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Problematis cuiusdam Diophantei evolutio

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PROBLEMATIS CVIVSDAM DIOPHANTEI EVOLVTIO.

Auctore

L. EVLERO.

1.

Cum olim istud problema Diophanteum tractassem, quo quearebantur tres numeri, quo 1°. summa 2°. summa productorum ex binis et 3°. productum omnium sint numeri quadrati, solutio tantis difficultatibus implicata videbatur, vt huius generis problemata adhuc difficiliora vix aggredi essem ausus. Multo autem difficilius esse problema, cuius enodationem hic suspicio, nemo dubitabit, qui eius solutionem tentare voluerit. Problema autem hoc ita se habet:

Inuenire quatuor numeros eius indolis, vt 1°. summa singulorum, 2°. summa factorum ex binis 3°. summa factorum ex ternis, et 4°. productum omnium sint numeri quadrati.

Vel quod eodem redit

Inuenire aequationem biquadraticam huius formae: $x^4 - A x^3 + B x^2 - C x + D = 0$, *quae omnes suas radices*

radices habeat rationales, et cuius insuper singuli coëfficientes A, B, C, D *sint numeri quadrati.*

2. Non dubito fore plerosque, qui mirabuntur, me in huiusmodi quaestionibus euoluendis, quas nunc quidem summi Geometrae auersari videntur, operam consumere; verum equidem fateri cogor, me ex huiusmodi inuestigationibus tantundem fere voluptatis capere, quam ex profundissimis Geometriae sublimioris speculationibus. Ac si plurimum studii et laboris impendi in quaestionibus grauioribus euoluendis, huiusmodi variatio argumenti quandam mihi haud ingratam recreationem afferre solet. Ceterum Analysis sublimior tantum debet Methodo Diophanteae, vt nefas videatur eam penitus repudiare.

3. Problema igitur propositum aggressurus, primum obseruo, solutionem eius generalem frustra tentari; postquam enim pluribus modis calculum instituissem, ac semper in formulas nullo pacto extricabiles incidissem, agnoui vix quicquam praestari posse, nisi vires nostras in solutionem quandam particularem intendamus. Sequenti ergo modo quatuor numeros quaesitos constituo:

$$Mab, Mbc, Mcd, Mda$$

vbi etsi quinque litterae sunt inductae, tamen haec potio ista limitatione restringitur, vt productum primi in tertium aequale sit producto secundi in quartum: quae restrictio vtique in se non est necessaria,

 vixque

vixque dubitare licet, quin etiam eiusmodi quaterni numeri quaesito satisfaciant, in quibus haec conditio locum non habeat; verum equidem nullam adhuc viam detegere valui, qua huiusmodi solutiones elicere liceret.

4. Hac igitur numerorum quaesitorum forma constituta, quatuor conditiones praescriptae sequentes aequationes suppeditant:

I. $M(ab + bc + cd + da) = \text{Quadrato}$

II. $M^2(abbc + bccd + cdda + daab + 2abcd) = \text{Quadr.}$

III. $M^3(abbccd + abccdd + aabcdd + aabbcd) = \text{Quadr.}$

IV. $M^4 aabbccdd = \text{Quadr.}$

vbi postrema conditio iam sponte impletur, neque vero hinc concludere licet, limitationem supra inductam esse necessariam; cum eadem conditio aeque obtineretur, si quis quatuor numerorum insuper per numerum quadratum quemcunque multiplicaretur, quo pacto solutio ab omni restrictione liberaretur, sed tum reliquae aequationes nullo modo resolui possent.

5. Restrictio autem adhibita hoc commodi nobis largitur, vt tertia aequatio hanc formam induat

$$M\,abcd\,(ab + bc + cd + da) = \text{Quadr.}$$

vnde cum ob primam iam quadratum esse debeat haec forma

$$M(ab + bc + cd + da),$$

necesse

necesse est, vt hoc productum $abcd$ quadrato aequetur. Praeterea autem vt tam primae quam tertiae conditioni satisfiat, capi oportet

$$M = ab + bc + cd + da$$

vel si haec summa factorem habeat quadratum puta ff sufficiet sumi

$$M = \frac{ab + bc + cd + da}{ff},$$

siquidem per se manifestum est, solutionem semper ad numeros integros reduci posse.

6. Hinc iam ratio est perspicua, cur initio quatuor quaesitis numeris factorem communem M tribuerim; eo igitur rite definito, vt sit

$$M = ab + bc + cd + da \text{ vel } M = \frac{ab + bc + cd + da}{ff}$$

duae tantum supersunt conditiones, quas impleri oportet; alteram scilicet modo elicui, qua esse debet

$$abcd = \text{Quadrato}$$

alteram aequatio secunda suppeditat, quae postulat ob factorem M^2 iam quadratum, vt sit

$$abbc + bccd + acdd + aabd + 2abcd = \text{Quadr.}$$

quae in hanc formam redigitur:

$$(aa + cc)bd + ac(bb + dd) + 2abcd = \text{Quadr.}$$

seu $bd(aa + cc) + (b + d)^2 ac = \text{Quadr.}$

7. Tota ergo quaestio ad inuentionem huiusmodi quatuor numerorum a, b, c, d est perducta, vt binis modo memoratis conditionibus satisfiat; vbi notari conuenit, inter binos numeros a et c similem

 ratio-

rationem intercedere atque inter binos b et d; atque totum negotium a sola ratione tam inter a et c quam inter b et d pendere. Quare vt pro quauis solutione minimos numeros obtineamus, tam numeros a et c quam b et d primos inter se statui oportet. Si enim communem haberent diuisorem, eo sublato conditioni vtrique aeque satisfieret.

8. Quia euolutio posterioris aequationis praecipuas difficultates inuoluit, ab ea inchoandum esse arbitror, ac primo quidem obseruo, etiamsi ea duas rationes $a:c$ et $b:d$ contineat, neutram tamen arbitrio nostro relinqui; vnde inprimis inquirendum est, cuiusmodi rationes pro alterutra accipi debeant, vt forma nostra quadratum reddi possit. Quod quo facilius perspiciatur, consideremus casum, quo loco alterius rationis ratio dupla poneretur, sit ergo $b:a = 2:1$, et haec forma $2aa + 2cc + 9ac$ quadratum reddi deberet; quod autem nunquam fieri posse facile intelligitur. Posito enim $a = p + q$ et $c = p - q$, prodit haec forma $13pp - 5qq$ quae nullo modo vnquam quadratum exhibere potest; idem euenit si poneretur $b:d = 3:1$; vnde patet, nonnisi certas rationum species pro alterutra rationum $a:c$ et $b:d$ assumi posse; reliquas vero omnes ab hac inuestigatione excludi.

9. Statim autem patet inter rationes huic scopo accommodatas primum locum obtinere rationes quadraticas; sit igitur $b:d = pp:qq$, et formula nostra

$$ppqq(aa+cc)+ac(pp+qq)^2$$

aeque-

aequetur huic quadrato $ppqqaa \pm \frac{2m}{n} pqac + \frac{mm}{nn} cc$

vnde fit $nn(pp+qq)^2 a + nnppqqc = 2mnpqa + mmc$.

ideoque $\frac{a}{c} = \frac{mm - nnppqq}{nn(pp+qq)^2 - 2mnpq}$

vel fit $m = \pm kpq$, vt habeamus has formulas satisfacientes.

$\frac{b}{d} = \frac{pp}{qq}$ et $\frac{a}{c} = \frac{(kk - nn)ppqq}{nn(pp+qq)^2 \pm 2knppqq}$ existente $k > n$.

10. Euoluamus casus simpliciores numerorum k et n et habebimus aequationis nostrae sequentes resolutiones.

Si fuerit $\frac{b}{d} = \frac{pp}{qq}$ erit

I. $\frac{a}{c} = \frac{3ppqq}{(pp+qq)^2 \pm 4ppqq}$; II. $\frac{a}{c} = \frac{8ppqq}{(pp+qq)^2 \pm 6ppqq}$

III. $\frac{a}{c} = \frac{5ppqq}{4(pp+qq)^2 \pm 12ppqq}$; IV. $\frac{a}{c} = \frac{15ppqq}{(pp+qq)^2 \pm 8ppqq}$

V. $\frac{a}{c} = \frac{7ppqq}{9(pp+qq)^2 \pm 24ppqq}$; VI. $\frac{a}{c} = \frac{24ppqq}{(pp+qq)^2 \pm 10ppqq}$

VII. $\frac{a}{c} = \frac{21ppqq}{4(pp+qq)^2 \pm 20ppqq}$; VIII. $\frac{a}{c} = \frac{16ppqq}{9(pp+qq)^2 \pm 30ppqq}$

IX. $\frac{a}{c} = \frac{9ppqq}{16(pp+qq)^2 \pm 40ppqq}$ etc.

11. Si iam pro litteris k, n, p, q eiusmodi valores inueniri possent, vt productum ac seu haec expressio

$$n(kk - nn)(n(pp+qq)^2 \pm 2kppqq)$$

fieret numerus quadratus, haberetur solutio problematis propositi, siquidem tum ob $bd = ppqq$ etiam formula $abcd$ foret quadratum. Verum haec inuestigatio nimis est molesta, quam vt eam suscipi conueniat; ac si forte succederet, ad maximos numeros certe perduceret. Quare consultum erit etiam

 alias

alias rationes pro $\frac{b}{d}$ contemplari, quae quidem alteri conditioni scilicet

$$bd(aa+cc)+ac(b+d)^2 = \text{Quadr.}$$

conuenire queant. At ob similem rationem fractionum $\frac{b}{d}$ et $\frac{a}{c}$ omnes valores hic pro $\frac{a}{c}$ eruti etiam viciſſim pro $\frac{b}{d}$ assumi poterunt, vnde denuo nouae huius generit fractiones elicientur.

