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Solutio problematis, quo duo quaeruntur numeri, quorum productum tam summa quam differentia eorum sive auctum sive minutum fiat quadratum

Leonhard Euler

SOLVTIO PROBLEMATIS,

QVO DVO QVAERVNTVR NVMERI, QVORVM PRODVCTVM TAM SVMMA, QVAM DIFFERENTIA EORVM, SIVE AVCTVM SIVE MINVTVM FIAT QVADRATVM.

Auctore

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1. Problema hoc mihi ante complures annos Berolini a Centurione quodam Prussico erat propositum, quod se, Lipsiae ab amico accepisse aiebat; neque vero se neque istum amicum solutionem vllo modo inuenire potuisse. Quaerebat igitur ex me vtrum hoc Problema possibile iudicarem nec ne? Statim quidem hoc problema mihi ob elegantiam mirifice placebat et quum facile summam solutionis difficultatem perspexissem, id omnino dignum iudicaui in quo vires meas exercerem. Tandem vero post plura tentamina solutionem sum adeptus, quae ita se habebat: Positis duobus numeris quaesitis A et B, inueni $A = \frac{13 \cdot 29^2}{8 \cdot 9^2} = \frac{10933}{648}$ et $B = \frac{5 \cdot 29^2}{3^2 \cdot 11^2} = \frac{4205}{3872}$.

2. Via autem qua ad hanc solutionem perveni, ita erat comparata, vt nullo modo mihi liceret, alias solutiones inde eruere; etiamsi nullus dubitandi locus relinqueretur,

retur, quin hoc problema innumerabiles admitteret solutiones. Nuper autem cum in hoc idem argumentum incidissem, casu prorsus fortuito methodus mihi se obtulit, infinitas solutiones huius Problematis eliciendi. Quod quum casui prorsus singulari sit acceptum referendum, quaestio haec omnino digna mihi est visa, quam accuratius perscrutarer. Quare primo quidem solutionem generalem proponam, deinde vero artificium illud, quod mihi infinitas solutiones suppeditauit, vberius euoluam.

Solutio Problematis generalis.

3. Si literae A et B denotent ambos numeros quaesitos, necesse est, vt sequentes quatuor formulae quadrata efficiantur:

$$\text{I. } AB + A + B = \square; \quad \text{II. } AB + A - B = \square;$$
$$\text{III. } AB - A + B = \square; \quad \text{IV. } AB - A - B = \square.$$

Quum autem statim pateat, hos numeros integros esse non posse, ob rationes mox perspiciendas, eos ita expressos assumo, vt sit $A = \frac{z}{x}$ et $B = \frac{z}{y}$, ita vt quatuor sequentes formulae ad quadrata reducendae habeantur:

$$\text{I. } \tfrac{z}{xy}(z + y + x) = \square; \quad \text{II. } \tfrac{z}{xy}(z + y - x) = \square;$$
$$\text{III. } \tfrac{z}{xy}(z - y + x) = \square; \quad \text{IV. } \tfrac{z}{xy}(z - y - x) = \square.$$

4. Quod si ergo factor communis fuerit quadratum, quatuor sequentes formulas quadrata effici oportet, quas qui-

quidem per ambiguitatem signorum ita duabus formulis comprehendere licet:

I. et II. $z + y \pm x = \square$; III. et IV. $z - y \pm x = \square$.

Quare quum in genere sit $aa + bb \pm 2ab = \square$ similique modo $cc + dd \pm 2cd = \square$; statuamus vt sequitur:

$$z + y = aa + bb;\quad x = 2ab$$
$$z - y = cc + dd;\quad x = 2cd.$$

Vt autem fiat $2ab = 2cd$, statuatur vtrumque $= 2pqrs = x$, sumaturque $a = pq$; $b = rs$; $c = pr$; et $d = qs$ eritque

$$z + y = aa + bb = ppqq + rrss \text{ et}$$
$$z - y = cc + dd = pprr + qqss$$ vnde colligitur
$$z = \frac{(pp + ss)(qq + rr)}{2} \text{ et } y = \frac{(pp - ss)(qq - rr)}{2}$$

tum vero erit

$$\text{I.}\quad z + y + x = (a + b)^2 = (pq + rs)^2$$
$$\text{II.}\quad z + y - x = (a - b)^2 = (pq - rs)^2$$
$$\text{III.}\quad z - y + x = (c + d)^2 = (pr + qs)^2$$
$$\text{IV.}\quad z - y - x = (c - d)^2 = (pr - qs)^2.$$

5. Superest igitur, vt etiam factor communis $\frac{z}{xy}$ quadratum reddatur, qui euolutus praebet hanc formulam:

$$\frac{z}{xy} = \frac{(pp + ss)(qq + rr)}{2pqrs\,(pp - ss)\,qq - rr)}$$

at vero in hoc efficiendo summa consistit difficultas; quodsi enim numerator in denominatorem ducatur, vt haec formula quadratum fieri debeat:

$$2pqrs(pp-ss)(qq-rr)(pp+ss)(qq+rr) = \square$$

singulae litterae ad quinque dimensiones assurgunt, cuiusmodi quaestiones in Analysi Diophantea adhuc non sunt tractari solitae; ceterum iam olim post plura tentamina reperi huic conditioni satisfieri, sumendo $p=13$, $s=11$, $q=16$, et $r=11$, vti periculum facienti mox patebit.

