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Leonhard Euler

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# DE INVENTIONE QVOTCVNQVE MEDIARVM PROPORTIONALIVM CITRA RADICVM EXTRACTIONEM.

Auctore

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## Propositio I.

1.

Si numeri $a$, $b$, $c$, $d$ etc. sint continue proportionales, etiam differentiae $b-a$, $c-b$, $d-c$ etc. erunt in proportione geometrica continua eiusdem exponentis; ac si prior series a proportione geometrica aberret, posterior multo magis aberrabit.

## Demonstratio.

Prior propositionis pars in elementis est demonstrata; pro altera autem parte ponamus exponentem rationis geometricae $=r$ vt secundum proportionem geometricam foret $b=ar$, $c=ar^2$, $d=ar^3$ etc. Sit autem $b=ar+z$ manente $c=arr$, ita vt $z$ sit error termini $b$ a proportione geometrica,

erunt-



eruntque differentiae $b-a=a(r-1)+z$ et $c-b=ar(r-1)-z$, quarum ratio est $\frac{ar(r-1)-z}{a(r-1)+z}=r-\frac{(r+1)z}{a(r-1)+z}$, cuius aberratio ab exponente $r$ vtique maior est, quam rationis $\frac{b}{a}=r+\frac{z}{a}$. Vnde etiam facile intelligitur, si seriei $a$, $b$, $c$, $d$ etc. plures termini a ratione geometrica $1:r$ aberrent, in serie differentiarum maiores errores inesse debere.

## Corollarium.

2. Vicissim ergo quantumuis series differentiarum a proportione geometrica aberrauerit, series terminorum ipsa propius ad hanc proportionem accedet.

## Propositio II.

3. Inter duos numeros datam rationem $1:r$ tenentes medium proportionalem inuenire sine extractione radicis.

## Solutio.

Sint numeri propositi A et A$r$, mediusque proportionalis prope saltem verus $=$B; vt hi tres numeri A, B, A$r$ progressionem a geometrica parum aberrantem constituant. Statuantur autem differentiae pro lubitu: $B-A=a$ et $Ar-B=b$, hincque colligitur $A(r-1)=a+b$ et $B(r-1)=ar+b$, ita vt sit

$$A=\frac{a+b}{r-1} \quad \text{et} \quad B=\frac{ar+b}{r-1}$$



nihil

nihil autem impedit quominus numeri A et B in ratione $1 : r - 1$ augeantur, vt in integris habeamus:

$$A = a + b \quad \text{et} \quad B = ar + b$$

vnde sumtis pro lubitu binis numeris $a$ et $b$, hi tres numeri

$$a + b; \quad ar + b; \quad ar + br$$

quorum primus est ad tertium in ratione data $1 : r$ eo propius ad progressionem geometricam accedent, quo minus numerorum assumtorum $a$ et $b$ ratio a ratione $1 : \sqrt{r}$ aberrauerit; seu fractio $\frac{ar + b}{a + b}$ propius ad valorem $\sqrt{r}$ accedet, quam fractio $\frac{b}{a}$. Quo igitur valorem medii proportionalis $\sqrt{r}$ inter 1 et $r$ accuratius obtineamus, statuamus $a + b = a'$ et $ar + b = b'$; atque fractio $\frac{a'r + b'}{a' + b'}$ adhuc propius valorem $\sqrt{r}$ exhibebit simili ergo modo si porro statuamus

$$a' + b' = a''; \quad a'' + b'' = a'''; \quad a''' + b''' = a'''' \text{ etc.}$$
$$a'r + b' = b''; \quad a''r + b'' = b'''; \quad a'''r + b''' = b'''' \text{ etc.}$$

fractiones $\frac{b''}{a''}$; $\frac{b'''}{a'''}$; $\frac{b''''}{a''''}$ etc. continuo accuratius valorem medii proportionalis $\sqrt{r}$ exprimentur.

## Coroll. 1.

4. Cum igitur quaestio sit de medio proportionali inter numeros $1 : r$ sumtis pro lubitu quobus numeris $a$ et $b$ formentur inde duae series

$a, a',$



$$a, \; a', \; a'', \; a''', \; a'''' \text{ etc.}$$
$$b, \; b', \; b'', \; b''', \; b'''' \text{ etc.}$$

hac lege vt sit

$$a' = a + b; \; a'' = a' + b'; \; a''' = a'' + b''; \; a'''' = a''' + b'''$$
$$b' = ar + b; \; b'' = a'r + b'; \; b''' = a''r + b''; \; b'''' = a'''r + b'''$$

et fractiones $\frac{b}{a}$, $\frac{b'}{a'}$, $\frac{b''}{a''}$, $\frac{b'''}{a'''}$, $\frac{b''''}{a''''}$ etc. continuo propius valorem quaesitum $\sqrt{r}$ exhibebunt.

## Coroll. 2.

5. Vel si constituatur progressio a ratione geometrica continua quantumuis aberrans, cuius termini alterni sint in ratione $1 : r$

$$a, \; b, \; ar, \; br, \; ar^2, \; br^2, \; ar^3, \; br^3, \; ar^4, \; br^4 \text{ etc.}$$

hinc binis terminis coniungendis noua formetur progressio:

$$a + b, \; b + ar, \; ar + br, \; br + ar^2, \; ar^2 + br^2, \; br^2 + ar^3, \text{ etc.}$$

haecque magis ad progressionem geometricam accedet.

## Coroll. 3.

6. Si hic denuo bini termini coniungantur, prodibit haec series:

$$a(r + 1) + 2b; \; 2ar + b(r + 1); \; ar(r + 1) + 2br; \; 2ar^2 + br(r + 1) \text{ etc.}$$

hincque porro simili modo istae

$$a(3r + 1) + b(r + 3); \; ar(r + 3) + b(3r + 1); \; ar(3r + 1) + br(r + 3) \text{ etc.}$$

$a(rr$

$$a(rr+6r+1)+b(4r+4);\ ar(4r+4)+b\,rr+6r+1);$$
$$ar(rr+6r+1)+br(4r+4)\ \text{etc.}$$
$$a(5rr+10r+1)+b(rr+10r+5);\ ar(rr+10r+5)$$
$$+b(5rr+10r+1);\ \text{etc.}$$

quae continuo propius ad progreſſionem geometricam conuergunt.

