1770

De summis serierum numeros Bernoullianos involventium

Leonhard Euler

Consideretur formula differentialis $P\,dt - Q\,du$, quae multiplicatore R integrabilis reddatur; ponaturque $dv = R(P\,dt - Q\,du)$, eritque v functio cognita ipsarum t et u, vnde vicissim t per u et v definiatur ac fiet $dt = \frac{Q\,du}{P} + \frac{dv}{PR}$. Quare formula proposita erit:

$\frac{MQ+NP}{P}du + \frac{M\,dv}{PR}$, vbi $\frac{MQ+NP}{P}$ est functio ipsarum u et v tantum sicque resolutio per n°. 4 absoluetur.

Scholion. 2.

25. De his integrationibus imprimis notandum est, eas esse generalissimas, dum functiones maxime generales cuiuspiam variabilis in integralia introducuntur, pro quibus adeo functiones nullo continuitatis vinculo contentas, quae vt supra videmus ex libero manus ductu nascuntur, assumere licet. Quin etiam omnium huius generis quaestionum criterium in hoc consistit, vt tales functiones prorsus ab arbitrio nostro pendentes in earum solutiones introducantur.

DE

DE SVMMIS SERIERVM NVMEROS BERNOVLLIANOS INVOLVENTIVM.

Auctore

L. EVLERO.

1.

Quantopere sint notatu digni numeri ab Inuentore *Bernoulliani* vocati, quippe quibus olim *Iacobus Bernoulli* in Arte coniectandi est vsus ad progressiones potestatum numerorum naturalium summandas, cum ab aliis, qui serierum doctrinam nouis inuentis locupletauerunt, tum etiam a me abunde est ostensum, vbi per eosdem numeros serierum potestatum reciprocarum summas expressas dedi. *Bernoullius* quidem progressionem horum numerorum ob calculi molestiam non vltra quintum terminum continuauit, qui sunt $\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}$, atque Auctori vsque ad vndecimas potestates summandas sufficiebant. Postquam autem satis concinnam huius progressionis legem detexissem, 17 eius primores terminos assignaui. Ipsos vero numeros *Bernoullianos* respectiue per numeros 6, 10, 14, 18, 22 etc. multiplico quo denominatores fiant simplicio-

 res,

res, terminosque huius nouae feriei litteris 𝔄, 𝔅, ℭ, 𝔇, 𝔈 etc. defignans, earum fequentes reperi valores:

$$\mathfrak{A} = 1$$
$$\mathfrak{B} = \frac{1}{3}$$
$$\mathfrak{C} = \frac{1}{3}$$
$$\mathfrak{D} = \frac{3}{5}$$
$$\mathfrak{E} = \frac{5}{3}$$
$$\mathfrak{F} = \frac{691}{105}$$
$$\mathfrak{G} = \frac{35}{1}$$
$$\mathfrak{H} = \frac{3617}{15}$$
$$\mathfrak{I} = \frac{43867}{21}$$
$$\mathfrak{K} = \frac{1222277}{55}$$
$$\mathfrak{L} = \frac{854513}{3}$$
$$\mathfrak{M} = \frac{1181820455}{273}$$
$$\mathfrak{N} = \frac{76977927}{1}$$
$$\mathfrak{O} = \frac{23749461029}{15}$$
$$\mathfrak{P} = \frac{8615841276005}{231}$$
$$\mathfrak{Q} = \frac{84802531453387}{85}$$
$$\mathfrak{R} = \frac{90219075042845}{3}$$

2. Contemplatio autem ferierum poteftatum reciprocarum, quarum fummas quoties exponens eft numerus par, per fimiles poteftates numeri π peri-

pheriam circuli, cuius diameter eft $=1$, referentis definiri poffe demonftraui, horum numerorum nexum multo clarius exhibuit. Si enim has fummas fequenti modo defignamus:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.} = A\pi^2$$
$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} = B\pi^4$$
$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \text{etc.} = C\pi^6$$
$$1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \text{etc.} = D\pi^8$$
$$1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{4^{10}} + \frac{1}{5^{10}} + \text{etc.} = E\pi^{10}$$

primum per hos numeros A, B, C, D etc. praecedentes 𝔄, 𝔅, ℭ, 𝔇 etc. ita determinari oftendi vt fit:

	ideoque
$\mathfrak{A} = \frac{1.2.3}{1} A$	$A = \frac{2^0 \mathfrak{A}}{1.2.3}$
$\mathfrak{B} = \frac{1.2.3.4.5}{2^2} B$	$B = \frac{2^2 \mathfrak{B}}{1.2.3.4.5}$
$\mathfrak{C} = \frac{1.2.3 \ldots 7}{2^4} C$	$C = \frac{2^4 \mathfrak{C}}{1.2.3 \ldots 7}$
$\mathfrak{D} = \frac{1.2.3 \ldots 9}{2^6} D$	$D = \frac{2^6 \mathfrak{D}}{1.2.3 \ldots 9}$
$\mathfrak{E} = \frac{1.2.3 \ldots 11}{2^8} E$	$E = \frac{2^8 \mathfrak{E}}{1.2.3 \ldots 11}$
$\mathfrak{F} = \frac{1.2.3 \ldots 13}{2^{10}} F$	$F = \frac{2^{10} \mathfrak{F}}{1.2.3 \ldots 13}$
etc.	etc.

3. Porro autem pro litterarum A, B, C, D etc. progreffione duplicem obferuaui legem, cuius ope quamlibet per praecedentes determinari licet. Prior lex quemlibet terminum per fingulos praecedentium ita definit, vt fit

$A = \frac{1}{1.2.3}$

$$A = \frac{1}{1 \cdot 2 \cdot 3}$$
$$B = \frac{A}{1 \cdot 2 \cdot 3} - \frac{2}{1 \cdot 2 \cdots 5}$$
$$C = \frac{B}{1 \cdot 2 \cdot 3} - \frac{A}{1 \cdot 2 \cdots 5} + \frac{3}{1 \cdot 2 \cdots 7}$$
$$D = \frac{C}{1 \cdot 2 \cdot 3} - \frac{B}{1 \cdot 2 \cdots 5} + \frac{A}{1 \cdot 2 \cdots 7} - \frac{4}{1 \cdot 2 \cdots 9}$$
$$E = \frac{D}{1 \cdot 2 \cdot 3} - \frac{C}{1 \cdot 2 \cdots 5} + \frac{B}{1 \cdot 2 \cdots 7} - \frac{A}{1 \cdot 2 \cdots 9} + \frac{5}{1 \cdot 2 \cdots 11}$$

Altera vero lex commodius quemuis terminum per producta ex binis praecedentibus sequenti modo exprimit:

$$5B = 2A^2 \quad \text{existente} \quad A = \tfrac{1}{6}$$
$$7C = 4AB$$
$$9D = 4AC + 2BB$$
$$11E = 4AD + 4BC$$
$$13F = 4AE + 4BD + 2CC$$
$$15G = 4AF + 4BE + 4CD$$
$$17H = 4AG + 4BF + 4CE + 2DD$$
etc.

vnde etiam mihi quidem has series tam longe continuare licuit.

4. His expositis hoc loco in summas plurium serierum, quorum termini istos numeros A, B, C, D, E etc. praeter alios factores, quorum lex per se est manifesta, inuoluunt, inquirere constitui, ita v mihi in genere proposita sit inuestigatio summae huius seriei

S=α

$$S = \alpha A x^2 + \beta B x^4 + \gamma C x^6 + \delta D x^8 + \varepsilon E x^{10} + \text{etc.}$$

dum litterae α, β, γ, δ etc. seriem quamcunque cognitam constituunt, eximias enim hinc enasci series, quarum summae omni attentione sint dignae pluribus speciminibus iam ostendi. Incipio igitur ab hac serie:

$$S = Ax^2 + Bx^4 + Cx^6 + Dx^8 + Ex^{10} + \text{etc.}$$

quam per priorem legem progressionis litterarum A, B, C, D etc. manifesto ex euolutione huius fractionis resultare manifestum est:

$$S = \frac{\frac{1}{1 \cdot 2 \cdot 3}x^2 - \frac{2}{1 \cdot 2 \cdots 5}x^4 + \frac{3}{1 \cdot 2 \cdots 7}x^6 - \frac{4}{1 \cdot 2 \cdots 9}x^8 + \text{etc.}}{1 - \frac{1}{1 \cdot 2 \cdot 3}x^2 + \frac{1}{1 \cdot 2 \cdots 5}x^4 - \frac{1}{1 \cdot 2 \cdots 7}x^6 + \frac{1}{1 \cdot 2 \cdots 9}x^8 - \text{etc.}}$$

cuius denominator exhibet valorem $\frac{\sin. x}{x}$; eiusque differentiale $\frac{dx \cos. x}{x} - \frac{dx \sin. x}{xx}$ per $\frac{-x}{2dx}$ multiplicatum ipsum praebet numeratorem, ita vt sit:

$$s = \frac{\sin. x - x \cos. x}{2 \sin. x} = \tfrac{1}{2} - \tfrac{1}{2} x \cot. x$$

hincque summa istius seriei

$$Ax^2 + Bx^4 + Cx^6 + Dx^8 + Ex^{10} + \text{etc.} = \tfrac{1}{2} - x \cot. x$$

vbi notari meretur si x euanescat, fore summam $= \frac{1}{6}xx$ ob $\cot. x = \frac{1 - \frac{1}{3}xx}{x}$.

5. Ponamus $xx = -yy$, totamque seriem negatiue exponamus, vt quaeratur haec summa:

$$s = Ayy - By^4 + Cy^6 - Dy^8 + Ey^{10} - \text{etc.}$$

atque

atque cum iam ſit per legem priorem:

$$s = \frac{\frac{1}{1\cdot2\cdot3}y^2 + \frac{2}{1\cdot2\ldots5}y^4 + \frac{3}{1\cdot2\ldots7}y^6 + \frac{4}{1\cdot2\ldots9}y^8 + \text{etc.}}{1 + \frac{1}{1\cdot2\cdot3}y^2 + \frac{1}{1\cdot2\ldots5}y^4 + \frac{1}{1\cdot2\ldots7}y^6 + \frac{1}{1\cdot2\ldots9}y^8 + \text{etc.}}$$

cuius denominator eſt $\frac{1}{2y}(e^y - e^{-y})$, eiusque differentiale $\frac{-dy}{2yy}(e^y - e^{-y}) + \frac{dy}{2y}(e^y + e^{-y})$, quod per $\frac{y}{2dy}$ multiplicatum dat numeratorem $= -\frac{1}{4y}(e^y - e^{-y}) + \frac{1}{4}(e^y + e^{-y})$ ita vt huius ſeriei ſumma ſit

$$s = \frac{y}{2}\cdot\frac{e^y + e^{-y}}{e^y - e^{-y}} - \frac{1}{2}.$$

Caſus hic notari meretur quo $y = 1$, haecque ſeries ſummatur:

$$A - B + C - D + E - \text{etc.} = \frac{1}{2}\cdot\frac{ee+1}{ee-1} - \frac{1}{2} = \frac{1}{ee-1}.$$

