



1770

De summis serierum numeros Bernoullianos involventium

Leonhard Euler

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2018-09-25

Recommended Citation

Euler, Leonhard, "De summis serierum numeros Bernoullianos involventium" (1770). *Euler Archive - All Works*. 393.

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Considerentur formula differentialis $P dt - Q du$, quae multiplicatore R integrabilis reddatur; ponaturque $d\psi = R(Pdt - Qdu)$, eritque ψ functio cognita ipsarum t et u , unde vicissim t per u et ψ definiatur ac fiet $dt = \frac{Q du}{P} + \frac{d\psi}{R}$. Quare formula proposita erit:

$\frac{MQ + NP}{P} du + \frac{M d\psi}{R}$, vbi $\frac{MQ + NP}{P}$ est functio ipsarum u et ψ tantum atque resolutio per n. 4 absoluetur.

Scholion 2.

25. De his integrationibus imprimis notanduma est, eas esse generalissimas, dum functiones maxime generales cuiuspiam variabilis in integra inductantur, pro quibus adeo functiones nullo continuatis vinculo contentas, quae vt supra videmus ex libero manus ductu nascuntur, assumere licet. Quin etiam omnium huius generis quaestionum criterium in hoc consistit, vt tales functiones prorsus ab arbitrio nostro pendentes ip earum solutiones introducantur.

DE

SVMMIS SERIERVM

NVMEROS BERNOVLIANOS

INVOLVENTIVM.

Auctore

L. EULER O.

I.

Quantopere sint notatu digni numeri ab Inventore *Bernoulliani* vocati, quippe quibus olim *Iacobus Bernoulli* in Arte coniectandi est vñs ad progressionem potestatum numerorum naturalium summandas, cum ab aliis, qui serierum doctrinam novis inuentis locupletauerunt, tum etiam a me abunde est osensum, vbi per eosdem numeros serierum potestatum reciprocaram summam expressas dedi. *Bernoullius* quidem progressionem horum numerorum ob calculi molestantiam non vltra quintum terminum continuavit, qui sunt $\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{10}$, $\frac{1}{17}$, $\frac{1}{26}$, atque Auctori vsque ad vñdecimas potestates summandas sufficiebant. Postquam autem factis concinnam huius progressionis legem detexissem, 17 eius primores terminos assignavi. Ipsos vero numeros *Bernoullianos* respectiue per numeros 6, 10, 14, 18, 22 etc. multiplico quo denominatores sunt simplici-

Tom. XIV. Nou. Comm. R res,

$$\begin{aligned}
 A &= \frac{1}{1.2.3} \\
 B &= \frac{A}{1.2.3} - \frac{2}{1.2.3.5} \\
 C &= \frac{B}{1.2.3} - \frac{A}{1.2.3.5} + \frac{3}{1.2.3.7} \\
 D &= \frac{C}{1.2.3} - \frac{B}{1.2.3.5} + \frac{A}{1.2.3.7} - \frac{4}{1.2.3.9} \\
 E &= \frac{D}{1.2.3} - \frac{C}{1.2.3.5} + \frac{B}{1.2.3.7} - \frac{A}{1.2.3.9} + \frac{5}{1.2.3.11}
 \end{aligned}$$

Altera vero lex commodius quemvis terminum per producta ex binis precedentibus sequenti modo exprimit:

$$\begin{aligned}
 5B &= 2A^2 \text{ existente } A = \frac{1}{6} \\
 7C &= 4AB \\
 9D &= 4AC + 2BB \\
 11E &= 4AD + 4BG \\
 13F &= 4AE + 4BD + 2CC \\
 15G &= 4AF + 4BE + 4CD \\
 17H &= 4AG + 4BF + 4CE + 2DD \\
 &\text{etc.}
 \end{aligned}$$

Vnde etiam mihi quidem has series tam longe continuare licuit.

4. His expofitis hoc loco in summas plurimum ferierum, quorum termini istos numeros A, B, C, D, E etc. praeter alios factores, quorum lex per se est manifesta, innolunt, inquirere confitui, ita v. mihi in genere propofita fit inueftigatio summae huius feriei

$$S = a$$

$S = aAx^2 + bBx^4 + cCx^6 + dDx^8 + eEx^{10} + \text{etc.}$
 dum litterae a, b, c, d, e etc. feriem quamcumque cognitam constituunt, eximias enim hinc enasci series, quarum summae omni attentione sunt digne pluribus speciminibus iam ostendi. Incipio igitur ab hac serie:

$$S = Ax^2 + Bx^4 + Cx^6 + Dx^8 + Ex^{10} + \text{etc.}$$

quam per priorem legem progressionis litterarum A, B, C, D etc. manifesto ex evolutione huius fractionis relatare manifestum est:

$$S = \frac{1}{1.2.3} x^2 + \frac{2}{1.2.3.5} x^4 + \frac{2}{1.2.3.7} x^6 + \frac{4}{1.2.3.9} x^8 + \text{etc.}$$

cuius denominator exhibet valorem $\frac{sin. x}{x}$; eiusque differentiale $\frac{d x \cos. x}{x} = -\frac{d x \sin. x}{x^2}$ per $\frac{2}{x^2}$ multiplicatum ipsum praebet numeratorem, ita vt fit:

$$f = \frac{sin. x - x \cos. x}{2 sin. x} = \frac{1}{2} - \frac{1}{2} x \cot. x$$

hincque summa istius feriei

$$\begin{aligned}
 Ax^2 + Bx^4 + Cx^6 + Dx^8 + Ex^{10} + \text{etc.} &= \frac{1}{2} x \cot. x \\
 \text{vbi notari meretur si } x, \text{ evanescat, fore summam} \\
 &= \frac{1}{2} x x \text{ ob cot. } x = \frac{1 - \frac{1}{2} x^2}{x}
 \end{aligned}$$

5. Ponamus $xx = -yy$, totamque feriem negatiue exponamus, vt quaeratur haec summa:

$$s = Ayy - By^4 + Cy^6 - Dy^8 + Ey^{10} - \text{etc.}$$

R 3

atque

atque cum iam sit per legem priorem :

$$s = \frac{1}{1 \cdot 3} y^2 + \frac{1}{1 \cdot 3 \cdot 5} y^4 + \frac{1}{1 \cdot 3 \cdot 5 \cdot 7} y^6 + \frac{1}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9} y^8 + \text{etc.}$$

cuius denominator est $\frac{1}{2} (e^2 - e^{-2})$, eiusque differentiale $= \frac{d}{2} (ex - e^{-x}) + \frac{d}{2} (e^2 + e^{-2})$, quod per $\frac{2}{2 \cdot 2} y$ multiplicatum dat numeratorem $= -\frac{1}{2} (e^2 - e^{-2}) + \frac{1}{2} (e^2 + e^{-2})$ ita ut huius seriei summa sit

$$s = \frac{y}{2} \cdot \frac{e^2 + e^{-2}}{e^2 - e^{-2}} = \frac{1}{2}$$

Casus hic notari meretur quo $y = x$, haecque series sumatur :

$$A - B + C - D + E - \text{etc.} = \frac{1}{2} \frac{e^e + 1}{e^e - 1} = \frac{1}{2} = \frac{1}{e^e - 1}$$

