1772

Evolutio formulae integralis $\int x^{f-1} dx (log\ x)^{m/n}$ integratione a valore x=0 ad x=1 extensa

Leonhard Euler

EVOLVTIO FORMVLAE INTEGRALIS

$$\int x^{f-1}\, dx\, (lx)^{\frac{m}{n}}$$

INTEGRATIONE A VALORE $x=0$ AD $x=1$ EXTENSA.

Auctore

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Theorema 1.

1.

Si n denotat numerum integrum positiuum quemcunque et formulae $\int x^{f-1}\, dx\, (1-x^g)^n$ integratio a valore $x=0$ vsque ad $x=1$ extendatur, erit eius valor:

$$=\frac{g^n}{f}\cdot\frac{1.\quad 2.\quad 3.\ \dots\dots\dots\ n}{(f+g)(f+2g)(f+3g)\dots\dots(f+ng)}.$$

Demonstratio.

Notum est in genere integrationem formulae $\int x^{f-1}dx\,(1-x^g)^m$ reduci posse ad integrationem huius $\int x^{f-1}\, dx\,(1-x^g)^{m-1}$ quoniam quantitates constantes A et B ita definire licet, vt fiat

$$\int x^{f-1}dx\,(1-x^g)^m = A\int x^{f-1}dx\,(1-x^g)^{m-1} + Bx^f(1-x^g)^m$$

M 2 sumtis

ſumtis enim differentialibus prodit haec aequatio:

$$x^{f-1}dx(1-x^g)^m = Ax^{f-1}dx(1-x^g)^{m-1} + Bfx^{f-1}dx(1-x^g)^m - Bmgx^{f+g-1}dx(1-x^g)^{m-1}$$

quae per $x^{f-1}dx(1-x^g)^{m-1}$ diuiſa dat:

$$1-x^g = A + Bf(1-x^g) - Bmgx^g \text{ ſeu}$$

$$1-x^g = A - Bmg + B(f+mg)(1-x^g)$$

quae aequatio vt conſiſtere poſſit, neceſſe eſt ſit

$$1 = B(f+mg) \quad \text{et } A = Bmg$$

vnde colligimus $B = \frac{1}{f+mg}$ et $A = \frac{mg}{f+mg}$.

Quocirca habebimus ſequentem reductionem generalem:

$$\int x^{f-1}dx(1-x^g)^m = \frac{mg}{f+mg}\int x^{f-1}dx(1-x^g)^{m-1} + \frac{1}{f+mg}\cdot x^f(1-x^g)^m$$

quae cum euaneſcat poſito $x=0$, ſiquidem ſit $f>0$, conſtantis additione haud eſt opus. Quare extenſo vtroque integrali vsque ad $x=1$, pars integralis poſtrema ſponte euaneſcit, eritque pro caſu $x=1$

$$\int x^{f-1}dx(1-x^g)^m = \frac{mg}{f+mg}\int x^{f-1}dx(1-x^g)^{m-1}.$$

Cum igitur ſumto $m=1$ ſit $\int x^{f-1}dx(1-x^g)^0 = \frac{1}{f}x^f = \frac{1}{f}$ poſito $x=1$, nanciſcimur pro eodem caſu $x=1$ ſequentes valores:

$$\int x^{f-1}dx(1-x^g)^1 = \frac{g}{f}\cdot\frac{1}{f+g}$$

$$\int x^{f-1}dx(1-x^g)^2 = \frac{g^2}{f}\cdot\frac{1}{f+g}\cdot\frac{2}{f+2g}$$

$$\int x^{f-1}dx(1-x^g)^3 = \frac{g^3}{f}\cdot\frac{1}{f+g}\cdot\frac{2}{f+2g}\cdot\frac{3}{f+3g}$$

hinc-

hincque pro numero quocunque integro positiuo n concludimus fore

$$\int x^{f-1}dx(1-x^g)^n = \frac{g^n}{f}\cdot\frac{1}{f+g}\cdot\frac{2}{f+2g}\cdot\frac{3}{f+3g}\cdots\cdots\frac{n}{f+ng}$$

si modo numeri f et g sint positiui.

Coroll. 1.

2. Hinc ergo viciſſim valor huiusmodi producti ex quotcunque factoribus formati, per formulam integralem exprimi poteſt, ita vt ſit

$$\frac{1.\;2.\;3.\ldots n}{(f+g)(f+2g)(f+3g)\ldots(f+ng)} = \frac{f}{g^n}\int x^{f-1}dx(1-x^g)^n$$

integrali hoc a valore $x=0$ vsque ad $x=1$ extenſo.

Coroll. 2.

3. Quodſi ergo huiusmodi habeatur progreſſio:

$$\frac{1}{f+g};\ \frac{1.\;2}{(f+g)(f+2g)};\ \frac{1.\;2.\;3}{(f+g)(f+2g)(f+3g)};\ \frac{1.\;2.\;3.\;4}{(f+g)(f+2g)(f+3g)(f+4g)};\ \text{etc.}$$

eius terminus generalis qui indici indefinito n conuenit commode hac forma integrali $\frac{f}{g^n}\int x^{f-1}dx(1-x^g)^n$ repraeſentatur, cuius ope ea progreſſio interpolari, eiusque termini indicibus fractis reſpondentes exhiberi poterunt.

Coroll. 3.

4. Si loco n ſcribamus $n-1$, habebimus:

$$\frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)}{(f+g)(f+2g)(f+3g) \ldots (f+(n-1)g)} = \frac{f}{g^{n-1}} \int x^{f-1} dx (1-x^g)^{n-1}$$

quae per $\frac{n}{f+ng}$ multiplicata praebet

$$\frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}{(f+g)(f+2g)(f+3g) \ldots (f+ng)} = \frac{f \cdot ng}{g^n (f+ng)} \int x^{f-1} dx (1-x^g)^{n-1}.$$

Scholion 1.

5. Hanc posteriorem formam immediate ex praecedente deriuare licuisset, cum modo demonstrauerimus esse:

$$\int x^{f-1} dx (1-x^g)^n = \frac{ng}{f+ng} \int x^{f-1} dx (1-x^g)^{n-1}$$

siquidem vtrumque integrale a valore $x = 0$ vsque ad $x = 1$ extendatur; quam integralium determinationem in sequentibus vbique subintelligi oportet. Deinde etiam perpetuo est tenendum, quantitates f et g esse positiuas, quippe quam conditionem demonstratio allata absolute postulat. Quod autem ad numerum n attinet, quatenus eo index cuiusque termini progressionis (§. 3.) designatur, nihil impedit, quominus eo numeri quicunque siue positiui siue negatiui denotentur, quandoquidem eius progressionis omnes termini etiam indicibus negatiuis respondentes per formulam integralem datam exhiberi censentur. Interim tamen probe tenendum est hanc reductionem

$$\int x^{f-1} dx (1-x^g)^m = \frac{mg}{f+mg} \int x^{f-1} dx (1-x^g)^{m-1}$$

non esse veritati consentaneam, nisi sit $m > 0$; quia alioquin

alioquin pars algebraica $\frac{1}{f+mg}x^f(1-x^g)^m$ non euanesceret posito $x=1$.

Scholion 2.

6. Huiusmodi series, quas transcendentes appellare licet, quia termini indicibus fractis respondentes sunt quantitates transcendentes, iam olim in Comment. Petrop. Tomo V. fusius sum prosecutus; vnde hoc loco non tam istas progressiones, quam eximias formularum integralium comparationes, quae inde deriuantur, diligentius sum scrutaturus. Cum scilicet ostendissem huius producti indefiniti $1.2.3\ldots n$ valorem hac formula integrali $\int dx\,(l\frac{1}{x})^n$ ab $x=0$ ad $x=1$ extensa exprimi, quae res quoties n est numerus integer positiuus per ipsam integrationem est manifesta, eos casus examini subieci, quibus pro n numeri fracti accipiuntur; vbi quidem ex ipsa formula integrali neutiquam patet, ad quodnam genus quantitatum transcendentium hi termini referri debeant. Singulari autem artificio eosdem terminos ad quadraturas magis cognitas reduxi, quod propterea maxime dignum videtur, vt maiori studio perpendatur.

Problema 1.

7. Cum demonstratum sit esse:

$$\frac{1.\quad 2.\quad 3\ldots\ldots n}{(f+g)(f+2g)(f+3g)\ldots\ldots(f+ng)}=\frac{f}{g^n}\int x^{f-1}dx(1-x^g)^n$$

inte-

integrali ab $x = 0$ ad $x = 1$ extenso; eiusdem producti casu quo $g = 0$ valorem per formulam integralem assignare.

Solutio.

Posito $g = 0$ in formula integrali membrum $(1 - x^g)^n$ euanescit, simul vero etiam denominator g^n, vnde quaestio huc redit vt fractionis $\frac{(1 - x^g)^n}{g^n}$ valor definiatur casu $g = 0$, quo tam numerator quam denominator euanescit. Hunc in finem spectetur g vt quantitas infinite parua, et cum sit $x^g = e^{g\,l\,x}$ fiet $x^g = 1 + g\,l\,x$ ideoque $(1 - x^g)^n = g^n(-l\,x)^n = g^n(l\frac{1}{x})^n$; ex quo pro hoc casu formula nostra integralis abit in $f\int x^{f-1}dx(l\frac{1}{x})^n$ ita vt iam habeatur

$$\frac{1.\ 2.\ 3\ldots\ldots n}{f^n} = f\int x^{f-1}dx\,(l\tfrac{1}{x})^n$$

seu $1.\ 2.\ 3\ldots\ldots n = f^{n+1}\int x^{f-1}dx(l\frac{1}{x})^n$.

Coroll. 1.

8. Quoties n est numerus integer positiuus, integratio formulae $\int x^{f-1}dx(l\frac{1}{x})^n$ succedit, eaque ab $x = 0$ ad $x = 1$ extensa reuera prodit id productum, cui istam formulam aequalem inuenimus. Sin autem pro n capiantur numeri fracti eadem formula integralis inseruiet huic progressioni hypergeometricae interpolandae:

1; 1. 2; 1. 2. 3; 1. 2. 3. 4; 1. 2. 3. 4. 5; etc.

seu 1; 2; 6; 24; 120; 720; 5040; etc.

Coroll.

Coroll. 2.

9 Si expressio modo inuenta per principalem diuidatur, orietur productum, cuius factores in progressione arithmetica quacunque progrediuntur:

$$(f+g)(f+2g)(f+3g)\ldots(f+ng)=f^n g^n.\frac{\int x^{f-1}dx\,(l\frac{1}{x})^n}{\int x^{f-1}dx(1-x^g)^n}$$

cuius ergo etiam valores, si n sit numerus fractus hinc assignare licebit.

Coroll. 3.

10. Cum sit

$$\int x^{f-1}dx(1-x^g)^n=\frac{ng}{f+ng}\int x^{f-1}dx(1-x^g)^{n-1}$$

erit etiam simili modo pro casu $g=0$.

$$\int x^{f-1}dx(l\frac{1}{x})^n=\frac{n}{f}\int x^{f-1}dx(l\frac{1}{x})^{n-1}$$

hincque per istas alteras formulas integrales:

$$1.\,2.\,3\ldots\ldots n=nf^n\int x^{f-1}dx(l\frac{1}{x})^{n-1} \text{ et}$$

$$(f+g)(f+2g)\ldots(f+ng)=f^{n-1}g^{n-1}(f+ng).\frac{\int x^{f-1}dx\,(l\frac{1}{x})^{n-1}}{\int x^{f-1}dx(1-x^g)^{n-1}}.$$

Scholion.

11. Cum inuenerimus esse:

$$1.\,2.\,3\ldots\ldots n=f^{n+1}\int x^{f-1}dx\,(l\frac{1}{x})^n$$

patet hanc formulam integralem non a valore quantitatis f pendere, quod etiam facile perspicitur ponendo $x^f=y$, vnde fit $fx^{f-1}dx=dy$, et $l\frac{1}{x}=$

 $-lx$

$-lx = -\frac{1}{f}ly = \frac{1}{f}l\frac{1}{y}$, ideoque $f^n(l\frac{1}{x})^n = (l\frac{1}{y})^n$, ita vt ſit

$$1.\ 2.\ 3 \ldots\ldots n = \int dy\,(l\tfrac{1}{y})^n$$

quae formula ex priori naſcitur ponendo $f = 1$. Pro interpolatione ergo huiusmodi formarum totum negotium huc reducitur, vt iſtius formulae integralis $\int dx\,(l\frac{1}{x})^n$ valores definiantur, quando exponens n eſt numerus fractus. Veluti ſi n ſit $= \frac{1}{2}$, aſſignari oportet valorem huius formulae $\int dx\,\sqrt{l\frac{1}{x}}$, quem olim iam oſtendi eſſe $= \frac{1}{2}\sqrt{\pi}$ denotante π circuli peripheriam cuius diameter $= 1$: pro aliis autem numeris fractis eius valorem ad quadraturas curuarum algebraicarum altioris ordinis reuocare docui. Quae reductio cum minime ſit obuia, atque tum ſolum locum habeat, quando formulae $\int dx\,(l\frac{1}{x})^n$ integratio a valore $x = 0$ ad $x = 1$ extenditur, ſingulari attentione digna videtur. Etſi autem iam olim hoc argumentum tractaui, tamen quia per plures ambages eo ſum perductus, idem hic reſumere et concinnius euoluere conſtitui.

Theorema 2.

12. Si formulae integrales a valore $x = 0$ vsque ad $x = 1$ extendantur et n denotet numerum integrum poſitiuum erit:

$$\frac{1.\ 2.\ 3 \ldots\ldots n}{(n+1)(n+2)(n+3)\ldots 2n} = \tfrac{1}{2}ng\int x^{f+ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}$$

quicunque numeri poſitiui loco f et g accipiantur.