Si hic pro litteris A, B, C, D etc. ipſae ſeries aſſumtae reſtituantur, et quatenus fieri poteſt, in ſummas colligantur, erit

$$\frac{1}{\pi\pi+1} + \frac{1}{4\pi\pi+1} + \frac{1}{9\pi\pi+1} + \frac{1}{16\pi\pi+1}\ \text{etc.} = \frac{1}{ee-1}.$$

6. Inuenta ſumma ſeriei

$$Ax^2 + Bx^4 + Cx^6 + Dx^8 + \text{etc} = \frac{1}{2} - \frac{1}{2}x\ \text{cot.}\ x$$

in qua ſimul alteram complecti licet

$$Ay^2 - By^4 + Cy^6 - Dy^8 + \text{etc.} = \frac{y}{2}\cdot\frac{e^{2y}+1}{e^{2y}-1} - \frac{1}{2}$$

tam ope differentiationis quam integrationis innumerabiles aliae inde deduci poſſunt quarum ſumma pariter

pariter aſſignari valet. Multiplicata ſcilicet illa ſerie per x^n differentiatio dabit

$$(n+2)Ax^{n+1} + (n+4)Bx^{n+3} + (n+6)Cx^{n+5} + (n+8)Dx^{n+7}\ \text{etc.}$$

$$= \frac{n}{2}x^{n-1} - \frac{(n+1)}{2}x^n\ \text{cot.}\ x + \frac{x^{n+1}}{2\ \text{ſin}\ x^2}\ \text{ſiue}$$

$$(n+2)Ax^2 + (n+4)Bx^4 + (n+6)Cx^6 + (n+8)Dx^8\ \text{etc.}$$

$$= \frac{n}{2} - \frac{1}{2}(n+1)x\ \text{cot.}\ x + \frac{xx}{2\ \text{ſin.}\ x^2}$$

ſin autem illa ſeries per $x^{n-1}dx$ multiplicata integretur, prodibit ſequens ſummatio:

$$\frac{A}{n+2}x^{n+2} + \frac{B}{n+4}x^{n+4} + \frac{C}{n+6}x^{n+6} + \frac{D}{n+8}x^{n+8} + \text{etc.}$$

$$= \frac{1}{2n}x^n - \frac{1}{2}\int x^n dx\ \text{cot.}\ x$$

quae ſumma vt cognita eſt ſpectanda, etiamſi formulae $\int x^n dx$ cot. x integrale euolui vel exprimi finite nequit. Quin etiam ambabus operationibus combinandis ac repetendis infinitae ſeries formae

$$\alpha Ax^2 + \beta Bx^4 + \gamma Cx^6 + \delta Dx^8 + \text{etc.}$$

obtinebuntur, vbi literae $\alpha, \beta, \gamma, \delta$ etc. ſint producta ex duabus pluribusue fractionibus, quarum tam numeratores quam denominatores progreſſiones arithmeticas conſtituant. Veluti ſi ſeries per differentiationem inuenta per $\frac{dx}{x}$ multiplicetur et integretur orietur:

$$\frac{n+2}{2}Ax^2 + \frac{n+4}{4}Bx^4 + \frac{n+6}{6}Cx^6 + \text{etc.} = \frac{n}{2}l\frac{x}{\text{ſin.}\ x} - \frac{x\ \text{coſ.}\ x}{2\ \text{ſin.}\ x} + \frac{1}{2}$$

ita

ita vt sit.

$$\tfrac{1}{2}Ax^2+\tfrac{1}{4}Bx^4+\tfrac{1}{6}Cx^6+\tfrac{1}{8}Dx^8+\text{etc.}=\tfrac{1}{2}l\frac{x}{\sin x}$$

7. Datur vero praeterea alia methodus omnino singularis ex serie inuenta alias innumerabiles eruendi quarum summa itidem assignari queat. Hunc in finem seriem principalem ita repraesento:

$$Aa^2x^2+Ba^4x^4+Ca^6x^6+Da^8x^8+\text{etc.}=\tfrac{1}{2}-\tfrac{1}{2}ax\cot.ax$$

eamque multiplico per eiusmodi formulam differentialem Xdx, vt si post integrationem ipsi x certus valor $x=f$ tribuatur, integrale $\int Xx^n dx$ valorem nanciscatur concinnum: scilicet vt fiat:

$$\int Xx^2dx=\alpha\int Xdx;\ \int Xx^4dx=\beta\int Xx^2dx;\ \int Xx^6dx=\gamma\int Xx^4dx \text{ etc.}$$

quo facto nascetur huiusmodi series:

$$\alpha Aa^2+\alpha\beta Ba^4+\alpha\beta\gamma Ca^6+\alpha\beta\gamma\delta Da^8+\text{etc.}=\tfrac{1}{2}-\frac{a\int Xxdx\cot.ax}{2\int Xdx}$$

vbi X ita accipi potest, vt $\alpha, \beta, \gamma, \delta$ etc. fiant vel numeri in arithmetica progressione procedentes, vel fractiones, quarum tam numeratores quam denominatores talem progressionem constituant. Veluti si sumatur.

$X=x^{m-1}(1-x^2)^k$ erit posito $x=1$ | ideoque

$$\begin{array}{l|l}
\int x^{m+1}dx(1-x^2)^k=\frac{m}{m+2k+2}\int x^{m-1}dx(1-x^2)^k & \alpha=\frac{m}{m+2k+2}\\
\int x^{m+3}dx(1-x^2)^k=\frac{m+2}{m+2k+4}\int x^{m+1}dx(1-x^2)^k & \beta=\frac{m+2}{m+2k+4}\\
\int x^{m+5}dx(1-x^2)^k=\frac{m+4}{m+2k+6}\int x^{m+3}dx(1-x^2)^k & \gamma=\frac{m+4}{m+2k+6}\\
\int x^{m+7}dx(1-x^2)^k=\frac{m+6}{m+2k+8}\int x^{m+5}dx(1-x^2)^k & \delta=\frac{m+6}{m+2k+8}
\end{array}$$

etc.

At

At si sumatur $Xdx=e^{-mxx}x^n dx$ erit posito post integrationem $x=\infty$

$$\begin{array}{l|l}
 & \text{ideoque}\\
\int e^{-mxx}x^{n+2}dx=\frac{n+1}{2m}\int e^{-mxx}x^n dx & \alpha=\frac{n+1}{2m}\\
\int e^{-mxx}x^{n+4}dx=\frac{n+3}{2m}\int e^{-mxx}x^{n+2}dx & \beta=\frac{n+3}{2m}\\
\int e^{-mxx}x^{n+6}dx=\frac{n+5}{2m}\int e^{-mxx}x^{n+4}dx & \gamma=\frac{n+5}{2m}
\end{array}$$

etc.

Sumto autem $Xdx=x^{n-1}dx(lx)^m$ fit posito $x=1$ post integrationem:

$$\int x^{n-1}dx(lx)^m=\pm\frac{1.2.3\ldots\ldots m}{n^{m+1}}$$

signo + valente si m sit numerus par, contra signo −.

8. His autem transformationibus, quae alibi fusius sunt expositae, hic non immoror, sed alium fontem, vnde huiusmodi series promanant, contemplabor quem olim iam mihi aperuit summatio progressionum generalis scilicet si seriei cuiuscunque terminus generalis, seu is qui indici x conuenit, ponatur $=X$, vt sit X functio quaecunque ipsius x, huiusque seriei terminus summatorius statuatur $=S$ reperi fore per numeros *Bernoullianos* $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc.

$$2S=2\int Xdx+X+\frac{\mathfrak{A}dX}{1.2.3dx}-\frac{\mathfrak{B}d^3X}{1.2\ldots5dx^3}+\frac{\mathfrak{C}d^5X}{1.2\ldots7dx^5}-\frac{\mathfrak{D}d^7X}{1.2\ldots9dx^7}+\text{etc.}$$

vnde

vnde ſi alteri numeri A, B, C, D etc. ad ſummas poteſtatum reciprocarum relati introducantur, deducimus:

$$2S = 2\int X\,dx + X + \frac{A\,dX}{1\,dx} - \frac{B\,d^3X}{2^2\,dx^3} + \frac{C\,d^5X}{2^4\,dx^5} - \frac{D\,d^7X}{2^6\,dx^7} + \text{etc.}$$

Quare ſi ſeriem cuius terminus generalis eſt X et ſummatorius $= S$ pro lubitu accipiamus habebimus hanc ſummationem:

$$\frac{A\,dX}{2\,dx} - \frac{B\,d^3X}{2^3\,dx^3} + \frac{C\,d^5X}{2^5\,dx^5} - \frac{D\,d^7X}{2^7\,dx^7} + \text{etc.} = S - \int X\,dx - \tfrac{1}{2}X.$$

Quaecunque ergo pro X ſumatur functio ipſius, conceſſa progreſſionis, cuius X eſt terminus generalis, ſummatione iſtius ſeriei litteras A, B, C, D etc. inuoluentis ſummam aſſignare poterimus, etiamſi forte eiusdem ſummatio ſecundum praecepta modo expoſita inſtituta ſummis difficultatibus ſit obnoxia.

9. Primum ergo ipſi X tribuamus eiusmodi valorem vt ſit $X = \frac{1}{x^n}$; vnde ſit

$$\frac{dX}{dx} = \frac{-n}{x^{n+1}};\quad \frac{d^3X}{dx^3} = \frac{-n(n+1)(n+2)}{x^{n+3}};\quad \frac{d^5X}{dx^5} = \frac{-n(n+1)\ldots\ldots(n+4)}{x^{n+5}}$$

et quia eſt $\int X\,dx = \frac{-1}{(n-1)x^{n-1}} + O$ atque

$$S = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \ldots\ldots + \frac{1}{x^n}.$$

habebi-

habebimus hanc ſeriem:

$$\frac{-nA}{2x^{n+1}} + \frac{n(n+1)(n+2)B}{2^3x^{n+3}} - \frac{n(n+1)(n+2)(n+3)C}{2^5x^{n+5}} + \text{etc.}$$

$$= S - \frac{1}{2x^n} + \frac{1}{(n-1)x^{n-1}} - O$$

vbi conſtantem O ex caſu quodam cognito definiri conuenit, quo ipſi x certus tribuitur valor. Ita poſito $x = \infty$, quoniam tum tota ſeries in nihilum abit conſtans hacc O exprimet ſummam huius ſeriei in infinitum continuatae $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \text{etc.}$ quam nouimus per quadraturam circuli π exhiberi poſſe quoties exponens n eſt numerus par. Hos ergo caſus primum euoluam.

Caſus I. quo $n = 2$.