6. Quodsi autem quocunque modo huiusmodi valores idonei pro literis p; q; r; s fuerint inuenti, solutio problematis inde ita adstruitur:

Posita formula $\frac{(pp+ss)(qq+rr)}{2pqrs(pp-ss)(qq-rr)} = \frac{M^2}{N^2}$, primo ambo numeri quaesiti, ita erunt expressi

$$A = \frac{(pp+ss)(qq+rr)}{4pqrs} \text{ et } B = \frac{(pp+ss)(qq+rr)}{(pp-ss)(qq-rr)}$$

tum vero conditionibus problematis ita satisfiet vt sit,

$$\text{I. } \sqrt{(AB+A+B)} = \frac{M}{N}(pq+rs)$$

$$\text{II. } \sqrt{(AB+A-B)} = \frac{M}{N}(pq-rs)$$

$$\text{III. } \sqrt{(AB-A+B)} = \frac{M}{N}(pr+qs)$$

$$\text{IV. } \sqrt{(AB-A-B)} = \frac{M}{N}(pr-qs).$$

Singularis Euolutio nostrae formulae, quae ad quadratum est reuocanda.

7. Quum omnis opera in hac formula reducenda frustra consumatur, quamdiu in ea tot diversae quantitates occurrunt, earumque singulae ad tot dimensiones assurgunt, ante omnia elaborandum est, vt diuersis factoribus de-

denominatoris communes diuisores concilientur; hunc in finem vsus sum sequentibus positionibus: $p+s=\alpha\beta$; $p-s=\varepsilon\zeta$; $q+r=\alpha\gamma$; et $q-r=\varepsilon\eta$, ita vt fiat $p=\frac{\alpha\beta+\varepsilon\zeta}{2}$; $s=\frac{\alpha\beta-\varepsilon\zeta}{2}$; $q=\frac{\alpha\gamma+\varepsilon\eta}{2}$ et $r=\frac{\alpha\gamma-\varepsilon\eta}{2}$; tum vero nostra conditio principalis postulat, vt sit:

$$\frac{(pp+ss)(qq+rr)}{2pqrs.\beta\gamma\zeta\eta.\alpha^2\varepsilon^2}=\frac{M^2}{N^2} \text{ siue}$$

$$\frac{(pp+ss)(qq+rr)}{2pqrs.\beta\gamma\zeta\eta}=\frac{M^2}{N^2}.\alpha^2\varepsilon^2.$$

8. Secundo constituatur ratio inter litteras r et s, quae sit vt $f:g$, eritque $f:g::\alpha\gamma-\varepsilon\eta:\alpha\beta-\varepsilon\zeta$ siue $g(\alpha\gamma-\varepsilon\eta)=f(\alpha\beta-\varepsilon\zeta)$, vnde colligitur $\alpha(f\beta-g\gamma)=\varepsilon(f\zeta-g\eta)$, quocirca ponamus: $\alpha=f\zeta-g\eta$; $\varepsilon=f\beta-g\gamma$; tum vero habebitur

$$p=\frac{2f\beta\zeta-g\beta\eta-g\gamma\zeta}{2};\ q=\frac{f\beta\eta+f\zeta\gamma-2g\gamma\eta}{2};$$

$$r=\frac{f(\gamma\zeta-\beta\eta)}{2} \text{ et } s=\frac{g(\gamma\zeta-\beta\eta)}{2}.$$

9. Vt adhuc plures factores in denominatore communes reddamus; faciamus insuper $q=h\beta\zeta$ vnde haec aequatio emergit: $2h\beta\zeta=f\beta\eta+f\zeta\gamma-2g\gamma\eta$ siue $\beta(2h\zeta-f\eta)=\gamma(f\zeta-2g\eta)$ quam ob rem ponamus $\beta=f\zeta-2g\eta$ et $\gamma=2h\zeta-f\eta$. Ex his autem valoribus porro colligimus: $\alpha=f\zeta-g\eta$; $\varepsilon=(ff-2gh)\zeta-fg\eta$;

$$p+s=(f\zeta-g\eta)(f\zeta-2g\eta)=ff\zeta\zeta-3fg\zeta\eta+2gg\eta\eta$$

$$p-s=\zeta((ff-2gh)\zeta-fg\eta)=(ff-2gh)\zeta\zeta-fg\zeta\eta$$

$$q+r=(f\zeta-g\eta)(2h\zeta-f\eta)=2fh\zeta\zeta-(ff+2hg)\zeta\eta+fg\eta\eta$$

$$q-r=\eta((ff-2gh)\zeta-fg\eta)=(ff-2hg)\zeta\eta-fg\eta\eta$$

 hinc-

hincque porro:

$$p = (ff - gh)\zeta\zeta - 2fg\zeta\eta + gg\eta\eta$$
$$s = gh\zeta\zeta - fg\zeta\eta + gg\eta\eta = g(h\zeta\zeta - f\zeta\eta + g\eta\eta)$$
$$q = fh\zeta\zeta - 2gh\zeta\eta = h\zeta(f\zeta - 2g\eta)$$
$$r = fh\zeta\zeta - ff\zeta\eta + fg\eta\eta = f(h\zeta\zeta - f\zeta\eta + g\eta\eta).$$

10. Denique hos valores ita determinemus, vt numerus p diuisor euadat formulae $qq + rr$, iam vero inuenitur:

$$qq + rr = ffgg\eta^4 - 2f^3g\eta^3\zeta + (f^4 + 2ffgh + 4gghh)\eta\eta\zeta\zeta - 2fh(ff + 2gh)\eta\zeta^3 + 2ffhh\zeta^4$$

quare quum sit $p = gg\eta\eta - 2fg\eta\zeta + (ff - gh)\zeta\zeta$, vt p fiat factor illius formulae, statuatur alter factor $ff\eta\eta + t\zeta\eta + u\zeta\zeta$ eritque productum:

$$\begin{array}{lllll} ffgg\eta^4 & -2f^3g\eta^3\zeta & +(f^4 - ffgh)\eta\eta\zeta\zeta & +t(ff-gh)\eta\zeta^3 & +u(ff-gh)\zeta^4 \\ & +tgg & -2tfg & -2ufg & \\ & & +ugg & & \end{array}$$