Si hic pro litteris A, B, C, D etc. ipsae series assumatae restantur, et quatenus fieri potest, in summas colligantur, erit

$$\frac{1}{\pi + 1} + \frac{1}{4\pi + 1} + \frac{1}{9\pi + 1} + \frac{1}{16\pi + 1} + \text{etc.} = \frac{1}{e^{\pi} - 1}$$

6. Inventa summa seriei

$$Ax^2 + Bx^4 + Cx^6 + Dx^8 + \text{etc.} = \frac{1}{2} - \frac{1}{2} x \cot. x$$

in qua simul alteram completi licet

$$Ay^2 - By^4 + Cy^6 - Dy^8 + \text{etc.} = \frac{1}{2} \frac{e^{2y} + 1}{e^{2y} - 1} - \frac{1}{2}$$

tam ope differentiationis quam integrationis innumerabiles aliae inde deduci possunt quarum summa pariter

pariter assignari valet. Multiplicata scilicet illa serie per x^2 differentiatio dabit

$$(n+2) Ax^{n+1} + (n+4) Bx^{n+3} + (n+6) Cx^{n+5} + (n+8) Dx^{n+7} \text{etc.}$$

$$= \frac{n}{2} x^{n-1} - \frac{(n+1)}{2} x^n \cot. x + \frac{x^{n+1}}{2 \sin x^2} \text{ sine}$$

$$(n+2) Ax^2 + (n+4) Bx^4 + (n+6) Cx^6 + (n+8) Dx^8 \text{etc.}$$

$$= \frac{n}{2} - \frac{1}{2} (n+1) x \cot. x + \frac{x}{2 \sin x^2}$$

sin autem illa series per $x^{n-1} dx$ multiplicata integretur, prodibit sequens summatio :

$$\frac{A}{n+2} x^{n+2} + \frac{B}{n+4} x^{n+4} + \frac{C}{n+6} x^{n+6} + \frac{D}{n+8} x^{n+8} + \text{etc.}$$

$$= \frac{1}{2} x^n - \frac{1}{2} \int x^n dx \cot. x$$

quae summa ut cognita est spectanda, etiam si formulae $\int x^n dx \cot. x$ integrale euolui vel exprimi finite nequit. Quin etiam ambabus operationibus combinandis ac rependendis infinitae series formae

$$\alpha Ax^2 + \beta Bx^4 + \gamma Cx^6 + \delta Dx^8 + \text{etc.}$$

obtinebuntur, vbi litterae $\alpha, \beta, \gamma, \delta$ etc. sint producta ex duabus pluribus fractionibus, quarum tam numeratores quam denominatores progressionis arithmeticas constituent. Veluti si series per differentiationem inventa per $\frac{d}{2x}$ multiplicetur et integretur orietur :

$$\frac{n+2}{2} Ax^2 + \frac{n+4}{4} Bx^4 + \frac{n+6}{6} Cx^6 + \text{etc.} = \frac{n}{2} \int \frac{x}{\sin x} - \frac{x \cos x}{2 \sin x^2} + \frac{1}{2}$$

ita

ita ut sit.

$$\frac{1}{2}A x^2 + \frac{1}{4}B x^4 + \frac{1}{6}C x^6 + \frac{1}{8}D x^8 + \text{etc.} = \frac{1}{m^2} x^m$$

7. Datur vero praeterea alia methodus omnino singularis ex serie inventa alias innumerabiles erudi quarum summa itidem assignari queat. Hunc in finem seriem principalem ita representento:

$$A a^2 x^2 + B a^4 x^4 + C a^6 x^6 + D a^8 x^8 + \text{etc.} = \frac{1}{2} a^2 \cot. dx$$

eamque multiplico per eiusmodi formulam differentialem $X dx$, ut si post integrationem ipsi x certus valor $x = f$ tribuatur, integrale $\int X x^m dx$ valorem nanciscatur concinnum: scilicet ut fiat:

$$\int X x^2 dx = a \int X dx; \int X x^4 dx = \frac{2}{3} \int X x^2 dx; \int X x^6 dx = \frac{3}{4} \int X x^4 dx \text{ etc.}$$

quo facto nascetur huiusmodi series:

$$a A a^2 + a^3 B a^4 + a^4 C a^6 + a^5 D a^8 + \text{etc.} = \frac{a \int X x^2 dx \cot. dx}{2 \int X x^2 dx}$$

vbi X ita accipi potest, ut a, β, γ, δ etc. fiant vel numeri in arithmetica progressionem procedentes, vel fractiones, quarum tam numeratores quam denominatores talem progressionem constituent.

Veluti si sumatur.

$$X = x^m - 1 (1 - x^2)^k \text{ erit posito } x = 1$$

ideoque

$$\begin{aligned} \int x^{m+1} dx (1-x^2)^k &= \frac{m}{m+2k+2} \int x^{m-1} dx (1-x^2)^k & a &= \frac{m}{m+2k+2} \\ \int x^{m+3} dx (1-x^2)^k &= \frac{m+2}{m+2k+4} \int x^{m+1} dx (1-x^2)^k & \beta &= \frac{m+2}{m+2k+4} \\ \int x^{m+5} dx (1-x^2)^k &= \frac{m+4}{m+2k+6} \int x^{m+3} dx (1-x^2)^k & \gamma &= \frac{m+4}{m+2k+6} \\ \int x^{m+7} dx (1-x^2)^k &= \frac{m+6}{m+2k+8} \int x^{m+5} dx (1-x^2)^k & \delta &= \frac{m+6}{m+2k+8} \end{aligned}$$

etc.

At

At si sumatur $X dx = e^{-m x} x^2 dx$ erit posito post integrationem $x = \infty$

$$\begin{aligned} \int e^{-m x} x^{n+2} dx &= \frac{n+1}{2m} \int e^{-m x} x^n dx & \text{ideoque} & \\ \int e^{-m x} x^{n+4} dx &= \frac{n+3}{4m} \int e^{-m x} x^{n+2} dx & a &= \frac{n+1}{2m} \\ \int e^{-m x} x^{n+6} dx &= \frac{n+5}{6m} \int e^{-m x} x^{n+4} dx & \beta &= \frac{n+3}{4m} \\ & & \gamma &= \frac{n+5}{6m} \end{aligned}$$

Sumto autem $X dx = x^{n-1} dx (x)^m$ sit posito $x = 1$ post integrationem:

$$\int x^{n-1} dx (x)^m = \frac{1.2.3 \dots m}{n^m + 1}$$

signo + valente si m sit numerus par, contra signo -.