10. Hoc ergo caſu $n = 2$ ſit conſtans $O = \frac{\pi\pi}{6} = A\pi^2$, poſitaque huius progreſſionis ſumma indefinite

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots + \frac{1}{x^2} = S$$

habebimus hanc ſummationem:

$$-\frac{2A}{2x^3} + \frac{2.3.4B}{2^3x^5} - \frac{2.3.4.5.6C}{2^5x^7} + \frac{2.3\ldots\ldots 8D}{2^7x^9} - \text{etc.}$$

$$= S - \frac{1}{2x^2} + \frac{1}{x} - A\pi^2 \text{ ſeu}$$

$$\frac{1.2A}{2^1x} - \frac{1.2.3.4B}{2^3x^3} + \frac{1.2\ldots\ldots 6C}{2^5x^5} - \frac{1.2\ldots\ldots 8D}{2^7x^7} + \text{etc.}$$

$$= A\pi^2x^2 - x + \tfrac{1}{2} - Sxx.$$

S 2 Cuius

Cuius ergo summa quoties x est numerus integer exhiberi potest. Ita obtinebimus:

$$\frac{1.2A}{2} - \frac{1.2.3.4B}{2^3} + \frac{1\ldots6C}{2^5} - \frac{1\ldots8D}{2^7} + \text{etc.} = A\pi^2 - \frac{1}{2}$$

$$\frac{1.2A}{4} - \frac{1\ldots4B}{4^3} + \frac{1\ldots6C}{4^5} - \frac{1\ldots8D}{4^7} + \text{etc.} = 4A\pi^2 - \frac{3}{2} - 4(1+\tfrac{1}{4})$$

$$\frac{1.2A}{6} - \frac{1\ldots4B}{6^3} + \frac{1\ldots6C}{6^5} - \frac{1\ldots8D}{6^7} + \text{etc.} = 9A\pi^2 - \frac{5}{2} - 9(1+\tfrac{1}{4}+\tfrac{1}{9})$$

$$\frac{1.2A}{8} - \frac{1\ldots4B}{8^3} + \frac{1\ldots6C}{8^5} - \frac{1\ldots8D}{8^7} + \text{etc.} = 16A\pi^2 - \frac{7}{2} - 16(1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}).$$

II. Inuestigemus iam easdem series methodo supra exposita, et cum ibi inuenissemus:

$$Aay - Ba^3y^3 + Ca^5y^5 - Da^7y^7 + \text{etc.} = \frac{1}{2}.\frac{e^{2ay}+1}{e^{2ay}-1} - \frac{1}{2ay}$$

multiplicemus per $e^{-y}y\,dy$, et integratione ita instituta vt integralia euanescant posito $y = 0$, statuamus $y = \infty$, sicque adipiscimur:

$$\int e^{-y}y^2dy = -e^{-y}y^2 - 2e^{-y}y - 2.1e^{-y} + 1.2 = 1.2$$

$$\int e^{-y}y^4dy = -e^{-y}(y^4 + 4y^3 + 4.3y^2 + 4.3.2.y + 4.3.2.1.) + 1.2.3.4 = 1.2.3.4$$

similique modo

$$\int e^{-y}y^6dy = 1.2.3.4.5.6;\ \int e^{-y}y^8dy = 1.2\ldots8.$$

Hinc itaque perueniemus ad hanc summationem

$$1.2Aa - 1\ldots4Ba^3 + 1\ldots6Ca^5 - \text{etc.} = \tfrac{1}{2}\int e^{-y}y\,dy.\frac{e^{2ay}+1}{e^{2ay}-1} - \frac{1}{2a}$$

Pona-

Ponamus nunc $a = \frac{1}{2}$ vt prodeat haec series:

$$\frac{1.2A}{2} - \frac{1.2.3.4B}{2^3} + \frac{1.2\ldots6C}{2^5} - \frac{1\ldots8D}{2^7} + \text{etc.}$$

cuius summam nouimus esse $= A\pi^2 - \frac{1}{2}$, nunc autem eandem ita expressam inuenimus:

$$\tfrac{1}{2}\int e^{-y}y\,dy.\frac{e^y+1}{e^y-1} - 1 = \tfrac{1}{2}\int y\,dy.\frac{1+e^{-y}}{e^y-1} - 1$$

si modo post integrationem ponatur $y = \infty$. Cuius veritas hoc modo ostendi potest: fit $e^{-y} = z$ et nunc integratione ita absoluta, vt integrale euanescat posito $z = 1$, statui oportet $z = 0$, quae substitutio praebet

$$\tfrac{1}{2}\int y\,dy.\frac{1+e^{-y}}{e^y-1} = \tfrac{1}{2}\int dz\, lz\frac{1+z}{1-z} = \int dz\,lz.(1 + z + z^2 + z^3 + z^4 + \text{etc.}).$$

Verum ob $\int z^{n-1}dz\,lz = \frac{z^n}{n}lz - \frac{z^n}{nn} + \frac{1}{nn}$ facto $z = 0$ fit $\int z^{n-1}dz\,lz = \frac{1}{nn}$, hincque per seriem

$$\tfrac{1}{2}\int y\,dy.\frac{1+e^{-y}}{e^y-1} = \tfrac{1}{1} + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \text{etc.} = A\pi\pi - \tfrac{1}{2}$$ vti oportet.

12. Facilius idem ostenditur ponendo $1 - z = v$ seu $z = 1 - v$, vt iam integralia a termino $v = 0$ vsque ad terminum $v = 1$ extendi debeant; tum autem nostra summa ita exprimetur $-\frac{1}{2}\int\frac{(2-v)\,dv}{v\,l(1-v)}$

$l(1-v)-1$, quam aequalem esse oportet ipsi $A\pi\pi-\frac{1}{2}$, ita vt sit

$$A\pi\pi=\tfrac{1}{2}-\tfrac{1}{2}\int\tfrac{2\,dv}{v}\,l(1-v)+\tfrac{1}{2}\int dv\,l(1-v)$$

at $\int dv\,l(1-v)=-(1-v)l(1-v)+(1-v)-1=-1$ sicque fit necesse est $A\pi\pi=-\int\frac{dv}{v}\,l(1-v)$, quod per se est manifestum. Cum enim sit

$$-l(1-v)=v+\tfrac{1}{2}v^2+\tfrac{1}{3}v^3+\tfrac{1}{4}v^4+\text{etc.}$$

erit integratione secundum legem praescriptam instituta:

$$-\int\tfrac{dv}{v}l(1-v)=1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}+\text{etc.}=A\pi^2.$$

13. Euoluamus etiam simili modo casum $a=\frac{1}{4}$ atque ostendi oportebit fore:

$$\tfrac{1}{2}\int e^{-y}y\,dy.\frac{e^{\frac{y}{2}}+1}{e^{\frac{y}{2}}-1}-2=4A\pi^2-\tfrac{1}{2}-4(1+\tfrac{1}{4})$$

seu $$\int e^{-y}y\,dy.\frac{e^{\frac{y}{2}}+1}{e^{\frac{y}{2}}-1}=8A\pi^2+1-8(1+\tfrac{1}{4})=8A\pi^2-9.$$

Ponamus $e^{\frac{-y}{2}}=1-v$, vt iam integrale a termino $v=0$ vsque ad $v=1$ extendi debeat, et habebimus ob $e^{-y}=(1-v)^2$, $y=-2l(1-v)$, et $dy=\frac{2\,dv}{1-v}$ hanc aequalitatem demonstrandam:

$$-4\int\frac{2-2v+vv}{v}dv\,l(1-v)=8A\pi^2-9$$

verum

verum vti iam obseruauimus est

$$\int dv\,l(1-v)=-1 \quad\text{et}\quad \int v\,dv\,l(1-v)=-\tfrac{3}{4}$$

vnde conficitur

$$-8\int\tfrac{dv}{v}l(1-v)-12+3=8A\pi^2-9 \;\text{ seu }\; \int\tfrac{dv}{v}l(1-v)=A\pi^2.$$

14. Simili modo si capiatur $a=\frac{1}{9}$ ostendi debet esse

$$\tfrac{1}{3}\int e^{-y}y\,dy.\frac{e^{\frac{y}{3}}+1}{e^{\frac{y}{3}}-1}-3=9A\pi^2-\tfrac{1}{2}-9(1+\tfrac{1}{4}+\tfrac{1}{9})$$

seu $$\int e^{-y}y\,dy.\frac{e^{\frac{y}{3}}+1}{e^{\frac{y}{3}}-1}=18A\pi^2+1-18(1+\tfrac{1}{4}+\tfrac{1}{9}).$$

Ponamus primo $e^{\frac{-y}{3}}=z$ vt sit $y=-3lz$ et $dy=\frac{-3\,dz}{z}$ habebimusque:

$$9\int zz\,dz\,lz.\frac{1+z}{1-z}=9\int dz\left(-zz-2z-2+\frac{2}{1-z}\right)lz$$

at est $\int zz\,dz\,lz=+\frac{1}{9}$, $\int z\,dz\,lz=+\frac{1}{4}$; $\int dz\,lz=+1$ vnde nostra formula integralis euadit

$$18\int\tfrac{dz}{1-z}lz-18(1+\tfrac{1}{4}+\tfrac{1}{9})+1$$

ita vt sit $\int\frac{dz}{1-z}lz=A\pi^2$ vti iam supra ostendimus atque hoc modo etiam sequentium casuum veritas euincetur.

15. Sin autem sumamus $a=1$, vt summanda sit haec series:

$$1.2A - 1.2.3.4B + 1 \ldots 6C - 1 \ldots 8D + \text{etc.}$$

quoniam fit $x = \frac{1}{2}$, ex §. 10 fummam affignare non licet, fiquidem valor progreffionis $S = 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{xx}$ quando terminorum numerus $= \frac{1}{2}$ non conftat. Altera vero methodus eius fummam praebet:

$$\tfrac{1}{2}\int e^{-y} y\,dy.\frac{1 + e^{-2y}}{1 - e^{-2y}} - \tfrac{1}{2}$$

quae pofito $e^{-y} = z$ in hanc formam transmutatur

$$\tfrac{1}{2}\int dz\,lz.\frac{1+zz}{1-zz} - \tfrac{1}{2} = \int \frac{dz\,lz}{1-zz} - \tfrac{1}{2}\int dz\,lz - \tfrac{1}{2}$$

et quia $\int dz\,lz = 1$, fiet ea

$$\int \frac{dz\,lz}{1-zz} - 1 = 1 + \tfrac{1}{9} + \tfrac{1}{25} + \tfrac{1}{49} + \text{etc.} - 1 = \tfrac{3}{4}A\pi\pi - 1.$$

Quod fi iam ponamus fumto $x = \frac{1}{2}$ fieri $S = \Delta$ eadem fumma reperitur $= \frac{1}{4}A\pi\pi - \frac{1}{4}\Delta$ vnde concludimus fore $\frac{3}{4}A\pi\pi - 1 = \frac{1}{4}A\pi\pi - \frac{1}{4}\Delta$ ideoque quantitas illa incognita $\Delta = 4 - 2A\pi\pi$ ex quo hanc progreffionem interpolare licebit

1	$4 - 2A\pi\pi$
$1 + \frac{1}{4}$	$4 - 2A\pi\pi + \frac{4}{9}$
$1 + \frac{1}{4} + \frac{1}{9}$	$4 - 2A\pi\pi + \frac{4}{9} + \frac{4}{25}$
$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$	$4 - 2A\pi\pi + \frac{4}{9} + \frac{4}{25} + \frac{4}{49}$
$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$	$4 - 2A\pi\pi + \frac{4}{9} + \frac{4}{25} + \frac{4}{49} + \frac{4}{81}$
etc.	