12. In genere quidem hic labor nimis foret taediosus, vnde casus primo simpliciores euoluam:

si $\frac{b}{d} = \frac{1}{1}$ erit $\frac{a}{c} = \frac{3}{8}; -\frac{4}{1}; \frac{4}{5}; \frac{5}{28}; -\frac{15}{4}; \frac{7}{12}; \frac{7}{65}; \frac{8}{33}$

si $\frac{b}{d} = \frac{4}{1}$ erit $\frac{a}{c} = \frac{3}{4}; \frac{12}{41}; \frac{32}{1}; \frac{32}{49}; \frac{5}{13}; \frac{5}{37}; -\frac{60}{7}; \frac{20}{19}$

si $\frac{b}{d} = \frac{9}{1}$ erit $\frac{a}{c} = \frac{27}{64}; \frac{27}{136}; \frac{36}{23}; \frac{36}{77}; \frac{45}{292}; \frac{108}{5}$

si $\frac{b}{d} = \frac{9}{4}$ erit $\frac{a}{c} = \frac{108}{35}; \frac{45}{61}; \frac{28}{73}; \frac{64}{49}; \frac{64}{289}$.

En ergo hic praeter expectationem duos casus, quibus pro a et c numeri quadrati prodierunt; vnde cum etiam b et d sint numeri quadrati duas iam sumus adepti problematis nostri solutiones.

13. En ergo duas problematis nostri solutiones; quarum prima ob $a = 64$; $b = 9$; $c = 40$, et $d = 4$ praebet:

$$M = 576 + 441 + 196 + 256 = 1469$$

sicque quatuor numeri quaesiti sunt

I. 1469. 196; II. 1469. 256; III. 1469. 441; IV. 1469; 576.

Altera ob $a = 64$; $b = 9$; $c = 289$; $d = 4$ dat

$$M = 576 + 2601 + 1156 + 256 = 4589$$

vnde

vnde alii quatuor numeri problemati ſatisfacientes ſunt

I. 4589. 256; II. 4589. 576; III. 4589. 1156; IV. 4589. 2601.

Has autem ſolutiones haud facile ex formula §. 11. data deriuare licuiſſet, etiamſi in ea contineantur.

14. Cum autem ſingulae fractiones pro $\frac{a}{c}$ inventae etiam pro $\frac{b}{d}$ vſurpari queant, euoluamus ſimpliciores, quae ſunt:

$$\frac{4}{3};\ \frac{5}{4};\ \frac{8}{3};\ \frac{12}{7};\ \frac{15}{5};\ \frac{20}{19};\ \frac{28}{5};\ \frac{32}{1};\ \frac{35}{8}\ \text{etc.}$$

Sit igitur primo $\frac{b}{d} = \frac{4}{3}$ et habebitur

$$12\,aa + 12\,cc + 49\,ac = \text{Quadr.}$$

cui ſatisfacit $\frac{a}{c} = 4$, ponatur ergo $\frac{a}{c} = 4 + x$

$$\begin{array}{l} \quad 192 + 96\,x + 12\,xx \\ + \ 12 \\ \quad 196 + 49\,x \\ \hline \quad 400 + 145\,x + 12\,xx = \square = (20 + xy)^2 \end{array}$$

ergo $145 + 12\,x = 40\,y + xyy$ et $x = \frac{145 - 40\,y}{yy - 12}$

hincque $\frac{a}{c} = \frac{4\,yy - 40\,y + 97}{yy - 12}$ ſeu poſito $y = \frac{m}{n}$

$$\frac{a}{c} = \frac{4\,mm - 40\,mn + 97\,nn}{mm - 12\,nn}$$

vnde ſequentes nouae fractiones idoneae ſimpliciores colliguntur

$$\frac{a}{c} = \frac{24}{1};\ \frac{37}{13};\ \frac{121}{24}.$$

15.

15. Statuatur simili modo $\frac{b}{d} = \frac{5}{4}$ fietque

$$20aa + 20cc + 81ac = \square$$

cui satisfacit $\frac{a}{c} = 1$ fit ergo $\frac{a}{c} = 1 + x$

$$\begin{array}{l} 20 + 40x + 20xx \\ 20 \\ 81 + 81x \\ \hline 121 + 121x + 20xx = \square = (11 + xy)^2 \end{array}$$

ergo $121 + 20x = 22y + xyy$ et $x = \frac{121 - 22y}{yy - 20}$

et $\frac{a}{c} = \frac{yy - 22y + 101}{yy - 20} = \frac{mm \pm 22mn + 101nn}{mm - 20nn}$

vnde elicitur $\frac{a}{c} = \frac{16}{5}$ ita vt sit $abcd$ quadratum.

16. Haec solutio nobis largitur quatuor numeros multo minores problemati satisfacientes. Cum enim habeamus:

$$a = 16,\ b = 5,\ c = 5,\ d = 4$$

erit factor communicis

$$M = \frac{80 + 25 + 20 + 64}{ff} = \frac{189}{ff},$$

vnde sumto $f = 3$ erit $M = 21$, et quatuor numeri problema soluentes erunt

I. 21. 20; II. 21, 25; III. 21 64, et IV. 21. 80.

quorum summa singulorum est $= 9.\ 21^2$

summa productorum ex binis $= 110^2.\ 21^2$

summa productorum ex ternis $= 4800^2\ 21^4$

Productum omnium $= 1600^2.\ 21^4$

ita

ita vt huius aequationis biquadraticae

$$x^4 - 9.21^2.x^3 + 110^2.21^2.xx - 4800^2.21^3x + 1600^2.24^4 = 0$$

radices sint

$$21.20;\ 21.25;\ 21.64;\ 21.80.$$

17. Ex cognita autem vna solutione, certa methodo aliae imo infinitae elici possunt; quod quo facilius ostendam, hac postrema solutione vtar, qua posito $\frac{b}{d} = \frac{5}{4}$ inuenimus in genere $\frac{a}{c} = \frac{yy - 22y + 101}{yy - 20}$ vnde vt $abcd$ fiat quadratum, reddi oportet hanc formam:

$$5(yy - 20)(yy - 22y + 101) = \text{Quadrato}$$

id quod euenit sumto $y = 5$. Statuatur ergo $y = z + 5$ et habebitur:

$$5(zz + 10z + 5)(zz - 12z + 16) = \square$$

seu $400 + 500z - 495zz - 10z^3 + 5z^4 = \square$

cui etiam satisfacit $z = 1$ et $y = 6$, vnde autem eadem solutio resultat.

18. Vt aliam solutionem eliciamus; fingamus radicem quadratam huius formae $20 + \frac{25}{2}z - \frac{521}{32}zz$, cuius quadratum

$$400 + 500z - 495zz - \frac{25.521}{32}z^3 + \frac{521^2}{32^2}z^4$$

illi formae aequatum praebet

$$\left(\frac{521^2}{32^2} - 5\right)z = \frac{25.521}{32} - 10$$

seu $z = \frac{32.12705}{266331} = \frac{32.1155}{24211} = \frac{32.105}{2201}$

 ideo-

ideoque $z = \frac{3360}{2201}$ et $y = \frac{14365}{2201}$, vnde pro a et c numeri enormes reſultant, quos euoluere operae non eſt pretium.

19. Vt autem plures ſolutiones deriuare liceat, ob caſum cognitum $z = 1$, ponamus $z = \frac{1}{1+v}$, et prodibit haec forma ad quadratum redigenda

$$\begin{array}{l} 400 + 1600v + 2400vv + 1600v^3 + 400v^4 \\ +500 + 1500v + 1500vv + 500v^3 \\ -495 - 990v + 495vv \\ -10 - 10v \\ +5 \\ \hline \end{array}$$

ſeu $400 + 2100v + 3405v^2 + 2100v^3 + 400v^4 = \square$

cuius radix poſita $= 20 + \frac{105}{2}v - 20vv$ dat

$$4205 - \frac{105^2}{4} + 4200v = 0$$

ſeu $v = -\frac{1159}{3360}$ et $1 + v = \frac{2201}{3360}$ vt ante.

Ob formam reciprocam erit etiam $v = -\frac{3360}{1159}$ et

$1 + v = -\frac{2201}{1159}$ et $z = -\frac{1159}{2201}$, hincque $y = \frac{9846}{2201}$

vnde autem non alia ſolutio obtinetur.

20. Quanquam autem hoc modo ex qualibet ſolutione aliae innumerae deduci poſſunt; tamen quia in primas caſu quaſi fortuito incidimus, methodus adhuc certa deſideratur, quae ad huius problematis ſolutionem perducat; cuius inuentio in analyſi Diophantea vtique maximi foret momenti. Verum antequam talem methodum expectare liceat, neceſſe videtur, vt natura huius formae

$$ac(xx + yy) + (a + c)^2 xy$$

ad

ad quadratum reducendae accuratius inuestigetur, et rationes pro $a:c$ assumendae, quibus resolutio succedit, explorentur, vnde hanc quaestionem perscrutandam propono.

Inuenire omnes valores idoneos pro ratione $a:c$ *substituendos, vt haec expressio:*

$$ac(xx+yy)+xy(aa+cc)+2acxy$$

quadrato aequalis reddi possit.