Demon-

Demonſtratio.

Cum ſupra (§. 4.) oſtenderimus eſſe:

$$\frac{1 \cdot 2 \cdot 3 \ldots\ldots n}{(f+g)(f+2g)\ldots(f+ng)} = \frac{f \cdot ng}{g^n(f+ng)} \int x^{f-1} dx (1-x^g)^{n-1}$$

habebimus ſi loco n ſcribamus $2n$

$$\frac{1 \cdot 2 \cdot 3 \ldots 2n}{(f+g)(f+2g)\ldots(f+2ng)} = \frac{f \cdot 2ng}{g^{2n}(f+2ng)} \int x^{f-1} dx (1-x^g)^{2n-1}$$

Diuidatur nunc prima aequatio per ſecundam, ac prodibit iſta tertia:

$$\frac{(f+(n+1)g)(f+(n+2)g)\ldots(f+2ng)}{(n+1)(n+2)\ldots\ldots 2n} = \frac{g^n(f+2ng)}{2(f+ng)} \cdot \frac{\int x^{f-1} dx (1-x^g)^{n-1}}{\int x^{f-1} dx (1-x^g)^{2n-1}}$$

At ſi in prima aequatione loco f ſcribatur $f+ng$, orietur haec aequatio quarta:

$$\frac{1 \cdot 2 \cdot 3 \ldots\ldots\ldots n}{(f+(n+1)g)(f+(n+2)g)\ldots(f+2ng)} = \frac{(f+ng)ng}{g^n(f+2ng)} \int x^{f+ng-1} dx (1-x^g)^{n-1}$$

Multiplicetur haec quarta aequatio per illam tertiam ac reperietur ipſa aequatio demonſtranda:

$$\frac{1 \cdot 2 \cdot 3 \ldots\ldots n}{(n+1)(n+2)(n+3)\ldots 2n} = \tfrac{1}{2} ng \int x^{f+ng-1} dx (1-x^g)^{n-1} \cdot \frac{\int x^{f-1} dx (1-x^g)^{n-1}}{\int x^{f-1} dx (1-x^g)^{2n-1}}.$$

Coroll. 1.

13. Si in prima aequatione ſtatuatur $f = n$ et $g = 1$ orietur idem productum:

$$\frac{1 \cdot 2 \cdot 3 \ldots\ldots n}{(n+1)(n+2)\ldots\ldots 2n} = \tfrac{1}{2} n \int x^{n-1} dx (1-x)^{n-1}$$

qua aequatione cum illa collata adipiscimur:

$$\frac{\int x^{n-1}\,dx\,(1-x)^{n-1}}{g\int x^{f+ng-1}\,dx\,(1-x^g)^{n-1}} = \frac{\int x^{f-1}\,dx\,(1-x^g)^{n-1}}{\int x^{f-1}\,dx\,(1-x^g)^{2n-1}}.$$

Coroll. 2.

14. Si in illa aequatione loco x scribamus x^g, fiet

$$\frac{1.\ 2.\ 3 \ldots\ldots n}{(n+1)(n+2)\ldots 2n} = \tfrac{1}{2}\,n\,g\int x^{ng-1}\,dx\,(1-x^g)^{n-1}$$

ita vt iam consequamur istam comparationem inter sequentes formulas integrales:

$$\int x^{ng-1}\,dx(1-x^g)^{n-1} = \int x^{f+ng-1}\,dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}\,dx(1-x^g)^{n-1}}{\int x^{f-1}\,dx(1-x^g)^{2n-1}}.$$

Coroll. 3.

15. Si in aequatione theorematis ponamus $g=0$ ob $(1-x^g)^m = g^m(l\frac{1}{x})^m$, potestates ipsius g se destruent orieturque haec aequatio:

$$\frac{1.\ 2.\ 3 \ldots n}{(n+1)(n+2)\ldots 2n} = \tfrac{1}{2}\,n\int x^{f-1}\,dx\,(l\tfrac{1}{x})^{n-1}\cdot\frac{\int x^{f-1}\,dx\,(l\frac{1}{x})^{n-1}}{\int x^{f-1}\,dx\,(l\frac{1}{x})^{2n-1}}$$

vnde colligimus

$$\frac{(\int x^{f-1}\,dx\,(l\frac{1}{x})^{n-1})^2}{\int x^{f-1}\,dx\,(l\frac{1}{x})^{2n-1}} = g\int x^{ng-1}\,dx\,(1-x^g)^{n-1}$$

seu ob

$\int x^{f-1}\,dx\,(l\frac{1}{x})^{n-1} = \frac{f}{n}\int x^{f-1}\,dx\,(l\frac{1}{x})^n$ hanc

$$\frac{2f}{n}\cdot\frac{(\int x^{f-1}\,dx\,(l\frac{1}{x})^n)^2}{\int x^{f-1}\,dx\,(l\frac{1}{x})^{2n}} = g\int x^{ng-1}\,dx\,(1-x^g)^{n-1}.$$

Coroll.

Coroll. 4.

16. Ponamus hic $f = 1$, $g = 2$ et $n = \frac{m}{2}$ vt m sit numerus integer positiuus, et ob $\int dx (l\frac{1}{x})^m = 1 . 2 . 3 \ldots m$ erit

$$\frac{4}{m} \cdot \frac{(\int dx (l\frac{1}{x})^{\frac{m}{2}})^2}{1 . 2 . 3 \ldots m} = 2 \int x^{m-1} dx (1 - x^2)^{\frac{m}{2} - 1}$$

hincque

$$\int dx (l\frac{1}{x})^{\frac{m}{2}} = \sqrt{1 . 2 . 3 \ldots m . \frac{m}{2} \int x^{m-1} dx (1 - x^2)^{\frac{m}{2} - 1}}$$

et sumendo $m = 1$ ob $\int \frac{dx}{\sqrt{(1 - xx)}} = \frac{\pi}{2}$ habebitur

$$\int dx \sqrt{l\frac{1}{x}} = \sqrt{\frac{1}{2} \int \frac{dx}{\sqrt{(1 - xx)}}} = \frac{1}{2} \sqrt{\pi}.$$

Scholion.

17. En ergo succinctam demonstrationem theorematis olim a me prolati, quod sit $\int dx \sqrt{l\frac{1}{x}} = \frac{1}{2}\sqrt{\pi}$, eamque ab interpolationis ratione, qua tum vsus fueram, libera. Deducta scilicet hic ea ex hoc theoremate quo inueni esse:

$$\frac{(\int x^{f-1} dx (l\frac{1}{x})^{n-1})^2}{\int x^{f-1} dx (l\frac{1}{x})^{2n-1}} = g \int x^{ng-1} dx (1 - x^g)^{n-1}$$

Principale autem theorema, vnde hoc est deductum ita se habet

$$g . \frac{\int x^{f-1} dx (1 - x^g)^{n-1} . \int x^{f+ng-1} dx (1 - x^g)^{n-1}}{\int x^{f-1} dx (1 - x^g)^{2n-1}} = \int x^{n-1} dx (1 - x)^{n-1}$$

vtrumque enim membrum per integrationem ab $x=0$ ad $x=1$ extensam euoluitur in hoc productum numericum:

$$\frac{1.\ 2.\ 3\ \ldots\ldots\ (n-1)}{(n+1)(n+2)\ldots\ldots(2n-1)}$$

Ac si alteri membro speciem latius patentem tribuere velimus, theorema ita proponi poterit vt sit:

$$g\frac{\int x^{f-1}dx(1-x^g)^{n-1}.\int x^{f+ng-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}$$
$$=k\int x^{nk-1}dx(1-x^k)^{n-1}$$

hicque si capiatur $g=0$, fit

$$\frac{(\int x^{f-1}dx(l\frac{1}{x})^{n-1})^2}{\int x^{f-1}dx(l\frac{1}{x})^{2n-1}}=k\int x^{nk-1}dx(1-x^k)^{n-1}.$$

Imprimis igitur notandum est, quod illa aequalitas subsistat, quicunque numeri loco f et g accipiantur casu quidem $f=g$, ea est manifesta, cum sit

$$\int x^{g-1}dx(1-x^g)^{n-1}=\frac{1-(1-x^g)^n}{ng}=\frac{1}{ng}$$

fiet enim

$$2g\int x^{ng+g-1}dx(1-x^g)^{n-1}=k\int x^{nk-1}dx(1-x^k)^{n-1}$$

et quia

$$\int x^{ng+g-1}dx(1-x^g)^{n-1}=\tfrac{1}{2}\int x^{ng-1}dx(1-x^g)^{n-1},$$

aequalitas est perspicua, quia k pro lubitu accipere licet. Eodem autem modo, quo ad hoc theorema perueni, ad alia similia pertingere licet.

Theo-

Theorema. 3.

18. Si sequentes formulae integrales a valore $x=0$ ad $x=1$ extendantur et n denotet numerum integrum positiuum quemcunque, erit

$$\frac{1.\ 2.\ 3\ \ldots\ldots n}{(2n+1)(2n+2)\ldots 3n}=\tfrac{2}{3}ng\int x^{f+2ng-1}dx(1-x^g)^{n-1}.\ \frac{\int x^{f-1}dx(1-x^g)^{2n-1}}{\int x^{f-1}dx(1-x^g)^{3n-1}}$$

quicunque numeri positiui pro f et g accipiantur.

Demonstratio.

In praecedente Theoremate iam vidimus esse:

$$\frac{1.\ 2.\ 3\ \ldots\ldots 2n}{(f+g)(f+2g)\ldots(f+2ng)}=\frac{f.\ 2ng}{g^{2n}(f+2ng)}\int x^{f-1}dx(1-x^g)^{2n-1}$$

simili autem modo, si in forma principali loco n scribamus $3n$ habebimus:

$$\frac{1.\ 2.\ 3\ \ldots\ 3n}{(f+g)(f+2g)\ldots(f+3ng)}=\frac{f.\ 3ng}{g^{3n}(f+3ng)}\int x^{f-1}dx(1-x^g)^{3n-1}$$

ex quo illa aequatio per hanc diuisa producit:

$$\frac{(f+(2n+1)g)(f+(2n+2)g)\ldots\ldots(f+3ng)}{(2n+1)(2n+2)\ \ldots\ldots\ldots\ 3n}=\frac{2g^n(f+3ng)}{3(f+2ng)}.\ \frac{\int x^{f-1}dx(1-x^g)^{2n-1}}{\int x^{f-1}dx(1-x^g)^{3n-1}}.$$

Verum si in aequatione principali (§. 4.) loco f scribamus $f+2gn$ adipiscimur hanc aequationem:

1. 2.

$$\frac{1 \,.\, 2 \,.\, 3 \,.\, \ldots \ldots \, n}{(f+(2n+1)g)(f+(2n+2)g) \ldots (f+3ng)} = \frac{(f+2ng).ng}{g^n (f+3ng)} \int x^{f+2ng-1} dx (1-x^g)^{n-1}.$$

Multiplicetur nunc haec aequatio per praecedentem, et orietur ipſa aequatio, quam demonſtrari oportet:

$$\frac{1 \,.\, 2 \,.\, 3 \,.\, \ldots \, n}{(2n+1)(2n+2) \ldots 3n} = \tfrac{2}{3} ng \int x^{f+2ng-1} dx (1-x^g)^{n-1}. \frac{\int x^{f-1} dx (1-x^g)^{2n-1}}{\int x^{f-1} dx (1-x^g)^{3n-1}}.$$

Coroll. 1.

18. Eundem valorem ex aequatione principali nanciſcimur ponendo $f = 2n$ et $g = 1$, ita vt ſit:

$$\frac{1 \,.\, 2 \,.\, 3 \ldots \ldots n}{(2n+1)(2n+2) \ldots 3n} = \tfrac{2}{3} n \int x^{2n-1} dx (1-x)^{n-1}$$

quae formula integralis loco x ſcribendo x^k transformatur in hanc $\frac{2}{3} n k \int x^{2nk-1} dx (1-x^k)^{n-1}$, ita vt ſit

$$g \int x^{f+2ng-1} dx (1-x^g)^{n-1}. \frac{\int x^{f-1} dx (1-x^g)^{2n-1}}{\int x^{f-1} dx (1-x^g)^{3n-1}} = k \int x^{2nk-1} dx (1-x^k)^{n-1}.$$

Coroll. 2.

20. Si hic ſtatuamus $g = 0$, ob $1 - x^g = g\, l\frac{1}{x}$ habebimus hanc aequationem:

$$\int x^{f-1} dx (l\tfrac{1}{x})^{n-1}. \frac{\int x^{f-1} dx (l\frac{1}{x})^{2n-1}}{\int x^{f-1} dx (l\frac{1}{x})^{3n-1}} = k \int x^{2nk-1} dx (1-x^k)^{n-1}$$

 cum

cum igitur ante inueniſſemus

$$\frac{\int x^{f-1}dx(l\frac{1}{x})^{n-1})^2}{\int x^{f-1}dx(l\frac{1}{x})^{2n-1}} = k\int x^{nk-1}dx(1-x^k)^{n-1}$$

habebimus has aequationes in ſe multiplicando:

$$\frac{(\int x^{f-1}dx(l\frac{1}{x})^{n-1})^3}{\int x^{f-1}dx(l\frac{1}{x})^{3n-1}} = k^2\int x^{nk-1}dx(1-x^k)^{n-1}.\int x^{2nk-1}dx(1-x^k)^{n-1}.$$

Coroll. 3.