et quoniam termini infinitefimi funt aequales, fit

$$1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16}\ \text{etc.} = 4(1 + \tfrac{1}{9} + \tfrac{1}{25} + \text{etc.}) - 2A\pi\pi$$

quae

quae aequalitas per fe eft manifefta cum fit

$$1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16}\ \text{etc.} = A\pi\pi \ \text{et}\ 1 + \tfrac{1}{9} + \tfrac{1}{25} + \text{etc.} = \tfrac{3}{4}A\pi\pi.$$

16. Confideremus rem in genere fitque $a = \frac{m}{n}$ vt fummanda fit haec feries infinita.

$$\frac{1.2Am}{n} - \frac{1.2.3.4Bm^3}{n^3} + \frac{1.2\ldots6Cm^5}{n^5} - \frac{1.2\ldots8Dm^7}{n^7} + \text{etc.}$$

ac, pofito $e^{-\frac{y}{n}} = z$, reperitur eius fumma

$$\tfrac{1}{2}\int e^{-y} y\,dy.\frac{1 + e^{-\frac{2my}{n}}}{1 - e^{-\frac{2my}{n}}} - \frac{n}{2m} = \frac{nn}{2}\int z^{n-1}\,dz\,lz.\frac{1 + z^{2m}}{1 - z^{2m}} - \frac{n}{2m}$$

quae reducitur ad hanc formam

$$nn\int \frac{z^{n-1}\,dz\,lz}{1 - z^{2m}} - \frac{nn}{2}\int z^{n-1}\,dz\,lz - \frac{n}{2m}.$$

Cum autem fit $\int z^{n-1}\,dz\,lz = \frac{1}{nn}$, per euolutionem primi membri nancifcimur hanc feriem illi aequalem

$$-\tfrac{1}{2} - \frac{n}{2m} + \frac{nn}{nn} + \frac{nn}{(2m+n)^2} + \frac{nn}{(4m+n)^2} + \frac{nn}{(6m+n)^2} + \text{etc.}$$

Verum ex §. 10 ob $2x = \frac{n}{m}$ feu $x = \frac{n}{2m}$ pofito

$$S = 1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \ldots + \tfrac{1}{x^2}$$

eadem fumma prodit $\frac{nn}{4mm}A\pi^2 - \frac{n}{2m} + \frac{1}{2} - \frac{nn}{4mm}S$ qua cum praecedente comparata colligitur.

$$S = A\pi\pi - \frac{4mm}{(2m+n)^2} - \frac{4mm}{(4m+n)^2} - \frac{4mm}{(6m+n)^2} - \frac{4mm}{(8m+n)^2} - \text{etc.}$$

ex quo valorem ipfius S affignare poterimus, quicunque numerus fractus pro x accipiatur veluti fi ftatuatur $x = \frac{\nu}{\mu}$ erit

S = A

$$S = A\pi^2 - \frac{\mu\mu}{(\mu+\nu)^2} - \frac{\mu\mu}{(2\mu+\nu)^2} - \frac{\mu\mu}{(3\mu+\nu)^2} - \frac{\mu\mu}{(4\mu+\nu)^2} - \text{etc.}$$

quae ſeries hoc modo immediate per x commodius exhibetur, vt ſit

$$S = A\pi^2 - \frac{1}{(x+1)^2} - \frac{1}{(x+2)^2} - \frac{1}{(x+3)^2} - \frac{1}{(x+4)^2} - \text{etc.}$$

17. Quod hic per tantas ambages inuenimus, ita obuium videtur, vt ſtatim immediate ex ſerie prima deriuari potuiſſet. Cum enim ſit

$$A\pi^2 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{x^2} + \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \text{etc.}$$

hinc vtique manifeſtum eſt fore

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{x^2} = A\pi^2 - \frac{1}{(x+1)^2} - \frac{1}{(x+2)^2} - \frac{1}{(x+3)^2} - \text{etc.}$$

Quia vero illa ſeriei ſumma $A\pi^2$ non eſt veritati conſentanea, niſi littera x denotet numeros integros quo quidem caſu concluſio eſt, perſpicua, eius certitudo pro caſibus quibus x eſt numerus fractus vel adeo irrationalis, maxime adhuc dubia relinquitur, et cum nunc quidem pateat, eam inter veritates eſſe referendam, hoc certe neutiquam ex iſto breui ratiocinio perſpicitur, ac niſi praecedentes rationes negotium confeciſſent, merito maximam haberemus dubitandi rationem. Nunc autem plena fiducia hoc ratiocinium multo latius extendere licet ita vt ſi fuerit

$$S = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \text{etc. in infinitum}$$

hinc

hinc tuto inferre queamus fore generatim

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \ldots + \frac{1}{x^n} = S - \frac{1}{(x+1)^n} - \frac{1}{(x+2)^n} - \frac{1}{(x+3)^n} - \text{etc.}$$

etiamſi x non fuerit numerus integer ſed fractus vel adeo irrationalis quicunque.

Caſus II. quo $n = 4$.

18. Poſito primo indefinite

$$S = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots + \frac{1}{x^4}$$

tum vero hac ſerie in infinitum continuata

$$O = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{etc. in infinitum}$$

vt ſit $O = B\pi^4$, habebimus hanc ſummationem

$$-\frac{4\,A}{2\,x^5} + \frac{4.5.6\,B}{2^3\,x^7} - \frac{4.\ldots 8\,C}{2^5\,x^9} + \frac{4.\ldots 10\,D}{2^7\,x^{11}} - \text{etc.}$$

$$= S - \frac{1}{2\,x^4} + \frac{1}{3\,x^3} - B\pi^4$$

quae per $-1.2.3\,x^4$ multiplicata abit in hanc

$$\frac{1.2\ldots 4\,A}{2\,x} - \frac{1.2\ldots 6\,B}{2^3\,x^3} + \frac{1.2\ldots 8\,C}{2^5\,x^5} - \frac{1.2\ldots 10\,D}{2^7\,x^7} + \text{etc.}$$

$$= 1.2.3\,B\pi^4 x^4 + 3 - 2x - 6Sx^4$$

ſicque quoties x eſt numerus integer, iſtius ſeriei ſumma exhiberi poteſt. Per numeros ergo 𝔄, 𝔅, ℭ etc. erit

$$\frac{4\,𝔄}{2\,x} - \frac{6\,𝔅}{2\,x^3} + \frac{8\,ℭ}{2\,x^5} - \frac{10\,𝔇}{2\,x^7} + \text{etc.} = 6B\pi^4 x^4 + 3 - 2x - 6Sx^4$$

19. Eiusdem autem ſeriei ſummam ex forma ſupra inuenta definire poſſumus, quae erat:

$$Aay - Ba^3y^3 + Ca^5y^5 - Da^7y^7 + \text{etc.} = \tfrac{1}{2}\cdot\frac{e^{2ay}+1}{e^{2ay}-1} - \frac{1}{2ay}$$

haec enim per $e^{-y}y^3\,dy$ multiplicata et integratione a termino $y=0$ vsque ad $y=\infty$ extensa praebet

$$1.2..4Aa - 1.2..6Ba^3 + 1.2..8Ca^5 - 1.2...10Da^7 + \text{etc.}$$

$$= \tfrac{1}{2}\int e^{-y}y^3\,dy.\ \frac{1+e^{-2ay}}{1-e^{-2ay}} - \frac{1}{a}$$

Hanc vero formulam integralem sine substitutione hoc modo euoluere licet: cum sit

$$\frac{1+e^{-2ay}}{1-e^{-2ay}} = 1 + 2e^{-2ay} + 2e^{-4ay} + 2e^{-6ay} + 2e^{-8ay} + \text{etc.}$$

multiplicetur per $\tfrac{1}{2}e^{-y}y^3\,dy$, et quoniam in genere est

$$\int e^{-my}y^3\,dy = -e^{-my}\left(\frac{y^3}{m} + \frac{3y^2}{m^2} + \frac{3.2.y}{m^3} + \frac{3.2.1}{m^4}\right) + \frac{3.2.1}{m^4} = \frac{1.2.3}{m^4}$$

summa illa transformatur in hanc seriem infinitam

$$-\frac{1}{a} + 3 + \frac{6}{(2a+1)^4} + \frac{6}{(4a+1)^4} + \frac{6}{(6a+1)^4} + \frac{6}{(8a+1)^4} + \text{etc.}$$

Hinc posito $a = \frac{1}{2x}$ vt prodeat prior series, erit etiam

$$-2x + 3 + \frac{6x^4}{(x+1)^4} + \frac{6x^4}{(x+2)^4} + \frac{6x^4}{(x+3)^4} + \frac{6x^4}{(x+4)^4} + \text{etc.}$$
$$= 6B\pi^4x^4 + 3 - 2x - 6Sx^4$$

ideoque per $6x^4$ diuidendo

$$B\pi^4 - S = \frac{1}{(x+1)^4} + \frac{1}{(x+2)^4} + \frac{1}{(x+3)^4} + \frac{1}{(x+4)^4} + \text{etc.}$$

prorsus vti supra iam animaduertimus.

Casus

Casus III. *quo* n *est numerus quicunque.*

20. Primum hic obseruo, si ponatur series infinita

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \text{etc.} = O$$

hanc seriem ex euolutione huius formulae integralis oriri $\int\frac{dz\,(lz)^{n-1}}{1-z}$, si integratio a termino $z=0$ vsque ad terminum $z=1$ extendatur. Cum enim hac lege obseruata fit

$$\int z^{m-1}dz(lz) = \frac{1}{m}z^m lz - \frac{1}{m^2}z^m = -\frac{1}{m^2}$$
$$\int z^{m-1}dz(lz)^2 = \frac{1}{m}z^m(lz)^2 - \frac{2}{m^2}z^m lz + \frac{2.1}{m^3}z^m = +\frac{1.2}{m^3}$$
$$\int z^{m-1}dz(lz)^3 = -\frac{1.2.3}{m^4}$$
$$\int z^{m-1}dz(lz)^4 = +\frac{1.2.3.4}{m^5}$$

etc.

erit in genere $\pm\int z^{m-1}dz(lz)^{n-1} = +\frac{1.2.3...(n-1)}{m^n}$

vbi signum superius $+$ valet si n sit numerus impar inferius vero $-$, si n sit numerus par. Quam ob rem euolutio formulae $\pm\int\frac{dz}{1-z}(lz)^{n-1}$ praebet hanc seriem sub eadem lege ambiguitatis:

$$1.2.3...(n-1)\left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \text{etc.}\right)$$

ita vt sit $O = \frac{\pm 1}{1.2.3.....(n-1)}\int\frac{dz}{1-z}(lz)^{n-1}$

21.