21. Ex superioribus iam satis liquet, rationem $a:c$ neutiquam pro lubitu accipi posse, sed eam certis conditionibus esse adstrictam, quas potissimum determinari oportet. Ad has conditiones explorandas statuamus:

$$ac(xx+yy)+xy(aa+cc)+2acxy=zz$$

quam aequationem in sequentes formas transfundere licet:

I. $(aa+cc)(xx+yy)=(a+c)^2(x+y)^2-2zz$

II. $(aa+4ac+cc)(xx+4xy+yy)=6zz+(a-c)^2(x-y)^2$

III. $(aa+cc)(xx+4xy+yy)=2zz+(a-c)^2(x+y)^2$

IV. $(aa+4ac)(xx+yy)=2zz+(a+c)^2(x+y)^2$.

22. Cum iam ex prima forma intelligamus, formulam $aa+cc$ factorem esse numeri huius formae $tt-2zz$, qui, vti constat, alios non admittit diuisores, nisi qui ipsi sint vel huius formae $AA-2BB$ vel huius $2AA-BB$, sequitur numerum $aa+cc$ in alterutra harum formarum contineri debere. Ex tertia autem forma intelligitur,

 eundem

eundem numerum $aa + cc$, cum sit diuisor formae $2zz + tt$, etiam in forma $2AA + BB$ contineri debere. Iam vero numeri formae $2AA - BB$ vel $AA - 2BB$ praeter binarium alios non habent diuisores primos, nisi qui in forma $8n \pm 1$ contineantur, et numeri formae $2AA + BB$ alios non habent diuisores primos praeter binarium, nisi qui vel in hac forma $8n + 1$ vel $8n + 3$ contineantur. Ex quo concluditur haec conditio, vt numerus $aa + cc$ alios praeter binarium non habeat diuisores primos, nisi qui sint formae $8n + 1$.

23. Simili modo cum altera formula $aa + 4ac + cc$ sit diuisor formae $6zz + tt$, quae alios diuisores praeter 2 et 3 non admittit primos, nisi qui in aliqua harum formularum:

$$24n + 1,\ 24n + 5,\ 24n + 7,\ 24n + 11$$

contineantur; tum vero quia eadem formula $aa + 4ac + cc$ etiam est diuisor formae $2zz + tt$, ea praeter 2 alios non admittit diuisores primos, nisi qui in alterutra harum formarum $8n + 1$ vel $8n + 3$ contineantur. Ex quibus coniunctis sequitur numerum $aa + 4ac + cc$ praeter 2 et 3 alios diuisores primos habere non posse, nisi qui contineantur vel in hac formula $24n + 1$ vel hac $24n + 11$.

24. Hinc e valoribus rationis $a:c$ primum omnes ii excluduntur, quibus numerus $aa + cc$, haberet diuisorem primum formae $8n + 5$; siquidem reliquae formae ineptae $8n + 3$ et $8n + 7$ sponte

ſponte excluduntur, propterea quod ſumma duorum quadratorum $aa + cc$ per tales numeros nunquam diuiſibilis exiſtit. Deinde etiam ii valores rationis $a : c$ excluduntur, quibus numerus $aa + 4ac + cc$, qui per ſe praeter 2 et 3 alios habere nequit diuiſores, niſi qui ſint huius formae $12n + 1$ vel huius formae $12n + 11$; haberet diuiſorem vel huius formae $24n + 13$ vel huius $24n + 23$. Quocirca ex rationibus pro $a : c$ adhibendis primo expungi debent omnes eae, quibus numerus $aa + cc$ diuidi poteſt per numerum primum formae $8n + 5$, deinde etiam eae, quibus numerus $aa + 4ac + cc$ admitteret diuiſorem formae $24n + 13$ vel $24n + 23$.

25 Quando autem ratio $a : c$ ita eſt comparata, vt numerus $aa + cc$ nullum habeat diuiſorem formae $8n + 5$; tum viciſſim certum eſt, eundem numerum tam in hac forma $2AA - BB$ quam hac $2AA + BB$ contineri. Ac ſi quoque numerus $aa + 4ac + cc$ nullum habet diuiſorem formae $24n + 13$ vel $24n + 23$; tum perinde certum eſt, eundem numerum tam in hac forma $2AA + BB$ quam iſta $6AA + BB$ contineri. Hac duplici regula obſeruata facili negotio omnes rationes, quas loco $a : c$ aſſumi non licet, excluduntur.

26. Facta autem hac excluſione, pro fractione $\frac{a}{c}$ ſequentes valores ſunt relicti:

$\frac{1}{1}$, $\frac{4}{1}$, $\frac{4}{3}$, $\frac{5}{4}$, $\frac{8}{3}$, $\frac{8}{7}$, $\frac{9}{1}$, $\frac{9}{4}$, $\frac{11}{4}$, $\frac{12}{5}$, $\frac{12}{7}$, $\frac{13}{5}$, $\frac{13}{12}$,
$\frac{15}{7}$, $\frac{15}{8}$, $\frac{16}{1}$, $\frac{16}{5}$, $\frac{16}{9}$, $\frac{16}{13}$, $\frac{17}{12}$, $\frac{18}{8}$, $\frac{19}{11}$, $\frac{20}{1}$, $\frac{20}{3}$, $\frac{20}{7}$, $\frac{20}{11}$,
$\frac{20}{13}$, $\frac{20}{19}$, $\frac{21}{5}$, $\frac{21}{20}$, $\frac{23}{8}$, $\frac{24}{1}$, $\frac{24}{5}$, $\frac{24}{7}$, $\frac{24}{11}$, $\frac{24}{19}$, $\frac{25}{1}$, $\frac{25}{4}$
$\frac{25}{9}$, $\frac{25}{12}$, $\frac{25}{16}$, $\frac{25}{17}$, $\frac{27}{11}$, $\frac{27}{20}$, $\frac{28}{5}$, $\frac{28}{13}$, $\frac{28}{15}$, $\frac{28}{23}$, $\frac{29}{5}$

vbi obseruari conuenit, reliquas rationes omnes frustra adhibitum iri; num autem hae omnes post exclusiones expositas relictae succedant; quaestio est maximi momenti, quae vix decidi posse videtur.

27. Hic prima ratio in praecedentibus nondum inuenta, est $\frac{8}{7}$, quae igitur an solutionem quaestionis admittat, videamus. Fieri nempe oportet:

$$56(xx+yy)+225\,xy=\square.$$

Ponatur $x=p+q$ et $y=p-q$, vt prodeat haec forma:

$$337\,pp-113\,qq=\square$$

quod an fieri possit, facilius exploratur, quam ex forma praecedente: satisfaciunt autem hi valores minimi $p=3$, et $q=4$, vnde colligitur $x=7$ et $y=-1$, seu $\frac{x}{y}=-7$, statuatur ergo $\frac{x}{y}=-\frac{7+v}{1}$, et prodit:

$$1225-559\,v+56\,vv=\square$$

vnde colligitur $v=\frac{70t-559}{tt-56}$ et $\frac{x}{y}=-\frac{7tt+70t-167}{tt-56}$, seu $\frac{x}{y}=\frac{7tt+70t+167}{56-tt}=\frac{7mm-14mn-nn}{20nn+12mn-mm}$.

28. Cum deinde etiam alios plures casus examinassem, inueni negotium semper succedere; ex quo asseuerare vix dubito, omnes istas fractiones post binas exclusiones ante memoratas relictas semper ita esse comparatas, vt loco rationis $a:c$ positae aequationem

$$ac(xx+yy)+(a+c)^2\,xy=\square$$

reso-

resolubilem reddant. Nunc igitur omnino operae foret pretium in indolem harum fractionum accuratius inquirere, earumque verum characterem indagare, quo eae ab omnibus reliquis fractionibus distinguuntur. Primo quidem patet, in iis omnes fractiones huius formae $\frac{pp}{qq}$ occurrere, quomodo autem reliquarum indoles sit comparata, altioris videtur indaginis.

29. Videamus autem, quomodo in genere numeri a et c comparati esse debeant, vt $aa+cc$ obtineat formam $AA-2BB$. Posito autem

$$aa+cc=AA-2BB \text{ erit } AA-aa=cc+2BB$$

ideoque tam $A+a$ quam $A-a$, vtpote diuisores formae $cc+2BB$, eiusdem formae numeri esse debent, vnde posito

$$A+a=pp+2qq, \text{ et } A-a=rr+2ss$$

fit $A=\frac{pp+2qq+rr+2ss}{2}$ et $a=\frac{pp+2qq-rr-2ss}{2}$

et ob $cc+2BB=(pp+2pp)(rr+2ss)$ erit

$$c=2qs+pr \text{ et } B=ps-qr.$$

Quocirca conditio praescripta impletur sumendo

$$a=pp-rr+2qq-2ss \text{ et } c=2pr+4qs$$

vnde fit

$$aa+cc=(pp+rr)^2+4(qq+ss)^2+4(pp-rr)(qq-ss)+16pqrs$$

quae forma non solum est

$$=(pp+rr+2qq+2ss)^2-2(2ps-2qr)^2$$

sed etiam

$$=(pp+rr-2qq-2ss)^2+2(2pq+2qs)^2.$$

Vnde

Vnde tam in hac forma $AA - 2BB$ quam ista $AA + 2BB$ continetur.