vbi primi termini iam congruunt, secundi vero dant $t = 0$, tertii $3ffgh + 4gghh = ugg$, vnde $u = \frac{3ffh}{g} + 4hh$; quarti porro praebent $u = \frac{h(ff + 2gh)}{g}$; quinti vero tandem dant $u = \frac{2ffhh}{ff - gh}$. Necesse igitur est, vt hi tres valores ipsius u inter se congruant, primus vero cum secundo collatus dat $3ffh + 4ghh = hff + 2ghh$, seu $2ffh + 2ghh = 0$, ideoque $ff + gh = 0$; at secundus tertio aequatus dat $f^4 - ffgh - 2gghh = 0$, siue $(ff + gh)(ff - 2gh) = 0$, vtrin-

vtrinque ergo conditioni satisfit vno eodemque valore $h = -\frac{ff}{g}$.

11. Quoniam igitur inuenimus $h = -\frac{ff}{g}$ reliqui valores sequenti modo exprimentur:

$$p = 2ff\zeta\zeta - 2fg\zeta\eta + gg\eta\eta$$
$$q = -\frac{ff}{g}.\zeta(f\zeta - 2g\eta) = 2ff\zeta\eta - \frac{f^3}{g}.\zeta\zeta$$
$$r = -\frac{f^3}{g}.\zeta\zeta - ff\zeta\eta + fg\eta\eta$$
$$s = -ff\zeta\zeta - fg\zeta\eta + gg\eta\eta,$$

vbi notatu dignum euenit, vt in valoribus p et s producta $f\zeta$ et $g\eta$, tamquam simplices quantitates occurrant, quod quidem in litteris q et r non accidit. Verum quia totum negotium, tantum in ratione q ad r versatur, hi ambo valores multiplicentur per $-\frac{g}{f}$, vt sit $q = ff\zeta\zeta - 2fg\zeta\eta$ et $r = ff\zeta\zeta + fg\zeta\eta - gg\eta\eta$; hanc ob rem vt formulas nostras in compendium redigamus atque adeo ad duas quantitates reuocemus, statuamus $f\zeta = m$ et $g\eta = n$, quo facto nostrae quatuor literae ita se habebunt:

$$p = 2mm - 2mn + nn;\ q = mm - 2mn = m(m - 2n);$$
$$s = -mm - mn + nn;\ r = mm + mn - nn.$$

12. Quoniam vero res eodem redit siue quaepiam litera positiue, siue negatiue accipiatur, ponamus

$$p = 2mm - 2mn + nn;\ q = mm - 2mn = m(m - 2n)$$
$$s = r = mm + mn - nn;\ \text{vnde fit}$$
$$p + s = 3mm - mn = m(3m - n)$$

 p —

$$p - s = mm - 3mn + 2nn = (m - n)(m - 2n)$$
$$q + r = 2mm - mn - nn = (m - n)(2m + n)$$
$$q - r = -3mn + nn = -n(3m - n).$$

Hic signum negationis in valore $q - r$, nihil plane turbat, tantum enim opus est litteras q et r inter se permutari, ita vt sit

$$p = 2mm - 2mn + nn;\ q = mm + mn - nn$$
$$s = mm + mn - nn;\ r = mm - 2mn = m(m - 2n)$$

vnde fit

$$p + s = 3mm - mn = m(3m - n)$$
$$p - s = mm - 3mn + 2nn = (m - n)(m - 2n)$$
$$q + r = 2mm - mn - nn = (2m + n)(m - n)$$
$$q - r = 3mn - nn \qquad = n(3m - n)$$

quibus valoribus in sequenti calculo vtemur.

13. His constitutis valoribus, pro numeratore nostrae fractionis habebimus:

$$pp + ss = 5m^4 - 6m^3n + 7mmnn - 6mn^3 + 2n^4, \text{ seu}$$
$$pp + ss = (mm + nn)(5mm - 6mn + 2nn) \text{ et}$$
$$qq + rr = 2m^4 - 2m^3n + 3mmnn - 2mn^3 + n^4, \text{ siue}$$
$$qq + rr = (mm + nn)(2mm - 2mn + nn)$$

vnde fractio nostra ad quadratum reducenda erit:

$$\frac{MM}{NN} = \frac{(5mm - 6mn + 2nn)\ (mm + nn)^2}{2n(2m + n).m^2\ (m - n)^2 (m - 2n)^2 (3m - n)^2 (mm + mn + nn)^2}$$

hincque colligimus:

$$\frac{M}{N} = \frac{mm + nn}{m(m - n)(m - 2n)(3m - n)(mm + mn - nn)} \cdot \sqrt{\frac{5mm - 6mn + 2nn}{2n(2m + n)}};$$

to-

totum ergo negotium huc est reductum, vt formula $\frac{5mm - 6mn + 2nn}{2n(2m+n)}$ quadratum efficiatur, id quod infinitis modis praestari posse manifestum est, statim atque vnicus casus innotuerit.

14. Quo haec forma tractabilior reddatur, ponamus $2m - n = l$, vt sit $n = 2m - l$ et formula ad quadratum reducenda erit: $\frac{mm - 2ml + 2ll}{(4m - 2l)(4m - l)}$, vbi productum ex numeratore in denominatorem euolutum quippe quod etiam quadratum esse debet, perducit ad hanc conditionem

$$16m^4 - 44m^3l + 58mmll - 28ml^3 + 4l^4 = \square$$

cuius quum ambo termini extremi iam sint quadrati per methodos satis cognitas facile est innumerabiles solutiones inuestigare; quem in finem ponamus $\frac{m}{l} = z$ vt habeamus hanc formulam $16z^4 - 44z^3 + 58zz - 28z + 4 = \square$; quae ponendo $z = y - 2$; transit in hanc:

$$16y^4 - 172y^3 + 706yy - 1300y + 900 = \square$$

vbi iterum ambo extremi termini sunt quadrata.