21. Simili modo haec ſeries ad datum quemvis terminum indefinite ſummari poterit, ſi enim ponatur:

$$S = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \dots + \frac{1}{m^n}$$

$$\text{erit } S = \frac{\pm 1}{1.2.3 \dots (n-1)} \int \frac{1-z^m}{1-z} dz (lz)^{n-1}$$

quae formula veritati eſt conſentanea ſiue m ſit numerus integer ſiue fractus: vnde cum ſit $\frac{\pm 1}{1.2 \dots (n-1)}$:

$$\int \frac{z^m dz}{1-z} (lz)^{n-1} = \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \text{etc.}$$

perſpicuum eſt fore:

$$S = O - \frac{1}{(m+1)^n} - \frac{1}{(m+2)^n} - \frac{1}{(m+3)^n} - \frac{1}{(m+4)^n} - \text{etc.}$$

Quocirca cum ſumto $m = \frac{1}{2}$ ſit

$$\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \text{etc.} = \frac{2^n}{1} + \frac{2^n}{3^n} + \frac{2^n}{5^n} + \frac{2^n}{7^n} + \text{etc.}$$

$= (2^n - 1) O$, hinc elicimus:

$$S = O - (2^n - 1) O + 2^n = 2^n - (2^n - 2) O$$

vnde valores interpolati ipſius S ita ſe habebunt

ſi ſit

ſi ſit	erit S
$m = 0$	0
$m = \frac{1}{2}$	$2^n - (2^n - 2) O$
$m = 1$	1
$m = 1\frac{1}{2}$	$2^n + \frac{2^n}{3^n} - (2^n - 2) O$
$m = 2$	$1 + \frac{1}{2^n}$
$m = 2\frac{1}{2}$	$2^n + \frac{2^n}{3^n} + \frac{2^n}{5^n} - (2^n - 2) O$
$m = 3$	$1 + \frac{1}{2^n} + \frac{1}{3^n}$
$m = 3\frac{1}{2}$	$2^n + \frac{2^n}{3^n} + \frac{2^n}{5^n} + \frac{2^n}{7^n} - (2^n - 2) O$

etc.

ſi ad ſingulos terminos addatur $(2^n - 2) O$, iique tum per 2^n diuidantur, habebitur interpolatio huius ſeriei

$$0,\ 1,\ 1 + \frac{1}{3^n},\ 1 + \frac{1}{3^n} + \frac{1}{5^n},\ 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n};\ \text{etc.}$$

22. Cum ſumma ſeriei infinitae O per quadraturam circuli ſeu litteram π ſit aſſignabilis, quoties exponens n fuerit numerus par, notari merentur ſequentium formularum integralium reductiones ad circuli quadraturam:

$\int \frac{dz}{1-z}$

$$\int\frac{dz}{1-z}\,lz = -1\cdot A\pi^2 \qquad = -\frac{2^0}{2\cdot3}\mathfrak{A}\pi^2$$
$$\int\frac{dz}{1-z}(lz)^3 = -1\cdot2\cdot3\,B\pi^4 \qquad = -\frac{2^2}{4\cdot5}\mathfrak{B}\pi^4$$
$$\int\frac{dz}{1-z}(lz)^5 = -1\cdot2\cdot3\cdot4\cdot5\,C\pi^6 = -\frac{2^4}{6\cdot7}\mathfrak{C}\pi^6$$
$$\int\frac{dz}{1-z}(lz)^7 = -1\cdot2\ldots7\,D\pi^8 = -\frac{2^6}{8\cdot9}\mathfrak{D}\pi^8$$
$$\int\frac{dz}{1-z}(lz)^9 = -1\cdot2\ldots9\,E\pi^{10} = -\frac{2^8}{10\cdot11}\mathfrak{E}\pi^{10}$$

etc.

Atque hinc eo magis eſt mirandum, quod nullam harum formularum $\int\frac{dz}{1-z}(lz)^2$, $\int\frac{dz}{1-z}(lz)^4$, $\int\frac{dz}{1-z}(lz)^6$ etc. nullo modo ad quampiam quadraturam cognitam reducere liceat, cum tamen ex hoc ordine prima formula $\int\frac{dz}{1-z}(lz)^0$ manifeſto per logarithmos abſoluatur.

23. Scribamus nunc in §. 9. m loco x, et aequationem ibi datam per $-1.2.\ldots(n-1)m^n$ multiplicemus, vt obtineamus hanc ſummationem:

$$\frac{1.2\ldots nA}{2m^2} - \frac{1.2\ldots(n+2)B}{2^3m^4} + \frac{1.2\ldots(n+4)C}{2^5m^6} - \frac{1.2\ldots(n+6)D}{2^7m^8} + \text{etc.}$$
$$= 1.2\ldots(n-1)(Om^n - Sm^n + \tfrac{1}{2} - \tfrac{m}{n-1})$$

ſimili autem modo quo ſupra ſumus vſi (19), ponendo $a = \frac{1}{2m}$ eiusdem ſeriei ſummam per ſequentem formulam integralem expreſſam inueniemus:

$$\tfrac{1}{2}\int e^{-y}y^{n-1}dy.\frac{1+e^{-y:m}}{1-e^{-y:m}} - 1.2\ldots(n-2)\;m$$

ita

ita vt ſit

$$\tfrac{1}{2}\int e^{-y}y^{n-1}dy.\frac{1+e^{-y:m}}{1-e^{-y:m}} = 1.2\ldots(n-1)(Om^n - Sm^n + \tfrac{1}{2})$$

atque ob $\frac{1+e^{-y:m}}{1-e^{-y:m}} = -1 + \frac{2}{1-e^{-y:m}}$ erit

$$\int\frac{e^{-y}y^{n-1}dy}{1-e^{-y:m}} = 1.2\ldots(n-1)m^n(O-S)$$

quae formula integralis ponendo $e^{-y:m} = z$ ad eam quam modo tractauimus, reducitur ſcilicet $\int\frac{dz}{1-z}(lz)^{n-1}$, ſiquidem eius integrale a termino $z=1$ vsque ad terminum $z=0$ extendatur.

Caſus IV. quo $n=1$.

24. Hic caſus peculiarem tractationem poſtulat, quia ſeriei $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$ etc. ſumma eſt infinita ſit ergo indefinite

$$S = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots\ldots + \tfrac{1}{x}$$

vt quia ob $X=\frac{1}{x}$ eſt $\int X dx = lx$, habebimus hanc ſummationem

$$\frac{-1A}{2x^2} + \frac{1.2.3B}{2^3x^4} - \frac{1.2\ldots5C}{2^5x^6} + \frac{1.2\ldots7D}{2^7x^8} - \text{etc.} = S - lx - \frac{1}{2x} - O$$

ſeu per $-x$ multiplicando:

$$\frac{1A}{2x} - \frac{1.2.3B}{2^3x^3} + \frac{1.2\ldots5C}{2^5x^5} - \frac{1.2\ldots7D}{2^7x^7} + \text{etc.} = (O-S)x + \tfrac{1}{2} + xlx$$

vbi conſtantem O ex caſu per ſe cognito definiri oportet. Veluti ſi ſumatur $x=1$, ob $S=1$ et $lx=0$, erit

 $O-\frac{1}{2}$

$$O-\tfrac{1}{2}=\tfrac{1A}{2}-\tfrac{1.2.3B}{2^3}+\tfrac{1.2.\dots.5C}{2^5}-\tfrac{1.2.\dots.7D}{2^7}+\text{etc.}$$

Quo autem iſte valor ipſius O facilius obtineatur, ponatur $x=10$, et cum fiat:

$$(O-1-\tfrac{1}{2}-\tfrac{1}{3}-\tfrac{1}{4}-\tfrac{1}{5}\dots-\tfrac{1}{10}).10+\tfrac{1}{2}+10l10=$$
$$\tfrac{1A}{20}-\tfrac{1.2.3B}{20^3}+\tfrac{1.2.\dots.5C}{20^5}-\tfrac{1.2.\dots.7D}{20^7}+\text{etc.}$$

hinc valor ipſius O per ſeriem maxime conuergentem eruitur:

$$O=0,5772156649015325$$

qui numerus eo maiori attentione dignus videtur, quod eum, cum olim in hac inueſtigatione multum ſtudii conſumſiſſem, nullo modo ad cognitum quantitatum genus reducere valui. Eo autem invento viciſſim ſeriei harmonicae ad quotcunque terminos continuatae ſumma facile aſſignatur, cum ſit

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\dots\dots+\tfrac{1}{x}=$$
$$O+lx+\tfrac{1}{2x}-\tfrac{1A}{2x^2}+\tfrac{1.2.3B}{2^3x^4}-\tfrac{1.2.\dots.5C}{2^5x^6}+\tfrac{1.2.\dots.7D}{2^7x^8}-\text{etc.}$$

25. Cum autem ſcripta littera m loco x ſeriei

$$\tfrac{1A}{2m}-\tfrac{1.2.3B}{2^3m^3}+\tfrac{1.2.\dots.5C}{2^5m^5}-\tfrac{1.2.\dots.7D}{2^7m^7}\ \text{etc.}$$

ſumma, quae modo prodiit $=(O-S)m+\tfrac{1}{2}+m\,lm$ exiſtente $S=1+\tfrac{1}{2}+\tfrac{1}{3}+\dots\tfrac{1}{m}$, etiam ex hac ſerie:

$$Aay-Ba^3y^3+Ca^5y^5-Da^7y^7+\text{etc.}=\tfrac{1}{2}.\frac{1+e^{-2ay}}{1-e^{-2ay}}-\frac{1}{2ay}$$
$$=-\tfrac{1}{2}$$

$=-\tfrac{1}{2}-\tfrac{1}{2ay}+\dfrac{1}{1-e^{-2ay}}$ definiri poſſit, hanc operationem ſuſcipiamus. Multiplicemus ſcilicet per $e^{-y}dy$ et integrale a termino $y=0$ vsque ad $y=\infty$ extendamus, et quia in genere eſt $\int e^{-y}y^{\mu}dy$ $=1.2.3\dots\mu$, nanciſcemur

$$1Aa-1.2.3Ba^3+1.2.\dots.5Ca^5-1.2.\dots.7Da^7+\text{etc.}$$
$$=-\tfrac{1}{2}-\tfrac{1}{2a}\int\frac{e^{-y}dy}{y}+\int\frac{e^{-y}dy}{1-e^{-2ay}}.$$

Quare poſito $a=\tfrac{1}{2m}$ erit

$$(O-S)m+\tfrac{1}{2}+m\,lm=-\tfrac{1}{2}-m\int\frac{e^{-y}dy}{y}+\int\frac{e^{-y}dy}{1-e^{-y:m}}$$

ideoque poſito $m=1$, fiet

$$O=-\int\frac{e^{-y}dy}{y}+\int\frac{e^{-y}dy}{1-e^{-y}}$$

ita vt numerus O quem quaerimus, duplicem quantitatem tranſcendentem inuoluat.

26. Transformemus has formulas ope ſubſtitutionis $e^{-y}=z$ fietque integralia a termino $z=1$ vsque ad $z=0$ extendendo

$$O=-\int\frac{dz}{lz}-\int\frac{dz}{1-z}=-\int\frac{dz(1-z+lz)}{(1-z)lz}.$$

Vel ſtatuamus $1-z=v$, vt iam integralia a termino $v=0$ vsque ad $v=1$ extendi debeant, provenietque

$$O=\int\frac{dv}{l(1-v)}+\int\frac{dv}{v}=\int\frac{v+l(1-v)}{v\,l(1-v)}dv.$$

 At

At maxima difficultas hic in eo consistit, quod vtraque pars seorsim euoluta praebeat numerum infinite magnum quae autem duo infinita necessario se mutuo ita tollere debent vt pro O obtineatur valor ille finitus supra assignatus.