30. Euoluamus simili modo alteram conditionem, quae postulat

$$aa + 4ac + cc = AA + 2BB,$$

et cum fiat

$$(a+2c)^2 - 3cc = AA + 2BB \text{ seu } (a+2c)^2 - AA = 2BB + 3cc$$

debet esse:

$$a + 2c + A = 2tt + 3uu \text{ et } a + 2c - A = xx + 6yy$$

ergo $a + 2c = \frac{2tt + 3uu + xx + 6yy}{2}$

tum vero ob $2BB + 3cc = (2tt + 3uu)(xx + 6yy)$ fit

$$B = tx - 3uy \text{ et } c = ux + 2ty, \text{ ideoque}$$

$$a = 2tt - 8ty + 6yy + 3uu - 4ux + xx$$

seu $a = 2(t-y)(t-3y) + (u-x)(3u-x)$

et $c = 2ux + 4ty$.

Vel sit $t = y + v$ et $x = u - z$ vt fiat

$$a = 2v(y - 2v) + z(z + 2u)$$

$$c = 4y(y + v) + 2u(u - z)$$

hocque modo simul alteri conditioni, quae esse debet $aa + 4ac + cc = 6AA + BB$, satisfit.

31. Quo igitur vtrique conditioni satisfiat, necesse est, vt ambo numeri a et c simul in sequentibus binis formulis contineantur:

$$a = (p-r)(p+r) + 2(q-s)(q+s); \quad c = 2pr + 4qs$$

$$a = (u-x)(3u-x) + 2(t-y)(t-3y); \quad c = 2ux + 4ty.$$

Noua

Noua ergo hinc naſcitur quaeſtio, quomodo hae binae geminae formulae ad eundem valorem ſint reducendae; ad quod neceſſe eſt, vt huic aequalitati ſatisfiat:

$$(ux+2ty)(pp-rr+2qq-2ss)=$$
$$(pr+2qs)(3uu-4ux+xx+2tt-8ty+6yy)$$

quoniam totum negotium in ratione $a:c$ verſatur.

Aliud Problema Diophanteum.

Inuenire quotcunque numeros, quorum quilibet in ſummam reliquorum multiplicatus producat numerum quadratum.

32. Sint numeri quaeſiti p, q, r, s etc. eorumque ſumma $=S$; requiritur ergo, vt omnes hae formulae:

$$p(S-p),\ q(S-q),\ (S-r),\ s(S-s) \text{ etc.}$$

ſint quadrata, quae cum ſint ſimiles, ſufficit pro vna poſuiſſe $p(S-p)=ffpp$, vnde fit $p=\frac{S}{1+ff}$. Quare numeri quaeſiti erunt

$$\frac{S}{1+ff},\ \frac{S}{1+gg},\ \frac{S}{1+hh},\ \frac{S}{1+kk} \text{ etc.}$$

dummodo eorum ſumma fiat $=S$; ſicque problema huc redit, vt quaerantur numeri quotcunque f, g, h, k etc. ita comparati, vt fiat

$$\frac{1}{1+ff}+\frac{1}{1+gg}+\frac{1}{1+hh}+\frac{1}{1+kk}+\text{etc.}=1,$$

33. Statuamus, quoniam hi numeri plerumque ſunt fracti,

 $f=\frac{a}{\alpha}$,

$$f = \frac{a}{\alpha},\quad g = \frac{b}{\beta},\quad h = \frac{c}{\gamma},\quad k = \frac{d}{\delta} \text{ etc.}$$

et quaestio huc redit, vt aliquot fractiones huiusmodi

$$\frac{\alpha\alpha}{aa+\alpha\alpha},\quad \frac{\beta\beta}{bb+\beta\beta},\quad \frac{\gamma\gamma}{cc+\gamma\gamma} \text{ etc.}$$

inueniantur, quorum summa vnitati aequetur; vbi obseruo, quemlibet denominatorem esse summam duorum quadratorum. Quodsi ergo talis denominator sit numerus primus, ex eo duae tantum eiusmodi nascuntur fractiones, scilicet

$$\frac{\alpha\alpha}{aa+\alpha\alpha} \text{ et } \frac{aa}{aa+\alpha\alpha},$$

quarum summa cum vnitati aequetur, euidens est, ambas simul capi non posse, nisi quaestio de duobus numeris instituatur, quorum alter in alterum ductus praebeat quadratum. Tum enim ob

$$\frac{aa}{aa+\alpha\alpha} + \frac{\alpha\alpha}{aa+\alpha\alpha} = 1,$$

sumto S. pro lubitu numeri satisfacientes erunt Maa et $M\alpha\alpha$, qui propterea casus nullam habet difficultatem.

34. Quando autem plures duobus numeri sunt inuestigandi, qui problemati conueniant; necesse est vt etiam casus, quibus denominatores sunt numeri compositi, euoluantur; siquidem inde plures fractiones huius indolis formari possunt; quarum cum binae itidem vnitati aequentur, sequente modo eas repraesentabo.

Denominator $D = (aa+\alpha\alpha)(bb+\beta\beta)$;

$$\frac{(ab-\alpha\beta)^2}{D} \qquad \frac{(a\beta+\alpha b)^2}{D}$$
$$\frac{(a\beta-\alpha b)^2}{D} \qquad \frac{(ab+\alpha\beta)^2}{D}$$

Deno-

Denominator $D = (aa + \alpha\alpha)(bb + \beta\beta)(cc + \gamma\gamma)$

$$\frac{(a\beta c + \alpha bc - ab\gamma + \alpha\beta\gamma)^2}{D} \quad \Bigg| \quad \frac{(a\beta\gamma + \alpha b\gamma + abc - \alpha\beta c)^2}{D}$$

$$\frac{(a\beta\gamma + \alpha b\gamma - abc + \alpha\beta c)^2}{D} \quad \Bigg| \quad \frac{(a\beta c + \alpha bc + ab\gamma - \alpha\beta\gamma)^2}{D}$$

$$\frac{(abc + \alpha\beta c - a\beta\gamma + \alpha b\gamma)^2}{D} \quad \Bigg| \quad \frac{(ab\gamma + \alpha\beta\gamma + a\beta c - \alpha bc)^2}{D}$$

$$\frac{(ab\gamma + \alpha\beta\gamma - a\beta c + \alpha bc)^2}{D} \quad \Bigg| \quad \frac{(abc + \alpha\beta c + a\beta\gamma - \alpha b\gamma)^2}{D}$$

35. Circa ordinem ſecundum annotaſſe iuuabit, eſſe

$$\frac{(ab - \alpha\beta)^2}{D} + \frac{(a\beta - \alpha b)^2}{D} = 1 - \frac{4ab\alpha\beta}{D} \quad \text{et}$$

$$\frac{(ab - \alpha\beta)^2}{D} + \frac{(ab + \alpha\beta)^2}{D} = -1 + \frac{aabb + \alpha\alpha\beta\beta - aa\beta\beta - \alpha\alpha bb}{D}.$$

Deinde in ordine tertio, ſi quatuor partes prioris columnae inuicem addantur, ſumma erit

$$2 - \frac{8(aa - \alpha\alpha)b\beta c\gamma}{(aa + \alpha\alpha)(bb + \beta\beta)(cc + \gamma\gamma)}.$$

Hinc non contemnenda ſubſidia peti poterunt pro quauis numerorum quaeſitorum multitudine, dum, ſi ſolutio in genere tentaretur, inſignes difficultates occurrerent. Quoniam igitur caſus duorum numerorum per ſe eſt perſpicuus, a caſu trium exordiar inde ad quatuor progreſſurus.

Caſus trium numeror.

36. Ponamus pro tribus numeris quaeſitis has fractiones:

$$\frac{aa}{aa + \alpha\alpha};\ \frac{(ab - \alpha\beta)^2}{(aa + \alpha\alpha)(bb + \beta\beta)};\ \frac{(a\beta - \alpha b)^2}{(aa + \alpha\alpha)(bb + \beta\beta)}$$

quarum ſumma eſt

$$\frac{aa}{aa + \alpha\alpha} + 1 - \frac{4a\alpha b\beta}{(aa + \alpha\alpha)(bb + \beta\beta)} \quad \text{vnitati aequanda}$$

 vnde

vnde fit

$$aa(bb+\beta\beta) = 4\,a\alpha b\beta \text{ hincque } \frac{a}{\alpha} = \frac{4b\beta}{bb+\beta\beta}.$$

Quare sumtis $a = 4b\beta$ et $\alpha = bb+\beta\beta$, numeri quaesiti ad integros perducti erunt:

$$aa(bb+\beta\beta);\ (ab-\alpha\beta)^2;\ (a\beta-\alpha b)^2.$$

Iam vero est $ab-\alpha\beta = 3bb\beta-\beta^3 = \beta(2bb-\beta\beta)$ et $a\beta-\alpha b = 3b\beta\beta - b^3 = b(3\beta\beta-bb)$.

Consequenter habebimus has formulas.

$$16bb\beta\beta(bb+\beta\beta);\ \beta\beta(3bb-\beta\beta)^2;\ bb(3\beta\beta-bb)^2$$

quarum quaelibet in summam reliquarum ducta producit quadratum.

37. Euoluamus hinc solutiones simpliciores, ponendo numeros minores loco b et β, quorum tantum ratio spectatur, ac si ambo sint impares, numeri quaesiti per 4 deprimantur:

		numeri quaesiti		
I.	$\frac{b}{\beta} = \frac{1}{1}$;	$p = 8$	; $q = 1$	; $r = 1$
II.	$\frac{b}{\beta} = \frac{2}{1}$;	$p = 320$	; $q = 121$	; $r = 4$
III.	$\frac{b}{\beta} = \frac{3}{1}$;	$p = 360$	; $q = 169$	; $r = 81$
IV.	$\frac{b}{\beta} = \frac{3}{2}$;	$p = 7488$	; $q = 2116$	; $r = 81$
V.	$\frac{b}{\beta} = \frac{4}{1}$;	$p = 4352$	; $q = 2209$	; $r = 2704$
VI.	$\frac{b}{\beta} = \frac{4}{3}$;	$p = 57600$	; $q = 13689$	; $r = 1936$
VII.	$\frac{b}{\beta} = \frac{5}{1}$;	$p = 2600$	; $q = 1369$	; $r = 12100$

38.