15. Ad hoc negotium expediendum, praestabit resolutionem nostrae aequationis siue prioris, siue posterioris in genere docere. Sit igitur proposita haec aequatio generalis:

$$\alpha\alpha z^4 - 2\beta z^3 + \gamma zz - 2\delta z + \varepsilon\varepsilon = \square;$$

atque pro idoneis valoribus ipsius z sequentes quatuor formulae per methodos consuetas reperiuntur:

I.

$$\text{I.}\quad z = \frac{2\alpha(\beta\varepsilon - \alpha\delta)}{2\alpha^3\varepsilon + \beta\beta - \alpha\alpha\gamma}$$

$$\text{II.}\quad z = \frac{2\alpha\varepsilon^3 + \delta\delta - \gamma\varepsilon\varepsilon}{2\varepsilon(\alpha\delta - \beta\varepsilon)}$$

$$\text{III.}\quad z = \frac{(2\alpha^3\varepsilon + \alpha\alpha\gamma - \beta\beta)(2\alpha^3\varepsilon - \alpha\alpha\gamma + \beta\beta)}{4\alpha\alpha(2\alpha^4\delta - \alpha\alpha\beta\gamma + \beta^3)}$$

$$\text{IV.}\quad z = \frac{4\varepsilon\varepsilon(2\beta\varepsilon^4 - \gamma\delta\varepsilon\varepsilon + \delta^3)}{(2\alpha\varepsilon^3 + \gamma\varepsilon\varepsilon - \delta\delta)(2\alpha\varepsilon^3 - \gamma\varepsilon\varepsilon + \delta\delta)}$$

vbi quum litterae α et ε pro lubitu tam positiue quam negatiue accipi queant, binae priores formulae geminos valores suppeditant.

16. Quemadmodum autem innumerabiles huius aequationis solutiones inueniri oporteat, sequenti modo calculus instituatur. Sit f valor quicunque per praecedentes formulas inuentus, ita vt nostra expressio

$$\alpha\alpha z^4 - 2\beta z^3 + \gamma zz - 2\delta z + \varepsilon\varepsilon,$$

posito $z = f$ fiat quadratum, sitque propterea

$$\alpha\alpha f^4 - 2\beta f^3 + \gamma ff - 2\delta f + \varepsilon\varepsilon = gg;$$

nunc igitur ponatur $z = x + f$ et nostra aequatio induet hanc formam:

$$\begin{array}{lllll} \alpha\alpha x^4 + 4\alpha\alpha x^3 & + 6\alpha\alpha ff & + 4\alpha\alpha f^3 & + gg = \square \\ - 2\beta & - 6\beta f \; xx & - 6\beta ff x & \\ & + \gamma & + 2\gamma f & \\ & & - 2\delta & \end{array}$$

quae aequatio breuitatis gratia ita repraesentetur:

$$a\,a\,x^4 - 2\,b\,x^3 + c\,x\,x - 2\,d\,x + e\,e = \square$$

ita vt sit $aa = \alpha\alpha$; $b = \beta - 2\alpha\alpha$; $c = \gamma - 6\beta f + 6\alpha\alpha ff$, $d = \delta - \gamma f + 3\beta ff - 2\alpha\alpha f^3$; ac denique $ee = gg$, vbi

sumi

sumi potest $a = \pm \alpha$ et $e = \pm g$. Tum vero quattuor noui valores pro z inueniuntur sequentes:

$$\text{I.}\quad z = f + \frac{2a(be - ad)}{2a^3e + bb - aac}$$

$$\text{II.}\quad z = f + \frac{2ae^3 + dd - cee}{2e(aa - be)}$$

$$\text{III.}\quad z = f + \frac{(2a^3e + aac - bb)(2a^3e - aac + bb)}{4aa(2a^4d - aabc + b^3)}$$

$$\text{IV.}\quad z = f + \frac{4ee(2be^4 - cdee + d^3)}{(2ae^3 + cee - dd)(2ae^3 - cee + dd)}$$

quoniam igitur quemcunque valorem pro z hoc modo inventum assumere licet, hinc numerus solutionum in infinitum augeri poterit.

17. Postquam autem pro z valor quicunque idoneus fuerit inuentus, qui sit $z = \frac{h}{k}$, ob $z = \frac{m}{l} = \frac{m}{2m - n}$, habebimus $m = h$ et $n = 2h - k$, ex quibus duobus numeris m et n reliquae quantitates sequenti modo determinantur:

$$p = 2mm - 2mn + nn;\quad q = mm + mn - nn;$$

$$s = mm + mn - nn;\quad r = mm - 2mn = m(m - 2n),$$

vbi notasse iuuabit esse:

$$pp + ss = (mm + nn)(5mm - 6mn + 2nn) \quad \text{et}$$

$$qq + rr = (mm + nn)(2mm - 2mn + nn) = (mm + nn)p,$$

atque hinc denique ambo nostri numeri quaesiti erunt

$$A = \frac{(mm + nn)^2 (5mm - 6mn + 2nn)}{4m(m - 2n)(mm + mn - nn)^2} \quad \text{et}$$

$$B = \frac{(mm + nn)^2 (5mm - 6mn + 2nn)(2mm - 2mn + nn)}{(3m - n)^2 (m - n)^2 mn(m - 2n)(2m + n)}.$$

18. Vt autem etiam innotescat, quemadmodum huiusmodi valores inuenti satisfaciant, ex binis numeris idoneis

m

m et n prodeat formula radicalis $\sqrt{\frac{5mm-6mn+2nn}{2n(2m+n)}}=\frac{\mu}{\nu}$, vnde colligitur $\frac{M}{N}=\frac{(mm+mnn)\mu}{\nu m(m-n)(m-2n)(3m-n)(mm+mn-nn)}$, tum vero quoniam supra litteras q et r permutauimus, quaternae formulae propositae, sequenti modo ad quadrata reducentur