21. Sine vlla reſtrictione hic ponere licet $f=1$; tum ergo ſumto $n=\frac{1}{3}$ et $k=3$ erit

$$\frac{(\int dx(l\frac{1}{x})^{-\frac{2}{3}})^3}{\int dx(l\frac{1}{x})^0} = 9\int dx(1-x^3)^{-\frac{2}{3}}.\int xdx(1-x^3)^{-\frac{2}{3}}$$

et ob $\int dx(l\frac{1}{x})^{-\frac{2}{3}} = 3\int dx(l\frac{1}{x})^{\frac{1}{3}}$ et $\int dx(l\frac{1}{x})^0 = 1$,

$$(\int dx(l\frac{1}{x})^{\frac{1}{3}})^3 = \int dx(1-x^3)^{-\frac{2}{3}}.\int xdx(1-x^3)^{-\frac{2}{3}}$$

tum vero ſumto $n=\frac{2}{3}$ et $k=3$ erit

$$\frac{(\int dx(l\frac{1}{x})^{-\frac{1}{3}})^3}{\int dx(l\frac{1}{x})} = 9\int xdx(1-x^3)^{-\frac{1}{3}}.\int x^3dx(1-x^3)^{-\frac{1}{3}}$$

ſeu $(\int dx(l\frac{1}{x})^{\frac{2}{3}})^3 = 4\int xdx(1-x^3)^{-\frac{1}{3}}\int x^3dx(1-x^3)^{-\frac{1}{3}}$.

Theorema generale.

22. Si ſequentes formulae integrales a valore $x=0$ vsque ad $x=1$ extendantur et n denotet numerum integrum poſitiuum quemcunque, erit

$$\frac{1.\ 2.\ 3\ \ldots\ldots n}{(\lambda n+1)(\lambda n+2)\ldots(\lambda+1)n}=\frac{\lambda}{\lambda+1}ng\int x^{f+\lambda ng-1}dx(1-x^g)^{n-1}.$$
$$\frac{\int x^{f-1}dx(1-x^g)^{\lambda n-1}}{\int x^{f-1}dx(1-x^g)^{(\lambda+1)n-1}}$$

quicunque numeri poſitiui pro litteris f et g accipiantur.

Demonſtratio.

Cum ſit vti ſupra oſtendimus:

$$\frac{1.\ 2\ \ldots\ldots\ n}{(f+g)(f+2g)\ldots\ldots(f+ng)}=\frac{f.\,ng}{g^n(f+ng)}\int x^{f-1}dx(1-x^g)^{n-1}$$

ſi hic loco n ſcribamus primo λn tum vero $(\lambda+1)n$ nanciſcemur has duas aequationes

$$\frac{1.\ 2\ \ldots\ldots\ \lambda n}{(f+g)(f+2g)\ldots\ldots(f+\lambda ng)}=\frac{f.\,\lambda ng}{g^{\lambda n}(f+\lambda ng)}\int x^{f-1}dx(1-x^g)^{\lambda n-1}$$

$$\frac{1.\ 2\ \ldots\ldots\ (\lambda+1)n}{(f+g)(f+2g)\ldots\ldots(f+(\lambda+1)ng)}=\frac{f.\,(\lambda+1)ng}{g^{(\lambda+1)n}(f+(\lambda+1)ng)}\int x^{f-1}dx(1-x^g)^{(\lambda+1)n-1}$$

quarum illa per hanc diuiſa praebet:

$$\frac{(f+\lambda ng+g)(f+\lambda ng+2g)\ldots\ldots(f+\lambda ng+ng)}{(\lambda n+1)(\lambda n+2)\ \ldots\ldots\ldots\ (\lambda n+n)}=g^n\frac{\lambda(f+\lambda ng+ng)}{(\lambda+1)(f+\lambda ng)}\cdot\frac{\int x^{f-1}dx(1-x^g)^{\lambda n-1}}{\int x^{f-1}dx(1-x^g)^{(\lambda+1)n-1}}.$$

At ſi in aequatione prima loco f ſcribamus $f+\lambda ng$ obtinebimus:

$$\frac{1 \cdot 2 \cdots\cdots n}{(f+\lambda ng+g)(f+\lambda ng+2g)\ldots(f+\lambda ng+ng)} = \frac{(f+\lambda ng)ng}{g^n(f+\lambda ng+ng)} \int x^{f+\lambda ng-1} dx (1-x^g)^{n-1}$$

quae duae aequationes in ſe ductae producunt ipſam aequalitatem demonſtrandam:

$$\frac{1 \cdot 2 \cdots\cdots n}{(\lambda n+1)(\lambda n+2)\ldots(\lambda n+n)} = \frac{\lambda n g}{\lambda+1} \int x^{f+\lambda ng-1} dx (1-x^g)^{n-1} \cdot \frac{\int x^{f-1} dx (1-x^g)^{\lambda n-1}}{\int x^{f-1} dx (1-x^g)^{(\lambda+1)n-1}}.$$

Coroll. 1.

23. Si in aequatione principali ſtatuamus $f = \lambda n$ et $g = 1$ reperiemus etiam:

$$\frac{1 \cdot 2 \cdots\cdots n}{(\lambda n+1)(\lambda n+2)\ldots(\lambda n+n)} = \frac{\lambda n}{\lambda+1} \int x^{\lambda n-1} dx (1-x)^{n-1}$$

quae forma loco x ſcribendo x^k abit in hanc:

$$\frac{\lambda n k}{\lambda+1} \int x^{\lambda n k-1} dx (1-x^k)^{n-1}$$

ita vt habeamus hoc theorema latiſſime patens:

$$g \int x^{f+\lambda ng-1} dx (1-x^g)^{n-1} \cdot \frac{\int x^{f-1} dx (1-x^g)^{\lambda n-1}}{\int x^{f-1} dx (1-x^g)^{\lambda n+n-1}} = k \int x^{\lambda nk-1} dx (1-x^k)^{n-1}.$$

Coroll. 2.

24. Hoc iam theorema locum habet, etiamſi n non ſit numerus integer, quin etiam cum nume-

rum λ pro lubitu accipere liceat, loco λn ſcribamus m, et perueniemus ad hoc theorema:

$$\frac{\int x^{f-1}\,dx(1-x^g)^{m-1}}{\int x^{f-1}\,dx(1-x^g)^{m+n-1}} = \frac{k\int x^{mk-1}\,dx(1-x^k)^{n-1}}{g\int x^{f+mg-1}\,dx(1-x^g)^{n-1}}.$$

Coroll. 3.

25. Si ponamus $g=0$; ob $1-x^g = g\,l\frac{1}{x}$, hoc theorema iſtam induet formam:

$$\frac{\int x^{f-1}\,dx(l\frac{1}{x})^{m-1}}{\int x^{f-1}\,dx(l\frac{1}{x})^{m+n-1}} = \frac{k\int x^{mk-1}\,dx(1-x^k)^{n-1}}{\int x^{f-1}\,dx(l\frac{1}{x})^{n-1}}$$

quae commodius ita repraeſentatur:

$$\frac{\int x^{f-1}\,dx(l\frac{1}{x})^{n-1}\cdot\int x^{f-1}\,dx(l\frac{1}{x})^{m-1}}{\int x^{f-1}\,dx(l\frac{1}{x})^{m+n-1}} = k\int x^{mk-1}\,dx(1-x^k)^{n-1}$$

vbi euidens eſt numeros m et n inter ſe permutari poſſe.

Scholion.

26. Duplicem ergo deteximus fontem, vnde innumerabiles formularum integralium comparationes haurire licet; alter fons §. 24. patefactus complectitur huiusmodi formulas integrales

$$\int x^{p-1}\,dx(1-x^g)^{q-1},$$

quas iam ante aliquod tempus pertractaui in obſeruationibus circa integralia formularum

$$\int x^{p-1}\,dx(1-x^n)^{\frac{q}{n}-1}$$

a vaa valore $x = 0$ vsque ad $x = 1$ extensa, vbi ostendi primo litteras p et q inter se permutari posse, vt sit

$$\int x^{p-1} dx (1-x^n)^{\frac{q}{n}-1} = \int x^{q-1} dx (1-x^n)^{\frac{p}{n}-1}$$

tum vero etiam esse

$$\int \frac{x^{p-1} dx}{(1-x^n)^{\frac{p}{m}}} = \frac{\pi}{n \sin. \frac{p\pi}{n}}$$

imprimis autem demonstraui esse:

$$\int \frac{x^{p-1} dx}{\sqrt[n]{(1-x^n)^{n-q}}} \cdot \int \frac{x^{p+q-1} dx}{\sqrt[n]{(1-x^n)^{n-r}}} = \int \frac{x^{p-1} dx}{\sqrt[n]{(1-x^n)^{n-r}}} \cdot \int \frac{x^{p+r-1} dx}{\sqrt[n]{(1-x^n)^{n-q}}}$$

in qua aequatione comparatio in §. 24. inuenta iam continetur; ita vt hinc nihil noui, quod non iam euolui, deduci queat. Alterum igitur fontem §. 25 indicatum hic potissimum inuestigandum suscipio, vbi cum sine vlla restrictione sumi queat $f = 1$, aequatio nostra primaria erit:

$$\frac{\int dx (l\frac{1}{x})^{n-1} \cdot \int dx (l\frac{1}{x})^{m-1}}{\int dx (l\frac{1}{x})^{m+n-1}} = k \int x^{mk-1} dx (1-x^k)^{n-1}$$

cuius beneficio valores formulae integralis $\int dx (l\frac{1}{x})^{\lambda}$ quando λ non est numerus integer ad quadraturas curuarum algebraicarum reuocare licebit; quandoquidem quoties λ est numerus integer, integratio habetur absoluta, quoniam est

$$\int dx (l\frac{1}{x})^{\lambda} = 1 \cdot 2 \cdot 3 \ldots\ldots \lambda.$$

Maximi autem momenti quaestio versatur circa eos casus,

casus, quibus λ est numerus fractus, quos ergo pro ratione denominationis hic successiue sum definiturus.

Problema 2.

27. Denotante i numerum integrum positiuum definire valorem formulae integralis $\int dx\,(l\frac{1}{x})^{\frac{i}{2}}$ integratione ab $x=0$ vsque ad $x=1$ extensa.

Solutio.

In aequatione nostra generali faciamus $m=n$ eritque

$$\frac{(\int dx\,(l\frac{1}{x})^{n-1})^2}{\int dx\,(l\frac{1}{x})^{2n-1}} = k\int x^{nk-1}\,dx\,(1-x^k)^{n-1}$$

Sit iam $n-1=\frac{i}{2}$, et ob $2n-1=i+1$ erit

$$\int dx\,(l\frac{1}{x})^{2n-1} = 1.\,2.\,3\,\ldots\ldots\,(i+1)$$

sumatur porro $k=2$ vt fit $nk-1=i+1$, fietque

$$\frac{(\int dx\,\surd\,(l\frac{1}{x})^i)^2}{1.\,2.\,3\,\ldots\,(i+1)} = 2\int x^{i+1}\,dx\,(1-x^2)^{\frac{i}{2}}$$

ideoque

$$\frac{\int dx\,\surd\,(l\frac{1}{x})^i}{\surd\,(1.\,2.\,3\,\ldots\,(i+1))} = \surd\,2\int x^{i+1}\,dx\,\surd\,(1-x^2)^i$$

vbi euidens est pro i numeros tantum impares sumi conuenire, quoniam pro paribus euolutio per se est manifesta.

Coroll. 1.

28. Omnes autem casus facile reducuntur ad $i=1$, vel adeo ad $i=-1$, dummodo enim $i+1$, non

non fit numerus negatiuus reductio inuenta locum habet. Pro hoc ergo casu erit:

$$\int \frac{dx}{\sqrt{l\frac{1}{x}}} = \sqrt{2}\int \frac{dx}{\sqrt{(1-xx)}} = \sqrt{\pi} \text{ ob } \int \frac{dx}{\sqrt{(1-xx)}} = \frac{\pi}{2}.$$

Coroll. 2.

29. Hoc autem casu principali expedito ob $\int dx(l\frac{1}{x})^n = n\int dx(l\frac{1}{x})^{n-1}$ habebimus,

$$\int dx\sqrt{l\frac{1}{x}} = \frac{1}{2}\sqrt{\pi}; \int dx(l\frac{1}{x})^{\frac{3}{2}} = \frac{1.3}{2.2}\sqrt{\pi}$$

atque in genere

$$\int dx(l\frac{1}{x})^{\frac{2n+1}{2}} = \frac{1}{2}.\frac{3}{2}.\frac{5}{2}.\frac{7}{2}\ldots\ldots\frac{(2n+1)}{2}\sqrt{\pi}.$$

Problema 3.

30. Denotante i numerum integrum positiuum definire valorem formulae integralis $\int dx(l\frac{1}{x})^{\frac{i}{3}-1}$ integratione ab $x=0$ ad $x=1$ extensa.

Solutio.

Inchoemus ab aequatione praecedentis problematis:

$$\frac{(\int dx(l\frac{1}{x})^{n-1})^2}{\int dx(l\frac{1}{x})^{2n-1}} = k\int x^{nk-1}dx(1-x^k)^{n-1}$$

atque in forma generali statuamus $m=2n$, vt habeatur:

$$\frac{\int dx(l\frac{1}{x})^{n-1}.\int dx(l\frac{1}{x})^{2n-1}}{\int dx(l\frac{1}{x})^{3n-1}} = k\int x^{2nk-1}dx(1-x^k)^{n-1}$$

ac

ac multiplicando has duas aequalitates adipiscimur:

$$\frac{(\int dx(l\frac{1}{x})^{n-1})^3}{\int dx(l\frac{1}{x})^{3n-1}} = kk\int x^{nk-1}dx(1-x^k)^{n-1}.\int x^{2nk-1}dx(1-x^k)^{n-1}.$$

Hic iam ponatur $n = \frac{i}{3}$ vt fit

$$\int dx(l\frac{1}{x})^{i-1} = 1.\ 2.\ 3\ \ldots\ldots\ (i-1)$$

sumaturque $k = 3$ ac prodibit

$$\frac{(\int dx\sqrt[3]{(l\frac{1}{x})^{i-3}})^3}{1.2.3\ldots(i-1)} = 9\int x^{i-1}dx\sqrt[3]{(1-x^3)^{i-3}}.\int x^{2i-1}dx\sqrt[3]{(1-x^3)^{i-3}}$$

vnde concludimus

$$\frac{\int dx\sqrt[3]{(l\frac{1}{x})^{i-3}}}{\sqrt[3]{1.\ 2.\ 3\ldots(i-1)}} = \sqrt[3]{9}\int\frac{x^{i-1}dx}{\sqrt[3]{(1-x^3)^{3-i}}}.\int\frac{x^{2i-1}dx}{\sqrt[3]{(1-x^3)^{3-i}}}.$$

Coroll. 1.