38. Aliae ſolutiones reperientur ex his formulis:

$$\frac{aa}{aa+\alpha\alpha};\ \frac{(ab-\alpha\beta)^2}{(aa+\alpha\alpha)(bb+\beta\beta)};\ \frac{(ab+\alpha\beta)^2}{(aa+\alpha\alpha)(bb+\beta\beta)}$$

quarum ſumma eſt

$$\frac{aa}{aa+\alpha\alpha}+1+\frac{aabb+\alpha\alpha\beta\beta-aa\beta\beta-\alpha\alpha bb}{(aa+\alpha\alpha)(bb+\beta\beta)}$$

quae cum vnitati aequari debeat, fiet

$2aabb+\alpha\alpha\beta\beta-\alpha\alpha bb=0$, hinc $\frac{aa}{\alpha\alpha}=\frac{bb-\beta\beta}{2bb}$

ſeu $\frac{bb}{\beta\beta}=\frac{\alpha\alpha}{\alpha\alpha-2aa}$ vnde $b=\alpha$ et $\beta=\sqrt{(\alpha\alpha-aa)}$.

Capiatur ergo:

$a=2mn;\ \alpha=mm+2nn;\ b=mm+2nn;\ \beta=mm-2nn$

eruntque tres numeri quaeſiti

$p=8\,mm\,nn(m^4+4n^4)$

$q=(mm+2mn-2nn)^2(mm+2nn)^2$

$r=(mm-2mn-2nn)^2(mm+2nn)^2$

vnde ſequentes ſolutiones deducuntur

I.	$p=40$;	$q=9$;	$r=81$
II.	$p=8.9.85$;	$q=121.169$;	$r=121$
III.	$p=8.4.65$;	$q=81.121$;	$r=81.9$
IV.	$p=8.36.145$;	$q=289.169$;	$r=289.121$
V.	$p=8.9.325$;	$q=361.529$;	$r=361.121$
VI.	$p=8.100.689$;	$q=1089.1369$;	$r=1089.9$
VII.	$p=8.16.1025$;	$q=1089.1521$;	$r=1089.529$
VIII.	$p=8.144.1105$;	$q=1681.2209$;	$r=1681.1$
IX.	$p=8.225.949$;	$q=1849.1369$;	$r=1849.529$.

39. Neque vero haec solutio generalis est putanda, sed potius innumerabiles aliae locum habent, quae in his geminis formulis non continentur. Pro generali enim solutione hanc aequationem resolui oporteret:

$$\frac{1}{1+xx}+\frac{1}{1+yy}+\frac{1}{1+zz}=1$$

vnde oritur $xxyyzz-xx-yy-zz-2=0$ hincque $zz=\frac{xx+yy+2}{xxyy-1}$, ita vt haec formula

$$(xxyy-1)(xx+yy+2)$$

in genere ad quadratum reduci debeat; quod quomodo sit efficiendum, non patet.

40. Interim ex solutione iam aliunde cognita ope huius formulae infinitae aliae elici possunt Diuidantur enim terni numeri inuenti veluti 40, 9, 81 per eorum summam 130, vt hae fractiones obtineantur:

$$\tfrac{4}{13};\quad \tfrac{9}{130};\quad \tfrac{81}{130},$$

quae cum generalibus comparatae praebent

$$x=\tfrac{3}{2};\quad y=\tfrac{11}{3};\quad z=\tfrac{7}{9},$$

quarum vna tantum $x=\frac{2}{3}$ pro cognita sumatur, pro binis reliquis vero haec aequatio resoluatur:

$$\tfrac{9}{4}yyzz-yy-zz-\tfrac{17}{4}=0 \quad \text{seu} \quad zz=\frac{4yy+17}{9yy-4}$$

vnde fit

$$(9yy-4)z=\sqrt{(9yy-4)(4yy+17)}.$$

Quia autem nouimus, satisfacere valorem $y=\frac{11}{3}$, statuamus $y=\frac{11+u}{3}$ sitque

3(9yy

$$3(9yy-4)z = \sqrt{(9.13+22u+uu)(49.13+88u+4uu)}$$

ita vt haec formula ad quadratum fit reducenda

$$273^2 + 22.1105u + 3041uu + 176u^3 + 4u^4$$

cuius radix fi ftatuatur $273 + \frac{85.11}{21}u \pm 2uu$ fit

$$\left(\frac{8.13.4489}{21^4} \mp 4.13.21\right)uu + 44\left(4 \mp \frac{85}{21}\right)u^3 = 0$$

et $u = -\frac{13(9978 \mp 9261}{11.21(84 \mp 85)}$ ficque

pro figno fuperiori $u = \frac{-13.283}{11.21}$ et $y = \frac{-1138}{693}$

pro figno inferiori $u = \frac{-1405}{2.1}$ et ob $y = +\frac{1138}{693}$

qui duo valores conueniunt et ob $y = \frac{1138}{693}$ fit

$$z = \sqrt{\frac{4.1138^2 + 17.693^2}{9.1138^2 - 4.693^2}} = \frac{3655}{8.11.40} = \frac{281}{345}.$$

vnde ternae fractiones prodeunt

$$\frac{4}{13};\quad \frac{480249}{13.136561};\quad \frac{57600}{136561}$$

quae in integris dant hos numeros:

$$\begin{aligned} p &= 4.136561 = 4.17.29.277 &&= 546244 \\ q &= 480249 = 693^2 &&= 480249 \\ r &= 13.57600 = 240^2 &&= 748800 \end{aligned}$$

hincque $p+q+r = 13.17.29.277 = 1775293$.

Hac ergo methodo folutiones particulares datae ad maiorem generalitatem euehuntur.

Cafus quatuor numerorum.

41. Statuamus quatuor fractiones:

$$\frac{aa}{aa+\alpha\alpha};\quad \frac{bb}{bb+\beta\beta};\quad \frac{(ab-\alpha\beta)^2}{(aa+\alpha\alpha)(bb+\beta\beta)};\quad \frac{(a\beta-\alpha b)^2}{(aa+\alpha\alpha)(bb+\beta\beta)}$$

quarum fumma eft

$$\frac{aa}{aa+\alpha\alpha} + \frac{bb}{bb+\beta\beta} + 1 - \frac{4a\alpha b\beta}{(aa+\alpha\alpha)(bb+\beta\beta)}$$

vnita-

vnitati aequanda; vnde fit:

$$2aabb + aa\beta\beta + \alpha\alpha bb = 4aa b\beta$$

ideoque $\frac{b}{\beta} = \frac{2a\alpha \pm \sqrt{(4aa\alpha\alpha - 2a^4 - aa\alpha\alpha)}}{2aa + \alpha\alpha}$

seu $\frac{b}{\beta} = \frac{2a\alpha \pm a\sqrt{(3\alpha\alpha - 2aa)}}{2aa + \alpha\alpha}$.

Quare litteras a et α ita accipi oportet, vt formula $3\alpha\alpha - 2aa$ quadratum euadat.

42. Hunc in finem ponamus:

$$\sqrt{(3\alpha\alpha - 2aa)} = \alpha + \frac{m}{n}(\alpha - a) \text{ fietque}$$

$$2mn\alpha + 2nna = 2mna + mm\alpha - mma.$$

Ergo $a = mm + 2mn - 2nn$

et $\alpha = mm + 2nn$

hinc $\alpha = a = -2mn + 4nn$ et

$$\sqrt{(3\alpha\alpha - 2aa)} = -mm + 4mn + 2nn.$$

Quocirca habebimus

vel $\frac{b}{\beta} = \frac{(mm + 2mn - 2nn)(3mm - 4mn + 2nn)}{2(mm + 2mn - nn)^2 + (mm + 2nn)^2} = \frac{mm + 2mn - 2nn}{nm + 4mn + 6nn}$

vel $\frac{b}{\beta} = \frac{(mm + 2mn - 2nn)(mm + 4mn + 6nn)}{2(mm + 2mn - 2nn)^2 + (mm + 2nn)^2} = \frac{mm + 2mn - 2nn}{3mm - 4mn + 2nn}$.