I. $\sqrt{(AB+A+B)}=\frac{M}{N}(pr+qs)=\frac{\mu}{\nu}\cdot\frac{(mm+nn)^2}{m(m-2n)(mm+mn-nn)}$

II. $\sqrt{(AB+A-B)}=\frac{M}{N}(pr-qs)=\frac{\mu}{\nu}\cdot\frac{(mm+nn)(m^4-8m^3n+6mmnn-n^4)}{m(m-n)(m-2n)(3-mn)(mm+mn-nn)}$

III. $\sqrt{(AB-A+B)}=\frac{M}{N}(pq+rs)=\frac{\mu}{\nu}\cdot\frac{(mm+nn)}{(m-n)(m-2n)}$

IV. $\sqrt{(AB-A-B)}=\frac{M}{N}(pq-rs)=\frac{\mu}{\nu}\cdot\frac{(mm+nn)}{m(3m-n)}$.

Aliae transformationes formulae resoluendae.

19. Quum tota quaestio huc sit perducta, vt ista formula (13) $\frac{5mm-6mn+2nn}{2n(2m+n)}$, siue $\frac{(2m-n)^2+(m-n)^2}{2n(2m+n)}$ ad quadratum reuocetur, ponamus $2m-n=t$ et $m-n=u$, ita vt sit $m=t-u$ et $n=t-2u$, hincque $2m+n=3t-4u$ atque nunc quadratum esse debeat $\frac{tt+uu}{(2t-4u)(3t-4u)}=\square$, siue $\frac{tt+uu}{(4u-2t)(4u-3t)}=\square$ circa quam formulam obseruo, numeratorem cum denominatore alios factores communes habere non posse praeter 2 et 5. Hinc igitur sequitur numeratorem $tt+uu$ vel ipsum quadratum esse debere vel duplum, vel quintuplum vel decuplum quadratum. Vnde quatuor casus resultant, quos singulos sequenti modo euoluamus.

20. Denotent litterae a et b binos cathetos trianguli rectanguli numerici, cuius hypothenusa sit $=c$, ita vt sit $aa+bb=cc$, nunc igitur pro primo casu faciamus

$tt+$

$tt + uu = cc$, quod fit sumendo $t = a$ et $u = b$, atque hoc casu necesse est, vt fiat $(4b - 2a)(4b - 3a) = \Box$.

Pro II^do Casu faciamus $tt + uu = 2cc$, quod fit sumendo $t = a - b$ et $u = a + b$, atque nunc necesse est vt sit $(a + 3b)(a + 7b) = \Box$.

Pro III^tio Casu faciamus $tt + uu = 5cc$, quod fit sumendo $t = a + 2b$ et $u = 2a - b$; tum enim ob $4u - 2t = 4a - 8b$ et $4u - 3t = 5a - 10b$, formula ad quadratum reducenda erit $(6a - 8b)(a - 2b) = \Box$, hoc est $(4b - 2a)(4b - 3a) = \Box$, quae cum Casu I^mo perfecte congruit.

Pro Casu denique IV^to, faciamus $tt + uu = 10 . cc$, quod fit sumendo $t = 3a + b$ et $u = a - 3b$, tum enim ob $4u - 2t = -14b - 2a$, et $4u - 3t = -5a - 15b$, formula ad quadratum reducenda erit $(3b + a)(7b + a) = \Box$, prorsus vti in casu secundo. Verum hic notandum est, casum tertium et quartum adhuc alio modo expediri posse. Si enim pro tertio ponamus $t = a + 2b$ et $u = b - 2a$, ob $4u - 2t = -10.a$ et $4u - 3t = -2b - 11.a$ formula ad quadratum reducenda erit $2a(11a + 2b) = \Box$.

Pro Casu quarto autem, si ponamus $t = 3a + b$ et $u = 3b - a$, ob $4u - 2t = 10b - 10a$ et $4u - 3t = 9b - 13.a$, formula ad quadratum reducenda est $(a - b)(13a - 9b) = \Box$. Verum plerumque quoties his duobus casibus satisfieri potest toties numeri t et u communi factore 5 praediti reperiuntur, ideoque ad nouas solutiones non perducunt.

21. His igitur duobus casibus postremis relictis, circa quatuor praecedentes omnino memoratu dignum est, quod primus et tertius, tum vero etiam secundus et quartus ad eandem formulam perduxerit, quare pro primo et tertio, si numeri a et b ita fuerint comparati, vt formula $(4b-2a)(4b-3a)$ fiat quadratum, tum duplici modo inde idonei valores pro t et u obtinentur; priori enim modo habebimus $t=a$ et $u=b$, altero vero modo $t=a+2b$ et $u=2a-b$. Simili modo pro casibus secundo et quarto, si fuerit formula $(3b+a)(7b+a)$ quadratum, tum etiam duo casus oriuntur, alter $t=a-b$ et $u=a+b$, alter vero $t=3a+b$ et $u=a-3b$. Operae igitur pretium erit has geminas resolutiones accuratius exponere.