8. His autem transformationibus, quae alibi susus sunt expositae, hic non immoror, sed alium fontem, unde huiusmodi series promanant, contemplabor quem olim iam mihi aperuit summatio progressionum generalis scilicet si seriei cuiuscunque terminus generalis, seu is qui indici x convenit, ponatur $= X$, ut sit X functio quaecunque ipsius x , huiusque seriei terminus summatorius statuatur $= S$ reperi fore per numeros *Bernoullianos* $\mathfrak{B}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc.

$$2S = 2 \int X dx + X + \frac{\mathfrak{B} dx}{1.2.3 dx} - \frac{\mathfrak{B} dx X}{1.2.3.4 dx^2} + \frac{\mathfrak{C} dx X}{1.2.3.4.5 dx^3} - \frac{\mathfrak{C} dx^2 X}{1.2.3.4.5.6 dx^4} + \text{etc.}$$

Tom. XIV. Nou. Comm.

S unde

vnde si alteri numeri A, B, C, D etc. ad summas potestatum reciprocarum relati introducantur, deducimus:

$$2S = 2 \int X dx + X^2 + \frac{AdX}{1dx} - \frac{Bd^2X}{2^2dx^2} + \frac{Cd^3X}{3^3dx^3} - \frac{Dd^4X}{4^4dx^4} + \text{etc.}$$

Quare si seriem cuius terminus generalis est X et summatorius = S pro lubitu accipiamus habebimus hanc summationem:

$$\frac{AdX}{2dx} - \frac{Bd^2X}{3^2dx^2} + \frac{Cd^3X}{5^3dx^3} - \frac{Dd^4X}{7^4dx^4} + \text{etc.} = S - \int X dx - \frac{1}{2}X.$$

Quaecunque ergo pro X sumatur functio ipsius, concessa progressionis, cuius X est terminus generalis, summatione istius seriei litteras A, B, C, D etc. innuolentis summam assignare poterimus, etiam si forte eiusdem summatio secundum praecepta modo exposita insitura summis difficultatibus sit obnoxia.

9. Primum ergo ipsi X tribuamus eiusmodi

valorem vt fit $X = \frac{1}{x^n}$; vnde fit

$$\frac{dX}{dx} = \frac{-n}{x^{n+1}}; \frac{d^2X}{dx^2} = \frac{-n(n+1)}{x^{n+2}}; \frac{d^3X}{dx^3} = \frac{-n(n+1)(n+2)}{x^{n+3}}; \frac{d^4X}{dx^4} = \frac{-n(n+1)(n+2)(n+3)}{x^{n+4}}$$

et quia est $\int X dx = \frac{-1}{(n-1)x^{n-1}} + O$ atque

$$S = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \dots + \frac{1}{x^n}$$

habebi-

habebimus hanc seriem:

$$\frac{-nA}{2x^{n+1}} + \frac{n(n+1)(n+2)B}{2^2x^{n+2}} - \frac{n(n+1)(n+2)(n+3)C}{2^3x^{n+3}} + \text{etc.}$$

$$= S - \frac{1}{2x^n} + \frac{1}{(n-1)x^{n-1}} - O$$

vbi constantem O ex casu quodam cognito definiti convenit, quo ipsi x certus tribuitur valor. Ita posito $x = \infty$, quoniam tum tota series in nihilum abit constans haec O exprimet summam huius seriei in infinitum continuatae $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \text{etc.}$ quam novimus per quadraturam circuli π exhiberi posse quoties exponens n est numerus par. Hos ergo casus primum caotiam.

Casus I. quo $n = 2$.

10. Hoc ergo casu $n = 2$ fit constans $O = \frac{\pi}{6}$ = A π^2 , posteaque huius progressionis summa indefinite

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{x^2} = S$$

habebimus hanc summationem:

$$- \frac{2A}{1 \cdot 3^3} + \frac{2 \cdot 3 \cdot 4 \cdot B}{2^2 \cdot 3^4} - \frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot C}{2^3 \cdot 3^5} + \frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot D}{2^4 \cdot 3^6} - \text{etc.}$$

$$= S - \frac{1}{2x^2} + \frac{1}{x} - A \pi^2 \text{ seu}$$

$$= A \pi^2 x^2 - x + \frac{1}{2} - S x x.$$

Cuius

Cuius ergo summa quoties x est numerus integer exhiberi potest. Ita obtinebimus :

$$\begin{aligned} \frac{1.2.A}{2} &= \frac{1.2.3.4.B}{2} + \frac{1.2.3.5.C}{2} - \frac{1.2.3.4.5.D}{2} + \text{etc.} = A\pi^2 - \frac{1}{2} \\ \frac{1.2.3.A}{4} &= \frac{1.2.3.4.B}{4} + \frac{1.2.3.5.C}{4} - \frac{1.2.3.4.5.D}{4} + \text{etc.} = 4A\pi^2 - \frac{1}{4} \\ \frac{1.2.3.4.A}{6} &= \frac{1.2.3.4.5.B}{6} + \frac{1.2.3.4.6.C}{6} - \frac{1.2.3.4.5.6.D}{6} + \text{etc.} = 9A\pi^2 - \frac{1}{6} \\ &\quad - 9(1 + \frac{1}{2} + \frac{1}{3}) \\ \frac{1.2.3.4.5.A}{8} &= \frac{1.2.3.4.5.6.B}{8} + \frac{1.2.3.4.5.7.C}{8} - \frac{1.2.3.4.5.6.7.D}{8} + \text{etc.} = 16A\pi^2 - \frac{1}{8} \\ &\quad - 16(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}). \end{aligned}$$

II. Inuestigemus iam eandem series methodo supra exposta, et cum ibi inuenissemus :

$$Agy - Bg^2y^2 + Cg^3y^3 - Dg^4y^4 + \text{etc.} = \frac{e^{2y} + 1}{2} - \frac{1}{2ay}$$

multiplicemus per $e^{-y}y dy$, et integratione ita in-
stituta ut integralia euaneſcant poſito $y = 0$, statua-
mus $y = \infty$, ſicque adificimur :

$$\begin{aligned} \int e^{-y}y^2 dy &= -e^{-y}y^2 - 2e^{-y}y - 2.1e^{-y} + 1.2 = 1.2 \\ \int e^{-y}y^3 dy &= -e^{-y}(y^3 + 3y^2 + 4.3y^2 + 4.3.2.y \\ &\quad + 4.3.2.1.) + 1.2.3.4 = 1.2.3.4 \\ &\quad \text{ſimilique modo} \end{aligned}$$