Retenta autem priori forma integralia more solito a termino $z = 0$ ad $z = 1$ extendamus, vt fit $O = \int \frac{dz}{lz} + \int \frac{dz}{1-z}$, et cum denotante i numerum infinitum fit $lz = i(z^{1:i} - 1)$, erit

$O = \int \frac{dz}{1-z} - i \int \frac{dz}{1 - z^{1:i}}$. Iam ponamus $z = u^i$, vt fiat

$$O = i \int \frac{u^{i-1} du}{1-u^i} - \int \frac{u^{i-1} du}{1-u}$$

quarum formularum euolutio praebet:

$$O = u^i + \tfrac{1}{2} u^{2i} + \tfrac{1}{3} u^{3i} \text{ etc.}$$
$$- \tfrac{1}{i} u^i - \tfrac{1}{i+1} u^{i+1} \ldots - \tfrac{1}{2i} u^{2i} - \tfrac{1}{2i+1} u^{2i+1} \ldots - \tfrac{1}{3i} u^{3i} - \tfrac{1}{3i+1} u^{3i+1} \text{ etc.}$$

et ponendo vti oportet $u = 1$, fit

$$O = +1 - \left(\tfrac{1}{i} + \tfrac{1}{i+1} + \tfrac{1}{i+2} \ldots + \tfrac{1}{2i-1}\right)$$
$$+ \tfrac{1}{2} - \left(\tfrac{1}{2i} + \tfrac{1}{2i+1} + \tfrac{1}{2i+2} \ldots + \tfrac{1}{3i-1}\right)$$
$$+ \tfrac{1}{3} - \left(\tfrac{1}{3i} + \tfrac{1}{3i+1} + \tfrac{1}{3i+2} \ldots + \tfrac{1}{4i-1}\right)$$
etc.

vbi notandum est harum progressionum harmonicarum primam $\frac{1}{i} + \frac{1}{i+1} + \frac{1}{i+2} \ldots + \frac{1}{2i-1}$ ob i numerum infinitum exprimere $l2$ secundam $l\frac{3}{2}$, tertiam

tertiam $l\frac{4}{3}$, etc. ita vt habeatur per seriem satis simplicem et regularem:

$$O = 1 - l2 + \tfrac{1}{2} - l\tfrac{3}{2} + \tfrac{1}{3} - l\tfrac{4}{3} + \tfrac{1}{4} - l\tfrac{5}{4} + \tfrac{1}{5} - l\tfrac{6}{5} + \text{ etc.}$$

27. Eandem hanc seriem ex prima statim forma deriuare licuisset, si enim ibi (24) ponatur $x = \infty$ fit $0 = S - lx - O$ ita vt fit

$$O = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} \ldots + \tfrac{1}{x} - lx$$

quia vero tam series $1 + \frac{1}{2} + \frac{1}{3} \ldots + \frac{1}{x}$ quam lx habet valorem infinitum, quo facilius posterior a priori auferri queat conueniet lx in totidem partes diuidere, quot prior series habet terminos, quod manifesto fit hoc modo

$$lx = l\tfrac{2}{1} + l\tfrac{3}{2} + l\tfrac{4}{3} + \ldots + l\tfrac{x}{x-1}$$

vnde series inuenta conficitur. Haec series nunc pluribus modis in alias formas transmutari potest, ex quibus valorem numeri O facile quam proxime saltem colligere licebit.

Primo enim cum fit $\frac{1}{n} - l\frac{n+1}{n} = \frac{1}{2n^2} - \frac{1}{3n^3} + \frac{1}{4n^4} - \frac{1}{5n^5} + \text{etc.}$ habebimus

$$O = \tfrac{1}{2}\left(1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \text{etc.}\right) - \tfrac{1}{3}\left(1 + \tfrac{1}{2^3} + \tfrac{1}{3^3} + \tfrac{1}{4^3} + \text{etc.}\right)$$
$$+ \tfrac{1}{4}\left(1 + \tfrac{1}{2^4} + \tfrac{1}{3^4} + \tfrac{1}{4^4} + \text{etc.}\right) - \tfrac{1}{5}\left(1 + \tfrac{1}{2^5} + \tfrac{1}{3^5} + \tfrac{1}{4^5} + \text{etc.}\right)$$
$$+ \tfrac{1}{6}\left(1 + \tfrac{1}{2^6} + \tfrac{1}{3^6} + \tfrac{1}{4^6} + \text{etc.}\right) - \tfrac{1}{7}\left(1 + \tfrac{1}{2^7} + \tfrac{1}{3^7} + \tfrac{1}{4^7} + \text{etc.}\right)$$
etc.

Deinde ob $\frac{1}{n-1} - l\frac{n}{n-1} = \frac{1}{2n^2} + \frac{2}{3n^3} + \frac{3}{4n^4} + \frac{4}{5n^5} + \frac{5}{6n^6} + \text{etc.}$

erit

erit etiam

$$O = \tfrac{1}{2}\left(\tfrac{1}{2^2}+\tfrac{1}{3^2}+\tfrac{1}{4^2}+\text{etc.}\right)+\tfrac{2}{3}\left(\tfrac{1}{2^3}+\tfrac{1}{3^3}+\tfrac{1}{4^3}+\tfrac{1}{5^3}+\text{etc.}\right)$$
$$+\tfrac{3}{4}\left(\tfrac{1}{2^4}+\tfrac{1}{3^4}+\tfrac{1}{4^4}+\text{etc.}\right)+\tfrac{4}{5}\left(\tfrac{1}{2^5}+\tfrac{1}{3^5}+\tfrac{1}{4^5}+\tfrac{1}{5^5}+\text{etc.}\right)$$
$$+\tfrac{5}{6}\left(\tfrac{1}{2^6}+\tfrac{1}{3^6}+\tfrac{1}{4^6}+\text{etc.}\right)+\tfrac{6}{7}\left(\tfrac{1}{2^7}+\tfrac{1}{3^7}+\tfrac{1}{4^7}+\tfrac{1}{5^7}+\text{etc.}\right)$$

etc.

28. Huius posterioris formae considerentur primo partes priores potestatibus paribus contentae; quae modo supra adhibito expressae ita se habebunt:

$$\tfrac{1}{2}(A\pi^2-1)+\tfrac{3}{4}(B\pi^4-1)+\tfrac{5}{6}(C\pi^6-1)+\text{etc.}$$

Contemplemur ergo seriem hanc:

$$P=(A\pi^2-1)x^2+(B\pi^4-1)x^4+(C\pi^6-1)x^6+\text{etc.}$$

eritque ex §. 7. $P=\frac{1}{2}-\frac{1}{2}\pi x\,\text{cot.}\,\pi x-\frac{xx}{1-xx}$, cuius valor casu $x=1$ reperitur ponendo $x=1-\omega$ euanescente ω, tum autem oritur $P=\frac{1}{2}+\frac{\pi(1-\omega)\,\text{cos.}\,\pi\omega}{2\,\text{sin.}\,\pi\omega}-\frac{1+2\omega+\omega\omega}{\omega(2-\omega)}$

seu $P=\frac{1}{2}+\frac{1-\omega}{2\omega}-\frac{1+2\omega}{\omega(2-\omega)}=\frac{1}{2}+\frac{\omega}{2\omega(2-\omega)}=\frac{3}{4}$, ita vt quod omnino notatu dignum videtur sit

$$A\pi^2-1+B\pi^4-1+C\pi^6-1+D\pi^8-1\ \text{etc.}=\tfrac{3}{4}.$$

Deinde per integrationem elicimus:

$$\int\frac{P\,dx}{x}=\tfrac{1}{2}(A\pi^2-1)xx+\tfrac{1}{4}(B\pi^4-1)x^4+\tfrac{1}{6}(C\pi^6-1)x^6+\text{etc.}$$

hincque $\int\frac{P\,dx}{x}=\frac{1}{2}lx-\frac{1}{2}l\,\text{sin.}\,\pi x+\frac{1}{2}l(1-xx)+\frac{1}{2}l\pi$ seu $\int\frac{P\,dx}{x}=\frac{1}{2}l(1-xx)-\frac{1}{2}l\frac{\text{sin.}\,\pi x}{\pi x}$. Ponatur nunc iterum $x=1-\omega$

fietque

fietque $\int\frac{P\,dx}{x}=\frac{1}{2}l\,2\omega-\frac{1}{2}l\frac{\text{sin.}\,\pi\omega}{\pi(1-\omega)}=\frac{1}{2}l2$ ita vt sit

$$\tfrac{1}{2}(A\pi\pi-1)+\tfrac{1}{4}(B\pi^4-1)+\tfrac{1}{6}(C\pi^6-1)+\text{etc.}=\tfrac{1}{2}l2$$

hac autem serie a superiori ablata relinquitur:

$$\tfrac{1}{2}(A\pi\pi-1)+\tfrac{3}{4}(B\pi^4-1)+\tfrac{5}{6}(C\pi^6-1)+\text{etc.}=\tfrac{3}{4}-\tfrac{1}{2}l2$$

ita vt sit

$$O=\tfrac{3}{4}-\tfrac{1}{2}l2+\tfrac{2}{3}\left(\tfrac{1}{2^3}+\tfrac{1}{3^3}+\tfrac{1}{4^3}+\text{etc.}\right)$$
$$+\tfrac{4}{5}\left(\tfrac{1}{2^5}+\tfrac{1}{3^5}+\tfrac{1}{4^5}+\text{etc.}\right)$$
$$+\tfrac{6}{7}\left(\tfrac{1}{2^7}+\tfrac{1}{3^7}+\tfrac{1}{4^7}+\text{etc.}\right)$$

etc.

29. Cum igitur numerus O duabus constet partibus, quarum prior est $\frac{3}{4}-\frac{1}{2}l2=0,403426409720027 3$, ipse autem numerus sit $O=\underline{0,5772156649015325}$ erit altera pars $=0,1737892551815052$ si ergo simili modo haec altera pars ad logarithmos vel quadraturam circuli reuocari posset, nihil amplius in hoc negotio desiderari posset. Haec autem pars altera ob

$$\tfrac{2}{3}z^3+\tfrac{4}{5}z^5+\tfrac{6}{7}z^7+\text{etc.}=2\int\frac{zz\,dz}{(1-zz)^2}=\frac{z}{1-zz}-\tfrac{1}{2}l\frac{1+z}{1-z}$$

sequenti forma exhiberi potest:

$$\tfrac{1}{2}\left(1+\tfrac{1}{3}-l\tfrac{3}{1}\right)+\tfrac{1}{4}\left(\tfrac{1}{2}+\tfrac{1}{4}-l\tfrac{4}{2}\right)+\tfrac{1}{6}\left(\tfrac{1}{3}+\tfrac{1}{5}-l\tfrac{5}{3}\right)+\text{etc.}$$

quae autem denotante i numerum infinitum sponte reducitur ad hanc expressionem

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots\ldots+\tfrac{1}{i}-li-\tfrac{3}{4}+\tfrac{1}{2}l2$$

ita

ita vt hinc nihil noui eliciatur, cum adiecta parte priori $\frac{3}{4}-\frac{1}{2}l2$ oriatur vt per se constat

$$O = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{i} - li.$$

Manet ergo quaestio magni momenti, cuiusnam indolis sit numerus iste O, et ad quodnam genus quantitatum sit referendus.