I. Si fuerit $(4b-2a)(4b-3a)=\square$, *existente* $aa+bb=cc$.

22. Hinc igitur primo statim deducimus fractionem supra (18) introductam $\frac{\mu}{\nu}=\frac{cc}{(4b-2a)(4b-3a)}$; deinde pro priori resolutione habebimus

$$t=a;\quad m=a-b$$
$$u=b;\quad n=a-2b$$
$$p=aa-2ab+2bb;\quad r=(a-b)(3b-a)$$
$$q=aa-ab-bb;\quad s=aa-ab-bb$$

$\frac{p}{s}=$

$$\frac{p}{s} = \frac{aa - 2ab + 2bb}{aa - ab - bb}; \quad \frac{q}{r} = \frac{aa - ab - bb}{(a-b)(3b-a)}$$

$$mm + nn = 2aa - 6ab - 5bb$$

pro altera vero solutione

$$t = a + 2b; \quad m = 3b - a;$$

$$u = 2a - b; \quad n = 4b - 3a;$$

$$p = 5(aa - 2ab + 2bb); \quad r = -5(a-b)\,3b - a)$$

$$q = -5(aa - ab - bb); \quad s = -5(aa - ab - bb)$$

$$\frac{p}{s} = \frac{aa - 2ab + 2bb}{aa - ab - bb}; \quad \frac{q}{r} = \frac{aa - ab - bb}{(a-b)(3b-a)}$$

unde manifestum est has duas solutiones a se inuicem non differre.

23. Speciales autem solutiones. quae ex hac formula primo intuitu deriuantur sunt sequentes

a	b	m	n	$\frac{p}{s}$	$\frac{q}{r}$
0	1	-1	-2	$\frac{2}{1}$	$\frac{1}{3}$
4	3	1	-2	$\frac{2}{1}$	$\frac{1}{1}$
12	5	7	2	$\frac{74}{59}$	$\frac{59}{21}$

quarum binae priores scopo nostro non conueniunt, tertia vero idoneam praebet solutionem atque adeo ab illa, quam olim iam inueni diuersam; quum enim sit $pp + ss = 8957 = 53 \cdot 169$ et $qq + rr = 3922 = 53 \cdot 74$ erunt ambo quaesiti numeri

$$A = \frac{160 \cdot 53^2 \cdot 74}{4 \cdot 74 \cdot 59^2 \cdot 21} = \frac{160 \cdot 53^2}{4 \cdot 21 \cdot 59^2}$$

$$B = \frac{169 \cdot 74 \cdot 53^2}{2 \cdot 16 \cdot 3 \cdot 5^2 \cdot 7 \cdot 19^2} = \frac{169 \cdot 37 \cdot 53^2}{16 \cdot 3 \cdot 5^2 \cdot 7 \cdot 19^2}.$$

24. Consideremus autem attentius hanc formulam: $(4b - 2a)(4b - 3a) = \square$ et quia numeri a et b, sunt ca-

theti trianguli rectanguli, atque euidens est, pro a sumi debere parem pro b vero imparem, statuamus $a = 2de$ et $b = dd - ee$, vt sit hypothenusa $c = dd + ee$, tum vero erit $4b - 2a = 4(dd - de - ee)$ et $4b - 3a = 4dd - 6de - 4ee$, quorum productum quum quadratum esse debeat, necesse est, vt vtriusque quadrans fiat quadratum, hoc est

$$\text{I}^{o}.\quad dd - de - ee = \square$$

$$\text{II}^{o}.\quad dd - \tfrac{3}{2}de - ee = \square,$$

vbi quum numerorum d et e alter debeat esse par, alter impar, etiam posterior numeris integris constat. Quod autem ad priorem attinet, quum sit $dd - de - ee = (d - \frac{1}{2}e)^2 - 5\frac{e^2}{4}$, ponamus $d - \frac{1}{2}e = rr + 5ss$ et $\frac{1}{2}e = 2rs$, tum enim fiet $dd - de - ee = (rr - 5ss)^2$; at vero habebimus $e = 4rs$ et $d = rr + 2rs + 5ss$ hincque $dd - ee = r^4 + 4r^3s - 2rrss + 20rs^3 + 25 \,.\, s^4$ et $de = 4r^3s + 8rrss + 20rs^3$, vnde altera conditio postulat: $r^4 - 2r^3s - 14rrss - 10rs^3 + 25s^4 = \square$.

25. Statuamus hic $\frac{r}{s} = z$, vt habeamus hanc formulam $z^4 - 2z^3 - 14zz - 10z + 25 = \square$, quae cum formula supra data (15) comparata praebet: $\alpha = \pm 1$; $\beta = 1$; $\gamma = -14$; $\delta = 5$; $\varepsilon = \pm 5$, vnde pro z quatuor sequentes expressiones

$$\text{I}^{o}.\quad z = \frac{2\alpha(\varepsilon - 5 \,.\, \alpha)}{2\alpha 3\varepsilon + 1 + 14} = \frac{2\alpha(\varepsilon - 5\alpha)}{2\alpha 3\varepsilon + 15} = \frac{2(\alpha\varepsilon - 5)}{2\alpha\varepsilon + 15}$$

hinc vel $z = 0$; vel $z = -4$

$$\text{II}^{o}.\quad z = \frac{50 \,.\, \alpha\varepsilon + 375}{2(5\alpha\varepsilon - 25)} = \frac{10 \,.\, \alpha\varepsilon + 75}{2(\alpha\varepsilon - 5)}$$ hincque

vel

vel $z = \infty$; vel $z = -\frac{5}{4}$

III°. $z = \frac{(2\alpha\varepsilon - 14 - 1)(2\alpha\varepsilon + 14 + 1)}{4(10 + 14 + 1)} = -\frac{125}{100} = -\frac{5}{4}$.

IV°. $z = \frac{100(1250 + 70 \cdot 25 + 5 \cdot 25)}{(50 \cdot \alpha\varepsilon - 15 \cdot 25)(50 \cdot \alpha\varepsilon + 15 \cdot 25)} = \frac{4 \cdot 25^2 \, 125}{25^2(2\alpha\varepsilon + 15)(2\alpha\varepsilon - 15)} = -4$.