Tandem numeri quaesiti habebuntur

$$p = aa(bb + \beta\beta);\ q = bb(aa + \alpha\alpha);\ r = (ab - \alpha\beta)^2$$

$$s = (a\beta - \alpha b)^2.$$

43. Cum sit $\alpha = mm + 2nn$, loco α alii numeri assumi nequeunt, nisi qui sint vel primi huius formae $8m + 1$ seu $8m + 3$; vel ex huiusmodi primis compositi. Simpliciores cum numeris a

et

et $\sqrt{(3\alpha\alpha - 2aa)}$ ipſis reſpondentibus in ſequenti tabella exhibeo:

$a=$	1	3	9	11	11	17	17	19	19
$\alpha=$	1	1	11	1	13	11	13	11	23
$\gamma=$	1	5	1	19	5	25	23	29	5
$\beta=$	3	11	323	123	459	531	627	603	1419
$b=$	3	1	187	3	221	99	143	99	759
vel $b=$	1	11	209	41	351	649	741	737	989
vel $\beta=$	1	1	19	3	17	9	11	9	33
$b=$	1	1	11	1	13	11	13	11	23
vel $\beta=$	3	11	17	41	27	59	57	67	43
$b=$	1	1	11	1	13	11	13	11	23

44. Cum ergo in genere ſit:

$$a = mm + 2mn - 2nn;\quad b = mm + 2mn - 2nn$$
$$\alpha = mm + 2nn \qquad \beta = mm + 4mn + 6mn$$
$$\text{vel } \beta = 3mm - 4mn + 2nn$$

erit

$$ab - \alpha\beta = -8nn(m+n)^2 \text{ vel } = -2mm(m-2n)^2$$
$$a\beta - \alpha b = 4n(m+n)(mm + 2mn - 2nn)$$
$$\text{vel } = 2m(m-2n)(mm + 2mn - 2nn).$$

Item $aa + \alpha\alpha = 2mm(m+n)^2 + 2nn(m-2n)^2$

et $bb + \beta\beta = 2(m+n)^2(m+2n)^2 + 2nn(m+4n)^2$

vel $= 2mm(2m-n)^2 + 2(m-n)^2(m-2n)^2$

vnde in numeris ſequentes nanciſcimur ſolutiones:

I.	$p=1$;	$q=1$;	$r=0$;	$s=0$
II.	$p=5$;	$q=1$;	$r=2$;	$s=2$
III.	$p=61$;	$q=5$;	$r=512$;	$s=32$
IV.	$p=841$;	$q=61$;	$r=225.450$;	$s=450$
V.	$p=121.205$;	$q=121.101$;	$r=16.32$;	$s=121.32$
VI.	$p=121.289$;	$q=121.101$;	$r=25.50$;	$s=121.50$
VII.	$p=121.2305$;	$q=121.241$;	$r=576.1152$;	$s=121.1152$
VIII.	$p=169.229$;	$q=169.145$;	$r=9.18$;	$s=169.18$
IX.	$p=169.449$;	$q=169.145$;	$r=64.128$;	$s=169.128.$

45. Formulae generales autem ita se habebunt

vel

$$p=(mm(m+n)^2+nn(m-2n)^2)(mm+2mn-3nn)^2$$
$$q=((m+n)^2(m+2n)^2+nn(m+4n)^2))(mm+2mn-2nn)^2$$
$$r=8nn(m+n)^2(mm+2mn-2nn)^2$$
$$s=8nn(m+n)^2 \cdot 4nn(m+n)^2$$

vel

$$p=(mm(m+n)^2+nn(m-2n)^2(mm+2mn-2nn)^2$$
$$q=(mm(2m-n)^2+(m-n)^2(m-2n)^2)(mm+2mn-2nn)^2$$
$$r=2mm(m-2n)^2(mm+2mn-2nn)^2$$
$$s=2mm(m-2n)^2 mm(m-2n)^2.$$

Vtroque casu quatuor numeri p, q, r, s ita sunt comparati, vt quilibet in summam trium reliquorum ductus producat numerum quadratum. Quanquam autem hinc innumerabiles solutiones deriuare licet, haec solutio nonnisi pro maxime particulari est habenda.

46.

46. Solutio autem generaliter instruitur, ponendo in genere pro quaternis fractionibus:

$$\frac{1}{1+zz},\quad \frac{1}{1+yy},\quad \frac{1}{1+xx},\quad \frac{1}{1+vv}$$

quarum summa cum vnitati esse debeat aequalis, orietur haec aequatio

$$vvxxyyzz = \begin{matrix} vvxx+vvyy+vvzz+2vv+2xx+3 \\ +yyzz+xxzz+xxyy+2yy+2zz \end{matrix}$$

cuius autem resolutio maximis difficultatibus est implicata. Verum si ex iam inuentis solutionibus, pro binis litteris x et v idonei valores accipiuntur, praeter valores reliquarum y et z cognitos innumerabiles alii assignari poterunt.

47. Vt hoc exemplo ostendam, assumam solutionem secundam his fractionibus $\frac{1}{2}$, $\frac{1}{10}$, $\frac{1}{5}$, $\frac{1}{5}$ contentam indeque statuo $v=2$ et $x=3$, reliquas autem, quae hoc exemplo sunt $y=1$ et $z=2$ vt incognitas specto. Habebimus ergo hanc aequationem

$$36yyzz = yyzz + 15yy + 15zz + 65$$

seu $7yyzz = 3yy + 3zz + 13$

ex qua prodit $zz = \frac{3yy+13}{7yy-3}$, ita vt haec formula $\frac{3yy+13}{7yy-3}$ quadrato aequari debeat, quod duobus casibus $y=1$ et $y=2$ euenire nouimus. Iam $7yy-3$ in genere fit quadratum ponendo $y = \frac{mm+3}{mm+4m-3}$, qui in $3yy+13$ substitutus dat

$$16m^4 + 104m^3 + 148mm - 312m + 144 = \Box$$

 cuius

cuius radix posita $4mm + 13m + 12$ dat $m = -\frac{16}{5}$

at radix posita $4mm - 13m + 12$ dat $m = \frac{9}{16}$

vtrinque reperitur $y = \frac{283}{37}$ et $z = \frac{254}{273}$.

48. Quanquam autem hoc modo ex inuenta quauis solutione continuo alias nouas elicere licet; tamen sic mox ad numeros praegrandes peruenitur; quod eo maius est incommodum, cum aliunde solutiones multo simpliciores obtineri queant; id quod quidem nulla certa methodo, sed mero tentamine praestatur: Considerantur scilicet plures fractiones huius formae $\frac{aa}{aa + \alpha\alpha}$, ex quibus quippe quatuor eligi oportet, quarum summa vnitati aequetur: Ita sumtis fractionibus, quarum denominatores in 130 continentur:

$$\frac{1}{2},\ \frac{1}{5},\ \frac{1}{10},\ \frac{4}{13},\ \frac{1}{26},\ \frac{1}{65},\ \frac{16}{65},\ \frac{9}{130},\ \frac{49}{130},$$

$$\frac{4}{5},\ \frac{9}{10},\ \frac{9}{13},\ \frac{25}{26},\ \frac{64}{65},\ \frac{49}{65},\ \frac{121}{130},\ \frac{81}{130}$$

binarum $\frac{9}{130}$ et $\frac{49}{130}$ summa est $\frac{29}{65}$, huic addatur $\frac{16}{65}$, proditque $\frac{45}{65} = \frac{9}{13}$, quae cum $\frac{4}{13}$ producit vnitatem. Ita quatuor fractiones

$$\frac{4}{13},\ \frac{16}{65},\ \frac{9}{130},\ \frac{49}{130}$$

praebent hos numeros

$$p = 40,\ q = 32,\ r = 9,\ s = 49.$$

Alio modo fit

$$\frac{9}{130} + \frac{1}{5} = \frac{35}{130} = \frac{7}{26},\ \text{porro}\ \frac{7}{26} + \frac{1}{26} = \frac{4}{13},$$

quae cum $\frac{9}{13}$ dat vnitatem, vnde ex fractionibus

$$\frac{9}{130},\ \frac{1}{5},\ \frac{1}{26},\ \frac{9}{13}$$

nascuntur hi numeri:

$$p = 9;\ q = 26;\ r = 5;\ s = 90$$

qui

qui vtique multo sunt minores, quam superiores certa ratione inuenti, primis quidem ibi exceptis, qui ob aequales numeros excludendi videntur.

49. Simili modo posita summa $p+q+r+s=170$ reperiuntur duae solutiones:

I. $p=1$, $q=10$, $r=34$, $s=125$.

II. $p=10$, $q=17$, $r=45$, $s=98$;

summa numerorum 290 dat

$p=1$, $q=40$, $r=121$, $s=128$.

Hinc itaque patet, casu quasi fortuito multo simpliciores numeros problemati satisfacientes reperiri, atque adeo hac ratione non difficulter quinque numeri assignari possunt, vt quilibet per reliquorum summam multiplicatus praebeat numerum quadratum, cuiusmodi sunt:

1, 40, 45, 58, 145

et 32, 61, 98, 169, 250.

Hocque modo etiam plures numeros huius indolis detegere licet, ad quos inueniendos nulla certa methodus adhuc est explorata.

Appendix.

50. Si problemati modo tractato haec conditio adiungatur, vt singuli numeri esse debeant quadrati; quaestionis quasi natura immutatur, quae ita enunciabitur:

Inuenire quotcunque numeros quadratos, vt summa omnium quolibet imminuta fiat numerus quadratus.

Sint numeri quadrati quaesiti

A^2, B^2, C^2, D^2 etc.

quorum summa ponatur $= S$, fierique debet

$$S - A^2 = P^2, \; S - B^2 = Q^2, \; S - C^2 = R^2 \text{ etc.}$$

vnde patet, S esse summam eiusmodi binorum quadratorum, quae pluribus modis in bina quadrata se distribui patiatur; seu posito $S = xx + yy$, hanc duorum quadratorum summam indefinite in alia bina quadrata secari oportet, quod in genere ita praestatur:

$$S = \left(\frac{2fx + (ff-1)y}{ff+1}\right)^2 + \left(\frac{(ff-1)x - 2fy}{ff+1}\right)^2 = xx + yy.$$

51. Pro casu ergo trium quadratorum poni debet:

$$A = x; \; B = \frac{2fx - (ff-1)y}{ff+1} \text{ et } C = \frac{2gx - (gg-1)y}{gg+1}$$

et summa quadratorum tum ipsi $xx + yy$ aequari. Quod cum in genere difficulter praestetur, in solutionem particularem inquiramus ponendo $g = \frac{f+1}{f-1}$, vnde fit

$$C = \frac{(ff-1)x - 2fy}{ff+1},$$

et haec oritur aequatio:

$$xx + xx + yy - \frac{8f(ff-1)}{(ff+1)^2}xy = xx + yy,$$

ex qua sequitur

$$x = \frac{8f(ff-1)}{(ff+1)^2}y \text{ seu } x = 8f(ff-1) \text{ et } y = (ff+1)^2$$

hincque quadratorum quaesitorum radices in integris

A = 8

$$A = 8f(ff-1)(ff+1)$$
$$B = 2f(3f^4 - 10ff + 3) = 2f(3ff-1)(ff-3)$$
$$C = (ff-1)(f^4 - 14ff + 1) = (ff-1)(ff+4f+1)(ff-4f+1)$$

vnde si $f = 2$ sequuntur hi numeri

$$A = 16.\ 3.\ 5;\quad B = 4.\ 11.\ 1;\quad C = 3.\ 13.\ 3$$

seu $A = 240$; $\quad B = 44$; $\quad C = 117$.