## Scholion.

7. Totum ergo negotium huc redit, vt binae ſeries $a, a', a'', a'''$ etc. $b, b', b'', b''', b''''$ etc. formentur quippe quarum termini homologi continuo propius rationem $1 : \sqrt{r}$ exhibebunt. Cum autem ſinguli termini poſt primos vtramque litteram $a$ et $b$ inuoluant ita vt quilibet terminus vtriusque hanc habiturus ſit formam $Ma+Nb$, primum obſeruo poſteriorem ſeriem ex priori oriri, ſi loco litterarum $a$ et $b$ ſcribantur $b$ et $ar$. Quare ſi prioris ſeriei terminus indici $n$ reſpondens fuerit $a^{(n)}=Ma+Nb$, poſterioris ſeriei terminus eidem indici $n$ reſpondens erit $b^{(n)}=Mb+Nar$, ita vt fractio $\frac{Mb+Nar}{Ma+Nb}$ eo exactius valorem $\sqrt{r}$ ſit expreſſura, quo maior fuerit exponens $n$, atque adeo ſumto $n=\infty$ verus valor $\sqrt{r}$ prodire debeat. Ita ſi exempli gratia capiatur $r=2$ ſeries illae binae ita ſe habebunt:

| | | | | | |
|---|---|---|---|---|---|
| $a$ | $a+b$ | $3a+2b$ | $7a+5b$ | $17a+12b$ | $41a+29b$ |
| $b$ | $2a+b$ | $4a+3b$ | $10a+7b$ | $24a+17b$ | $58a+41b$ |
| $2a$ | $2a+2b$ | $5a+4b$ | $14a+10b$ | $34a+24b$ | $82a+58b$ |

cui



cui tertiam ſubſcripſi primae duplam, quippe cuius ope continuatio facillime inſtituitur. Quicunque ergo numeri hic pro $a$ at $b$ accipiantur, ſeries media continebit ſatis prope media proportionalia inter primam et tertiam, vti facile perſpicitur. Simili modo ſumto $r=3$, hae ſeries ita ſe habebunt

| | | | | | |
|---|---|---|---|---|---|
| $a$ | $a+b$ | $4a+2b$ | $10a+6b$ | $28a+16b$ | $76a+44b$ |
| $b$ | $3a+b$ | $6a+4b$ | $18a+10b$ | $48a+28b$ | $132a+76b$ |
| $3a$ | $3a+3b$ | $12a+6b$ | $30a+18b$ | $84a+48b$ | $228a+132b$ |

eritque ergo ex vltimis $\frac{132a+76b}{76a+44b}=\frac{33a+19b}{19a+11b}$ eo exactius $=\sqrt{3}$, quo propius ratio $\frac{b}{a}$ eo accedat, ita ſumto $b=2$ et $a=1$ fit admodum exacte $\sqrt{3}=\frac{71}{41}$ et nunc ſumto $b=71$ et $a=41$ exactiſſime erit $=\frac{2702}{1560}=\frac{1351}{780}=\sqrt{3}$ errore ne millioneſimam quidem partem vnitatis attingente.

## Propoſitio III.

8. Inueſtigare legem progreſſionis binarum illarum ſerierum $a, a', a'', a'''$ etc. et $b, b', b'', b'''$ etc. quarum termini homologi continuo propius rationem $1 : \sqrt{r}$ exprimunt.

## Solutio.

Quoniam nouimus omnes terminos binas litteras $a$ et $b$ ita complecti, vt in forma $Ma+Nb$ contineantur, ac ſi pro priori ſtatuatur in genere $a^{(n)}=Ma+Nb$ tum pro poſteriori fore $b^{(n)}=Mb$ $+Nar$

$+Nar$, hinc lex progressionis suppeditat terminos sequentes:

$$a^{(n+1)} = (M+Nr)a + (M+N)b \text{ et}$$
$$b^{(n+1)} = (M+Nr)b + (M+N)ra$$

ex quo in legem progressionis vtriusque seriei inquiri oportet. Quo igitur scrutemur, quemadmodum primae seriei quilibet terminus definiatur, consideremus hanc seriem sub ista forma generaliori

$$a + a'x^1 + a''x^2 + a'''x^3 + a''''x^4 +$$

cuius in infinitum continuatae summa fingatur $Pa + Qb$ et quoniam altera ex hac nascitur, dum loco $a$ et $b$ scribitur $b$ et $ra$ erit

$$b + b'x + b''x^2 + b'''x^3 + b''''x^4 + \text{ etc.} = Pb + Qra$$

Addantur hae series inuicem, et quia est

$$a + b = a';\ a' + b' = a'';\ a'' + b'' = a''' \text{ etc. erit}$$
$$a' + a''x + a'''x^2 + a''''x^3 \text{ etc.} = (P+Qr)a + (P+Q)b$$

multiplicemus hanc seriem per $x$ et a prima subtrahamus prodibitque

$$a = Pa + Qb - (P+Qr)ax - (P+Q)bx$$

Quoniam vero quantitates P et Q a litteris $a$ et $b$ non pendent, hinc duae resultant aequationes

$$1 = P - Px - Qrx \text{ et } 0 = Q - Px - Qx$$

vnde deducimus has determinationes

$$P = \frac{1-x}{(1-x)^2 - rxx} = \frac{1-x}{1-2x-(r-1)xx} \text{ et}$$
$$Q = \frac{Px}{1-x} = \frac{x}{1-2x-(r-1)xx}.$$