31. Bini hic occurrunt casus principales, a quibus reliqui omnes pendent, ponendo scilicet vel $i = 1$ vel $i = 2$, qui sunt:

$$\text{I. } \int\frac{dx}{\sqrt[3]{(l\frac{1}{x})^2}} = \sqrt[3]{9}\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}.\int\frac{x\,dx}{\sqrt[3]{(1-x^3)^2}}$$

$$\text{II. } \int\frac{dx}{\sqrt[3]{l\frac{1}{x}}} = \sqrt[3]{9}\int\frac{x\,dx}{\sqrt[3]{(1-x^3)}}.\int\frac{x^3\,dx}{\sqrt[3]{(1-x^3)}}$$

quae posterior forma ob $\int\frac{x^3\,dx}{\sqrt[3]{(1-x^3)}} = \frac{1}{3}\int\frac{dx}{\sqrt[3]{(1-x^3)}}$

abit

abit in

$$\int \frac{dx}{\sqrt[3]{l\frac{1}{x}}} = \sqrt[3]{3}\int \frac{dx}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x\,dx}{\sqrt[3]{(1-x^3)}}.$$

Coroll. 2.

32. Si vti in obseruationibus meis ante allegatis breuitatis gratia ponamus $\int \frac{x^{p-1}\,dx}{\sqrt[3]{(1-x^3)^{3-q}}} = (\frac{p}{q})$, atque vt ibi pro hac classe $(\frac{2}{1}) = \frac{\pi}{3\,\text{sin.}\,\frac{\pi}{3}} = \alpha$, tum vero $(\frac{1}{1}) = \int \frac{dx}{\sqrt[3]{(1-x^3)^2}} = A$, erit

I. $\int \frac{dx}{\sqrt[3]{(l\frac{1}{x})^2}} = \sqrt[3]{9(\frac{1}{1})(\frac{2}{1})} = \sqrt[3]{9\alpha A}$

II. $\int \frac{dx}{\sqrt[3]{(l\frac{1}{x})^1}} = \sqrt[3]{3(\frac{1}{2})(\frac{2}{2})} = \sqrt[3]{\frac{3\alpha\alpha}{A}}.$

Coroll. 3.

33. Pro casu ergo priori habebimus,

$\int dx\sqrt[3]{(l\frac{1}{x})^{-2}} = \sqrt[3]{9\alpha A}$; $\int dx\sqrt[3]{l\frac{1}{x}} = \frac{1}{3}\sqrt[3]{9\alpha A}$ et

$\int dx\sqrt[3]{(l\frac{1}{x})^{3n+1}} = \frac{1}{3}\cdot\frac{4}{3}\cdot\frac{7}{3}\cdots\cdots\frac{3n+1}{3}\sqrt[3]{9\alpha A}$

pro altero vero casu

$\int dx\sqrt[3]{(l\frac{1}{x})^{-1}} = \sqrt[3]{\frac{3\alpha\alpha}{A}}$; $\int dx\sqrt[3]{(l\frac{1}{x})^2} = \frac{2}{3}\sqrt[3]{\frac{3\alpha\alpha}{A}}$ et

$\int dx\sqrt[3]{(l\frac{1}{x})^{3n-1}} = \frac{2}{3}\cdot\frac{5}{3}\cdot\frac{8}{3}\cdots\cdots\frac{3n-1}{3}\sqrt[3]{\frac{3\alpha\alpha}{A}}.$

Problema 4.

34. Denotante i numerum integrum positiuum definire valorem formulae integralis $\int dx(l\frac{1}{x})^{\frac{i}{4}-1}$ integratione ab $x=0$ ad $x=1$ extensa.

Solutio.

In solutione problematis praecedentis perducti sumus ad hanc aequationem

$$\frac{(\int dx(l\frac{1}{x})^{n-1})^2}{\int dx(l\frac{1}{x})^{2n-1}}=kk\int\frac{x^{nk-1}dx}{(1-x^k)^{1-n}}\cdot\int\frac{x^{2nk-1}dx}{(1-x^k)^{1-n}}$$

forma generalis autem sumendo $m=3n$ praebet

$$\frac{\int dx(l\frac{1}{x})^{n-1}\int dx(l\frac{1}{x})^{3n-1}}{\int dx(l\frac{1}{x})^{4n-1}}=k\int\frac{x^{3nk-1}dx}{(1-x^k)^{1-n}}$$

quibus coniungendis adipiscimur,

$$\frac{(\int dx(l\frac{1}{x})^{n-1})^4}{\int dx(l\frac{1}{x})^{4n-1}}=k^3\int\frac{x^{nk-1}dx}{(1-x^k)^{1-n}}\cdot\int\frac{x^{2nk-1}dx}{(1-x^k)^{1-n}}\cdot\int\frac{x^{3nk-1}dx}{(1-x^k)^{1-n}}$$

Sit nunc $n=\frac{i}{4}$ et sumatur $k=4$ fietque

$$\frac{\int dx(l\frac{1}{x})^{\frac{i}{4}-1}}{\sqrt[4]{1.2.3\ldots(i-1)}}=\sqrt[4]{4^3}\int\frac{x^{i-1}dx}{\sqrt[4]{(1-x^4)^{4-i}}}\cdot\int\frac{x^{2i-1}dx}{\sqrt[4]{(1-x^4)^{4-i}}}\cdot\int\frac{x^{3i-1}dx}{\sqrt[4]{(1-x^4)^{4-i}}}$$

Coroll.

Coroll 1.

35. Si igitur ſit $i = 1$, habebimus

$$\int dx \sqrt[4]{(l\tfrac{1}{x})^{-3}} = \sqrt[4]{4^3} \int \frac{dx}{\sqrt[4]{(1-x^4)^3}} \cdot \int \frac{x\,dx}{\sqrt[4]{(1-x^4)^3}} \cdot \int \frac{xx\,dx}{\sqrt[4]{(1-x^4)^3}}$$

quae expreſſio ſi littera P deſignetur erit in genere

$$\int dx \sqrt[4]{(l\tfrac{1}{x})^{4n-3}} = \tfrac{1}{4} \cdot \tfrac{5}{4} \cdot \tfrac{9}{4} \ldots\ldots\ldots \tfrac{4n-3}{4} \cdot P.$$

Coroll 2.

36. Pro altero caſu principali ſumamus $i = 3$ eritque

$$\int dx \sqrt[4]{(l\tfrac{1}{x})^{-1}} = \sqrt[4]{2 \cdot 4^3} \int \frac{x^2\,dx}{\sqrt[4]{(1-x^4)}} \cdot \int \frac{x^5\,dx}{\sqrt[4]{(1-x^4)}} \cdot \int \frac{x^8\,dx}{\sqrt[4]{(1-x^4)}}$$

ſeu facta reductione ad ſimpliciores formas

$$\int dx \sqrt[4]{(l\tfrac{1}{x})^{-1}} = \sqrt[4]{8} \int \frac{xx\,dx}{\sqrt[4]{(1-x^4)}} \cdot \int \frac{x\,dx}{\sqrt[4]{(1-x^4)}} \cdot \int \frac{dx}{\sqrt[4]{(1-x^4)}}$$

quae expreſio ſi littera Q deſignetur erit generatim

$$\int dx \sqrt[4]{(l\tfrac{1}{x})^{4n-1}} = \tfrac{3}{4} \cdot \tfrac{7}{4} \cdot \tfrac{11}{4} \ldots\ldots\ldots \tfrac{4n-1}{4} \cdot Q.$$

Scholion.

37. Si formulam integralem $\int \frac{x^{p-1}\,dx}{\sqrt[4]{(1-x^4)^{4-q}}}$ hoc ſigno $(\frac{p}{q})$ indicemus, ſolutio problematis ita ſe habebit

$$\int dx \sqrt[4]{(l\tfrac{1}{x})^{i-4}} = \sqrt[4]{1 \cdot 2 \cdot 3 \ldots\ldots (i-1) \cdot 4^3 (\tfrac{i}{1})(\tfrac{2i}{2})(\tfrac{3i}{3})}$$

 et

et pro binis caſibus euolutis fit

$$P = \sqrt[4]{4^3 \left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)} \text{ et } Q = \sqrt[4]{8 \left(\tfrac{3}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right)}.$$

Statuamus nunc pro iis formulis quae a circulo pendent:

$$\left(\tfrac{3}{1}\right) = \frac{\pi}{4 \,\text{fin.}\, \frac{\pi}{4}} = \alpha \text{ et } \left(\tfrac{2}{2}\right) = \frac{\pi}{4 \,\text{fin.}\, \frac{2\pi}{4}} = \beta$$

pro tranſcendentibus autem altioris ordinis

$$\left(\tfrac{2}{1}\right) = \int \frac{x\,dx}{\sqrt[4]{(1 - x^4)^3}} = \int \frac{dx}{\sqrt[2]{(1 - x^4)}} = A$$

quippe a qua omnes reliquae pendent ac reperimus,

$$P = \sqrt[4]{4^3 \tfrac{\alpha\alpha}{\beta}.\, AA} \text{ et } Q = \sqrt[4]{4.\, \alpha\alpha\beta.\, \tfrac{1}{AA}}$$

vnde patet eſſe $PQ = 4\alpha = \frac{\pi}{\text{fin.}\, \frac{\pi}{4}}$. Cum autem ſit $\alpha = \frac{\pi}{2\sqrt{2}}$ et $\beta = \frac{\pi}{4}$ erit $P = \sqrt[4]{32\, \pi AA}$ et $Q = \sqrt[4]{\frac{\pi^3}{8 AA}}$ et $\frac{P}{Q} = \frac{4A}{\sqrt{\pi}}$.

Problema 5.

38. Denotante i numerum integrum poſitiuum definire valorem formulae integralis $\int dx \sqrt[5]{(l \tfrac{1}{x})^{i-5}}$ integratione ab $x = 0$ ad $x = 1$ extenſa.

Solutio.

Ex praecedentibus ſolutionibus iam ſatis eſt perſpicuum pro hoc caſu tandem peruentum iri ad hanc formam

$\int dx$

$$\frac{\int dx \sqrt[5]{(l\frac{1}{x})^{i-5}}}{\sqrt[5]{1.2.3\ldots(i-1)}} = \sqrt[5]{5^4} \int \frac{x^{i-1}\,dx}{\sqrt[5]{(1-x^5)^{5-i}}} . \int \frac{x^{2i-1}\,dx}{\sqrt[5]{(1-x^5)^{5-i}}} . \int \frac{x^{3i-1}\,dx}{\sqrt[5]{(1-x^5)^{5-i}}} . \int \frac{x^{4i-1}\,dx}{\sqrt[5]{(1-x^5)^{5-i}}}$$

quae formulae integrales ad classem quintam dissertationis meae supra allegatae sunt referendae. Quare si modo ibi recepto signum $(\frac{p}{q})$ denotet hanc formulam $\int \frac{x^{p-1}\,dx}{\sqrt[5]{(1-x^5)^{5-q}}}$, valorem quaesitum ita commodius exprimere licebit, vt sit

$$\int dx \sqrt[5]{(l\frac{1}{x})^{i-5}} = \sqrt[5]{1.2.3\ldots(i-1)\,5^4 (\frac{i}{i})(\frac{2i}{i})(\frac{3i}{i})(\frac{4i}{i})}$$

vbi quidem sufficit ipsi i valores quinario minores tribuisse: quando autem numeratores quinarium superant tenendum est esse:

$(\frac{5+m}{i}) = \frac{m}{m+i}(\frac{m}{i})$ tum vero porro

$(\frac{10+m}{i}) = \frac{m}{m+i} . \frac{m+5}{m+i+5}(\frac{m}{i})$

$(\frac{15+m}{i}) = \frac{m}{m+i} . \frac{m+5}{m+i+5} . \frac{m+10}{m+i+10} . (\frac{m}{i})$

Deinde vero pro hac classe binae formulae quadraturam circuli inuoluunt quae sint.

$$(\frac{4}{1}) = \frac{\pi}{5\,\text{sin.}\,\frac{\pi}{5}} = \alpha \text{ et } (\frac{3}{2}) = \frac{\pi}{5\,\text{sin.}\,\frac{2\pi}{5}} = \beta$$

duae autem quadraturas altiores continent quae ponantur:

$$\left(\tfrac{3}{1}\right) = \int \frac{x\,x\,d\,x}{\sqrt[5]{(1-x^5)^4}} = \int \frac{d\,x}{\sqrt[5]{(1-x^5)^2}} = \mathrm{A} \text{ et}$$

$$\left(\tfrac{2}{2}\right) = \int \frac{x\,d\,x}{\sqrt[5]{(1-x^5)^2}} = \mathrm{B}$$

atque ex his valores omnium reliquarum formularum huius claſſis aſſignaui ſcilicet:

$$\left(\tfrac{5}{1}\right) = 1;\ \left(\tfrac{5}{2}\right) = \tfrac{1}{2};\ \left(\tfrac{5}{3}\right) = \tfrac{1}{3};\ \left(\tfrac{5}{4}\right) = \tfrac{1}{4};\ \left(\tfrac{5}{5}\right) = \tfrac{1}{5}$$

$$\left(\tfrac{4}{1}\right) = \alpha;\ \left(\tfrac{4}{2}\right) = \tfrac{\beta}{\mathrm{A}};\ \left(\tfrac{4}{3}\right) = \tfrac{\beta}{2\,\mathrm{B}};\ \left(\tfrac{4}{4}\right) = \tfrac{\alpha}{3\,\mathrm{A}}$$

$$\left(\tfrac{3}{1}\right) = \mathrm{A};\ \left(\tfrac{3}{2}\right) = \beta;\ \left(\tfrac{3}{3}\right) = \tfrac{\beta\beta}{\alpha\,\mathrm{B}}$$

$$\left(\tfrac{2}{1}\right) = \tfrac{\alpha\,\mathrm{B}}{\beta};\ \left(\tfrac{2}{2}\right) = \mathrm{B};$$

$$\left(\tfrac{1}{1}\right) = \tfrac{\alpha\,\mathrm{A}}{\beta}.$$

Coroll. 1.