Ex valore $z = -4$ oriuntur valores $r = 4$; $s = -1$; $d = 13$; $e = -16$ hincque $a = 416$ et $b = 87$, vnde oritur $\frac{p}{s} = \frac{23362}{25859}$, et $\frac{q}{r} = \frac{25859}{10199}$; at ex valore $z = -\frac{5}{4}$, habemus $r = 5$; $s = -4$; $d = 65$; $e = -80$, qui per quinarium ad terminos minores reducti praebent vt ante, $d = 13$ et $e = -16$, vbi notasse iuuabit ex his valoribus a et b praegrandes numeros pro p, q, r, s esse produturos.

26. At circa binas illas formulas notasse iuuabit, vtramque etiam quadrato negatiuo aequari posse, verum tum solutio eadem exsurgit, nisi quod valores pro a et b fiant negatiui. Ceterum hic notari conuenit, vltimae aequationi etiam valorem $z = -3$ satisfacere; etiamsi eum non per methodum consuetam detexerimus, inde autem fit $r = 3$ et $s = -1$; hincque porro $d = 2$ et $e = -3$; vnde fiat $a = -12$ et $b = -5$, quem casum iam supra euoluimus.

II. Si fuerit $(3b + a)(7b + a) = \square$.

27. Hic statim apparet sumi debere $a = dd - ee$ et $b = 2de$, vt fiat $c = dd + ee$ tum ergo sequentes duae formulae quadrata esse debent $dd + 6de - ee = \square$ et $dd + 14de - ee = \square$.

Quum

Quum prior sit $= (d + 3e)^2 - 10ee$; si ponamus $\zeta\eta = 10$, ac statuamus $d + 3e = \zeta rr + \eta ss$ et $e = 2rs$ fiet illa formula $= (\zeta\zeta rr - \eta ss)^2$, tum autem erit $d = \zeta rr - 6rs + \eta ss$ et $e = 2rs$; hinc ergo pro altera formula, quae est $(d + 7e)^2 - 50 . ee$, erit $d + 7e = \zeta rr + 8rs + \eta ss$ ideoque haec formula abit in $\zeta\zeta r^4 + 16\zeta r^3 s - 116 rrss + 16\eta rs^3 + \eta\eta s^4 = \square$, vnde per methodum supra indicatam infinitae solutiones inueniri possunt; vbi notasse iuuabit esse vel $\zeta = 1$ et $\eta = 10$, vel $\zeta = 2$ et $\eta = 5$.

28. Quum autem idonei valores pro a et b fuerint inuenti, duplici modo inde litterae t et u definiri poterunt. Priore modo fit $t = a - b$ et $u = a + b$, hinc $m = t - u = -2b$ et $n = -a - 3b$, ideoque $p = mm + (m - n)^2 = aa + 2ab + 5bb$; $q = s = mm + n(m - n) = -aa - 4ab + bb$ et $r = m(m - 2n) = -4b(a + 2b)$ ita vt sit

$$\frac{p}{s} = \frac{aa + 2ab + 5bb}{aa + 4ab - bb}; \text{ et } \frac{q}{r} = \frac{aa + 4ab - bb}{4b(a + 2b)}.$$

Posteriore vero modo fit $t = 3a + b$ et $u = a - 3b$, vnde $m = 2a + 4b$ et $n = a + 7b$, hincque porro ob $m - n = a - 3b$, fit $p = 5(aa + 2ab + 5bb)$ $q = s = 5(aa + 4ab - bb)$ et $r = 5 . 4b(a + 2b)$ sicque patet hunc posteriorem casum ad priorem redire.

29. Simpliciores autem solutiones, quas facili negotio diuinando elicere licet sunt sequentes:

a	b	m	n	$\frac{p}{s}$	$\frac{q}{r}$
1	0	-0	-1	$\frac{1}{1}$	$\frac{1}{0}$
-3	4	-8	-9	$\frac{13}{11}$	$\frac{11}{16}$
-35	12	-24	-1	$\frac{1106}{699}$	$\frac{699}{528}$.

Hic secundus casus praebet illam ipsam solutionem, quam iam olim dederam. His autem duabus formulis pertractatis adiungamus insuper binas postremas supra (20) inuentas.

III. *Si fuerit* $2a(11a + 2b) = \square$.

30. Casus simpliciores, qui statim se offerunt sunt:

a	b	m	n	$\frac{p}{s}$	$\frac{q}{r}$
0	1	1, 1	0, 0	$\frac{2}{1}$	$\frac{1}{1}$
4	3	15, 3	20, 4	$\frac{2}{1}$	$\frac{1}{3}$
16	-63	-15, -3	80, 16	$\frac{74}{68}$	$\frac{69}{21}$

vbi ex datis a et b, fit $t = a + 2b$ et $u = b - 2a$ hincque, vt ante $m = t - u = 3a + b$ et $n = t - 2u = 5a$. Hae solutiones autem iam in superioribus continentur.

IV. *Si fuerit* $(a - b)(13a - 9b) = \square$.

31. Inuentis idoneis valoribus pro a et b, erit $t = 3a + b$ et $u = 3b - a$, hinc $m = 4a - 2b = 2(2a - b)$ et $n = 5(a - b)$, atque ob $m - n = 3b - a$, atque $m - 2n = 2(4b - 3a)$ habebimus $\frac{p}{s} = \frac{17aa - 22ab + 13bb}{11aa + 4ab - 11bb}$ et $\frac{q}{r} = \frac{11aa + 4ab - 11bb}{4(6aa - 11ab + 4bb)}$. Solutiones autem simpliciores hinc oriundae sunt

a	b	m	n	$\frac{p}{s}$	$\frac{q}{r}$
0	1	-2	-5	$\frac{13}{11}$	$\frac{11}{16}$
4	$+3$	10, 2	5, 1	$\frac{1}{1}$	$\frac{1}{0}$

vbi memoratu dignum euenit, quod statim primum tenta-

men

men quo $a = 0$ et $b = 1$, praebeat solutionem iam dudum inuentam.