Quo-



Quocirca prior series $a + a'x + a''x^2 + a'''x^3 +$ etc. nascitur ex euolutione huius fractionis $\frac{a(1-x)+bx}{1-2x-(r-1)xx}$, posterior vero $b + b'x + b''x^2 + b'''x^3 +$ etc. ex euolutione huius $\frac{b(1-x)+arx}{1-2x-(r-1)xx}$ ita vt vtraque sit series recurrens secundi ordinis, scala relationis existente $2, (r-1)$, hincque pro serie priori $a, a', a'', a''', a''''$ etc. fit primo $a' = a + b$, tum vero $a'' = 2a' + (r-1)a;\ a''' = 2a'' + (r-1)a';\ a'''' = 2a''' + (r-1)a''$ etc. ex hac vero nascitur altera ponendo $a = b$ et $b = ra$. Hinc adeo huius seriei terminum generalem definire licet ad quod valores quantitatum P et Q in fractiones simplices resolui oportet. Cum igitur denominatoris communis factor sit $1 - x - x\sqrt{r} = 1 - x(1+\sqrt{r})$, pro quantitate P statuatur fractio simplex inde nata $= \frac{\mathfrak{A}}{1-x(1+\sqrt{r})}$, ac demonstraui fore $\mathfrak{A} = \frac{1-x}{1-x+x\sqrt{r}}$ posito $1 - x = x\sqrt{r}$, vnde fit $\mathfrak{A} = \frac{1}{2}$, pro altero autem factore tantum $\sqrt{r}$ negatiue accipi opus est, ita vt fit

$$P = \frac{1}{2}\cdot\frac{1}{1-x(1+\sqrt{r})} + \frac{1}{2}\cdot\frac{1}{1-x(1-\sqrt{r})}$$

Simili modo pro Q si fractio partialis ex denominatoris factore $1 - x(1+\sqrt{r})$ nata ponatur $\frac{\mathfrak{A}}{1-x(1+\sqrt{r})}$, reperitur $\mathfrak{A} = \frac{x}{1-x+x\sqrt{r}}$ posito $1 - x = x\sqrt{r}$ indeque $\mathfrak{A} = \frac{1}{2\sqrt{r}}$. Quare ipsi $\sqrt{r}$ binos valores tribuendo fit

$$Q = \frac{1}{2\sqrt{r}}\cdot\frac{1}{1-x(1+\sqrt{r})} - \frac{1}{2\sqrt{r}}\cdot\frac{1}{1-x(1-\sqrt{r})}$$

sicque summa prioris seriei $a + a'x + a''x^2 + a'''x^3 +$ etc. erit

$$= \frac{a\sqrt{r}+b}{2\sqrt{r}}\cdot\frac{1}{1-x(1+\sqrt{r})} + \frac{a\sqrt{r}-b}{2\sqrt{r}}\cdot\frac{1}{1-x(1-\sqrt{r})}$$



cum

cum nunc ex vtraque parte progreſſio naſcatur geometrica prodit noſtrae ſeriei terminus generalis

$$\frac{a\sqrt{r}+b}{2\sqrt{r}}(1+\sqrt{r})^n x^n+\frac{a\sqrt{r}-b}{2\sqrt{r}}(1-\sqrt{r})^n x^n$$

ita vt ſit indefinite.

$$a^{(n)}=\frac{a\sqrt{r}+b}{2\sqrt{r}}.(1+\sqrt{r})^n+\frac{a\sqrt{r}-b}{2\sqrt{r}}(1-\sqrt{r})^n$$

et pro altera ſerie

$$b^{(n)}=\frac{b+a\sqrt{r}}{2}(1+\sqrt{r})^n+\frac{b-a\sqrt{r}}{2}(1-\sqrt{r})^n$$

## Coroll.

9. Ex hoc termino generali demum plene conuincimur, fore ſumto exponente $n$ infinito $\frac{b^{(n)}}{a^{(n)}}=\sqrt{r}$, cum enim tum poteſtas $(1-\sqrt{r})^n$ prae priori $(1+\sqrt{r})^n$ euaneſcat, erit vtique

$$\frac{b^{(n)}}{a^{(n)}}=\frac{b\sqrt{r}+ar}{a\sqrt{r}+b}=\sqrt{r}.$$

Vnde ſimul patet quo maior capiatur exponens $n$, eo propius ad veritatem accedi.

## Scholion.

10. Eadem quidem veritas etiam hac ratione oſtendi poteſt. Poſito generatim $a^{(n)}=Ma+Nb$, erit $b^{(n)}=Mb+Nar$, et termini ſequentes:

$$a^{(n+1)}=(M+Nr)a+(M+N)b \text{ et } b^{(n+1)}=(M+Nr)b+(M+N)ar$$

Iam



Iam caſu $n=\infty$, nullum dubium ſupereſſe poteſt, quin ſit $\frac{b^{(n+1)}}{a^{(n+1)}}=\frac{b^{(n)}}{a^{(n)}}$; vnde neceſſe eſt ſit:

$$\frac{(M+Nr)b+(M+N)ar}{(M+Nr)a+(M+N)b}=\frac{Mb+Nar}{Ma+Nb}$$

Statuatur hic valor $=v$, et quia tum ſit

$$\frac{M}{N}=\frac{bv-ar}{b-av} \text{ et } \frac{M}{N}=\frac{arv+bv-br-ar}{b+ar-av-bv} \text{ ſeu}$$

$$\frac{M}{N}-v=\frac{a(vv-r)}{b-av}=\frac{(a+b)(vv-r)}{b+ar-(a+b)v}$$

vnde manifeſto ſequitur $vv-r=0$ et $v=\sqrt{r}$. Simul vero etiam quantitatum M et N haec relatio perſpicitur, quod ſumto $n=\infty$ fiat $\frac{M}{N}=v=\sqrt{r}$.