39. Sumto exponente $i = 1$ erit:

$$\int d\,x \sqrt[5]{\left(l\tfrac{1}{x}\right)^{-4}} = \sqrt[5]{5^4 \left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)} = \sqrt[5]{5^4 \cdot \tfrac{\alpha^3}{\beta^2}\,\mathrm{A}^2\,\mathrm{B}}$$

vnde in genere concludimus fore denotante n numerum integrum quemcunque

$$\int dx \sqrt[5]{\left(l\tfrac{1}{x}\right)^{5n-4}} = \tfrac{1}{5} \cdot \tfrac{6}{5} \cdot \tfrac{11}{5} \cdots \tfrac{5n-4}{5} \cdot \sqrt[5]{5^4 \cdot \tfrac{\alpha^3}{\beta^2}\,\mathrm{A}^2\,\mathrm{B}}.$$

Coroll. 2.

40. Sit nunc $i = 2$ et cum prodeat:

$$\int d\,x \sqrt[5]{\left(l\tfrac{1}{x}\right)^{-3}} = \sqrt[5]{1 \cdot 5^4 \left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right)\left(\tfrac{8}{2}\right)}$$

ob $\left(\tfrac{6}{2}\right) = \tfrac{1}{3}\left(\tfrac{1}{2}\right) = \tfrac{1}{3}\left(\tfrac{2}{1}\right)$ et $\left(\tfrac{8}{2}\right) = \tfrac{3}{5}\left(\tfrac{3}{2}\right)$

erit

erit haec expreſſio

$$\sqrt[5]{5^3 \left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)} = \sqrt[5]{5^3 . \alpha \beta . \frac{BB}{A}} \text{ et in genere}$$

$$\int dx \sqrt[5]{(l\tfrac{1}{x})^{5n-3}} = \frac{2}{5} . \frac{7}{5} . \frac{12}{5} \ldots \frac{5n-3}{5} \sqrt[5]{5^3 . \alpha \beta . \frac{BB}{A}}.$$

Coroll 3.

51. Sit $i = 3$ et forma inuenta:

$$\int dx \sqrt[5]{(l\tfrac{1}{x})^{-2}} = \sqrt[5]{2 . 5^4 \left(\frac{3}{3}\right)\left(\frac{6}{3}\right)\left(\frac{9}{3}\right)\left(\frac{12}{3}\right)} \text{ ob}$$

$$\left(\frac{6}{3}\right) = \frac{1}{4}\left(\frac{3}{1}\right); \quad \left(\frac{9}{3}\right) = \frac{4}{7}\left(\frac{4}{3}\right); \quad \left(\frac{12}{3}\right) = \frac{2}{5} . \frac{7}{10}\left(\frac{3}{2}\right)$$

abit in $\sqrt[5]{2 . 5^2 \left(\frac{3}{3}\right)\left(\frac{3}{1}\right)\left(\frac{4}{3}\right)\left(\frac{3}{2}\right)} = \sqrt[5]{5^2 . \frac{\beta^4}{\alpha} . \frac{A}{BB}}$

vnde in genere colligitur:

$$\int dx \sqrt[5]{(l\tfrac{1}{x})^{5n-2}} = \frac{3}{5} . \frac{8}{5} . \frac{13}{5} \ldots \frac{5n-2}{5} \sqrt[5]{5^2 . \frac{\beta^4}{\alpha} . \frac{A}{BB}}.$$

Coroll. 4.

42. Poſito denique $i = 4$ forma noſtra:

$$\int dx \sqrt[5]{(l\tfrac{1}{x})^{-1}} = \sqrt[5]{6 . 5^4 \left(\frac{4}{4}\right)\left(\frac{8}{4}\right)\left(\frac{12}{4}\right)\left(\frac{16}{4}\right)} \text{ ob}$$

$$\left(\frac{8}{4}\right) = \frac{3}{7}\left(\frac{4}{3}\right); \quad \left(\frac{12}{4}\right) = \frac{2}{6} . \frac{7}{11}\left(\frac{4}{2}\right); \quad \left(\frac{16}{4}\right) = \frac{1}{5} . \frac{6}{10} . \frac{11}{15}\left(\frac{4}{1}\right)$$

transformabitur in hanc:

$$\sqrt[5]{6 . 5 \left(\frac{4}{4}\right)\left(\frac{4}{3}\right)\left(\frac{4}{2}\right)\left(\frac{4}{1}\right)} = \sqrt[5]{5 . \frac{\alpha \alpha \beta \beta}{AAB}}$$

ita vt ſit in genere

$$\int dx \sqrt[5]{(l\tfrac{1}{x})^{5n-1}} = \frac{4}{5} . \frac{9}{5} . \frac{14}{5} \ldots \frac{5n-1}{5} \sqrt[5]{5 . \alpha \alpha \beta \beta . \frac{1}{AAB}}.$$

Scho-

Scholion.

43. Si valorem formulae integralis $\int dx \left(l\frac{1}{x}\right)^{\lambda}$ hoc signo $[\lambda]$ repraesentemus, casus hactenus euoluti praebent:

$$[-\tfrac{4}{5}] = \sqrt[5]{5^{4}\cdot\tfrac{\alpha^{3}}{\beta^{2}}\cdot A^{2}B}\,;\quad [+\tfrac{1}{5}] = \tfrac{1}{5}\sqrt[5]{5^{4}\cdot\tfrac{\alpha^{3}}{\beta^{2}}\cdot A^{2}B}$$

$$[-\tfrac{3}{5}] = \sqrt[5]{5^{3}\cdot\alpha\beta\cdot\tfrac{BB}{A}}\,;\quad [+\tfrac{2}{5}] = \tfrac{2}{5}\sqrt[5]{5^{3}\cdot\alpha\beta\cdot\tfrac{BB}{A}}$$

$$[-\tfrac{2}{5}] = \sqrt[5]{5^{2}\cdot\tfrac{\beta^{4}}{\alpha}\cdot\tfrac{A}{BB}}\,;\quad [+\tfrac{3}{5}] = \tfrac{3}{5}\sqrt[5]{5^{2}\cdot\tfrac{\beta^{4}}{\alpha}\cdot\tfrac{A}{BB}}$$

$$[-\tfrac{1}{5}] = \sqrt[5]{5\cdot\alpha^{2}\beta^{2}\cdot\tfrac{1}{AAB}}\,;\quad [+\tfrac{4}{5}] = \tfrac{4}{5}\sqrt[5]{5\cdot\alpha^{2}\beta^{2}\cdot\tfrac{1}{AAB}}$$

vnde binis, quarum indices simul sumti fiunt $= 0$ coniungendis colligimus.

$$[+\tfrac{1}{5}]\cdot[-\tfrac{1}{5}] = \alpha = \frac{\pi}{5\sin.\frac{\pi}{5}}$$

$$[+\tfrac{2}{5}]\cdot[-\tfrac{2}{5}] = 2\beta = \frac{2\pi}{5\sin.\frac{2\pi}{5}}$$

$$[+\tfrac{3}{5}]\cdot[-\tfrac{3}{5}] = 3\beta = \frac{3\pi}{5\sin.\frac{3\pi}{5}}$$

$$[+\tfrac{4}{5}]\cdot[-\tfrac{4}{5}] = 4\alpha = \frac{4\pi}{5\sin.\frac{4\pi}{5}}.$$

Ex antecedente autem problemate simili modo deducimus:

$$[-\tfrac{3}{4}] = P = \sqrt[4]{4^{5}\cdot\tfrac{\alpha\alpha}{\beta}\cdot AA}\,;\quad [+\tfrac{1}{4}] = \tfrac{1}{4}\sqrt[4]{4^{3}\cdot\tfrac{\alpha\alpha}{\beta}\cdot AA}$$

$$[-\tfrac{1}{4}] = Q = \sqrt[4]{4\cdot\alpha\alpha\beta\cdot\tfrac{1}{AA}}\,;\quad [+\tfrac{3}{4}] = \tfrac{3}{4}\sqrt[4]{4\cdot\alpha\alpha\beta\cdot\tfrac{1}{AA}}$$

hinc-

hincque

$$[+\tfrac{1}{4}].\,[-\tfrac{1}{4}] = \alpha = \frac{\pi}{4\,\text{fin.}\,\frac{\pi}{4}}$$

$$[+\tfrac{3}{4}].\,[-\tfrac{3}{4}] = 3\,\alpha = \frac{3\,\pi}{4\,\text{fin.}\,\frac{3\,\pi}{4}}$$

vnde in genere hoc Theorema adipifcimur quod fit

$$[\lambda].\,[-\lambda] = \frac{\lambda\,\pi}{\text{fin.}\,\lambda\,\pi}$$

cuius ratio ex methodo interpolandi olim expofita ita reddi poteft:

$$\text{cum fit } [\lambda] = \frac{1^{1-\lambda}.\,2^{\lambda}}{1+\lambda}.\,\frac{2^{1-\lambda}.\,3^{\lambda}}{2+\lambda}.\,\frac{3^{1-\lambda}.\,4^{\lambda}}{3+\lambda}\ \text{etc.}$$

$$\text{erit } [-\lambda] = \frac{1^{1+\lambda}.\,2^{-\lambda}}{1-\lambda}.\,\frac{2^{1+\lambda}.\,3^{-\lambda}}{2-\lambda}.\,\frac{3^{1+\lambda}.\,4^{-\lambda}}{3-\lambda}\ \text{etc.}$$

hincque

$$[\lambda].\,[-\lambda] = \frac{1.\,1}{1-\lambda\lambda}.\,\frac{2.\,2}{4-\lambda\lambda}.\,\frac{3.\,3}{9-\lambda\lambda}\ \text{etc.} = \frac{\lambda\,\pi}{\text{fin.}\,\lambda\,\pi}$$

vti alibi demonftraui.

Problema 6 generale.

44. Si litterae i et n denotent numeros integros pofitiuos definire valorem formulae integralis $\int dx\,(l\tfrac{1}{x})^{\frac{i-n}{n}}$ feu $\int dx\,\sqrt[n]{(l\tfrac{1}{x})^{i-n}}$, integratione ab $x=0$ ad $x=1$ extenfa.

Solutio.

Methodus hactenus vsitata quaesitum valorem sequenti modo per quadraturas curuarum algebraicarum expressum exhibebit:

$$\frac{\int dx \sqrt[n]{(l\frac{1}{x})^{i-n}}}{\sqrt[n]{1.2.3.(i-1)}} = \sqrt[n]{n^{n-1}} \int \frac{x^{i-1}\,dx}{\sqrt[n]{(1-x^n)^{n-i}}} \cdot \int \frac{x^{2i-1}\,dx}{\sqrt[n]{(1-x^n)^{n-i}}} \cdots \int \frac{x^{(n-1)i-1}\,dx}{\sqrt[n]{(1-x^n)^{n-i}}}.$$

Quod si iam breuitatis gratia formulam integralem $\int \frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ hoc charactere $(\frac{p}{q})$, formulam vero $\int dx \sqrt[n]{(l\frac{1}{x})^m}$ isthoc $[\frac{m}{n}]$ designemus, ita vt $[\frac{m}{n}]$ valorem huius producti indefiniti 1. 2. 3 .... z denotet existente $z = \frac{m}{n}$, succinctius valor quaesitus hoc modo expressus prodibit:

$$[\tfrac{i-n}{n}] = \sqrt[n]{1.2.3\ldots(i-1)\,n^{n-1}.\,(\tfrac{i}{i})(\tfrac{2i}{i})(\tfrac{3i}{i})\ldots\ldots(\tfrac{ni-i}{i})}$$

vnde etiam colligitur

$$[\tfrac{i}{n}] = \tfrac{i}{n}\sqrt[n]{1.2.3\ldots(i-1)\,n^{n-1}.\,(\tfrac{i}{i})(\tfrac{2i}{i})(\tfrac{3i}{i})\ldots(\tfrac{ni-i}{i})}.$$

Hic semper numerum i ipso n minorem accepisse sufficiet quoniam pro maioribus notum est esse:

$$[\tfrac{i+n}{n}] = \tfrac{i+n}{n}[\tfrac{i}{n}];\ \text{item}\ [\tfrac{i+2n}{n}] = \tfrac{i+n}{n}.\tfrac{i+2n}{n}[\tfrac{i}{n}]\ \text{etc.}$$

hocque modo tota inuestigatio ad eos tantum casus reducitur, quibus fractionis $\frac{i}{n}$ numerator i denominatore n est minor. Praeterea vero de formulis integrategralibus $\int \frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^n)^{n-q}}} = (\frac{p}{q})$, ſequentia notaſſe iuvabit:

I. Litteras p et q inter ſe eſſe permutabiles vt ſit $(\frac{p}{q}) = (\frac{q}{p})$.

II. Si alteruter numerorum p vel q ipſi exponenti n aequetur, valorem formulae integralis fore algebraicum ſcilicet:

$$(\frac{n}{p}) = (\frac{p}{n}) = \frac{1}{p}. \text{ ſeu } (\frac{n}{q}) = (\frac{q}{n}) = \frac{1}{q}.$$

III. Si ſumma numerorum $p + q$ ipſi exponenti n aequatur; formulae integralis $(\frac{p}{q})$ valorem per circulum exhiberi poſſe, cum ſit:

$$(\frac{p}{n-p}) = (\frac{n-p}{p}) = \frac{\pi}{n \,\text{ſin.}\, \frac{p\pi}{n}} \text{ et } (\frac{q}{n-q}) = (\frac{n-q}{q}) = \frac{\pi}{n \,\text{ſin.}\, \frac{q\pi}{n}}.$$

IV. Si alteruter numerorum p vel q maior ſit exponente n, formulam integralem $(\frac{p}{q})$ ad aliam revocari poſſe, cuius termini ſint ipſo n minores, quod ſit ope huius reductionis

$$(\frac{p+n}{q}) = \frac{p}{p+q} (\frac{p}{q}).$$

V. Inter plures huiusmodi formulas integrales talem relationem intercedere vt ſit:

$$(\frac{p}{q})(\frac{p+q}{r}) = (\frac{p}{r})(\frac{p+r}{q}) = (\frac{q}{r})(\frac{q+r}{p})$$

cuius ope omnes reductiones reperiuntur quas in obſeruationibus circa has formulas expoſui.