Si terminus generalis $X = lx$.

30. Hic ergo erit $\int X\,dx = x\,lx - x$ et ob $\frac{dX}{dx} = \frac{1}{x}$ fiet porro $\frac{d^3X}{dx^3} = \frac{1.2}{x^3}$; $\frac{d^5X}{dx^5} = \frac{1.2.3.4}{x^5}$; $\frac{d^7X}{dx^7} = \frac{1.2\ldots 6}{x^7}$ etc. Vnde si ponamus indefinite:

$$S = l1 + l2 + l3 + l4 + \ldots\ldots + lx$$

habebimus hanc summationem

$$\frac{A}{2x} - \frac{1.2\,B}{2^3x^3} + \frac{1.2.3.4\,C}{2^5x^5} - \frac{1.2\ldots 6\,D}{2^7x^7} + \text{etc.} = S - \tfrac{1}{2}lx - x\,lx + x - O$$

quae constans ita esse debet comparata, vt vni ipsius x valori satisfaciat. Sit ergo $x = 1$ et cum sit $S = l1 = 0$ erit

$$-O + 1 = \frac{A}{2} - \frac{1.2\,B}{2^3} + \frac{1.2.3.4\,C}{2^5} - \frac{1.2\ldots\ldots 6\,D}{2^7} + \text{etc.}$$

Cum igitur sit

$$Aay - Ba^3y^3 + Ca^5y^5 - Da^7y^7 + \text{etc.} = \tfrac{1}{2}.\frac{1+e^{-2ay}}{1-e^{-2ay}} - \frac{1}{2ay}$$

ob $\int e^{-y}y^n dy = 1.2\ldots n$ multiplicemus per $e^{-y}\frac{dy}{y}$ et integratio suppeditabit

$$Aa - 1.2Ba^3 + 1.2.3.4Ca^5 - \text{etc.} = \tfrac{1}{2}\int e^{-y}\frac{1+e^{-2ay}}{1-e^{-2ay}}\cdot\frac{dy}{y} - \frac{1}{2a}\int e^{-y}\frac{dy}{yy}$$

$$= \int\frac{e^{-y}\,dy}{y(1-e^{-2ay})} - \tfrac{1}{2}\int\frac{e^{-y}\,dy}{y} - \frac{1}{2a}\int e^{-y}\frac{dy}{yy}$$

inte-

gralibus his a termino $y = 0$ vsque ad $y = \infty$ extensis.

31. Statuamus nunc $a = \frac{1}{2x}$, et obtinebimus hanc aequationem:

$$-O + S + x - \tfrac{1}{2}lx - x\,lx = \int\frac{e^{-y}dy}{y(1-e^{-y:x})} - \tfrac{1}{2}\int\frac{e^{-y}dy}{y} - x\int e^{-y}\frac{dy}{yy}$$

in quibus integrationibus quantitatem x vt constantem spectari oportet. Quare sumto $x = 1$ fiet

$$-O + 1 = \int\frac{e^{-y}dy}{y(1-e^{-y})} - \tfrac{1}{2}\int\frac{e^{-y}dy}{y} - \int\frac{e^{-y}dy}{yy}$$

et quoniam est $-\int\frac{e^{-y}dy}{yy} = \frac{e^{-y}}{y} + \int\frac{e^{-y}dy}{y}$ erit

$$-O + 1 = \int\frac{e^{-y}dy}{y(1-e^{-y})} + \tfrac{1}{2}\int\frac{e^{-y}dy}{y} + \frac{e^{-y}}{y} - \frac{e}{o}.$$

Hic si ponatur $e^{-y} = z$ et integralia a termino $z = 0$ vsque ad $z = 1$ extendantur, reperitur:

$$-O + 1 = \tfrac{1}{2}\int\frac{dz}{lz} - \int\frac{dz}{(1-z).z} - \int\frac{dz}{(lz)^2}$$

neque vero hinc natura huius numeri O cognosci potest cum tamen aliunde constet eum esse $= \frac{1}{2}l2\pi$, sicque partim per logarithmos partim per circuli peripheriam π determinari. Quemadmodum ergo iste valor eruatur operae pretium erit accuratius perpendisse.

32. Quoniam a Wallisio inuenta est haec aequalitas

$$\frac{\pi}{2} = \frac{2.2}{1.3}\cdot\frac{4.4}{3.5}\cdot\frac{6.6}{5.7}\cdot\frac{8.8}{7.9}\cdot\frac{10.10}{9.11}\ \text{etc.}$$

erit

erit logarithmis sumendis

$$\tfrac{1}{2}l\tfrac{\pi}{2} = l2 - l3 + l4 - l5 + l6 - l7 + l8 - l9 + \text{etc.}$$

seu hoc modo per duplicem seriem

$$\tfrac{1}{2}l\tfrac{\pi}{2} = l2 + l4 + l6 + l8 + l10 + l12 + \text{etc.} \ldots + l2x + \tfrac{1}{2}l2(x+1)$$
$$- l3 - l5 - l7 - l9 - l11 - l13 - \text{etc.} - l(2x+1)$$

siquidem vtraque series in infinitum quidem sed tamen parem terminorum numerum continuetur, seu ipsi x vtrique idem valor tribuatur: quae duplex series etiam hoc modo exhiberi potest

$$\tfrac{1}{2}l\tfrac{\pi}{2} = l2 + l4 + l6 + l8 + \ldots\ldots + l2x - \tfrac{1}{2}l2x$$
$$- l1 - l3 - l5 - l7 - \ldots\ldots - l(2x-1)$$

At ex ipsa nostra serie sumto x infinito habemus

$$l1 + l2 + l3 + l4 \ldots + lx = O - x + (x + \tfrac{1}{2})lx$$

vnde si $xl2$ seu ad quemlibet terminum $l2$ addatur fit

$$l2 + l4 + l6 + l8 + \ldots + l2x = O - x + xl2 + (x + \tfrac{1}{2})lx$$

Deinde si ibi loco x scribamus $2x$ prodit

$$l1 + l2 + l3 + l4 + \ldots + l2x = O - 2x + (2x + \tfrac{1}{2})l2 + (2x + \tfrac{1}{2})lx$$

a qua si illa auferatur relinquitur:

$$l1 + l3 + l5 + \ldots + l(2x-1) = -x + (x + \tfrac{1}{2})l2 + xlx$$

quae summae si in illa forma loco vtriusque seriei substituantur orietur haec aequatio:

$$\left.\begin{array}{l}\tfrac{1}{2}l\tfrac{\pi}{2} + \tfrac{1}{2}l2x = O - x + xl2 + (x + \tfrac{1}{2})lx \\ + x - (x + \tfrac{1}{2})l2 - xlx\end{array}\right\} = O - \tfrac{1}{2}l2 + \tfrac{1}{2}lx$$

vnde

vnde concluditur $O = \tfrac{1}{2}l\tfrac{\pi}{2} + \tfrac{1}{2}l2x + \tfrac{1}{2}l2 - \tfrac{1}{2}lx = \tfrac{1}{2}l2\pi$ seu $O = 0,91893853320467274178032 97$.

33. Cum ergo sit $O = \tfrac{1}{2}l2\pi$ hinc vicissim colligimus fore

$$\tfrac{1}{2}\int\frac{dz}{lz} - \int\frac{dz}{(1-z)lz} - \int\frac{dz}{(lz)^2} = 1 - \tfrac{1}{2}l2\pi$$

sicque patet has tres integrationes, siquidem a termino $z = 0$ ad terminum $z = 1$ extendantur perduci ad quantitatem $l2\pi$ quod quomodo per calculum ostendi possit, haud liquet, vnde haec inuestigatio eo maiori attentione digna videtur. Facile quidem perspicitur esse

$-\int\frac{dz}{(lz)^2} = \frac{z}{lz} - \int\frac{dz}{lz}$, ita vt sit $\frac{z}{lz} - \tfrac{1}{2}\int\frac{dz}{lz} - \int\frac{dz}{(1-z)lz} = 1 - \tfrac{1}{2}l2\pi$

Parum quoque lucramur ponendo $z = v^i$ et $lz = i(1-v)$ existente i numero infinito consequimur autem hanc aequationem.

$$\frac{-v^i}{i(1-v)} + \tfrac{1}{2}\int\frac{v^{i-1}dv}{1-v} + \int\frac{v^{i-1}dv}{(1-v)(1-v^i)} = 1 - \tfrac{1}{2}l2\pi$$

quae integralia pariter ab $v = 0$ vsque ad $v = 1$ extendi debent.

34. Euolutio harum formularum nihil aliud suppeditat nisi quod statim ex prima aequatione sumendo numerum x infinitum concludi potest, quia enim tum series litteras A. B. C. D etc. complectens euanescit, habebimus

$$O = \tfrac{1}{2}l2\pi = l1 + l2 + l3 + \ldots + lx - (x + \tfrac{1}{2})lx + x$$

vbi

vbi cum feries $l1 + l2 \ldots + lx$ conftet x terminis quaelibet reliquarum partium $(x + \frac{1}{2})\,lx$ et x in feriem totidem terminorum conuertatur. Ac pofterior quidem x totidem terminos vnitati aequales praebet, prior vero $(x + \frac{1}{2})\,lx$ fequenti modo euoluitur:

$$\tfrac{1}{2}\,l2\pi = l1 + l2 + l3 \ldots + l(x-1) + lx$$
$$+ 1 + 1 + 1 \ldots + 1 \qquad + 1$$
$$-\tfrac{3}{2}\,l1 - \tfrac{5}{2}\,l2 - \tfrac{7}{2}\,l3 \ldots - (x-\tfrac{1}{2})\,l(x-1) - (x+\tfrac{1}{2})\,lx$$
$$+\tfrac{3}{2}\,l1 + \tfrac{5}{2}\,l2 \ldots + (x-\tfrac{3}{2})\,l(x-2) + (x-\tfrac{1}{2})\,l(x-1)$$

vnde colligitur haec feries fatis concinna:

$$\tfrac{1}{2}\,l2\pi = 1 - (\tfrac{3}{2}\,l\tfrac{2}{1} - 1) - (\tfrac{5}{2}\,l\tfrac{3}{2} - 1) - (\tfrac{7}{2}\,l\tfrac{4}{3} - 1) - (\tfrac{9}{2}\,l\tfrac{5}{4} - 1)\ \text{etc.}$$

quae commodius hac forma exhibetur:

$$1 - \tfrac{1}{2}\,l2\pi = \tfrac{3}{2}\,l\tfrac{2}{1} - 1 + \tfrac{5}{2}\,l\tfrac{3}{2} - 1 + \tfrac{7}{2}\,l\tfrac{4}{3} - 1 + \tfrac{9}{2}\,l\tfrac{5}{4} - 1\ \text{etc.}$$

