit $v = \Omega$, et posito $p = y^\lambda q$, fiet $v = y^n Q$, hincque $Q = \frac{v}{y^n}$

statua-

ſtatuatur nunc $y^n = \frac{v}{z}$, vt fiat $Q = z$, vnde quum Q ſit functio ipſius q, per reſolutionem aequationum q aequabitur certae cuipiam functioni ipſius z, quae ſit Z, ita vt ſit

$$q = Z, \quad \text{hinc} \quad p = y^\lambda Z$$

et $dy = p\,dx = y^\lambda Z\,dx$, vnde fit $dx = \frac{dy}{y^\lambda Z}$,

quum autem ſit $y^n = \frac{v}{z}$, erit

$$y = \frac{v^{\frac{1}{n}}}{z^{\frac{1}{n}}} \text{ et } dy = \tfrac{1}{n}\left(\frac{dv.\,v^{\frac{1}{n}-1}}{z^{\frac{1}{n}}} - \frac{v^{\frac{1}{n}}\,dz}{z^{\frac{1}{n}+1}}\right),$$

ex quo habebitur

$$dx = \tfrac{1}{n}.\,\frac{v^{\frac{1}{n}-\frac{\lambda}{n}-1}\,dv}{z^{\frac{1-\lambda}{n}}.\,Z} - \tfrac{1}{n}.\,\frac{v^{\frac{1-\lambda}{n}}\,dz}{Z.\,z^{\frac{1-\lambda}{n}+1}},$$

quod multiplicatum per $n.\,v^{\frac{\lambda-1}{n}}$ praebebit

$$n\,dx.\,v^{\frac{\lambda-1}{n}} = \frac{dv}{v.\,z^{\frac{1-\lambda}{n}}.\,Z} - \frac{dz}{z^{\frac{1-\lambda}{n}+1}.\,Z};$$

hic vltimi membri integrale manifeſto eſt certa poteſtas ipſius z, quam ſaltem per quadraturas exhibere licet; penultimum vero membrum ob

$$Z = q = \frac{p}{y^\lambda}, \text{ abit in } \frac{dv\,y^\lambda}{p\,v.\,z^{\frac{1-\lambda}{n}}}$$

et ſubſtituto pro z eius valore $\frac{v}{y^n}$, transformatur in

$$\frac{y\,dv}{p.\,v^{\frac{1-\lambda}{n}+1}},$$

quare

quare primo membro per reductionem integrando, consequimur

$$n x . v^{\frac{\lambda-1}{n}} - \int (\lambda-1) x . v^{\frac{\lambda-1}{n} - 1} d v = \int \frac{y\, d v .\, v^{\frac{\lambda-1}{n} - 1}}{p}$$

$$- \int \frac{d z .\, z^{\frac{\lambda-1}{n} - 1}}{Z},$$

atque hinc colligitur

$$\int d v .\, v^{\frac{\lambda-1}{n} - 1} ((\lambda-1) x + \tfrac{y}{p}) = n x . v^{\frac{\lambda-1}{n}} + \int \frac{d z . z^{\frac{\lambda-1}{n} - 1}}{Z},$$

quum igitur sit $d v = d \Omega$, patet formulam propositam $d \Omega$ integrabilem reddi, si multiplicetur per

$$(\Omega)^{\frac{\lambda-1}{n} - 1} ((\lambda - 1) x + \tfrac{y}{p}),$$

qui idcirco est multiplicator quaesitus.

34. Quum in problemate penultimo formulae $d \Omega$ multiplicator sit

$$\frac{p x - (\lambda + 1) y}{(\Omega)^{\frac{\lambda+1}{n} + 1}}, \text{ huius formulae } \frac{d \Omega}{(\Omega)^{\frac{\lambda+1}{n} + 1}},$$

quae etiam est verum differentiale, quippe cuius integrale est, $\frac{n}{\lambda+1} . \frac{1}{(\Omega)^{\frac{\lambda+1}{n}}}$, multiplicator erit $p x - (\lambda + 1) y$, simili modo quum in vltimo problemate formulae $d \Omega$ multiplicator sit inuentus $(\Omega)^{\frac{\lambda-1}{n} - 1} ((\lambda - 1) x + \frac{y}{p})$; erit huius formulae

quippe

$d\Omega\,(\Omega)^{\frac{\lambda-1}{n}-1}$, quae etiam est verum differentiale, quippe cuius integrale est $\frac{n}{\lambda-1}.(\Omega)^{\frac{\lambda-1}{n}}$, multiplicator erit $(\lambda-1)\,x+\frac{y}{p}$; hi duo casus ob simplicitatem multiplicatoris inprimis notatu digni videntur, ex quo operae pretium erit, in genere omnes formulas differentiales inuestigare, quibus talis multiplicator conueniat, quem in finem hoc Lemma praemittimus.

Lemma.

35. Si formulae differentialis $d\Omega$ multiplicator fuerit V, tum vicissim formulae differentialis dV multiplicator erit Ω, quum enim sit

$$\int \Omega\, dV = V\,\Omega - \int V.\, d\Omega,$$

quoniam formula $V\,d\Omega$ per hypothesin est integrabilis; necesse est, etiam formulam $\int \Omega\, dV$ esse integrabilem.