52. Ad casum autem quatuor quadratorum progrediamur, quandoquidem tum problema fit difficillimum, vt solutio adeo simplicissima iam ad maximos numeros exsurgat. Faciamus ergo

$$A = x;\quad B = \frac{2fx - (ff-1)y}{1+ff};\quad C = \frac{(ff-1)x - 2fy}{1+ff};$$
$$D = \frac{2px - (pp-1)y}{pp+1}$$

et cum sit

$$BB + CC = xx + yy - \frac{4f(ff-1)xy}{(1+ff)^2},$$

posito breuitatis ergo $\frac{4f(ff-1)}{(ff+1)^2} = g$ prodit haec aequatio:

$$xx + \frac{4ppxx - 4p(pp-1)xy + (pp-1)^2yy}{(pp+1)^2} - 2gxy = 0 \text{ seu}$$

$$(pp-1)^2 yy = 2g(pp+1)^2 xy - 4ppxx + 4p(pp-1)xy - (pp+1)^2 xx \quad \text{hincque}$$

$$\frac{(pp-1)^2 y}{x} = g(pp+1)^2 + 2p(pp-1) \pm \sqrt{(gg(pp+1)^4 + 4gp(pp-1)(pp+1)^2 + 4pp(pp-1)^2 - (pp-1)^2(pp+1)^2 - 4pp(pp-1)^2)}$$
$$= g(pp+1)^2 + 2p(pp-1) \pm (pp+1)\sqrt{(gg(pp+1)^2 + 4gp(pp-1) - (pp-1)^2)}.$$

53. Haec formula rationalis reddenda insigni molestia premi videtur, quam autem ponendo $p = \frac{q+1}{q-1}$ tollere licet. Facilior vero redditur solutio,

tio, si pro primo numero sumatur $A = y$ vnde fit:

$$4ppxx = 2g(pp+1)^2 xy - (pp-1)^2 yy + 4p(pp-1)xy - (pp+1)^2 yy$$ hincque

$$\frac{4ppx}{y} = g(pp+1)^2 + 2p(pp-1) \pm (pp+1)\sqrt{(gg(pp+1)^2 + 4gp(pp-1) - 4pp)}$$

vbi quantitas rationalis reddenda est

$$ggp^4 + 4gp^3 + (2gg-4)pp - 4gp + gg$$

cuius radix posita $gpp + 2p + g$ dat $p = -g$ ita vt sit

$$\frac{4ggx}{y} = g(gg+1)^2 - 2g(gg-1) \pm (gg+1)(g^3-g) \text{ seu}$$

$$\frac{4gx}{y} = (gg+1)^2 - 2(gg-1) \pm (gg+1)(gg-1) \text{ Ergo}$$

vel $\frac{4gx}{y} = 2(g^4+1)$ vel $\frac{4gx}{y} = 4$.

54. Euoluamus primo posteriorem solutionem vtpote simpliciorem, et ob $\frac{y}{x} = \frac{g}{1}$ et $p = -g$ habebitur:

$A = g$; $B = \frac{2f - g(ff-1)}{ff+1}$; $C = \frac{ff-1-2fg}{ff+1}$; $D = \frac{-2g - g(gg-1)}{gg+1}$ seu $D = -g$; forent ergo duo quadrata A^2 et D^2 inter se aequalia scilicet $A = D = g = \frac{4f(ff-1)}{(ff+1)^2}$, et pro reliquis

$$B = \frac{2f(f^4 - 6ff + 1)}{(ff+1)^3} \text{ et } C = \frac{(ff-1)(f^4-6ff+1)}{(ff+1)^3}$$

quae radices per $(ff+1)^3$ multiplicando ad numeros integros reuocatae fient

$A = D = 4f(ff-1)(ff+1)$; $B = 2f(f^4 - 6ff + 1)$;
$C = (ff-1)(f^4 - 6ff + 1)$

vnde

vnde ſumto $f = 2$ oritur haec ſolutio:

$A = 8.3.5;\ D = 8.3.5;\ B = 4.7;\ C = 3.7$

ſeu $A = 120;\quad D = 120;\quad B = 28;\quad C = 21.$

55. Si aequalitas duorum numerorum minus placet, euoluamus alteram ſolutionem $\frac{x}{y} = \frac{g^4+1}{2g}$ vnde fit $x = g^4+1$, $y = 2g$ et ob $p = -g$; erit

$A = 2g;\ B = \frac{2f(g^4+1) - 2g(ff-1)}{ff+1};\ C = \frac{(ff-1)(g^4+1) - 4fg}{ff+1}$

et $D = \frac{-2g(g^4+1) - 2(gg-1)g}{gg+1} = 2g^3$ ſeu

$$A = 2g(ff+1)$$
$$B = 2f(g^4+1) - 2g(ff-1)$$
$$C = (ff-1)(g^4+1) - 4fg$$
$$D = 2g^3(ff+1)$$

vbi $g = \frac{4f(ff-1)}{(ff+1)^2}$, ſeu ponatur $g = \frac{m}{n}$ et omnibus ad integros reductis fiet

$$A = 2mn^3(ff+1)$$
$$B = 2f(m^4+n^4) - 2mn^3(ff-1)$$
$$C = (ff-1)(m^4+n^4) - 4fmn^3$$
$$D = 2m^3n(ff+1)$$

Hinc ſumto $f = 2$ vt ſit $g = \frac{24}{25} = \frac{m}{n}$ erunt quatuor quadratorum radices:

$$A = 2^4.3.5^7 \quad = 3750000$$
$$B = 2^2.7.22843 = 639604$$
$$C = 3^3.7.13219 = 832797$$
$$D = 2^{10}.3^3.5^3 \quad = 3456000.$$

56. Ob hos numeros tam grandes problema eo magis eſt attentione dignum, quamobrem operae pretium videtur, adhuc aliam eius ſolutionem etſi particularem proponere. Poſitis igitur quatuor quadratis quaeſitis vv, xx, yy, zz, primo has duas tantum conditiones conſidero:

$$vv+yy+zz=\square \quad \text{et} \quad xx+yy+zz=\square$$

quibus vt ſatisfaciam, aſſumo binos numeros a et α vt ſit $aa+\alpha\alpha=AA$, ac ſtatuo

$$vv+yy+zz=\frac{Av+\alpha x}{a} \quad \text{et}$$

$$xx+yy+zz=\frac{Ax+\alpha v}{a}$$

vt vtrinque eadem prodeat aequatio

$$aa(yy+zz)=\alpha\alpha(vv+xx)+2\alpha Avx$$

ſimili modo pro binis reliquis conditionibus pono

$$yy+vv+xx=\frac{Ay-az}{\alpha}$$

$$zz+vv+xx=\frac{Az-ay}{\alpha}$$

prodibitque hinc

$$\alpha\alpha(vv+xx)=aa(yy+zz)-2Aayz$$

quae duae aequationes additae dant

$$\alpha vx=ayz; \text{ hincque } z=\frac{\alpha vx}{ay}$$

qui valor in priori ſubſtituatur fietque

$$aayy+\frac{\alpha\alpha vvxx}{yy}-\alpha\alpha vv-\alpha\alpha xx-2\alpha Avx=0$$

ſeu $\alpha\alpha xx(vv-yy)=2\alpha Avxyy+\alpha\alpha vvyy-aay^4$

et $\alpha x=\frac{Avyy\pm y\sqrt{(AAvvyy+\alpha\alpha v^4-\alpha\alpha vvyy-aavvy^2+aay^4)}}{vv-yy}$

quae

quae ob $AA = \alpha\alpha + aa$ abit in

$$\frac{\alpha x}{y} = \frac{Avy \pm \surd(\alpha\alpha v^4 + aay^4)}{vv - yy}.$$

57. Ponatur $v = y(1+s)$ et cum fiat

$$\surd(\alpha\alpha v^4 + aay^4) = yy\surd(AA + 4\alpha\alpha s + 6\alpha\alpha ss + 4\alpha\alpha s^3 + \alpha\alpha s^4)$$

statuatur haec radix $= A + \frac{2\alpha\alpha}{A}s + \alpha ss$ eritque

$$6\alpha\alpha ss + 4\alpha\alpha s^3 = (\tfrac{4\alpha^4}{AA} + 2\alpha A)ss + \tfrac{4\alpha^3}{A}s^3$$

hincque $s = \frac{A^3 - 3\alpha AA + 2\alpha^3}{2\alpha A(A-\alpha)} = \frac{AA - 2\alpha A - 2\alpha\alpha}{2\alpha A}$.