## Propoſitio IV.

11. Inter duos numeros datam rationem $1:r$ tenentes duos medios proportionales in rationalibus proxime exhibere.

## Solutio.

Sumantur bini numeri quicunque $a$ et $ar$ in ratione data inter eosque capiantur duo medii quicunque $b$ et $c$ atque quantumuis relatio $a:b:c:ar$ a ratione geometrica diſcrepet, inde alias propius eo accedentes hoc modo eliciemus. Quaerantur alii ſimiles quaterni numeri $A:B:C:Ar$ quorum illi ſint differentiae, ita vt ſit $B-A=a$; $C-B=b$ et $Ar-C=c$, hincque $B=A+a$; $C=A+a+b$; et $Ar=A+a+b+c$ ſeu $A=\frac{a+b+c}{r-1}$; $B=\frac{ar+b+c}{r-1}$; $C=\frac{ar+br+c}{r-1}$



qui

qui per $r-1$ multiplicati praebebunt hos quaternos numeros

$$a' = a+b+c;\ b' = b+c+ar;\ c' = c+ar+br;\ a'r = ar+br+cr$$

qui iam multo propius ad proportionem geometricam continuam accedent. Simili ergo modo hinc alii noui $a''$, $b''$, $c''$, $a''r$ deriuabuntur sumendo:

$$a'' = a'+b'+c';\ b'' = b'+c'+a'r;\ c'' = c'+a'r+b'r$$

hincque denuo alii, qui continuo propius proposito satisfacient. Totum ergo negotium reducitur ad formationem trium progressionum:

I. $a, a', a'', a''', a'''' \ldots\ldots a^{(n)}$

II. $b, b', b'', b''', b'''' \ldots\ldots b^{(n)}$

III. $c, c', c'', c''', c'''' \ldots\ldots c^{(n)}$

quarum lex est satis simplex quae quo vlterius continuentur, eo propius quaterni numeri

$$a^{(n)} : b^{(n)} : c^{(n)} : ra^{(n)}$$

proportionem geometricam continuam exhibebunt, etiamsi initio assumti $a, b, c, ra$ plurimum aberraverint.

## Coroll. 1.

12. De his tribus seriebus primum obseruo singulos earum terminos huiusmodi formam $La+Mb+Nc$ esse habituros, ita vt quantitates L, M, N litteras pro arbitrio assumtas $a, b, c$ non inuoluant, sed a sola ratione proposita $1:r$ pendeant.

Coroll.



## Coroll. 2.

13. Deinde si primae seriei terminus quicunque fuerit $a^{(n)} = La+Mb+Nc$, euidens est pro serie secunda fore $b^{(n)} = Lb+Mc+Nra$, et pro serie tertia $c^{(n)} = Lc+Mra+Nrb$. Vnde sufficit harum trium serierum primae indolem explorauisse.

## Scholion.

14. Harum obseruationum ope inuentio duarum mediarum proportionalium, quae quidem in rationalibus proxime satisfaciat, expedite instituitur. Sint enim exempli causa inter duos numeros rationem duplam tenentes duo medii proportionales inuestigandi, et operatio numerica sumendis pro litteris $a$, $b$, $c$ numeris 1, 1, 1 ob $r=2$ ita se habebit

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $a$ .. 1 | 3 | 12 | 46 | 177 | 681 | 2620 | 10080 |
| $b$ .. 1 | 4 | 15 | 58 | 223 | 858 | 3301 | 12700 |
| $c$ .. 1 | 5 | 19 | 73 | 281 | 1081 | 4159 | 16001 |
| $2a$ .. 2 | 6 | 24 | 92 | 354 | 1362 | 5240 | 20160 |
| $2b$ .. 2 | 8 | 30 | 116 | 446 | 1716 | 6602 | 25400 |

Hic quatuor postremi numeri

$$10080 : 12700 : 16001 : 20160$$

tam parum a proportione geometrica recedunt, vt si inter extremos per extractionem radicis cubicae duo medii proportionales quaerantur, ii ne parte quidem

quidem decies millesima a veritate aberrent; est enim:

$$\frac{12700}{10080} = \sqrt[3]{\frac{2048 \cdot 51000}{1024192512}} = \sqrt[3]{\left(2 - \frac{2074}{1024 \cdot 52512}\right)}$$

ideoque $= \sqrt[3]{2} - \frac{2074}{3 \cdot 1024192512 \sqrt[3]{4}}$, vnde cum fiat $12700 = 10080 \sqrt[3]{2} - \frac{1}{23906}$ error infra particulam decies millesimam vnitatis subsistit, ipsa autem fractio $\frac{12700}{10080}$ tantum particula $\frac{1}{10080 \cdot 23906} = \frac{1}{240972480}$ hoc est minore quam vicies millionesima vnitatis a vero valore $\sqrt[3]{2}$ deficit, tantam autem praecisionem ope logarithmorum attingere non licet. Vnde intelligitur, quantum vsum haec methodus praestare queat in radicibus cuiusuis dignitatis proxime exprimendis.

## Propositio V.

15. Inuestigare legem harum trium progressionum:

$a, a', a''$ etc. $\quad b, b', b''$, etc. $\quad c, c', c''$ etc.

quarum termini homologi $a^{(n)} : b^{(n)} : c^{(n)}$ continuo propius proportionem $1 : \sqrt[3]{r} : \sqrt[3]{r^2}$ exprimunt.

### Solutio.

Cum omnes termini ex ternis primo assumtis $a, b$ et $c$ ita componantur, vt sit $a^{(n)} = La + Mb + Nc$, erit



erit ex earum indole $b^{(n)} = Lb + Mc + Nra$ et $c^{n} = Lc + Mra + Nrb$ at lex progressionis praebet sequentes terminos:

$$a^{(n+1)} = (L+Mr+Nr)a + (L+M+Nr)b + (L+M+N)c$$
$$b^{(n+1)} = (L+Mr+Nr)b + (L+M+Nr)c + (L+M+N)ra$$
$$c^{n+1} = (L+Mr+Nr)c + (L+M+Nr)ra + (L+M+N)rb.$$

Hinc si generalius statuamus:

$$a + a'x + a''x^2 + a'''x^3 + \text{etc.} = Pa + Qb + Rc \text{ erit}$$
$$b + b'x + b''x^2 + b'''x^3 + \text{etc.} = Pb + Qc + Rra$$
$$c + c'x + c''x^2 + c'''x^3 + \text{etc.} = Pc + Qra + Rrb.$$