Problema.

36. Inuenire omnes formulas differentiales $d\Omega$, quibus conueniat multiplicator $\alpha y + p x$, denotante α numerum quemcunque.

Solutio.

Quum ob $dy = p\,dx$, fit

$$d(\alpha y + p x) = (\alpha + 1)\,p\,dx + x\,dp,$$

huius formulae multiplicatorem oportet esse Ω, ex

 qua

qua conditione quantitatem Ω determinare licet. Sit igitur propofita haec formula differentialis

$$(\alpha + 1) p\, dx + x\, dp,$$

pro qua habetur multiplicator

$M = 1$ eritque $v = \alpha y + p x$,

alter vero multiplicator erit

$N = x^{\alpha}$, fietque tum $u = p.\, x^{\alpha + 1}$,

quare fi Z denotet functionem quamcunque binarum variabilium

$$v = \alpha y + p x \quad \text{et} \quad u = p.\, x^{\alpha + 1},$$

in genere multiplicator noftrae formulae erit,

$$M \left(\frac{dZ}{dv}\right) + N \left(\frac{dZ}{du}\right),$$

quamobrem, quum fit

$$\Omega = \left(\frac{dZ}{dv}\right) + x^{\alpha} \left(\frac{dZ}{du}\right),$$

eius differentiale $d\Omega$ continebit omnes formulas differentiales, quibus conuenit multiplicator $\alpha y + p x$.

37. Si pro Z fumatur functio quaecunque ipfius u, erit

$$\left(\frac{dZ}{dv}\right) = 0$$

et $\left(\frac{dZ}{du}\right)$ erit functio quaedam ipfius u quae fit $f : u$, hinc noftrum erit Ω

$$x^{\alpha}.\, f : u = x^{\alpha} f : x^{\alpha + 1},$$

quae forma probe conuenit cum problemate §. 32. vbi multiplicator erat $p x - (\lambda + 1) y$, ita, vt fit $\alpha = -(\lambda + 1)$. Deinde quod hic eft $d\Omega$,

ibi

ibi erat $\dfrac{dv}{v^{\frac{\lambda+1}{n}}+1}$,

ficque quod hic eft Ω;

ibi erat $\dfrac{n}{\lambda+1} \cdot \dfrac{1}{v^{\frac{\lambda+1}{n}}}$,

at vero ibi erat

$$v = x^n Q = x^n \Phi : \frac{p}{x^\lambda},$$

ita vt inde fit

$$\frac{1}{v^{\frac{\lambda+1}{n}}} = x^{-\lambda+1} \cdot \Delta : \left(\frac{p}{x^\lambda}\right),$$

quam formam in illa contineri, fatis manifeftum eft. Hinc ergo patet illud problema effe cafum maxime particularem huius problematis, huiusque folutionem infinities latius patere.

Problema.

38. Inuenire omnes formulas differentiales $d\Omega$, quibus conueniat multiplicator $\alpha x + \frac{y}{p}$.

Solutio.

Quum ob $dx = \frac{dy}{p}$, fit

$$d.(\alpha x + \tfrac{y}{p}) = (\alpha + 1)\tfrac{dy}{p} - \tfrac{y\,dp}{pp},$$

confideremus hoc ipfum differentiale tanquam formulam propofitam, cuius multiplicator Ω fit inveftigandus, et quum primus multiplicator fit $M = 1$, erit

erit $v = x x + \frac{y}{p}$, alter autem multiplicator colligitur $N = y^{\alpha}$, ex quo fit $u = \frac{y^{\alpha+1}}{p}$, quare ſi Z denotet functionem quamcunque binarum variabilium

$$v = a x + \frac{y}{p} \text{ et } u = \frac{y^{\alpha+1}}{p},$$

expreſſio generalis pro multiplicatore quaeſito erit

$$\Omega = \left(\frac{dZ}{dv}\right) + y^{\alpha}\left(\frac{dZ}{du}\right),$$

vbi notandum, ſi pro Z ſumatur tantum functio vnicae variabilis u, tum hanc ſolutionem ad caſum problematis §. 33. perducere.

39. Quae hactenus de inueſtigatione multiplicatorum ſunt tradita, inſignem praeſtant vſum in reſolutione aequationum differentialium ſecundi gradus, quum enim ob $dy = p\,dx$, omnes aequationes huius ordinis ad hanc formam redigere liceat,

$$P\,dp + Q\,dx + R\,dy = 0,$$

manifeſtum eſt, ſi vnus huius formulae multiplicator innoteſcat; tum ſtatim aequationem ſemel integratam, quae erit differentialis primi ordinis, obtineri, quam deinceps ſecundum praecepta cognita tractari conueniet; at vero ſi bini eius formulae multiplicatores fuerint cogniti, tum ſtatim aequationem finitam ſeu bis integratam elicere licebit, ita vt non opus ſit integratione repetita, quam operationem ſequenti problemate doceamus.

Pro-

Problema.