32. Quod si pro vlteriore huius formulae euolutione ponamus $a = 2de$ et $b = dd - ee$, fiet $a - b = ee + 2de - dd$ siue mutandis signis, vt $(b - a)(9b - 13a) = \square$, erit $b - a = dd - 2de - ee$ et $9b - 13a = 9dd - 26 . de - 9ee$, reddamus nunc priorem quadratum, quae quum sit $(d-e)^2 - 2ee$, statuamus $d - e = rr + 2ss$ et $e = 2rs$, tum enim fiet $dd - 2de - ee = (rr - 2ss)^2$, tum vero alter factor ob $dd - ee = r^4 + 4r^3s + 8rrss + 8rs^3 + 4s^4$, erit $9r^4 - 16r^3s - 68rrss - 32rs^3 + 36 . s^4$, vbi casus primo intuitu se offerentes sunt 1°. $r = 1$, $s = 0$, 2°. $r = 0$, $s = 1$, 3°. $r = 1$ et $s = -1$, 4°. $r = 2$ et $s = -1$; 5°. $r = 1$ et $s = 2$.

33. Pro horum casuum primo habemus $d = 1$ et $e = 0$; hinc $a = 0$ et $b = 1$, qui iam occurrit, pro secundo habemus $d = 2$ et $e = 0$, hinc $a = 0$ et $b = 1$, qui a praecedente non differt. At pro tertio habemus $d = 1$ et $e = -2$, hinc $a = -4$ et $b = -3$, qui supra iam est tractatus, pro quarto habemus $d = 2$ et $e = -4$ siue $d = 1$ et $e = -2$, vnde fit $a = -4$ et $b = -3$ vt praecedens, pro quinto denique habemus $d = 13$ et $e = 4$, hinc $a = 104$ et $b = 153$, ex quibus numeri praegrandes pro quaesitis A et B resultant, quibus non immoramur.

34.

34. Imprimis autem quoque notatu dignus est casus, quo inuenimus $\frac{p}{s} = \frac{2}{1}$ et $\frac{q}{r} = \frac{1}{3}$, siue $\frac{q}{r} = \frac{3}{1}$, vnde deducuntur numeri quaesiti $A = \frac{25}{12}$ et $B = \frac{25}{12}$ ita vt ambo numeri quaesiti hoc casu fiant aequales, quod quidem scopo problematis minus conuenit. Si enim numeri aequales desiderentur ob eorum differentiam euanescentem quaestio huc rediret, vt inueniatur numerus A, ita vt tam $AA + 2A$, quam $AA - 2A$ fiat quadratum, quod quidem est facillimum, statuatur enim $AA = \frac{aa+bb}{nn}$ et $2A = \frac{2ab}{nn}$, fiet vtique $\sqrt{(AA+2A)} = \frac{a+b}{n}$ et $\sqrt{(AA-2A)} = \frac{a-b}{n}$; verum nunc requiritur vt $aa + bb$ sit quadratum, quem in finem ponamus, $a = pp - qq$ et $b = 2pq$, vt fiat $A = \frac{pp+qq}{n}$, est vero etiam $A = \frac{2pq(pp-qq)}{nn}$, vnde fit $n(pp+qq) = 2pq(pp-qq)$ et $n = \frac{2pq(pp-qq)}{pp+qq}$, ita vt numerus quaesitus in genere sit $A = \frac{(pp+qq)^2}{2pq(pp-qq)}$, tales ergo numeri sunt sequentes: $A = \frac{25}{12}$; 2°. $A = \frac{169}{60}$; 3°. $A = \frac{289}{120}$; 4°. $A = \frac{625}{168}$ etc.

35. Pro solutionibus autem ad quaestionem propositam accommodatis, duae in numeris non nimis magnis notatu dignae videntur, quarum prior est ea ipsa, quam iam dudum inueni, qua erat $A = \frac{13 \cdot 29^2}{8 \cdot 9^2}$ et $B = \frac{5 \cdot 29^2}{32 \cdot 11^2}$, siue $A = \frac{10933}{648}$ et $B = \frac{4205}{3872}$ vnde

$$\sqrt{(AB + A + B)} = \frac{7 \cdot 29 \cdot 37}{16 \cdot 9 \cdot 11}$$
$$\sqrt{(AB + A + B)} = \frac{29^2}{16 \cdot 3 \cdot 11}$$
$$\sqrt{(AB - A + B)} = \frac{29^2}{16 \cdot 9}$$
$$\sqrt{(AB - A - B)} = \frac{29}{48}.$$

Pro altera vero solutione orta ex valoribus:

$$\frac{p}{s} = \frac{74}{59} \text{ et } \frac{q}{r} = \frac{59}{21} \text{ obtinemus:}$$

$$A = \frac{13^2 \cdot 53^2}{4 \cdot 21 \cdot 59^2} \text{ et } B = \frac{13^2 \cdot 37 \cdot 53^2}{16 \cdot 3 \cdot 5^2 \cdot 7 \cdot 19^2}$$

vnde $\sqrt{(AB + A + B)} = \frac{13 \cdot 53}{8 \cdot 3 \cdot 7}$

$$\sqrt{(AB + A - B)} = \frac{13 \cdot 53^2}{8 \cdot 3 \cdot 5 \cdot 7 \cdot 19}$$

$$\sqrt{(AB - A + B)} = \frac{13 \cdot 53^2}{8 \cdot 3 \cdot 7 \cdot 59}$$

$$\sqrt{(AB - A - B)} = \frac{13 \cdot 41 \cdot 47 \cdot 53}{8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 59}.$$

OBSER-