Addantur hae series inuicem, et quia est

$a + b + c = a'$, $a' + b' + c' = a''$, $a'' + b'' + c'' = a'''$ erit

$$a' + a''x + a'''x^2 + \text{etc.} = (P+Qr+Rr)a + (P+Q+Rr)b + (P+Q+R)c$$

quae per $x$ multiplicata et a prima subtracta dat

$$a = Pa + Qb + Rc - (P+Qr+Rr)ax - (P+Q+Rr)bx - (P+Q+r)cx.$$

Quia autem quantitates P, Q, R litteras $a$, $b$, $c$ non inuoluunt, hinc nascuntur tres aequationes:

| | | | |
|---|---|---|---|
| I. | $1 = P - Px - Qrx - Rrx$ | | $1 = P - Q - Q(r-1)x$ |
| II. | $0 = Q - Px - Qx - Rrx$ | hincque | $0 = Q - R - R(r-1)x$ |
| III. | $0 = R - Px - Qx - Rx$ | | $1 = P - R - (Q+R)(r-1)x.$ |

Pro faciliori resolutione statuamus $1 - x + rx = z$ et sequens combinatio praebebit

$$\begin{array}{ll} \mathrm{I}-\mathrm{II} & 1 = P - Qz \\ \mathrm{II}-\mathrm{III} & 0 = Q - Rz \end{array} \quad \text{hinc} \quad \begin{array}{l} Q = Rz \\ P = 1 + Rzz \end{array}$$

vnde fit

III. $0 = R(1-x) - Px - Qx = R(1 - x - xz - xzz) - x$

ideoque $R = \frac{x}{1 - x(1 + z + zz)}$, at est $x = \frac{z-1}{r-1}$.

Ergo $R = \frac{z-1}{r-1-(z-1)(1+z+zz)} = \frac{z-1}{r-z^3}$ sicque prodit

$$P = \frac{r-zz}{r-z^3}; \quad Q = \frac{zz-z}{r-z^3}; \quad R = \frac{z-1}{r-z^3}$$

quarum formarum cum denominator sit

$$r - 1 - 3(r-1)x - 3(r-1)^2 x^2 - (r-1)^3 x^3$$

seu $(r-1)(1 - 3x - 3(r-1)x^2 - (r-1)^2 x^3)$

perspicuum est nostras tres progressiones esse recurrentes scala relationis existente $3, 3(r-1), (r-1)^2$, ita vt sit

$$a^{(n)} = 3a^{(n-1)} + 3(r-1)a^{(n-2)} + (r-1)^2 a^{(n-3)}.$$

Nunc pro terminis generalibus harum progressionum fractiones P, Q, R in simplices resolui oportet: quoniam autem denominatoris factor simplex est $\sqrt[3]{r} - z$ simul vicem binorum reliquorum gerens, siquidem $\sqrt[3]{r}$ tres inuoluit valores diuersos, sufficit hunc vnicum factorem considerasse. Sit ergo ex fractione R fractio simplex oriunda $= \frac{A}{\sqrt[3]{r} - z}$, et numera-



merator erit $A = \frac{z-1}{\sqrt[3]{r^2} + z\sqrt[3]{r} + zz}$ posito $z = \sqrt[3]{r}$, vnde fit $A = \frac{\sqrt[3]{r} - 1}{3\sqrt[3]{rr}}$, ideoque

$$P = \frac{1}{3\sqrt[3]{r^2}} \cdot \frac{r - \sqrt[3]{r^2}}{\sqrt[3]{r} - z} \text{ etc. } Q = \frac{1}{3\sqrt[3]{r^2}} \cdot \frac{\sqrt[3]{r^2} - \sqrt[3]{r}}{\sqrt[3]{r} - z} \text{ etc. } R = \frac{1}{3\sqrt[3]{r^2}} \cdot \frac{\sqrt[3]{r} - 1}{\sqrt[3]{r} - z} \text{ etc.}$$

Restituatur pro $z$ valor $1 + (r-1)x$, sitque $\frac{r-1}{\sqrt[3]{r} - 1} = s$, ac fiet

$$P = \frac{1}{3} \cdot \frac{1}{1-sx} + \text{etc. } Q = \frac{1}{3\sqrt[3]{r}} \cdot \frac{1}{1-sx} + \text{etc. } R = \frac{1}{3\sqrt[3]{r^2}} \cdot \frac{1}{1-sx} + \text{etc.}$$

Hinc cum sit $a + a'x + a''x^2 + \text{etc.} = Pa + Qb + Rc$ sequitur fore

$$a^{(n)} = \tfrac{1}{3} s^n \left(a + \frac{b}{\sqrt[3]{r}} + \frac{c}{\sqrt[3]{rr}}\right) + \ldots\ldots + \ldots\ldots$$

vbi duo membra omissa ex primo ita formantur vt loco $\sqrt[3]{r}$ scribatur $\frac{-1 \pm \sqrt{-3}}{2}\sqrt[3]{r}$, id quod etiam de $s = \frac{r-1}{\sqrt[3]{r} - 1}$ est intelligendum. Deinde vero pro binis reliquis seriebus habebitur:

$$b^{(n)} = \tfrac{1}{3} s^n \left(a\sqrt[3]{r} + b + \frac{c}{\sqrt[3]{r}}\right) + \ldots\ldots + \ldots\ldots$$

$$c^{(n)} = \tfrac{1}{3} s^n \left(a\sqrt[3]{r^2} + b\sqrt[3]{r} + c\right) + \ldots\ldots + \ldots\ldots$$

## Coroll. I.

16. Si $n$ sit numerus praegrandis, bina membra omissa prae primis hic appositis euanescunt; ex



quo

quo perſpicuum eſt tum fore $a^{(n)} : b^{(n)} : c^{(n)} = 1 : \sqrt[3]{r} : \sqrt[3]{rr}$; in quo ipſo tota vis methodi hic traditae conſiſtit.

## Coroll. 2.

17. Natura harum ſerierum recurrentium tertii ordinis ideo imprimis notari meretur; quod fractiones principales P, Q, R tam concinne in fractiones ſimplices reſoluere licuit, atque ex terminis generalibus inde deriuatis natura harum ſerierum facile perſpicitur.