Quare $\frac{v}{y} = \frac{AA - 2\alpha\alpha}{2\alpha A}$ et radix illa

$$= A + \frac{\alpha(AA - 2\alpha A - 2\alpha\alpha)}{AA} + \frac{(AA - 2\alpha A - 2\alpha\alpha)^2}{4\alpha AA}$$

$$= A + \frac{(AA - 2\alpha A + 2\alpha\alpha)(AA - 2\alpha A - 2\alpha\alpha)}{4\alpha AA}$$

$$= \frac{A^4 + 4\alpha\alpha AA - 4\alpha^4}{4\alpha AA}.$$

Porro est $vv - yy = \frac{(AA + 2\alpha A - 2\alpha\alpha)(AA - 2\alpha A - 2\alpha\alpha)}{4\alpha\alpha AA}yy$

hincque $\frac{(AA + 2\alpha A - 2\alpha\alpha)(AA - 2\alpha A - 2\alpha\alpha)}{4\alpha AA}\cdot\frac{x}{y}$

$$= \frac{AA - 2\alpha\alpha}{2\alpha} \pm \frac{(A^4 + 4\alpha\alpha AA - 4\alpha^4)}{4\alpha AA}$$

$=$ vel $\frac{A^4 - 8\alpha\alpha AA + 4\alpha^4}{4\alpha AA} = \frac{(AA + 2\alpha A - 2\alpha\alpha)(AA - 2\alpha A - 2\alpha\alpha)}{4\alpha AA}$

vel $\frac{3A^4 - 4\alpha^4}{4\alpha AA}$.

Consequenter habebimus

vel $\frac{x}{y} = 1$

vel $\frac{x}{y} = \frac{3A^4 - 4\alpha^4}{(AA - 2\alpha\alpha)^2 - 4\alpha\alpha AA}$

denique est $\frac{z}{y} = \frac{AA - 2\alpha\alpha}{2\alpha A}\cdot\frac{x}{y}$ ob $\frac{v}{y} = \frac{AA - 2\alpha\alpha}{2\alpha A}$.

58. Duas igitur adepti sumus solutiones, quarum prior ita se habet: sumto $y = 2\,\alpha\,a\,A$

$$v = a\,(AA - 2\,\alpha\,\alpha)$$
$$x = 2\,\alpha\,a\,A$$
$$y = 2\,\alpha\,a\,A$$
$$z = \alpha\,(AA - 2\,\alpha\,\alpha)$$

Vnde sumendo $\alpha = 3$, $a = 4$ et $A = 5$ prodit solutio simplicissima

$$v = 28;\; x = 120;\; y = 120;\; z = 21.$$

Altera autem solutio in numeris integris dat

$$v = a(AA - 2\,\alpha\alpha)(AA + 2\,\alpha A - 2\alpha\alpha)(AA - 2\alpha A - 2\,\alpha\alpha)$$
$$x = 2\,a\alpha\,A\,(3\,A^4 - 4\,\alpha^4)$$
$$y = 2\,a\alpha\,A\,(AA + 2\,\alpha A - 2\,\alpha\alpha)(AA - 2\,\alpha A - 2\,\alpha\alpha)$$
$$z = \alpha(AA - 2\,\alpha\alpha)(3\,A^4 - 4\,\alpha^4).$$

Vnde sumtis $\alpha = 3$, $a = 4$, $A = 5$ solutio simplicissima emergit

$$v = 4 \cdot 7 \cdot 37 \cdot 23 \;=\; 23828$$
$$x = 8 \cdot 3 \cdot 5 \cdot 1551 = 186120$$
$$y = 8 \cdot 3 \cdot 5 \cdot 37 \cdot 23 = 102120$$
$$z = 3 \cdot 7 \cdot 1551 \;=\; 32571$$

quorum numerorum quadrata sunt

$$vv = 567773584$$
$$xx = 34640654400$$
$$yy = 10428494400$$
$$zz = 1060870041$$

repe-

reperiturque:

$$xx+yy+zz=214779^2;\ vv+yy+zz=109805^2$$
$$vv+xx+zz=190445^2;\ vv+xx+yy=213628^2$$
at $vv+xx+yy+zz=25.\ 1201.\ 1555297.$

59. Quo ratio harum formularum clarius perſpiciatur, notari conuenit eſſe:

$$3A^4-4\alpha^4=-(AA+2aA-2aa)(AA-2aA-2aa)$$

vnde erit

$$v=a(AA-2aa)(AA+2\alpha A-2\alpha\alpha)(AA-2\alpha A-2\alpha\alpha)$$
$$z=\alpha(AA-2\alpha\alpha)(AA+2aA-2aa)(AA-2aA-2aa)$$
$$x=2a\alpha A(AA+2\alpha A-2\alpha\alpha)(AA-2\alpha A-2\alpha\alpha)$$
$$y=2a\alpha A(AA+2aA-2aa)(AA-2aA-2aa)$$

ſicque patet, numeros a et α inter ſe permutari, vt natura rei poſtulat. Quod facilius ex his formulis perſpicietur:

$$v=a(aa-\alpha\alpha)(3\alpha^4+6aa\alpha\alpha-a^4)$$
$$z=\alpha(aa-\alpha\alpha)(3a^4+6aa\alpha\alpha-\alpha^4)$$
$$x=2a\alpha A(3a^4+6aa\alpha\alpha-\alpha^4)$$
$$y=2a\alpha A(3\alpha^4+6aa\alpha\alpha-a^4).$$

Hinc eſt in genere:

$$vv+xx+yy=a^2(a^6+13a^4\alpha\alpha+11aa\alpha^4+7\alpha^6)^2$$
$$xx+yy+zz=\alpha^2(\alpha^6+13aa\alpha^4+11a^4\alpha\alpha+7a^6)^2$$
$$vv+yy+zz=A^2(a^6-a^4\alpha\alpha+15a^2\alpha^4+\alpha^6)^2$$
$$vv+xx+zz=A^2(\alpha^6-a^2\alpha^4+15a^4\alpha^2+a^6)^2$$

et ſumma omnium $xx + yy + zz + vv =$
$A^2(a^{12} + 34a^{10}\alpha^2 + 175a^8\alpha^4 + 92a^6\alpha^6 + 175a^4\alpha^8 + 34a^2\alpha^{10} + \alpha^{12})$
quae in hos factores reſoluitur:
$A^2(a^4 + 6a^2\alpha^2 + \alpha^4)(a^8 + 28a^6\alpha^2 + 6a^4\alpha^4 + 28a^2\alpha^6 + \alpha^8)$.

60. Neque tamen hae formulae minimos numeros ſuppeditant; ſequenti enim modo minores reperiuntur. Vt formula

$$\alpha\alpha v^4 + aay^4$$

fiat quadratum, ſumtis ſimilibus numeris b et β, vt ſit

$$bb + \beta\beta = BB,$$

ſtatuatur

$$\alpha vv = \beta M \text{ et } ayy = bM,$$

ſeu $\frac{vv}{yy} = \frac{a\beta}{\alpha b}$, vt fiat

$$\sqrt{(\alpha\alpha v^4 + aay^4)} = BM = \frac{aB}{b}yy,$$

vbi neceſſe eſt, vt $\frac{a}{\alpha}.\frac{\beta}{b}$ ſit quadratum. Sit ergo

$$\frac{a}{\alpha}.\frac{\beta}{b} = \frac{mm}{nn}, \text{ eritque } \frac{v}{y} = \frac{m}{n} \text{ tum}$$

$$\frac{x}{y} = \frac{\frac{Am}{n} \pm \frac{aB}{b}}{\alpha\left(\frac{a\beta}{\alpha b} - 1\right)} = \frac{Abm + aBn}{a\beta n - \alpha bn} \text{ et}$$

$$\frac{z}{y} = \frac{\alpha m}{\alpha n}.\frac{x}{y} = \frac{\beta(Abm \pm aBn)}{bm(a\beta - \alpha b)}.$$

Iam ponatur

$$a = 21,\ \alpha = 20,\ A = 29,\ b = 35,\ \beta = 12 \text{ et } B = 37,$$

eritque $\frac{mm}{nn} = \frac{21}{20}.\frac{12}{35} = \frac{9}{25}$,

vt

vt fit $m = 3$ et $n = 5$, vnde colligitur:

$$\frac{v}{y} = \frac{3}{5};\ \frac{x}{y} = \frac{29.35.3 \pm 37.21.5}{-5.4.7.16} = \frac{3(29 \pm 37)}{64}$$

et $\frac{z}{y} = \frac{3(29 \pm 37)}{16.7}$.

Pro figno fuperiori ergo erit

$$\frac{v}{y} = \frac{3}{5};\ \frac{x}{y} = \frac{3}{8} \text{ et } \frac{z}{y} = \frac{3}{14},$$

vnde in integris

$$\begin{array}{l|l}
v = 8.37 = 168 & \sqrt{(xx+yy+zz)} = 305 \\
x = 3.5.7 = 105 & \sqrt{(vv+yy+zz)} = 332 \\
y = 8.5.7 = 280 & \sqrt{(vv+xx+zz)} = 207 \\
z = 4.3.5 = 60 & \sqrt{(vv+xx+yy)} = 343
\end{array}$$

$$vv + xx + yy + zz = 121249 = 29.37.113.$$

Huiusmodi autem formulae generales funt:

$$v = 4fg(f+g)(3f-g)(3ff+gg)$$
$$y = 4fg(f-g)(3f+g)(3ff+gg)$$
$$z = (ff-g)(9ff-gg)(3ff+gg)$$
$$z = 2fg(ff-gg)(9ff-gg).$$

OB-