40. Propofita aequatione differentiali:

$$P\,dp + Q\,dx + R\,dy = 0,$$

fi eius duo multiplicatores innotefcant M et N, eius aequationem finitam bis integratam inuenire.

Solutio.

Quum M et N fint multiplicatores cogniti, ponamus:

$$M(P\,dp + Q\,dx + R\,dy) = dv$$

et $$N(P\,dp + Q\,dx + R\,dy) = du,$$

hinc habebuntur quantitates v et u, quomodocunque ex ternis variabilibus p, x et y conflatae. Ob priorem igitur multiplicatorem fiet $v = a$ et ob pofteriorem $u = b$, vbi a et b funt binae conftantes, vtraque integratione ingreffae; quum igitur duae habeantur aequationes finitae inter ternas variabiles x, y et p, fi inde p eliminetur, prodibit aequatio finita inter binas coordinatas x et y, vel quod eodem redit, inde determinari poterunt binae harum litterarum per tertiam.

41. Hanc methodum ex praecedentibus per plura exempla facile illuftrare liceret, verum inftar omnium nobis erit, problema initio huius differtationis commemoratum, quo linea curua aequatione inter binas coordinatas x et y exprimenda quaeritur, cuius radius ofculi aequetur formulae $n\sqrt{(xx+yy)}$.

Exemplum.

Propofita aequatione differentiali fecundi gradus

$$\frac{d p}{(1+p p)^{\frac{3}{2}}}-\frac{n d x}{\sqrt{(x x+y y)}}=0,$$

inuenire aequationem finitam inter x et y, ope duplicis multiplicatoris. Primum multiplicatorem iam fupra obferuauimus effe $x+p y$, vnde fit

$$d v=\frac{d p(x+p y)}{(1+p p)^{\frac{3}{2}}}-\frac{n(x d x+y d y)}{\sqrt{(x x+y y)}},$$

adeoque erit

$$v=\frac{p x-y}{\sqrt{(1+p p)}}-n \sqrt{(x x+y y)}=a.$$

Pro altero multiplicatore, quoniam is tam facile non liquet, tam eum, quam integrale inde natum fimul inueftigemus, ponamus primo $y=x z$, et ob $d y=p d x$ fiet

$$p d x=z d x+x d z \text{ hincque } \frac{d x}{x}=\frac{d z}{p-z},$$

quo valore fubftituto, formula noftra erit

$$\frac{d p}{(1+p p)^{\frac{3}{2}}}-\frac{n d z}{(p-z) \sqrt{(1+z z)}}=0,$$

quare pofito porro $z=\frac{p+q}{1-p q}$, vnde fit

$$p-z=-\frac{q(1+p p)}{1-p q}, \quad \sqrt{(1+z z)}=\frac{\sqrt{(1+p p)(1+q q)}}{1-p q},$$

et $d z=\frac{d p(1+q q)+d q(1+p p)}{(1-p q)^2}$

trans-

transformatur noſtra formula in hanc

$$\frac{dp}{(1+pp)^{\frac{3}{2}}}+\frac{n\,dp\,(1+qq)+n\,dq\,(1+pp)}{q\,(1+pp)^{\frac{3}{2}}\sqrt{(1+qq)}}$$

$$=\frac{(q\sqrt{(1+qq)}+n(1+qq))\,dp+n\,dq\,(1+pp)}{q\,(1+pp)^{\frac{3}{2}}\sqrt{(1+qq)}},$$

vel in

$$\frac{q+n\sqrt{(1+qq)}}{q\sqrt{(1+pp)}}\left(\frac{dp}{1+pp}+\frac{n\,dq}{q+n\sqrt{(1+qq)}}\right),$$

hinc igitur patet, pro altero multiplicatore ſumi debere

$$M=\frac{q\sqrt{(1+pp)}}{q+n\sqrt{(1+qq)}},$$

tumque fit

$$du=\frac{dp}{1+pp}+\frac{n\,dq}{q+n\sqrt{(1+qq)}}$$

quae formula eſt integrabilis fietque inde

$$u=\int\frac{dp}{1+pp}+\int\frac{n\,dq}{q+n\sqrt{(1+qq)}}=b$$

ex qua vel q per p, vel p per q determinabitur, tum vero quum ſit

$$z=\frac{p+q}{1-pq}=\frac{y}{x}$$

inde fit

$$y=\frac{(p+q)\,x}{1-pq},$$

qui valor in prima aequatione ſubſtitutus praebet

$$\frac{-qx\sqrt{(1+pp)}}{(1-pq)}-nx\frac{\sqrt{(1+pp)(1+qq)}}{1-pq}=a,$$

nunc

nunc igitur inueniri poteſt x, vnde colligitur

$$x = \frac{-a(1 - pq)}{(q + n\sqrt{(1 + qq)})\sqrt{(1 + pp)}},$$

ſicque quia datur p per q, vel q per p, etiam x eodem modo definitur, deinde vero erit

$$y = \frac{-a(p + q)}{(q + n\sqrt{(1 + qq)})\sqrt{(1 + pp)}}$$

quae eſt ſolutio completa problematis.

OBSER-