Coroll. 1.

45. Si hoc modo ope reductionis n°. IV. indicatae formam inuentam ad ſingulos caſus accommodemus, eos ſequenti ratione ſimpliciſſime exhibere poterimus. Ac primo quidem pro caſu $n = 2$, quo nulla opus eſt reductione habebimus:

$$[\tfrac{1}{2}] = \tfrac{1}{2} \sqrt[2]{2 (\tfrac{1}{1})} = \tfrac{1}{2} \sqrt[2]{\frac{\pi}{\text{ſin.}\ \frac{\pi}{2}}} = \tfrac{1}{2} \sqrt{\pi}.$$

Coroll. 2.

46. Pro caſu $n = 3$ habebimus has reductiones:

$$[\tfrac{1}{3}] = \tfrac{1}{3} \sqrt[3]{3^2 . (\tfrac{1}{1})(\tfrac{2}{1})}$$

$$[\tfrac{2}{3}] = \tfrac{2}{3} \sqrt[3]{3 . 1 . (\tfrac{2}{2})(\tfrac{1}{2})}.$$

Coroll. 3.

47. Pro caſu $n = 4$ hae tres reductiones obtinentur:

$$[\tfrac{1}{4}] = \tfrac{1}{4} \sqrt[4]{4^3 (\tfrac{1}{1})(\tfrac{2}{1})(\tfrac{3}{1})}$$

$$[\tfrac{2}{4}] = \tfrac{2}{4} \sqrt[4]{4^2 . 2 . (\tfrac{2}{2})^2 (\tfrac{4}{2})} = \tfrac{1}{2} \sqrt[2]{4 (\tfrac{2}{2})} \text{ ob } (\tfrac{4}{2}) = \tfrac{1}{2}$$

$$[\tfrac{3}{4}] = \tfrac{3}{4} \sqrt[4]{4 . 1 . 2 (\tfrac{3}{3})(\tfrac{2}{3})(\tfrac{1}{3})}$$

cum in media ſit $(\tfrac{2}{2}) = (\frac{2}{4-2}) = \frac{\pi}{4}$ erit vtique vt ante

$$[\tfrac{2}{4}] = [\tfrac{1}{2}] = \tfrac{1}{2} \sqrt{\pi}.$$

Coroll.

Coroll. 4.

48. Sit nunc $n = 5$, et prodeunt hae quatuor reductiones:

$$\left[\tfrac{1}{5}\right] = \tfrac{1}{5}\sqrt[5]{5^4}\cdot\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)$$

$$\left[\tfrac{2}{5}\right] = \tfrac{2}{5}\sqrt[5]{5^3\cdot 1}\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)$$

$$\left[\tfrac{3}{5}\right] = \tfrac{3}{5}\sqrt[5]{5^2\cdot 1\cdot 2}\left(\tfrac{3}{3}\right)\left(\tfrac{1}{3}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{2}{3}\right)$$

$$\left[\tfrac{4}{5}\right] = \tfrac{4}{5}\sqrt[5]{5\cdot 1\cdot 2\cdot 3}\left(\tfrac{4}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{1}{4}\right).$$

Coroll. 5.

49. Sit $n = 6$, et habebimus has reductiones:

$$\left[\tfrac{1}{6}\right] = \tfrac{1}{6}\sqrt[6]{6^5}\cdot\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)\left(\tfrac{5}{1}\right)$$

$$\left[\tfrac{2}{6}\right] = \tfrac{2}{6}\sqrt[6]{6^4\cdot 2}\left(\tfrac{2}{2}\right)^2\left(\tfrac{4}{2}\right)^2\left(\tfrac{6}{2}\right) = \tfrac{1}{3}\sqrt[3]{6^2}\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)$$

$$\left[\tfrac{3}{6}\right] = \tfrac{3}{6}\sqrt[6]{6^3\cdot 3\cdot 3}\left(\tfrac{3}{3}\right)^3\left(\tfrac{6}{3}\right)^2 = \tfrac{1}{2}\sqrt[2]{6}\left(\tfrac{3}{3}\right)$$

$$\left[\tfrac{4}{6}\right] = \tfrac{4}{6}\sqrt[6]{6^2\cdot 2\cdot 4\cdot 2}\left(\tfrac{4}{4}\right)^2\left(\tfrac{2}{4}\right)^2\left(\tfrac{6}{4}\right) = \tfrac{2}{3}\sqrt[3]{6\cdot 2}\left(\tfrac{4}{4}\right)\left(\tfrac{2}{4}\right)$$

$$\left[\tfrac{5}{6}\right] = \tfrac{5}{6}\sqrt[6]{6\cdot 1\cdot 2\cdot 3\cdot 4}\left(\tfrac{5}{5}\right)\left(\tfrac{4}{5}\right)\left(\tfrac{3}{5}\right)\left(\tfrac{2}{5}\right)\left(\tfrac{1}{5}\right).$$

Coroll. 6.

50. Posito $n = 7$ sequentes sex prodeunt aequationes:

$$\left[\tfrac{1}{7}\right] = \tfrac{1}{7}\sqrt[7]{7^6}\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)\left(\tfrac{5}{1}\right)\left(\tfrac{6}{1}\right)$$

$$\left[\tfrac{2}{7}\right] = \tfrac{2}{7}\sqrt[7]{7^5\cdot 1}\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right)$$

$$\left[\tfrac{3}{7}\right]=\tfrac{3}{7}\sqrt[7]{7^4.1.2\left(\tfrac{3}{3}\right)\left(\tfrac{6}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{5}{3}\right)\left(\tfrac{1}{3}\right)\left(\tfrac{4}{3}\right)}$$

$$\left[\tfrac{4}{7}\right]=\tfrac{4}{7}\sqrt[7]{7^3.1.2.3\left(\tfrac{4}{4}\right)\left(\tfrac{1}{4}\right)\left(\tfrac{5}{4}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{6}{4}\right)\left(\tfrac{3}{4}\right)}$$

$$\left[\tfrac{5}{7}\right]=\tfrac{5}{7}\sqrt[7]{7^2.1.2.3.4\left(\tfrac{5}{5}\right)\left(\tfrac{3}{5}\right)\left(\tfrac{1}{5}\right)\left(\tfrac{6}{5}\right)\left(\tfrac{4}{5}\right)\left(\tfrac{2}{5}\right)}$$

$$\left[\tfrac{6}{7}\right]=\tfrac{6}{7}\sqrt[7]{7.1.2.3.4.5\left(\tfrac{6}{6}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{4}{6}\right)\left(\tfrac{3}{6}\right)\left(\tfrac{2}{6}\right)\left(\tfrac{1}{6}\right)}.$$

Coroll. 7.

51. Sit $n=8$, et ſeptem hae reductiones impetrabuntur.

$$\left[\tfrac{1}{8}\right]=\tfrac{1}{8}\sqrt[8]{8^7\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)\left(\tfrac{5}{1}\right)\left(\tfrac{6}{1}\right)\left(\tfrac{7}{1}\right)}$$

$$\left[\tfrac{2}{8}\right]=\tfrac{2}{8}\sqrt[8]{8^6.2\left(\tfrac{2}{2}\right)^2\left(\tfrac{4}{2}\right)^2\left(\tfrac{6}{2}\right)^2\left(\tfrac{8}{2}\right)}=\tfrac{1}{4}\sqrt[4]{8^3\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right)}$$

$$\left[\tfrac{3}{8}\right]=\tfrac{3}{8}\sqrt[8]{8^5.1.2\left(\tfrac{3}{3}\right)\left(\tfrac{6}{3}\right)\left(\tfrac{1}{3}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{7}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{5}{3}\right)}$$

$$\left[\tfrac{4}{8}\right]=\tfrac{4}{8}\sqrt[8]{8^4.4.4.4\left(\tfrac{4}{4}\right)^4\left(\tfrac{8}{4}\right)^3}=\tfrac{1}{2}\sqrt[2]{8\left(\tfrac{4}{4}\right)}$$

$$\left[\tfrac{5}{8}\right]=\tfrac{5}{8}\sqrt[8]{8^3.1.2.3.4\left(\tfrac{5}{5}\right)\left(\tfrac{2}{5}\right)\left(\tfrac{7}{5}\right)\left(\tfrac{4}{5}\right)\left(\tfrac{1}{5}\right)\left(\tfrac{6}{5}\right)\left(\tfrac{3}{5}\right)}$$

$$\left[\tfrac{6}{8}\right]=\tfrac{6}{8}\sqrt[8]{8^2.4.2.6.4.2\left(\tfrac{6}{6}\right)^2\left(\tfrac{4}{6}\right)^2\left(\tfrac{2}{6}\right)^2\left(\tfrac{8}{6}\right)}=\tfrac{3}{4}\sqrt[4]{8.2.4\left(\tfrac{6}{6}\right)\left(\tfrac{4}{6}\right)\left(\tfrac{2}{6}\right)}$$

$$\left[\tfrac{7}{8}\right]=\tfrac{7}{8}\sqrt[8]{8.1.2.3.4.5.6\left(\tfrac{7}{7}\right)\left(\tfrac{6}{7}\right)\left(\tfrac{5}{7}\right)\left(\tfrac{4}{7}\right)\left(\tfrac{3}{7}\right)\left(\tfrac{2}{7}\right)\left(\tfrac{1}{7}\right)}.$$

Scholion.

52. Superfluum foret hos caſus vlterius euoluere cum ex allatis ordo iſtarum formularum ſatis perſpiciatur. Si enim in formula propoſita $\left[\frac{m}{n}\right]$ numeri m et n ſint inter ſe primi lex eſt manifeſta, cum fiat

$$\left[\tfrac{m}{n}\right]=\tfrac{m}{n}\sqrt[n]{n^{n-m}.1.2\ldots(m-1).\left(\tfrac{1}{m}\right)\left(\tfrac{2}{m}\right)\left(\tfrac{3}{m}\right)\ldots\left(\tfrac{n-1}{m}\right)}$$

ſin

fin autem hi numeri m et n communem habeant diuiforem expediet quidem fractionem $\frac{m}{n}$ ad minimam formam reduci et ex cafibus praecedentibus quaefitum valorem peti, interim tamen etiam operatio hoc modo inftitui poterit. Cum expreffio quaefita certe hanc habeat formam.

$$[\tfrac{m}{n}] = \tfrac{m}{n}\sqrt[n]{n^{n-m}}\,\mathrm{P}.\,\mathrm{Q}$$

vbi Q eft productum ex $n-1$ formulis integralibus P vero productum ex aliquot numeris abfolutis, primum pro illo producto Q inueniendo, continuetur haec formularum feries $(\frac{m}{m})(\frac{2m}{m})(\frac{3m}{m})$ donec numerator fuperet exponentem n, eiusque loco exceffus fupra n fcribatur, qui fi ponatur $=\alpha$, vt iam formula noftra fit $(\frac{\alpha}{m})$; hic ipfe numerator α dabit factorem producti P tum hinc formularum feries porro ftatuatur $(\frac{\alpha}{m})(\frac{\alpha+m}{m})(\frac{\alpha+2m}{m})$ etc. donec iterum ad numeratorem exponente n maiorem perueniatur, formulaque prodeat $(\frac{n+\beta}{m})$ cuius loco fcribi oportet $(\frac{\beta}{m})$, fimulque hinc factor β in productum P inferatur, ficque progredi conueniet, donec pro Q prodierint $n-1$ formulae. Quae operationes quo facilius intelligantur, cafum formulae $[\frac{9}{12}] = \frac{9}{12}\sqrt[12]{12^3}\,\mathrm{P}.\,\mathrm{Q}$ hoc modo euoluamus, vbi inueftigatio litterarum Q et P ita inftituetur.