$\int e^{-y}y^6 dy = 1.2.3.4.5.6$; $\int e^{-y}y^4 dy = 1.2 \dots 8.$
Hinc itaque perueniemus ad hanc ſummationem

$$1.2.Ag - 1.4.Bg^2 + 1.6.Cg^3 - \text{etc.} = \frac{1}{2} \int e^{-y}y^6 dy \cdot \frac{e^{2y} + 1}{e^{2y} - 1} - \frac{1}{2a}$$

Pona-

Ponamus nunc $a = \frac{1}{2}$. ut prodeat haec ſeries :

$$\frac{1.1.A}{2} = \frac{1.2.3.4.B}{2} + \frac{1.2.3.5.C}{2} - \frac{1.2.3.4.5.D}{2} + \text{etc.}$$

cuius ſummam nouimus eſſe $= A\pi^2 - \frac{1}{2}$, nunc au-
tem eandem ita expreſſam inuenimus :

$$\frac{1}{2} \int e^{-y}y dy \cdot \frac{e^2 + 1}{e^2 - 1} - 1 = \frac{1}{2} \int y dy \cdot \frac{1 + e^{-y}}{e^y - 1} - 1$$

ſi modo poſt integrationem ponatur $y = \infty$. Cuius
veritas hoc modo oſtendi potest: ſit $e^{-y} = z$ et
nunc integratione ita abſoluta, ut integrale euane-
ſcat poſito $z = 1$, ſtatui oportet $z = 0$, quae ſub-
ſtitutio praebet

$$\frac{1}{2} \int y dy \cdot \frac{1 + e^{-y}}{e^y - 1} = \frac{1}{2} \int dz \cdot \frac{1 + z}{1 - z} = \int dz/z \cdot (1 + z + z^2 + z^3 + \dots) + z^4 + \text{etc.}$$

Verum ob $\int z^{n-1} dz/z = \frac{z^n}{n} - \frac{z^n}{nn} + \frac{1}{nn}$ facta $z = 0$

fit $\int z^{n-1} dz/z = \frac{1}{n}$, hincque per ſeriem

$$\frac{1}{2} \int y dy \cdot \frac{1 + e^{-y}}{e^y - 1} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \text{etc.} = A\pi^2 - \frac{1}{2}$$

oportet.

12. Facillius idem oſtenditur ponendo $x - z = \psi$
ſeu $z = 1 - \psi$, ut iam integralia a termino $\psi = 0$
vsque ad terminum $\psi = 1$ extendi debeant; tum
autem noſtra ſumma ita exprimitur $-\frac{1}{2} \int \frac{1 - \psi}{1 - \psi} d\psi$

$1/(1-\psi)^{-1}$, quam aequalem esse oportet ipsi $A\pi\pi^{-\frac{1}{2}}$, ita ut sit

$$A\pi\pi = \frac{1}{2} \int_0^{\frac{d^2}{v}} 1/(1-\psi) + \frac{1}{2} \int d\psi/(1-\psi)$$

at $\int d\psi/(1-\psi) = -(1-\psi)/(1-\psi) + (1-\psi)^{-1} = -1$

sicque sit necesse est $A\pi\pi = -\int \frac{d^2}{v} 1/(1-\psi)$; quod

per se est manifestum. Cum enim sit

$$-1/(1-\psi) = \psi + \frac{1}{2}\psi^2 + \frac{1}{3}\psi^3 + \frac{1}{4}\psi^4 + \text{etc.}$$

erit integratione secundum legem praescriptam in-
finita:

$$-\int \frac{d^2}{v} 1/(1-\psi) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.} = A\pi^2.$$

13. Euoluamus etiam simili modo casum $a = \frac{1}{2}$ atque ostendi oportebit fore:

$$\frac{1}{2} \int e^{-y} y dy \cdot \frac{e^{\frac{y}{2}} + 1}{e^{\frac{y}{2}} - 1} - 2 = 4A\pi^2 - \frac{1}{2} - 4(1 + \frac{1}{2})$$

seu $\int e^{-y} y dy \cdot \frac{e^{\frac{y}{2}} + 1}{e^{\frac{y}{2}} - 1} = 8A\pi^2 + 1 - 8(1 + \frac{1}{2}) = 8A\pi^2 - 9.$

Ponamus $e^{-\frac{y}{2}} = 1 - \psi$, ut iam integrale a termino $\psi = 0$ usque ad $\psi = 1$ extendi debeat, et habebimus quod $e^{-\frac{y}{2}} = (1-\psi)^2$, $y = -2/(1-\psi)$, et $dy = \frac{2}{1-\psi} d\psi$. hanc aequalitatem demonstrandam:

$$-4 \int \frac{1-\psi + \psi^2 + \psi^3}{\psi} d\psi/(1-\psi) = 8A\pi^2 - 9$$

verum

verum uti iam observauimus est

$$\int d\psi/(1-\psi) = -1 \quad \text{et} \quad \int \psi d\psi/(1-\psi) = -\frac{1}{2}$$

unde conficitur

$$-8 \int \frac{d^2}{v} 1/(1-\psi) - 12 + 3 = 8A\pi^2 - 9 \quad \text{seu} \quad \int \frac{d^2}{v} 1/(1-\psi) = A\pi^2.$$

14. Simili modo si capiatur $a = \frac{1}{3}$ ostendi debet esse

$$\frac{1}{3} \int e^{-y} y dy \cdot \frac{e^{\frac{y}{3}} + 1}{e^{\frac{y}{3}} - 1} - 3 = 9A\pi^2 - \frac{1}{3} - 9(1 + \frac{1}{3} + \frac{1}{9})$$

seu $\int e^{-y} y dy \cdot \frac{e^{\frac{y}{3}} + 1}{e^{\frac{y}{3}} - 1} = 18A\pi^2 + 1 - 18(1 + \frac{1}{3} + \frac{1}{9}).$

Ponamus primo $e^{-\frac{y}{3}} = z$ ut sit $y = -3 \log z = -\frac{3 dx}{z}$ habebimusque:

$$9 \int z dz \log z \cdot \frac{1 + z^{\frac{1}{3}}}{1 - z^{\frac{1}{3}}} = 9 \int dz (-z z^{-2} z^{-2} + \frac{z^{-2}}{1 - z^{\frac{1}{3}}}) \log z$$

at est $\int z z dz \log z = +\frac{1}{2} \log z$, $\int z dz \log z = +\frac{1}{2} \log z$, $\int dz \log z = +1$

Unde nostra formula integralis euadit

$$18 \int \frac{d^2}{v} \log z - 18(1 + \frac{1}{3} + \frac{1}{9}) + 1$$

ita ut sit $\int \frac{d^2}{v} \log z = A\pi^2$ uti iam supra ostendimus atque hoc modo etiam sequentium casuum veritas euincitur.