Terminus generalis huius feriei eft $\frac{x}{2}\,l\frac{x+1}{x-1} - 1$, qui in hanc feriem euoluitur: $\frac{1}{3x^2} + \frac{1}{5x^4} + \frac{1}{7x^6} + \frac{1}{9x^8} + \text{etc.}$ ex quo per infinitas feries habebimus:

$$1 - \tfrac{1}{2}\,l2\pi = \tfrac{1}{3\cdot 3^2} + \tfrac{1}{5\cdot 3^4} + \tfrac{1}{7\cdot 3^6} + \tfrac{1}{5\cdot 3^8} + \text{etc.}$$
$$\tfrac{1}{3\cdot 5^2} + \tfrac{1}{5\cdot 5^4} + \tfrac{1}{7\cdot 5^6} + \tfrac{1}{5\cdot 5^8} + \text{etc.}$$
$$\tfrac{1}{3\cdot 7^2} + \tfrac{1}{5\cdot 7^4} + \tfrac{1}{7\cdot 7^6} + \tfrac{1}{5\cdot 7^8} + \text{etc.}$$
$$\tfrac{1}{3\cdot 9^2} + \tfrac{1}{5\cdot 9^4} + \tfrac{1}{7\cdot 9^6} + \tfrac{1}{5\cdot 9^8} + \text{etc.}$$

etc.

vbi cum feries poteftatum reciprocarum primo termino truncatarum occurrant, erit

$$1 - \tfrac{1}{2}\,l2\pi = \tfrac{1}{3}(\tfrac{2^2-1}{2^2}A\pi^2 - 1) + \tfrac{1}{5}(\tfrac{2^4-1}{2^4}B\pi^4 - 1) + \tfrac{1}{7}(\tfrac{2^6-1}{2^6}C\pi^6 - 1) + \text{etc.}$$

fupra autem inueneramus:

$$1 - \tfrac{1}{2}\,l2\pi = \tfrac{A}{2} - \tfrac{1\cdot 2\,B}{2^3} + \tfrac{1\cdot 2\cdot 3\cdot 4\,C}{2^5} - \tfrac{1\cdot 2\cdots 6\,D}{2^7} + \text{etc.}$$

35. Huiusmodi relationes eo maiorem attentionem merentur, quo magis funt abfconditae, vnde operae erit pretium feriem modo inuentam accuratius euoluere. Hunc in finem eam generaliori forma complectar, ftatuens:

$$P = \tfrac{1}{3}A\pi^2u^2 + \tfrac{1}{5}B\pi^4u^4 + \tfrac{1}{7}C\pi^6u^6 + \tfrac{1}{9}D\pi^8u^8 + \text{etc.}$$
$$Q = \tfrac{1}{3}A\tfrac{\pi^2u^2}{2^2} + \tfrac{1}{5}B\tfrac{\pi^4u^4}{2^4} + \tfrac{1}{7}C\tfrac{\pi^6u^6}{2^6} + \tfrac{1}{9}D\tfrac{\pi^8u^8}{2^8} + \text{etc.}$$
$$R = \tfrac{1}{3}u^2 + \tfrac{1}{5}u^4 + \tfrac{1}{7}u^6 + \tfrac{1}{9}u^8 + \text{etc.} = \tfrac{1}{2u}\,l\tfrac{1+u}{1-u} - 1$$

vt pofito $u = 1$ fit $1 - \frac{1}{2}\,l2\pi = P - Q - R$

Iam ad valores litterarum P et Q definiendos fumo aequationem fupra §. 6 datam

$$Ax^2 + Bx^4 + Cx^6 + Dx^8 + \text{etc.} = \tfrac{1}{2} - \tfrac{1}{2}x\,\text{cot.}\,x$$

vnde per integrationem fit

$$\tfrac{1}{3}Ax^3 + \tfrac{1}{5}Bx^5 + \tfrac{1}{7}Cx^7 + \text{etc.} = \tfrac{1}{2}x - \tfrac{1}{2}\int\tfrac{x\,dx\,\text{cof.}\,x}{\text{fin.}\,x}$$

$$\text{feu}\ \tfrac{1}{3}Ax^2 + \tfrac{1}{5}Bx^4 + \tfrac{1}{7}Cx^6 + \text{etc.} = \tfrac{1}{2} - \tfrac{1}{2x}\int\tfrac{x\,dx\,\text{cof.}\,x}{\text{fin.}\,x}.$$

Hinc pofito primo $x = \pi u$ tum vero $x = \frac{\pi u}{2}$ deducimus

$$P = \tfrac{1}{2} - \tfrac{1}{2\pi u}\int\tfrac{\pi\pi u\,du\,\text{cof.}\,\pi u}{\text{fin.}\,\pi u} = \tfrac{1}{2} - \tfrac{\pi}{2}\int\tfrac{u\,du\,\text{cof.}\,\pi u}{\text{fin.}\,\pi u}\ \text{ob}\ u = 1$$

$$Q = \tfrac{1}{2} - \tfrac{1}{\pi u}\int \frac{\pi\pi u\,du\,\text{cof.}\,\frac{1}{2}\pi u}{4\,\text{fin.}\,\frac{1}{2}\pi u} = \tfrac{1}{2} - \tfrac{\pi}{4}\int \frac{u\,du\,\text{cof.}\,\frac{1}{2}\pi u}{\text{fin.}\,\frac{1}{2}\pi u}$$

et ob $\text{fin.}\,\pi u = 2\,\text{fin.}\,\frac{1}{2}\pi u\,\text{cof.}\,\frac{1}{2}\pi u$ et $\text{cof.}\,\pi u = \text{cof.}\,\frac{1}{2}\pi u^2 - \text{fin.}\,\frac{1}{2}\pi u^2$ fiet

$$P - Q = \tfrac{\pi}{4}\int \frac{u\,du\,(\text{cof.}\,\frac{1}{2}\pi u^2 - \text{cof.}\,\frac{1}{2}\pi u^2 + \text{fin.}\,\frac{1}{2}\pi u^2)}{\text{fin.}\,\frac{1}{2}\pi u\,\text{cof.}\,\frac{1}{2}\pi u} = \tfrac{\pi}{4}\int \frac{u\,du\,\text{fin.}\,\frac{1}{2}\pi u}{\text{cof.}\,\frac{1}{2}\pi u}$$

ita vt fit $1 - \frac{1}{2}l2\pi = \frac{\pi}{4}\int \frac{u\,du\,\text{fin.}\,\frac{1}{2}\pi u}{\text{cof.}\,\frac{1}{2}\pi u} - \frac{1}{2}l\frac{1+u}{1-u} + 1$

seu $l2\pi = \int \frac{-\frac{1}{2}\pi u\,du\,\text{fin.}\,\frac{1}{2}\pi u}{\text{cof.}\,\frac{1}{2}\pi u} + l\frac{1+u}{1-u} - l\frac{1+u}{1-u} + u\,l\,\text{cof.}\,\frac{1}{2}\pi u - \int du\,l\,\text{cof.}\,\frac{1}{2}\pi u$

fiquidem integratione abfoluta ponatur $u = 1$.

36. Statuamus nunc angulum $\frac{1}{2}\pi u = \Phi$, feu $u = \frac{2\Phi}{\pi}$, vt integrale a termino $\Phi = 0$ vsque ad $\Phi = \frac{\pi}{2} = 90°$ extendi oporteat, ac praecedens aequatio abibit in hanc formam

$$l2\pi = l\frac{\pi + 2\Phi}{\pi - 2\Phi} + l\,\text{cof.}\,\Phi - \tfrac{2}{\pi}\int d\Phi\,l\,\text{cof.}\,\Phi \text{ fiue ob } \Phi = \tfrac{\pi}{2}$$

$$l2\pi = l2\pi - l\frac{\pi - 2\Phi}{\text{cof.}\,\Phi} - \tfrac{2}{\pi}\int d\Phi\,l\,\text{cof.}\,\Phi$$

vbi fractio $\frac{\pi - 2\Phi}{\text{cof.}\,\Phi}$ cafu $\Phi = \frac{\pi}{2}$ abit in $\frac{2}{\text{fin.}\,\Phi} = 2$, ita vt fit $l2\pi = l2\pi - l2 - \frac{2}{\pi}\int d\Phi\,l\,\text{cof.}\,\Phi$ feu $\int d\Phi\,l\,\text{cof.}\,\Phi = -\frac{\pi l2}{2}$.

Quod fi ergo demonftrari poffet formulae integralis $\int d\Phi\,l\,\text{cof.}\,\Phi$ valorem a termino $\Phi = 0$ ad $\Phi = 90° = \frac{\pi}{2}$ extenfum reuera quantitati $-\frac{\pi l2}{2}$ aequari, omnino per aliam viam id affequeremur quod ante per ambages circa valorem litterae $Q = \frac{1}{2}l2\pi$ conclufimus. Quoniam vero nunc de hoc valore fumus

certi, infigne confecuti fumus hoc theorema quod fit integrali a valore $\Phi = 0$ vsque ad $\Phi = \frac{\pi}{2} = 90°$ extenfum $\int d\Phi\,l\,\text{cof.}\,\Phi = -\frac{\pi l2}{2}$

vel fi ponamus $\text{cof.}\,\Phi = v$, vt termini integrationis fint $v = 1$ et $v = 0$ oftendendum eft fore

$$\int \frac{dv\,lv}{\sqrt{(1 - vv)}} = \frac{\pi l2}{2}$$

vnde hoc integrali in feriem euoluto erit

$$\frac{\pi l2}{2} = 1 + \frac{1}{2.3^2} + \frac{1.3}{2.4.5^2} + \frac{1.3.5}{2.4.6.7^2} + \frac{1.3.5.7}{2.4.6.8.9^2}$$

quae feries viciffim reducitur ad $\int \frac{ds}{s}$ Ang. fin. s ponendo poft integrationem $s = 1$: haec vero porro pofito $s = \text{fin.}\,\Phi$ ad hanc

$\int \frac{\Phi\,d\Phi\,\text{cof.}\,\Phi}{\text{fin.}\,\Phi} = \Phi\,l\,\text{fin.}\,\Phi - \int d\Phi\,l\,\text{fin.}\,\Phi = \frac{\pi l2}{2}$, feu $\int d\Phi\,l\,\text{fin.}\,\Phi = -\frac{\pi l2}{2}$

quae cum fuperiori congruit.

36. Quod autem fit $\int d\Phi\,l\,\text{fin.}\,\Phi = -\frac{\pi l2}{2}$ hoc modo demonftratur. Cum fit $\frac{\text{cof.}\,\Phi}{\text{fin.}\,\Phi} = 2\,\text{fin.}\,2\Phi + 2\,\text{fin.}\,4\Phi + 2\,\text{fin.}\,6\Phi + 2\,\text{fin.}\,8\Phi$ etc. erit

$l\,\text{fin.}\,\Phi = -\text{cof.}\,2\Phi - \frac{1}{2}\text{cof.}\,4\Phi - \frac{1}{3}\text{cof.}\,6\Phi - \frac{1}{4}\text{cof.}\,8\Phi \text{ etc.} - l2$

ergo $\int d\Phi\,l\,\text{fin.}\,\Phi = -\Phi\,l2 - \frac{1}{2}\text{fin.}\,2\Phi - \frac{1}{2.2^2}\text{fin.}\,4\Phi - \frac{1}{2.3^2}\text{fin.}\,6\Phi - \frac{1}{2.4^2}\text{fin.}\,8\Phi$ etc.

iam facto $\Phi = \frac{\pi}{2}$ fit $\int d\Phi\,l\,\text{fin.}\,\Phi = -\frac{\pi}{2}l2$.