Pro Q. . . . $(\frac{9}{9})(\frac{6}{9})(\frac{3}{9})(\frac{12}{9})(\frac{9}{9})(\frac{6}{9})(\frac{3}{9})(\frac{12}{9})(\frac{9}{9})(\frac{6}{9})(\frac{3}{9})$

Pro P 6. 3 9. 6. 3 9. 6. 3

ficque

ſicque reperitur:

$$Q = \left(\tfrac{9}{9}\right)^3 \left(\tfrac{6}{9}\right)^3 \left(\tfrac{3}{9}\right)^3 \left(\tfrac{12}{9}\right)^2 \quad \text{et}$$

$$P = 6^3 .\ 3^3 .\ 9^2 .$$

Cum igitur ſit $\left(\frac{12}{9}\right) = \frac{1}{9}$ ſit $PQ = 6^3 . 3^3 \left(\frac{9}{9}\right) \left(\frac{6}{9}\right)^3 \left(\frac{3}{9}\right)^3$ ideoque

$$\left[\tfrac{9}{12}\right] = \tfrac{3}{4} \sqrt[4]{12 . 6 . 3 . \left(\tfrac{9}{9}\right)\left(\tfrac{6}{9}\right)\left(\tfrac{3}{9}\right)}.$$

Theorema.

53. Quicunque numeri integri poſitiui litteris m et n indicentur, erit ſemper ſignandi modo ante expoſito:

$$\left[\tfrac{m}{n}\right] = \tfrac{m}{n} \sqrt[n]{n^{n-m} . 1 . 2 . 3 \ldots (m-1) \left(\tfrac{1}{m}\right)\left(\tfrac{2}{m}\right)\left(\tfrac{3}{m}\right) \ldots \left(\tfrac{n-1}{m}\right)}.$$

Demonſtratio.

Pro caſu, quo m et n ſunt numeri inter ſe primi, veritas theorematis in antecedentibus eſt euicta, quod autem etiam locum habeat, ſi illi numeri m et n commune diuiſore gaudeant, inde quidem non liquet: verum ex hoc ipſo, quod pro caſibus, quibus m et n ſunt numeri primi, veritas conſtat, tuto concludere licet, theorema in genere eſſe verum. Minime quidem diffiteor hoc concludendi genus prorſus eſſe ſingulare, ac plerisque ſuſpectum videri debere. Quare quo nullum dubium relinquatur quoniam pro caſibus, quibus numeri m et n inter ſe ſunt compoſiti, geminam expreſſionem ſumus nacti, vtriusque conſenſum pro caſibus ante euolutis oſtendiſſe iuuabit. Inſigne autem iam ſup-

peditat

peditat firmamentum cafus $m = n$, quo forma noftra manifefto vnitatem producit.

Coroll. 1.

54. Primus cafus confenfus demonftrationem poftulans eft quo $m = 2$ et $n = 4$, pro quo fupra §. 47. inuenimus

$$\left[\tfrac{2}{4}\right] = \tfrac{2}{4}\sqrt[4]{4^2}\cdot\left(\tfrac{2}{2}\right)^2 \text{ nunc autem vi theorematis eft}$$

$$\left[\tfrac{2}{4}\right] = \tfrac{2}{4}\sqrt[4]{4^2}\cdot 1\left(\tfrac{1}{2}\right)\left(\tfrac{2}{2}\right)\left(\tfrac{3}{2}\right)$$

vnde comparatione inftituta fit $\left(\tfrac{2}{2}\right) = \left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)$ cuius veritas in Obferuationibus fupra allegatis eft confirmata.

Coroll. 2.

55. Si $m = 2$ et $n = 6$, ex fuperioribus (49) eft

$$\left[\tfrac{2}{6}\right] = \tfrac{2}{6}\sqrt[6]{6^4}\cdot\left(\tfrac{2}{2}\right)^2\left(\tfrac{4}{2}\right)^2 \text{ nunc vero per theorema}$$

$$\left[\tfrac{2}{6}\right] = \tfrac{2}{6}\sqrt[6]{6^4}\cdot 1\left(\tfrac{1}{2}\right)\left(\tfrac{2}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{5}{2}\right)$$

ideoque neceffe eft fit

$$\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right) = \left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right)$$

cuius veritas indidem patet.

Coroll. 3.

56. Si $m = 3$ et $n = 6$, peruenitur ad hanc aequationem:

$$\left(\tfrac{3}{3}\right)^2 = 1.\,2\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{5}{3}\right)$$

at ſi $m = 4$ et $n = 6$ fit ſimili modo:

$$2^2 \left(\tfrac{4}{4}\right)\left(\tfrac{2}{4}\right) = 1.2.3 \left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{5}{4}\right)$$

ſeu $\left(\tfrac{4}{4}\right)\left(\tfrac{2}{4}\right) = \tfrac{3}{2}\left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{5}{4}\right)$

quod etiam verum deprehenditur.

Coroll. 4.

57. Caſus $m = 2$ et $n = 8$ praebet hanc aequalitatem:

$$\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right) = \left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right)\left(\tfrac{7}{2}\right)$$

at caſus $m = 4$ et $n = 8$ hanc:

$$\left(\tfrac{4}{4}\right)^3 = 1.2.3 \left(\tfrac{1}{4}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{5}{4}\right)\left(\tfrac{6}{4}\right)\left(\tfrac{7}{4}\right)$$

caſus denique $m = 6$ et $n = 8$ iſtam

$$2.4 \left(\tfrac{6}{8}\right)\left(\tfrac{4}{8}\right)\left(\tfrac{2}{8}\right) = 1.3.5 \left(\tfrac{1}{8}\right)\left(\tfrac{3}{8}\right)\left(\tfrac{5}{8}\right)\left(\tfrac{7}{8}\right)$$

quae etiam veritati ſunt conſentaneae.

Scholion.

58. In genere autem ſi numeri m et n communem habeant factorem 2, et formula propoſita ſit $\left[\tfrac{2m}{2n}\right] = \left[\tfrac{m}{n}\right]$ quia eſt;

$$\left[\tfrac{m}{n}\right] = \tfrac{m}{n} \sqrt[n]{n^{n-m}.\, 1.2.3 \ldots (m-1) \left(\tfrac{1}{m}\right)\left(\tfrac{2}{m}\right)\left(\tfrac{3}{m}\right) \ldots \left(\tfrac{n-1}{m}\right)}$$

erit eadem ad exponentem $2n$ reducta:

$$\tfrac{m}{n} \sqrt[2n]{2n^{2n-2m}.\, 2^2.4^2.6^2 \ldots (2m-2)^2 \left(\tfrac{2}{2m}\right)^2\left(\tfrac{4}{2m}\right)^2\left(\tfrac{6}{2m}\right)^2 \ldots \left(\tfrac{2n-2}{m}\right)^2}$$

Per theorema vero eadem expreſſio fit

$$\tfrac{m}{n} \sqrt[2n]{2n^{2n-2m}.\, 1.2.3 \ldots (2m-1) \left(\tfrac{1}{2m}\right)\left(\tfrac{2}{2m}\right)\left(\tfrac{3}{2m}\right) \ldots \left(\tfrac{2n-1}{2m}\right)}$$

vnde

vnde pro exponente $2n$ erit

$$\frac{2.4.6\ldots(2m-2)(\frac{2}{2m})(\frac{4}{2m})(\frac{6}{2m})\ldots(\frac{2n-2}{2m})}{1.3.5\ldots(2m-1)(\frac{1}{2m})(\frac{3}{2m})(\frac{5}{2m})\ldots(\frac{2n-1}{2m})}=$$

Simili modo si communis diuisor sit 3 pro exponente $3n$ reperietur

$$3^2.6^2.9^2\ldots(3m-3)^2(\frac{3}{3m})^2(\frac{6}{3m})^2(\frac{9}{3m})^2\ldots(\frac{3n-3}{3m})^2=$$
$$1.2.4.5\ldots(3m-2)(3m-1)(\frac{1}{3m})(\frac{2}{3m})(\frac{4}{3m})(\frac{5}{3m})\ldots(\frac{3n-1}{3m})$$

quae aequatio concinnius ita exhiberi potest:

$$\frac{1.\ 2.\ 4.\ 5.\ 7.\ 8.\ 10\ldots(3m-2)(3m-1)}{3^2.\quad 6^2.\quad 9^2\ldots(3m-3)^2}=$$
$$\frac{(\frac{3}{3m})^2\qquad(\frac{6}{3m})^2\ldots(\frac{3n-3}{3m})^2}{(\frac{1}{3m})(\frac{2}{3m})(\frac{4}{3m})(\frac{5}{3m})(\frac{7}{3m})\ldots(\frac{3n-2}{3m})(\frac{3n-1}{3m})}.$$

In genere autem si communis diuisor sit d et exponens dn habebitur.

$$[d.2d.3d\ldots(dm-d)(\frac{d}{dm})(\frac{2d}{dm})(\frac{3d}{dm})\ldots(\frac{dn-d}{dm})]^d=$$
$$1.\ 2.\ 3.\ 4\ldots(dm-1)(\frac{1}{dm})(\frac{2}{dm})(\frac{3}{dm})\ldots(\frac{dn-1}{dm})$$

quae aequatio facile ad quosuis casus accommodari potest vnde sequens Theorema notari meretur.

Theorema.

59. Si α fuerit diuisor communis numerorum m et n haecque formula $(\frac{p}{q})$ denotet valorem inte-

 gralis

gralis $\int \frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ ab $x = 0$ vsque ad $x = 1$ extensi, erit

$$[\alpha . 2\alpha . 3\alpha \ldots\ldots (m-\alpha)(\tfrac{\alpha}{m})(\tfrac{2\alpha}{m})(\tfrac{3\alpha}{m})\ldots\ldots(\tfrac{n-\alpha}{m})]^{\alpha} =$$

$$1 . 2 . 3 \ldots\ldots (m-1)(\tfrac{1}{m})(\tfrac{2}{m})(\tfrac{3}{m})\ldots\ldots(\tfrac{n-1}{m}).$$

Demonſtratio.

Ex praecedente ſcholio veritas huius theorematis perſpicitur, cum enim ibi diuiſor communis eſſet $= d$, binique numeri propoſiti dm et dn horum loco hic ſcripſi m et n loco diuiſoris eorum autem d litteram α quam diuiſoris rationem aequalitas enunciata ita complectitur, vt in progreſſione arithmetica α, 2α, 3α, etc. continuata occurrere aſſumantur ipſi numeri m et n ideoque etiam $m-\alpha$ et $n-\alpha$. Ceterum fateri cogor hanc demonſtrationem vtpote inductioni potiſſimum innixam, neutiquam pro rigoroſa haberi poſſe: cum autem nihilominus de eius veritate ſimus conuicti, hoc theorema eo maiori attentione dignum videtur, interim tamen nullum eſt dubium, quin vberior huiusmodi formularum integralium euolutio tandem perfectam demonſtrationem ſit largitura. quod autem iam ante nobis hanc veritatem perſpicere licuerit, inſigne hinc ſpecimen analyticae inueſtigationis elucet.

Coroll.

Coroll. 1.

60. Si loco signorum adhibitorum ipsas formulas integrales substituamus, theorema nostrum ita se habebit vt sit:

$$\alpha . 2\alpha . 3\alpha \ldots (m-\alpha) \int \frac{x^{\alpha-1}\,dx}{\sqrt[n]{(1-x^n)^{n-m}}} \cdot \int \frac{x^{2\alpha-1}\,dx}{\sqrt[n]{(1-x^n)^{n-m}}} \cdots\cdots \int \frac{x^{n-\alpha-1}\,dx}{\sqrt[n]{(1-x^n)^{n-m}}} =$$

$$\sqrt[\alpha]{1 . 2 . 3 \ldots (m-1)} \int \frac{dx}{\sqrt[n]{(1-x^n)^{n-m}}} \cdot \int \frac{x\,dx}{\sqrt[n]{(1-x^n)^{n-m}}} \cdots\cdots \int \frac{x^{n-2}\,dx}{\sqrt[n]{(1-x^n)^{n-m}}}.$$

Coroll. 2.

61. Vel si ad abbreuiandum statuamus $\sqrt[n]{(1-x^n)^{n-m}} = X$ erit

$$\alpha . 2\alpha . 3\alpha \ldots (m-\alpha) \int \frac{x^{\alpha-1}\,dx}{X} \cdot \int \frac{x^{2\alpha-1}\,dx}{X} \cdots\cdots \int \frac{x^{n-\alpha-1}\,dx}{X} =$$

$$\sqrt[\alpha]{1 . 2 . 3 \ldots (m-1)} \int \frac{dx}{X} \cdot \int \frac{x\,dx}{X} \cdot \int \frac{x^2\,dx}{X} \cdots\cdots \int \frac{x^{n-2}\,dx}{X}.$$

Theorema generale.

62. Si binorum numerorum m et n diuisores communes sint α, β, γ etc. formulaque $\left(\frac{p}{q}\right)$ denotet valorem integralis $\int \frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ ab $x = 0$ ad $x = 1$ extensi sequentes expressiones ex huiusmodi formulis integralibus formatae inter se erunt aequales:

$$[\alpha.\ 2\alpha.\ 3\alpha \ldots\ldots (m-\alpha)(\tfrac{\alpha}{m})(\tfrac{2\alpha}{m})(\tfrac{3\alpha}{m})\ldots\ldots(\tfrac{n-\alpha}{m})]^{\alpha} =$$
$$[\beta.\ 2\beta.\ 3\beta \ldots\ldots (m-\beta)(\tfrac{\beta}{m})(\tfrac{2\beta}{m})(\tfrac{3\beta}{m})\ldots\ldots(\tfrac{n-\beta}{m})]^{\beta} =$$
$$[\gamma.\ 2\gamma.\ 3\gamma \ldots\ldots (m-\gamma)(\tfrac{\gamma}{m})(\tfrac{2\gamma}{m})(\tfrac{3\gamma}{m})\ldots\ldots(\tfrac{n-\gamma}{m})]^{\gamma} \text{ etc.}$$

Demonſtratio.