15. Sin aystem sumamus $a = 1$, ut sum-
manda sit haec series:

1. 2 A - 1. 2. 3. 4 B + 1 ... 6 C - 1 ... 8 D + etc.

quoniam fit $x = \frac{1}{2}$, ex §. 10. summam assignare non licet, siquidem valor progressionis $S = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2x}$ quando terminorum numerus $= \frac{1}{2}$ non constat. Altera vero methodus eius summam praebet:

$$\int \frac{1}{2} e^{-xy} dy \cdot \frac{1 + e^{-2y}}{1 - e^{-2y}} - \frac{1}{2}$$

quae posito $e^{-y} = z$ in hanc formam transmutatur

$$\int \frac{dz}{z} \cdot \frac{1+z^2}{1-z^2} - \frac{1}{2} = \int \frac{dz}{z} \cdot \frac{1+z^2}{1-z^2} - \frac{1}{2}$$

et quia $\int dz/z = 1$, fiet ea

$$\int \frac{dz}{1-z^2} - 1 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \text{etc.} - 1 = \frac{1}{2} A \pi \pi - 1.$$

Quod si iam ponamus summo $x = \frac{1}{2}$ feri $S = A$ eadem summa reperitur $= \frac{1}{2} A \pi \pi - \frac{1}{2} A$ unde concludimus fore $\frac{1}{2} A \pi \pi - 1 = \frac{1}{2} A \pi \pi - \frac{1}{2} A$ ideoque quantitas illa incognita $\Delta = 4 - 2 A \pi \pi$ ex quo hanc progressionem interpolare licebit.

1	4 - 2 A π π
1 + 1/4	4 - 2 A π π + 1/4
1 + 1/4 + 1/9	4 - 2 A π π + 1/3 + 1/4
1 + 1/4 + 1/9 + 1/16	4 - 2 A π π + 1/3 + 1/4 + 1/16
1 + 1/4 + 1/9 + 1/16 + 1/25	4 - 2 A π π + 1/3 + 1/4 + 1/16 + 1/25
etc.	

et quoniam termini infinitesimi sunt aequales, fit $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$ etc. $= 4(1 + \frac{1}{4} + \frac{1}{9} + \dots)$ $= 2 A \pi \pi$ quae

quae aequitas per se est manifesta cum sit

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$$
 etc. $= A \pi \pi$ et $1 + \frac{1}{4} + \frac{1}{9} + \dots = \frac{1}{2} A \pi \pi$.

16. Consideremus rem in genere sitque $a = \frac{m}{n}$ vt summanda sit haec series infinita.

$$\frac{1 \cdot 2 \cdot 3 \dots m}{n} - \frac{1 \cdot 2 \cdot 3 \dots B m^2}{n^2} + \frac{1 \cdot 2 \cdot 3 \dots C m^3}{n^3} - \frac{1 \cdot 2 \cdot 3 \dots D m^4}{n^4} + \text{etc.}$$

ac posito $e^{-y} = z$, reperitur eius summa

$$\int \frac{1}{2} e^{-xy} dy \cdot \frac{1 + e^{-2y}}{1 - e^{-2y}} - \frac{1}{2} = \int \frac{dz}{z} \cdot \frac{1+z^2}{1-z^2} - \frac{1}{2} = \int \frac{dz}{z} \cdot \frac{1+z^2}{1-z^2} - \frac{1}{2}$$

quae reducitur ad hanc formam

$$\int \frac{z^{2n-1} dz}{1-z^{2n}} - \frac{1}{2} = \int z^{2n-1} dz/z - \frac{1}{2}$$

Cum autem sit $\int z^{2n-1} dz/z = \frac{1}{2n}$, per evolutionem primi membri nascimur hanc seriem illi aequalem

$$\frac{1}{2} - \frac{1}{2m} + \frac{1}{2n} + \frac{1}{2(m+n)^2} + \frac{1}{2(m+n)^2} + \frac{1}{2(m+n)^2} + \dots$$

Verum ex §. 10 ob $2x = \frac{n}{m}$ seu $x = \frac{n}{2m}$ posito

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{x^2}$$

eadem summa prodit $\frac{n}{m} A \pi^2 - \frac{n}{2m} + \frac{1}{2} - \frac{n}{4m} S$ qua cum praecedente comparata colligitur.

$$S = A \pi^2 - \frac{1}{2m} + \frac{1}{2} - \frac{n}{4m} S$$
 etc.

ex quo valorem ipsius S assignare poterimus, quicunque numerus factus pro x accipitur veluti si statuitur $x = \frac{1}{m}$ erit

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$$S = A\pi^2 - \frac{\mu\mu}{(\mu+\nu)^2} - \frac{\mu\mu}{(2\mu+\nu)^2} - \frac{\mu\mu}{(3\mu+\nu)^2} - \frac{\mu\mu}{(4\mu+\nu)^2} - \text{etc.}$$

quae series hoc modo immediate per x commodius exhibetur, vt sit

$$S = A\pi^2 - \frac{1}{(x+1)^2} - \frac{1}{(x+2)^2} - \frac{1}{(x+3)^2} - \frac{1}{(x+4)^2} - \text{etc.}$$

17. Quod hic per tantas ambages inuenimus, ita obuuium uideretur, vt statim immediate ex serie prima derivari possidet. Cum enim sit

$$A\pi^2 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{x^2} + \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \text{etc.}$$

hinc utique manifestum est fore

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{x^2} = A\pi^2 - \frac{1}{(x+1)^2} - \frac{1}{(x+2)^2} - \frac{1}{(x+3)^2} - \text{etc.}$$

Quia vero illa seriei summa $A\pi^2$ non est ueritati consentanea, nisi littera x denoter numeros integros quo quidem casu conclusio est, perspicua, eius certitudo pro casibus quibus x est numerus fractus vel adeo irrationalis, maxime adhuc dubia relinquitur, et cum nunc quidem pateat, eam inter ueritates esse referendam, hoc certe neutiquam ex isto breui ratiocinio perspicitur, ac nisi praecedentes rationes negotium conficissent, merito maximam haberemus dubitandi rationem. Nunc autem plena fiducia hoc ratiocinium multo lauius extendere licet ita ut si fuerit

$$S = 1 + \frac{1}{2^\pi} + \frac{1}{3^\pi} + \frac{1}{4^\pi} + \frac{1}{5^\pi} + \text{etc. in infinitum}$$

hinc

hinc tuto inferre queamus fore generatim

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \dots + \frac{1}{x^n} = S - \frac{1}{(x+1)^n} - \frac{1}{(x+2)^n} - \frac{1}{(x+3)^n} - \text{etc.}$$

etiam si x non fuerit numerus integer sed fractus vel adeo irrationalis quicunque.