## Scholion.

18. Hinc ratio iſtam methodum ad plures medias proportionales extendendi ita iam eſt manifeſta, vt ſuperfluum foret omnia ratiocinia, quibus operationes vſurpatae innituntur, repetere. Quam ob rem inuentionem plurium mediarum proportionalium inter duos numeros datam rationem $1 : r$ tenentes, nunc quidem ſatis ſuccincte exponere atque adeo binas propoſitiones cuique caſui tribuendas commode in vnam contrahere poterimus.

## Propoſitio VI.

19. Inter duos numeros rationem datam $1 : r$ tenentes, tres medias continue proportionales in rationalibus proxime exhibere.

Solutio.



## Solutio.

Sumtis in ratione data duobus numeris $a$ et $ra$ inter eos pro lubitu tres medii conſtituantur $b, c, d$, vt habeantur hi quinque numeri quantumvis a ſcopo aberrantes:

$$a : b : c : d : ra.$$

Hinc formentur alii hac lege vt ſit

$$a' = a + b + c + d$$
$$b' = b + c + d + ra$$
$$c' = c + d + ra + rb$$
$$d' = d + ra + rb + rc$$

qui conſtituent progreſſionem iam multo propius ad ſcopum attingentem hanc:

$$a' : b' : c' : d' : ra'$$

ex quibus porro eadem lege alii noui quaerantur, indeque denuo alii, quo pacto continuo propius ad proportionem geometricam continuam accedetur, ita vt aberratio tandem omni aſſignabili minor euadat.

Singulae porro harum ſerierum

$$a, a', a'', a''', a'''' \text{ etc.}$$
$$b, b', b'', b''', b'''' \text{ etc.}$$
$$c, c', c'', c''', c'''' \text{ etc.}$$
$$d, d', d'', d''', d'''' \text{ etc.}$$

ſunt recurrentes quarti ordinis ſecundum ſcalam relationis: $4, 6(r-1), 4(r-1)^2, (r-1)^3$.



Deni-

Denique harum ſcrierum termini generales, poſito breuitatis gratia $\frac{r-1}{\sqrt[4]{r}-1}=1+\sqrt[4]{r}+\sqrt[4]{r^2}+\sqrt[4]{r^3}=s$ erunt

$$a^{(n)}=\frac{a\sqrt[4]{r^3}+b\sqrt[4]{r^2}+c\sqrt[4]{r}+d}{4\sqrt[4]{r^3}}s^n+\text{etc.}$$

$$b^{(n)}=\frac{a\sqrt[4]{r^3}+b\sqrt[4]{r^2}+c\sqrt[4]{r}+d}{4\sqrt[4]{r^2}}s^n+\text{etc.}$$

$$c^{(n)}=\frac{a\sqrt[4]{r^3}+b\sqrt[4]{r^2}+c\sqrt[4]{r}+d}{4\sqrt[4]{r}}s^n+\text{etc.}$$

$$d^{(n)}=\frac{a\sqrt[4]{r^3}+b\sqrt[4]{r^2}+c\sqrt[4]{r}+d}{4}s^n+\text{etc.}$$

quarum expreſſionum prima tantum membra appoſui, dum ex his reliqua facile formantur, loco $\sqrt[4]{r}$ eius ternos reliquos valores ſubſtituendo.

Ceterum ſi ſtatuatur

$$a+a'x+a''x^2+a'''x^3+a''''x^4+\text{etc.}=\mathrm{P}a+\mathrm{Q}b+\mathrm{R}c+\mathrm{S}d$$

ac breuitatis gratia fiat $1-x+rx=z$ reperitur vt ante:

$$\mathrm{P}=\frac{r-z^3}{r-z^4};\ \mathrm{Q}=\frac{z^3-z^2}{r-z^4};\ \mathrm{R}=\frac{z^2-z}{r-z^4};\ \mathrm{S}=\frac{z-1}{r-z^4}$$

vnde ſimul reliquarum ſimilium ſerierum a litteris $b$, $c$, $d$ incipientium ſummae exhibentur.

Exem-



## Exemplum.

20. Inter duos numeros rationem duplam tenentes, tres medii proportionales ſequenti modo reperiuntur:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $a$ . . . | 1 | 4 | 22 | 116 | 613 | 3240 | 17124 | 90504 |
| $b$ . . . | 1 | 5 | 26 | 138 | 729 | 3853 | 20364 | 107628 |
| $c$ . . . | 1 | 6 | 31 | 164 | 867 | 4582 | 24217 | 127992 |
| $d$ . . . | 1 | 7 | 37 | 195 | 1031 | 5449 | 28799 | 152209 |
| $2a$ . . . | 2 | 8 | 44 | 232 | 1226 | 6480 | 34248 | 181008 |

vbi vltimi numeri tam prope progreſſionem geometricam in ratione $1:\sqrt[4]{2}$ procedentem conſtituunt, quam fieri poteſt, numeris non maioribus adhibendis. Ita ſatis exacte erit $\frac{107628}{90504}=\sqrt[4]{2}$ ſeu per 12 reducendo $\frac{8969}{7542}=\sqrt[4]{2}$ cuius error longe infra partem millioneſimam vnitatis ſubſiſtit.

## Scholion 1.

21. In Muſicis ſimilis quaeſtio de vndecim mediis proportionalibus inter rationem duplam inveniendis tractari ſolet, vt hinc omnia ſemitonia vnius *Octauae* inter ſe aequalia reddantur; quod temperamentum etſi principiis harmoniae aduerſatur, tamen non abs re fore arbitror eius ſolutionem ex iisdem principiis petitam hic apponere:

| | | | | | | |
|---|---|---|---|---|---|---|
| A .. | 1 | 12 | 210 | 3532 | 59879 | 998592 |
| B .. | 1 | 13 | 222 | 3742 | 62911 | 1057971 |
| H .. | 1 | 14 | 235 | 3964 | 66653 | 1120882 |
| C .. | 1 | 15 | 249 | 4199 | 70617 | 1187535 |
| Cs .. | 1 | 16 | 264 | 4448 | 74816 | 1258152 |
| D .. | 1 | 17 | 280 | 4712 | 79264 | 1332968 |
| Ds .. | 1 | 18 | 297 | 4992 | 83976 | 1412232 |
| E .. | 1 | 19 | 315 | 5289 | 88968 | 1496208 |
| F .. | 1 | 20 | 334 | 5604 | 94257 | 1585176 |
| Fs .. | 1 | 21 | 354 | 5938 | 99861 | 1679433 |
| G .. | 1 | 22 | 375 | 6292 | 105799 | 1779294 |
| Gs .. | 1 | 23 | 397 | 6667 | 112091 | 1885093 |
| $a$ ... | 2 | 24 | 420 | 7064 | 118758 | 1997184 |

vltima columna tam parum a progreſſione geometrica recedit, vt error ne ad millioneſimam partem aſſurgere ſit cenſendus.