Ex praecedente Theoremate huius veritas manifeſto ſequitur cum quaelibet harum expreſſionum ſeorſim aequetur huic:

$$1.\ 2.\ 3 \ldots\ldots (m-1)(\tfrac{1}{m})(\tfrac{2}{m})(\tfrac{3}{m})\ldots\ldots\ldots(\tfrac{n-1}{m})$$

quae vnitati vtpote minimo communi diuiſori numerorum m et n conuenit. Tot igitur huiusmodi expreſſiones inter ſe aequales exhiberi poſſunt, quot fuerint diuiſores communes binorum numerorum m et n.

Coroll. 1.

63. Cum ſit haec formula $(\frac{n}{m}) = \frac{1}{m}$, ideoque $m(\frac{n}{m}) = 1$; expreſſiones noſtrae aequales ſuccinctius hoc modo repraeſentari poſſunt:

$$[\alpha.\ 2\alpha.\ 3\alpha \ldots\ldots m(\tfrac{\alpha}{m})(\tfrac{2\alpha}{m})(\tfrac{3\alpha}{m})\ldots\ldots(\tfrac{n}{m})]^{\alpha} =$$
$$[\beta.\ 2\beta.\ 3\beta \ldots\ldots m(\tfrac{\beta}{m})(\tfrac{2\beta}{m})(\tfrac{3\beta}{m})\ldots\ldots(\tfrac{n}{m})]^{\beta} =$$
$$[\gamma.\ 2\gamma.\ 3\gamma \ldots\ldots m(\tfrac{\gamma}{m})(\tfrac{2\gamma}{m})(\tfrac{3\gamma}{m})\ldots\ldots(\tfrac{n}{m})]^{\gamma}.$$

Etſi enim hic factorum numerus eſt auctus, tamen ratio compoſitionis facilius in oculos incurrit.

Coroll.

Coroll. 2.

64. Si ergo ſit $m = 6$ et $n = 12$ ob horum numerorum diuiſores communes 6, 3, 2, 1 quatuor ſequentes formae inter ſe aequales habebuntur:

$$[6(\tfrac{6}{6})(\tfrac{12}{6})]^6 = [3.6(\tfrac{3}{6})(\tfrac{6}{6})(\tfrac{9}{6})(\tfrac{12}{6})]^3 =$$
$$[2.4.6(\tfrac{2}{6})(\tfrac{4}{6})(\tfrac{6}{6})(\tfrac{8}{6})(\tfrac{10}{6})(\tfrac{12}{6})]^2 =$$
$$1.2.3.4.5.6(\tfrac{1}{6})(\tfrac{2}{6})(\tfrac{3}{6})\ldots(\tfrac{12}{6}).$$

Coroll. 3.

65. Si vltima cum penultima combinetur, naſcetur haec aequatio:

$$\frac{1.3.5}{2.4.6} = \frac{(\tfrac{2}{6})(\tfrac{4}{6})(\tfrac{6}{6})(\tfrac{8}{6})(\tfrac{10}{6})(\tfrac{12}{6})}{(\tfrac{1}{6})(\tfrac{3}{6})(\tfrac{5}{6})(\tfrac{7}{6})(\tfrac{9}{6})(\tfrac{11}{6})}$$

vltima autem cum antepenultima comparata praebet:

$$\frac{1.2.4.5}{3.3.6.6} = \frac{(\tfrac{3}{6})(\tfrac{3}{6})(\tfrac{6}{6})(\tfrac{6}{6})(\tfrac{9}{6})(\tfrac{9}{6})(\tfrac{12}{6})(\tfrac{12}{6})}{(\tfrac{1}{6})(\tfrac{2}{6})(\tfrac{4}{6})(\tfrac{5}{6})(\tfrac{7}{6})(\tfrac{8}{6})(\tfrac{10}{6})(\tfrac{11}{6})}.$$

Scholion.

66. Infinitae igitur hinc conſequuntur relationes inter formulas integrales formae:

$$\int \frac{x^{p-n}\,dx}{\sqrt[n]{(1-x^n)^{n-q}}} = (\tfrac{p}{q})$$

quae eo magis ſunt notatu dignae, quod ſingulari prorſus methodo ad eas hic ſumus perducti. Ac ſi quis de earum veritate adhuc dubitet, obſeruationes meas circa has formulas integrales conſulat, indeque

pro

pro quouis casu oblato de veritate facile conuincetur. Etsi autem illa tractatio huic confirmandae inseruit, tamen relationes hic erutae eo maioris sunt momenti, quod in iis certus ordo cernitur, eaeque per omnes classes, quantumuis exponentem n accipere lubeat, facili negotio continuentur, in priori vero tractatione calculus pro classibus altioribus continuo fiat operosior et intricatior.

SVPPLEMENTVM
continens demonstrationem
Theorematis §. 53. propositi.

Demonstrationem hanc altius peti conuenit; sumatur scilicet aequatio §. 25. data, quae posito $f = 1$ et mutatis litteris est:

$$\frac{\int dx\,(l\tfrac{1}{x})^{\nu-1}.\int dx\,(l\tfrac{1}{x})^{\mu-1}}{\int dx\,(l\tfrac{1}{x})^{\mu+\nu-1}} = \varkappa\int\frac{x^{\varkappa\mu-1}\,dx}{(1-x^{\varkappa})^{1-\nu}},$$

eaque per reductiones notas hac forma repraesentetur:

$$\frac{\int dx\,(l\tfrac{1}{x})^{\nu}.\int dx\,(l\tfrac{1}{x})^{\mu}}{\int dx\,(l\tfrac{1}{x})^{\mu+\nu}} = \frac{\varkappa\mu\nu}{\mu+\nu}\int\frac{x^{\varkappa\mu-1}\,dx}{(1-x^{\varkappa})^{1-\nu}}.$$

Statuatur nunc $\nu = \frac{m}{n}$ et $\mu = \frac{\lambda}{n}$ tum vero $\varkappa = n$ vt habeamus;

$$\frac{\int dx\,(l\tfrac{1}{x})^{\frac{m}{n}}.\int dx\,(l\tfrac{1}{x})^{\frac{\lambda}{n}}}{\int dx\,(l\tfrac{1}{x})^{\frac{\lambda+m}{n}}} = \frac{\lambda m}{\lambda+m}\int\frac{x^{\lambda-1}\,dx}{\sqrt[n]{(1-x^{n})^{n-m}}}$$

qua

quae breuitatis gratia, more supra vsitato, ita concinne referatur:

$$\frac{[\frac{m}{n}]\,[\frac{\lambda}{n}]}{[\frac{\lambda+m}{m}]} = \frac{\lambda m}{\lambda+m} \cdot \left(\frac{\lambda}{m}\right)$$

Iam loco λ successiue scribantur numeri $1, 2, 3, 4 \ldots n$ omnesque hae aequationes, quarum numerus est $= n$ in se inuicem ducantur, et aequatio resultans erit:

$$[\tfrac{m}{n}]^n \cdot \frac{[\frac{1}{n}][\frac{2}{n}][\frac{3}{n}] \ldots \ldots [\frac{n}{n}]}{[\frac{m+1}{n}][\frac{m+2}{n}][\frac{m+3}{n}] \ldots [\frac{m+n}{n}]} =$$

$$m^n \cdot \frac{1}{m+1} \cdot \frac{2}{m+2} \cdot \frac{3}{m+3} \ldots \ldots \frac{n}{m+n} \left(\tfrac{1}{m}\right)\left(\tfrac{2}{m}\right)\left(\tfrac{3}{m}\right) \ldots \left(\tfrac{n}{m}\right) =$$

$$m^n \cdot \frac{1 \cdot 2 \cdot 3 \ldots \ldots m}{(n+1)(n+2)(n+3) \ldots (n+m)} \left(\tfrac{1}{m}\right)\left(\tfrac{2}{m}\right)\left(\tfrac{3}{m}\right) \ldots \left(\tfrac{n}{m}\right).$$

Simili autem modo pars prior transformetur vt fit

$$[\tfrac{m}{n}]^n \cdot \frac{[\frac{1}{n}][\frac{2}{n}][\frac{3}{n}] \ldots \ldots [\frac{m}{n}]}{[\frac{n+1}{n}][\frac{n+2}{n}][\frac{n+3}{n}] \ldots [\frac{n+m}{n}]}$$

cuius conuenientia cum forma praecedente multiplicando per crucem, vt aiunt, sponte se prodit. Cum vero ex natura harum formularum fit

$$[\tfrac{n+1}{n}] = \tfrac{n+1}{n}[\tfrac{1}{n}];\ [\tfrac{n+2}{n}] = \tfrac{n+2}{n}[\tfrac{2}{n}];\ [\tfrac{n+3}{n}] = \tfrac{n+3}{n}[\tfrac{3}{n}] \text{ etc.}$$

ob harum formularum numerum $= m$, euadet haec prior pars:

$$[\tfrac{m}{n}]^n \cdot \frac{n^m}{(n+1)(n+2)(n+3) \ldots (n+m)}$$

quae cum aequalis sit parti alteri ante exhibitae:

$$m^n \cdot \frac{1 \cdot 2 \cdot 3 \ldots \ldots m}{(n+1)(n+2)(n+3) \ldots (n+m)} \left(\tfrac{1}{m}\right)\left(\tfrac{2}{m}\right)\left(\tfrac{3}{m}\right) \ldots \left(\tfrac{n}{m}\right)$$

adipiſcimur hanc aequationem:

$$\left[\frac{m}{n}\right]^n = \frac{m^n}{n^m} . 1 . 2 . 3 \ldots m \left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\ldots\ldots\left(\frac{n}{m}\right)$$

ita vt ſit

$$\left[\frac{m}{n}\right] = m \sqrt[n]{\frac{1 . 2 . 3 \ldots m}{n^m}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\ldots\left(\frac{n}{m}\right)}$$

quae cum propoſita in §. 53. ob $\left(\frac{n}{m}\right) = \frac{1}{m}$ omnino congruit, ex quo eius veritas nunc quidem ex principiis certiſſimis eſt euicta.

Demonſtratio Theorematis

§. 59. propoſiti.

Etiam hoc Theorema firmiori demonſtratione indiget, quam ex aequalitate ante ſtabilita:

$$\frac{\left[\frac{m}{n}\right] . \left[\frac{\lambda}{n}\right]}{\left[\frac{\lambda + m}{n}\right]} = \frac{\lambda m}{\lambda + m}\left(\frac{\lambda}{m}\right)$$

ita adorno. Exiſtente α communi diuiſore numerorum m et n, loco λ ſucceſſiue ſcribantur numeri α, 2α, 3α etc. vsque ad n, quorum multitudo eſt $= \frac{n}{\alpha}$ atque omnes aequalitates hoc modo reſultantes in ſe inuicem ducantur, vt prodeat haec aequatio

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} . \frac{\left[\frac{\alpha}{n}\right]\left[\frac{2\alpha}{n}\right]\left[\frac{3\alpha}{n}\right]\ldots\ldots\ldots\ldots\left[\frac{n}{n}\right]}{\left[\frac{m+\alpha}{n}\right]\left[\frac{m+2\alpha}{n}\right]\left[\frac{m+3\alpha}{n}\right]\ldots\ldots\left[\frac{m+n}{n}\right]} =$$

$$m^{\frac{n}{\alpha}} . \frac{\alpha}{m+\alpha} . \frac{2\alpha}{m+2\alpha} . \frac{3\alpha}{m+3\alpha} \ldots \frac{n}{m+n}\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\ldots\left(\frac{n}{m}\right).$$

Iam

Iam prior pars in hanc formam ipſi aequalem transmutetur:

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} \cdot \frac{\left[\frac{\alpha}{n}\right]\left[\frac{2\alpha}{n}\right]\left[\frac{3\alpha}{n}\right]\ldots\ldots\ldots\left[\frac{m}{n}\right]}{\left[\frac{n+\alpha}{n}\right]\left[\frac{n+2\alpha}{n}\right]\left[\frac{n+3\alpha}{n}\right]\ldots\left[\frac{n+m}{n}\right]}$$

quae ob $\left[\frac{n+\alpha}{n}\right] = \frac{n+\alpha}{n}\left[\frac{\alpha}{n}\right]$ ſicque de ceteris reducitur ad hanc:

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} \cdot \frac{n}{n+\alpha} \cdot \frac{n}{n+2\alpha} \cdot \frac{n}{n+3\alpha} \ldots\ldots \frac{n}{n+m}.$$

Poſterior vero aequationis pars ſimili modo transformatur in:

$$m^{\frac{n}{\alpha}} \cdot \frac{\alpha}{n+\alpha} \cdot \frac{2\alpha}{n+2\alpha} \cdot \frac{3\alpha}{n+3\alpha} \ldots \frac{m}{n+m} \left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\ldots\left(\frac{n}{m}\right)$$

vnde enaſcitur haec aequatio:

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} n^{\frac{m}{\alpha}} = m^{\frac{n}{\alpha}} \cdot \alpha .\ 2\alpha .\ 3\alpha \ldots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\ldots\left(\frac{n}{m}\right)$$

hincque

$$\left[\frac{m}{n}\right] = m \sqrt[n]{\frac{1}{m^n}} \left(\alpha .\ 2\alpha .\ 3\alpha \ldots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\ldots\left(\frac{n}{m}\right)\right)^{\alpha}$$

quae expreſſio cum praecedente comparata praebet hanc aequationem:

$$\left(\alpha .\ 2\alpha ,\ 3\alpha \ldots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\ldots\left(\frac{n}{m}\right)\right)^{\alpha} =$$

$$1.\ 2.\ 3 \ldots m\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\ldots\left(\frac{n}{m}\right)$$

quod de omnibus diuiſoribus communibus binorum numerorum m et n eſt intelligendum.