Casus II. quo $n = 4$.

18. Posito primo indefinite

$$S = 1 + \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots + \frac{1}{x^4}$$

tum uero hac serie in infinitum continuata

$$O = 1 + \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{etc. in infinitum}$$

vt sit $O = B\pi^4$, habebimus hanc summationem

$$\frac{1^4 A}{2^4} + \frac{1^4 \cdot 5 \cdot 6 B}{2^5 x^7} - \frac{1^4 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 C}{2^7 x^9} + \frac{1^4 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 D}{2^9 x^{11}} - \text{etc.}$$

$$= S - \frac{1}{2^4} + \frac{1}{2^5 x^3} - B\pi^4$$

quae per $1 \cdot 2 \cdot 3 x^4$ multiplicata abit in hanc

$$\frac{1^4 \cdot 2 \cdot 3 \cdot 4 A}{2^4 x} - \frac{1^4 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 B}{2^5 x^3} + \frac{1^4 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 C}{2^7 x^5} - \frac{1^4 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 D}{2^9 x^7} + \text{etc.}$$

$$= 1 \cdot 2 \cdot 3 B\pi^4 x^4 + 3 - 2x - 6Sx^4$$

sequae quod x est numerus integer, istius seriei summa exhiberi potest. Per numeros ergo \mathfrak{A} , \mathfrak{B} , \mathfrak{C} etc. erit

$$\frac{4\mathfrak{A}}{2x} - \frac{6\mathfrak{B}}{2x^3} + \frac{1\mathfrak{C}}{2x^5} - \frac{10\mathfrak{D}}{2x^7} + \text{etc. } 1 \cdot 6 B\pi^4 x^4 + 3 - 2x - 6Sx^4$$

19. Eiusdem autem seriei summam ex forma supra inuenta definire possumus, quae erat:

T 2

A ay

$$Aay - Ba^2y^2 + Ca^3y^3 - Da^4y^4 + \text{etc.} = \frac{e^{2ay} + 1}{e^{2ay} - 1} - \frac{1}{2ay}$$

haec enim per $e^{-y}y^2 dy$ multiplicata et integratione a termino $y = 0$ vsque ad $y = \infty$ extensa praebet

$$1.2..4Aa - 1.2..6Ba^2 + 1.2..8Ca^3 - 1.2..10Da^4 + \text{etc.}$$

$$= \frac{1}{2} \int e^{-y} y^2 dy \cdot \frac{1 + e^{-2ay}}{1 - e^{-2ay}} - \frac{1}{2}$$

Hanc vero formulam integram sine substitutione hoc modo euoluere licet: cum sit

$$\frac{1 + e^{-2ay}}{1 - e^{-2ay}} = 1 + 2e^{-2ay} + 2e^{-4ay} + 2e^{-6ay} + 2e^{-8ay} + \text{etc.}$$

multiplicetur per $\frac{1}{2} e^{-y} y^2 dy$, et quoniam in genere est

$$\int e^{-my} y^2 dy = -e^{-my} \left(\frac{y^2}{m} + \frac{2y}{m^2} + \frac{2y^2}{m^3} + \frac{2y^3}{m^4} + \frac{2y^4}{m^5} + \text{etc.} \right)$$

summa illa transformatur in hanc seriem infinitam

$$-\frac{1}{2} + 3 + \frac{6}{(2a+1)^2} + \frac{6}{(4a+1)^2} + \frac{6}{(6a+1)^2} + \frac{6}{(8a+1)^2} + \text{etc.}$$

Hinc posito $a = \frac{1}{2x}$ vt prodeat prior series, erit etiam

$$-2x + 3 + \frac{6x^2}{(x+1)^2} + \frac{6x^2}{(x+3)^2} + \frac{6x^2}{(x+5)^2} + \frac{6x^2}{(x+7)^2} + \text{etc.}$$

$= 6B\pi^2 x^2 + 3 - 2x - 6Sx^2$

$$\text{ideoque per } 6x^2 \text{ dividendo}$$

$$B\pi^2 - S = \frac{1}{(x+1)^2} + \frac{1}{(x+3)^2} + \frac{1}{(x+5)^2} + \frac{1}{(x+7)^2} + \text{etc.}$$

prorsus vt supra iam animaduertimus.

Casus

Casus III. quo n est numerus quicumque.

20. Primum hic obseruo, si ponatur series infinita

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \text{etc.} = O$$

hanc seriem ex evolutione huius formulae integralis

$$\text{origi } \int \frac{dz (1z)^{n-1}}{1-z}, \text{ si integratio a termino } z = 0$$

vsque ad terminum $z = 1$ extendatur. Cum enim huc lege obseruata sit

$$\int z^{m-1} dz (1z) = \frac{1}{m} z^m 1z - \frac{1}{m} z^m = -\frac{1}{m}$$

$$\int z^{m-1} dz (1z)^2 = \frac{1}{m} z^m (1z)^2 - \frac{2}{m^2} z^m 1z + \frac{2}{m^3} z^m = +\frac{1}{m^2}$$

$$\int z^{m-1} dz (1z)^3 = -\frac{1}{m^3}$$

$$\int z^{m-1} dz (1z)^4 = +\frac{1}{m^4}$$

etc.

erit in genere $\pm \int z^{m-1} dz (1z)^{n-1} = + \frac{1.2.3 \dots (n-1)}{m^n}$

vbi signum superius + valet si n sit numerus impar inferius vero -, si n sit numerus par. Quam ob rem euolutio formulae $\pm \int \frac{dz}{1-z} (1z)^{n-1}$ praebet hanc seriem sub eadem lege ambiguitatis:

$$1.2.3 \dots (n-1) \left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \text{etc.} \right)$$

ita vt sit $O = \frac{1}{1.2.3 \dots (n-1)} \int \frac{dz}{1-z} (1z)^{n-1}$

21. Simili modo haec series ad datum quemvis terminum indefinite summari poterit, si cum ponatur:

$$S = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \dots + \frac{1}{m^n}$$

$$\text{erit } S = \frac{1}{1 \cdot 2 \cdot 3 \dots (n-1)} \int \frac{1-x^m}{1-x} dz (1z)^{n-1}$$

quae formula veritati est contentanea siue m sit numerus integer siue fractus: vnde cum sit $\frac{1}{1 \cdot 2 \cdot 3 \dots (n-1)}$:

$$\int \frac{x^m dz}{1-x} (1z)^{n-1} = \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \text{etc.}$$

perficuum est fore:

$$S = 0 - \frac{1}{(m+1)^n} - \frac{1}{(m+2)^n} - \frac{1}{(m+3)^n} - \frac{1}{(m+4)^n} - \text{etc.}$$