## Scholion.

22. Quae hactenus ſunt tradita facile ad quotcunque medios proportionales inueniendos in genere accommodari poſſunt, in quo negotio hoc imprimis notari meretur, quod ſeries numerorum, quibus ſolutio continetur non ſolum ſint recurrentes, ſed etiam denominator fractionum, ex quibus naſcuntur ſemper in factores reſolui queat, ad quemcunque etiam gradum aſcendat: vnde egregia exempla aequationum altioris gradus ſolutionem admittentium colliguntur, quibus coniectura mea circa formam



formam radicum cuiusque gradus olim prolata pulcherrime confirmatur. Verum methodus hic expoſita multo latius extendi poteſt, quemadmodum in ſequente propoſitione ſum oſtenſurus; ita vt inde adhuc maiora ſubſidia in Analyſin redundatura videantur.

## Propoſitio VII.

23. Methodum multo latius patentem exhibere, cuius ope inter duos numeros datam rationem $1 : r$ tenentes quotcunque medii proportionales in rationalibus proxime inueniri queant.

## Solutio.

Inter duos numeros $a$ et $ar$ datam rationem tenentes vt ante totidem medii pro lubitu accipiantur, quot medios proportionales aſignari oportet. Ponamus autem quatuor medios inueniri debere, quoniam hinc vis methodi clarius perſpicietur, quam ſi rem generaliter tractare velimus. Conſtituta ergo pro hoc caſu ad lubitum tali progreſſione

$$a : b : c : d : e : ar : br : cr : dr : er : ar^2 \text{ etc.}$$

ſumantur pro arbitrio quinque indices $\alpha, \beta, \gamma, \delta, \epsilon$ per quos inde noua ſimilis progreſſio formetur:

$$a' : b' : c' : d' : e' : a'r : b'r : c'r : d'r : e'r : a'r^2 \text{ etc.}$$

hac lege vt ſit

 $a' =$

$$a' = \alpha a + \beta b + \gamma c + \delta d + \varepsilon e$$
$$b' = \alpha b + \beta c + \gamma d + \delta e + \varepsilon a r$$
$$c' = \alpha c + \beta d + \gamma e + \delta a r + \varepsilon b r$$
$$d' = \alpha d + \beta e + \gamma a r + \delta b r + \varepsilon c r$$
$$e' = \alpha e + \beta a r + \gamma b r + \delta c r + \varepsilon d r$$

Tum vero per eosdem indices ex hac progreſſiore denuo alia formetur noua, atque ita porro, vt hac ratione ſequentes ſeries obtineantur

$$a + a'x + a''x^2 + a'''x^3 + \text{etc.} = Pa + Qb + Rc + Sd + Te$$
$$b + b'x + b''x^2 + b'''x^3 + \text{etc.} = Pb + Qc + Rd + Se + Tar$$
$$c + c'x + c''x^2 + c'''x^3 + \text{etc.} = Pc + Qd + Re + Sar + Tbr$$
$$d + d'x + d''x^2 + d'''x^3 + \text{etc.} = Pd + Qe + Rar + Sbr + Tcr$$
$$e + e'x + e''x^2 + e'''x^3 + \text{etc.} = Pe + Qar + Rbr + Scr + Tdr$$

vnde ex lege praeſcripta valores litterarum P, Q, R, S, T quae a litteris arbitrariis $a, b, c, d, e$ ſunt immunes, et tantum ab indicibus $\alpha, \beta, \gamma, \delta, \varepsilon$ vna cum quantitate $x$ et ratione propoſita $1 : r$ pendent, ita determinantur vt ſit:

$$\frac{P-1}{x} = \alpha P + \beta T r + \gamma S r + \delta R r + \varepsilon Q r$$
$$\frac{Q}{x} = \alpha Q + \beta P + \gamma T r + \delta S r + \varepsilon R r$$
$$\frac{R}{x} = \alpha R + \beta Q + \gamma P + \delta T r + \varepsilon S r$$
$$\frac{S}{x} = \alpha S + \beta R + \gamma Q + \delta P + \varepsilon T r$$
$$\frac{T}{x} = \alpha T + \beta S + \gamma R + \delta Q + \varepsilon P$$





ex quibus aequalitatibus quidem valores harum litterarum admodum perplexi eliciuntur, ita vt denominator communis huiusmodi formam ſit habiturus:

$$1 - Ax - Bx^2 - Cx^3 - Dx^4 - Ex^5$$

indicium praebens ſeries illas eſſe recurrentes ex eadem ſcala relationis oriundas. Verum quod hic potiſſimum eſt notandum, hunc denominatorem ſemper in factores ſimplices reſoluere licet, qui inter ſe ita erunt ſimiles, vt ex quinis ipſius $\sqrt[5]{r}$ valoribus ſimili modo formentur. Scilicet ſi breuitatis gratia ponatur

$$\alpha + \beta\sqrt[5]{r} + \gamma\sqrt[5]{r^2} + \delta\sqrt[5]{r^3} + \varepsilon\sqrt[5]{r^4} = s$$

vbi etiam $s$ quinos valores diuerſos ſortitur erit $1 - sx$ factor ſimplex illius denominatoris, ſimul omnes quinque in ſe inuoluens. Hinc ergo ſingulas fractiones, quibus litterae illae P, Q, R, S, T exprimuntur in quinque fractiones ſimplices reſoluere licebit quae ita concinne expreſſae reperiuntur:

$$P = \frac{1}{s(1-sx)} + \ldots + \ldots + \ldots + \ldots$$
$$Q = \frac{1}{s(1-sx)\sqrt[5]{r}} + \ldots + \ldots + \ldots + \ldots$$
$$R = \frac{1}{s(1-sx)\sqrt[5]{r^2}} + \ldots + \ldots + \ldots + \ldots$$
$$S = \frac{1}{s(1-sx)\sqrt[5]{r^3}} + \ldots + \ldots + \ldots + \ldots$$
$$T = \frac{1}{s(1-sx)\sqrt[5]{r^4}} + \ldots + \ldots + \ldots + \ldots$$

vbi quaterna membra punctis indicata ex primis, ipsi $\sqrt[5]{r}$ quatuor reliquos valores tribuendo, sunt supplenda.