Quocirca cum sumto $m = \frac{1}{2}$ sit

$$\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \text{etc.} = \frac{2^n}{1} + \frac{2^n}{3^n} + \frac{2^n}{5^n} + \frac{2^n}{7^n} + \text{etc.}$$

$= (2^n - 1) O$, hinc elicimus:

$$S = 0 - (2^n - 1) O + 2^n = 2^n - (2^n - 2) O$$

vnde valores interpolati ipsius S ita se habebunt

¶ sit

si sit	erit S
$m = 0$	0
$m = \frac{1}{2}$	$2^n - (2^n - 2) O$
$m = 1$	1
$m = 1\frac{1}{2}$	$2^n + \frac{2^n}{3^n} - (2^n - 2) O$
$m = 2$	$1 + \frac{1}{2^n}$
$m = 2\frac{1}{2}$	$2^n + \frac{2^n}{3^n} + \frac{2^n}{5^n} - (2^n - 2) O$
$m = 3$	$1 + \frac{1}{2^n} + \frac{1}{3^n}$
$m = 3\frac{1}{2}$	$2^n + \frac{2^n}{3^n} + \frac{2^n}{5^n} + \frac{2^n}{7^n} - (2^n - 2) O$
	etc.

si ad singulos terminos addatur $(2^n - 2) O$, ique- rum per 2^n dividantur, habebitur interpolatio huius seriei

$$b, 1, 1 + \frac{1}{3^n}, 1 + \frac{1}{3^n} + \frac{1}{5^n}, 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n}; \text{ etc.}$$

22. Cum summa seriei infinitae O per quadraturam circuli seu litteram π sit assignabilis, quoties exponens n fuerit numerus par, notari merentur sequentium formularum integralium reductiones ad circuli quadraturam:

$$\int \frac{dz}{1-z^2}$$

$$\begin{aligned} \int_{1-\frac{d}{z}}^{\frac{d}{z}} lz &= -1. A \pi^1 &= -\frac{2^1}{2 \cdot 1} \pi^1 \\ \int_{1-\frac{d}{z}}^{\frac{d}{z}} (lz)^2 &= -1. 2. 3 B \pi^4 &= -\frac{2^2 \cdot 3}{4 \cdot 3} \pi^4 \\ \int_{1-\frac{d}{z}}^{\frac{d}{z}} (lz)^3 &= -1. 2. 3. 4. 5 C \pi^6 &= -\frac{2^3 \cdot 4 \cdot 5}{6 \cdot 7} \pi^6 \\ \int_{1-\frac{d}{z}}^{\frac{d}{z}} (lz)^4 &= -1. 2. 3. 4. 5. 6. 7 D \pi^8 &= -\frac{2^4 \cdot 5 \cdot 6 \cdot 7}{8 \cdot 9} \pi^8 \\ \int_{1-\frac{d}{z}}^{\frac{d}{z}} (lz)^5 &= -1. 2. \dots 9 E \pi^{10} &= -\frac{2^5}{10 \cdot 11} \pi^{10} \\ && \text{etc.} \end{aligned}$$

Atque hinc eo magis est mirandum, quod nullam naturam formularum $\int_{1-\frac{d}{z}}^{\frac{d}{z}} (lz)^2$, $\int_{1-\frac{d}{z}}^{\frac{d}{z}} (lz)^4$, $\int_{1-\frac{d}{z}}^{\frac{d}{z}} (lz)^6$ etc. nullo modo ad quampiam quadraturam cognitam reducere liceat, cum tamen ex hoc ordine prima formula $\int_{1-\frac{d}{z}}^{\frac{d}{z}} (lz)^0$ manifesto per logarithmos absoluitur.

23. Scribamus nunc in §. 9. m loco x , et aequationem ibi datam per $-1. 2. \dots (n-1)m^2$ multiplicemus, vt obtineamus hanc summationem:

$$\frac{1.2 \dots 1A}{2 m^2} - \frac{1.2 \dots 1A + 2B}{2^3 m^2} + \frac{1.2 \dots 1A + 4C}{2^5 m^2} - \frac{1.2 \dots 1A + 6D}{2^7 m^2} + \text{etc.}$$

$$= 1. 2. \dots (n-1)(Om^n - Sm^n + \frac{1}{2} - \frac{m}{n-1})$$

simili autem modo quo supra sumus vñ (§. 9), ponendo $a = \frac{1}{2m}$ eiusdem ferèi summam per sequentem formulam integram expressam invenimus:

$$\frac{1}{2} \int e^{-y} y^{n-1} dy \cdot \frac{1 + e^{-y:m}}{1 - e^{-y:m}} = 1. 2. \dots (n-2) m$$

ita

ita vt fit

$$\frac{1}{2} \int e^{-y} y^{n-1} dy \cdot \frac{1 + e^{-y:m}}{1 - e^{-y:m}} = 1. 2. \dots (n-1)(Om^n - Sm^n + 2)$$

atque ob $\frac{1 + e^{-y:m}}{1 - e^{-y:m}} = 1 + \frac{2}{1 - e^{-y:m}}$ erit

$$\int \frac{e^{-y} y^{n-1} dy}{1 - e^{-y:m}} = 1. 2. \dots (n-1)m^2(O-S)$$

quae formula integralis ponendo $e^{-y:m} = z$ ad eam quam modo tractauimus, reducitur scilicet $\int_{1-\frac{d}{z}}^{\frac{d}{z}} (lz)^{n-1}$, siquidem eius integrale a termino $z = 1$ vsque ad terminum $z = 0$ extendatur.

Casus IV. quo $n = 1$.

24. Hic casus peculiarem tractationem postulat, quia ferèi $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ etc. summa est infinita sit ergo indefinite

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{x}$$

vt quia ob $X = \frac{1}{x}$ est $\int X dx = lx$, habebimus hanc summationem

$$\frac{-1A}{2 x^2} + \frac{1.2 \cdot 3B}{2^3 x^2} - \frac{1.2 \dots 5C}{2^5 x^2} + \frac{1.2 \dots 7D}{2^7 x^2} - \text{etc.} = S - lx - \frac{1}{2x} - O$$

feu per $-x$ multiplicando:

$$\frac{1A}{2x} - \frac{1.2 \cdot 3B}{2^3 x^2} + \frac{1.2 \dots 5C}{2^5 x^2} - \frac{1.2 \dots 7D}{2^7 x^2} + \text{etc.} = (O-S)x + \frac{1}{2} + lx$$

vbi constantem O ex casu per se cognito definiti oportet. Veluti si sumatur $x = 1$, ob $S = 1$ et $lx = 0$, erit

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$$O - \frac{1}{2} = \frac{1A}{2} - \frac{1.2.3B}{2^3} + \frac{1.2.3.4.5C}{2^5} - \frac{1.2.3.4.5.6.7D}{2^7} + \text{etc.}$$