Hinc iam quinque serierum a numeris arbitrariis $a, b, c, d, e$ incipientium termini generales formari possunt, qui etiam ponendo breuitatis gratia

$$a\sqrt[5]{r^4} + b\sqrt[5]{r^3} + c\sqrt[5]{r^2} + d\sqrt[5]{r} + e = k$$

(vbi quoque quantitas $k$ quinque valores inuoluere est existimanda) sequenti modo concinne exprimuntur:

$$a^{(n)} = \frac{k}{5\sqrt[5]{r^4}} s^n + \ldots + \ldots + \ldots + \ldots$$
$$b^{(n)} = \frac{k}{5\sqrt[5]{r^3}} s^n + \ldots + \ldots + \ldots + \ldots$$
$$c^{(n)} = \frac{k}{5\sqrt[5]{r^2}} s^n + \ldots + \ldots + \ldots + \ldots$$
$$d^{(n)} = \frac{k}{5\sqrt[5]{r}} s^n + \ldots + \ldots + \ldots + \ldots$$
$$e^{(n)} = \frac{k}{5} s^n + \ldots + \ldots + \ldots + \ldots$$

vbi quaterna membra omissa simili modo vt supra ex primis constitui oportet.

Hinc iam id in quo cardo rei versatur, intelligitur scilicet si series illae in infinitum continuentur, vt exponens $n$ in infinitum excrescat, tum respectu eius membri, in quo ipsi $\sqrt[5]{r}$ valor realis positiuus tribuitur, reliqua euanescere, sicque manifesto numeros $a^{(n)} : b^{(n)} : c^{(n)} : d^{(n)} : e^{(n)} : ra^{(n)}$ progressionem geometricam constituere. Verum hic probe est notandum, illam euanescentiam locum non habere nisi indices $\alpha, \beta, \gamma, \delta, \varepsilon$ sint positiui, quemadmodum hinc etiam casus supra tractatus resultat, si hi indices vnitati aequales statuantur.



## Scholion.

24. Circa hanc solutionem generalem obseruari conuenit, quod si valores litterarum P, Q, R, S, T ex formulis inuentis euoluantur, earumque denominator communis ad nihilum redigatur, vt posito $x = \frac{1}{z}$ huiusmodi prodeat aequatio quinti gradus:

$$z^5 - Az^4 - Bz^3 - Cz^2 - Dz - E = 0$$

tum huius aequationis radicem fore

$$z = \alpha + \beta\sqrt[5]{r} + \gamma\sqrt[5]{r^2} + \delta\sqrt[5]{r^3} + \varepsilon\sqrt[5]{r^4}$$

in qua forma simul omnes quinque radices contineantur si modo pro $\sqrt[5]{r}$ eius quinque valores successiue substituantur. Cum igitur hae radices eam ipsam habeant formam, quam olim coniectura eram assecutus, hinc multo confidentius affirmare poterimus, omnium aequationum cuiuscunque gradus radices eo modo exprimi, quem coniectura mea indicat. Quodsi indices

indices $\alpha, \beta, \gamma, \delta, \varepsilon$ vnitati aequentur aequatio quinti gradus fit ex ſuperioribus:

$$z^5 - 5z^4 - 10(r-1)z^3 - 10(r-1)^2 z^2 - 5(r-1)^3 z - (r-1)^4 = 0$$

cuius radix erit $z = 1 + \sqrt[5]{r} + \sqrt[5]{r^2} + \sqrt[5]{r^3} + \sqrt[5]{r^4}$. Seu poſito $z = y + 1$ erit huius aequationis

$$y^5 = 10ry^3 + 10r(r+1)y^2 + 5r(rr+r+1)y + r(r^3+r^2+r+1)$$

radix $y = \sqrt[5]{r} + \sqrt[5]{r^2} + \sqrt[5]{r^3} + \sqrt[5]{r^4}$.

DE



# DE INTEGRATIONE AEQVATIONIS DIFFERENTIALIS:

$$a^n d^n y + b.a^{n-1} d^{n-1} y dx + c.a^{n-2} d^{n-2} y dx^2 + \ldots + r y dx^n = X dx^n \quad \text{(I)}$$

Auctore

*AND. IOH. LEXELL.*

## 1.

Ex occaſione aequationis differentialis $a^4 d^4 y - y dx^4 = 0$, cuius integrali inueniendo nuper eram intentus, incidi in methodum eam integrandi, haud inelegantem, quippe quae non ſolum eo nomine ſe commendabat, quod pro ſpeciali iſto exemplo videbatur eſſe facillima; ſed etiam quia optimo cum ſucceſſu applicari poſſet, ad perficiendam integrationem aequationis differentialis ſupra propoſitae, quae formae eſt generalis et innumeras ſub ſe complectitur. Deinceps vero cum perſpicerem, a ſummis Mathematicis, inprimis ab Illuſtr. *Eulero* in Tom. VII. Miſcellan. Berolin. et Tom. III. Nov. Comment. Acad. Imper. Petropol. antea iam traditas fuiſſe, hanc aequationem integrandi Methodos, merito dubius haeſi, vtrum quae ad hanc materiem illuſtrandam meditatus eram, publicae luci committerem, an non ſatius foret, eadem penitus

tus