Quo autem iste valor ipsius O facilius obtineatur, ponatur $x = 10$, et cum fiat:

$$(O - 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \frac{1}{32} - \frac{1}{64} - \frac{1}{128} - \frac{1}{256} - \frac{1}{512} - \frac{1}{1024} - \frac{1}{2048} - \frac{1}{4096} - \frac{1}{8192} - \frac{1}{16384} - \frac{1}{32768} - \frac{1}{65536} - \frac{1}{131072} - \frac{1}{262144} - \frac{1}{524288} - \frac{1}{1048576} - \frac{1}{2097152} - \frac{1}{4194304} - \frac{1}{8388608} - \frac{1}{16777216} - \frac{1}{33554432} - \frac{1}{67108864} - \frac{1}{134217728} - \frac{1}{268435456} - \frac{1}{536870912} - \frac{1}{1073741824} - \frac{1}{2147483648} - \frac{1}{4294967296} - \frac{1}{8589934592} - \frac{1}{17179869184} - \frac{1}{34359738368} - \frac{1}{68719476736} - \frac{1}{137438953472} - \frac{1}{274877906944} - \frac{1}{549755813888} - \frac{1}{1099511627776} - \frac{1}{2199023255552} - \frac{1}{4398046511104} - \frac{1}{8796093022208} - \frac{1}{17592186044416} - \frac{1}{35184372088832} - \frac{1}{70368744177664} - \frac{1}{140737488355328} - \frac{1}{281474976710656} - \frac{1}{562949953421312} - \frac{1}{1125899906842624} - \frac{1}{2251799813685248} - \frac{1}{4503599627370496} - \frac{1}{9007199254740992} - \frac{1}{18014398509481984} - \frac{1}{36028797018963968} - \frac{1}{72057594037927936} - \frac{1}{144115188075855872} - \frac{1}{288230376151711744} - \frac{1}{576460752303423488} - \frac{1}{1152921504606846976} - \frac{1}{2305843009213693952} - \frac{1}{4611686018427387904} - \frac{1}{9223372036854775808} - \frac{1}{18446744073709551616} - \frac{1}{36893488147419103232} - \frac{1}{73786976294838206464} - \frac{1}{147573952589676412928} - \frac{1}{295147905179352825856} - \frac{1}{590295810358705651712} - \frac{1}{1180591620717411303424} - \frac{1}{2361183241434822606848} - \frac{1}{4722366482869645213696} - \frac{1}{9444732965739290427392} - \frac{1}{18889465931478580854784} - \frac{1}{37778931862957161709568} - \frac{1}{75557863725914323419136} - \frac{1}{151115727451828646838272} - \frac{1}{302231454903657293676544} - \frac{1}{604462909807314587353088} - \frac{1}{1208925819614629174706176} - \frac{1}{2417851639229258349412352} - \frac{1}{4835703278458516698824704} - \frac{1}{9671406556917033397649408} - \frac{1}{19342813113834066795298816} - \frac{1}{38685626227668133590597632} - \frac{1}{77371252455336267181195264} - \frac{1}{154742504910672534362390528} - \frac{1}{309485009821345068724781056} - \frac{1}{618970019642690137449562112} - \frac{1}{1237940039285380274899124224} - \frac{1}{2475880078570760549798248448} - \frac{1}{4951760157141521099596496896} - \frac{1}{9903520314283042199192993792} - \frac{1}{19807040628566084398385987584} - \frac{1}{39614081257132168796771975168} - \frac{1}{79228162514264337593543950336} - \frac{1}{158456325028528675187087900672} - \frac{1}{316912650057057350374175801344} - \frac{1}{633825300114114700748351602688} - \frac{1}{1267650600228229401496703205376} - \frac{1}{2535301200456458802993406410752} - \frac{1}{5070602400912917605986812821504} - \frac{1}{10141204801825835211973625643008} - \frac{1}{20282409603651670423947251286016} - \frac{1}{40564819207303340847894502572032} - \frac{1}{81129638414606681695789005144064} - \frac{1}{162259276829213363391578010288128} - \frac{1}{324518553658426726783156020576256} - \frac{1}{649037107316853453566312041152512} - \frac{1}{1298074214633706907132624082305024} - \frac{1}{2596148429267413814265248164610048} - \frac{1}{5192296858534827628530496329220096} - \frac{1}{10384593717069655257060992658440192} - \frac{1}{20769187434139310514121985316880384} - \frac{1}{41538374868278621028243970633760768} - \frac{1}{83076749736557242056487941267521536} - \frac{1}{166153499473114484112975882535043072} - \frac{1}{332306998946228968225951765070086144} - \frac{1}{664613997892457936451903530140172288} - \frac{1}{1329227995784915872903807060280344576} - \frac{1}{2658455991569831745807614120560689152} - \frac{1}{5316911983139663491615228241121378304} - \frac{1}{10633823966279326983230456482242756608} - \frac{1}{21267647932558653966460912964485513216} - \frac{1}{42535295865117307932921825928971026432} - 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At maxima difficultas hic in eo consistit, quod utraque pars seorsim evoluta praebet numerum infinitae magnitudinis quae autem duo infinita necessario se mutuo ita tollere debent ut pro O obtineatur valor ille finitus supra assignatus.

Retenta autem priori forma integralia more solito a termino $z = 0$ ad $z = 1$ extendamus, ut sit $O = \int_0^1 \frac{dz}{1-z^2} + \int_1^z \frac{dz}{1-z^2}$, et cum denotante i numerum infinitum sit $1/z = i(z^{i-1} - 1)$, erit

$$O = \int_{i-2z}^{\frac{dz}{1-z^2}} - i \int_{1-z^{i-1}}^{\frac{dz}{1-z^2}}$$

Iam ponamus $z = u^i$, ut fiat

$$O = i \int \frac{u^{i-1} du}{1-u^2} - \int \frac{u^{i-1} du}{1-u}$$

quarum formularum evolutio praebet:

$$O = u^{\frac{1}{2}} + \frac{1}{2} u^{\frac{3}{2}} + \frac{1}{2} u^{\frac{5}{2}} + \dots + \frac{1}{2} u^{\frac{2i-1}{2}}$$

$$- \frac{1}{i} u^{-\frac{1}{i}} + \frac{1}{2i} u^{-\frac{2}{i}} + \dots - \frac{1}{2i} u^{-\frac{1}{i}} + \frac{1}{2i} u^{-\frac{2}{i}} + \dots$$

et ponendo vii oportet $u = 1$, fit

$$O = 1 - \left(\frac{1}{2} + \frac{1}{2i} + \frac{1}{2i^2} + \dots + \frac{1}{2i^{i-1}}\right) + \frac{1}{2} - \left(\frac{1}{2} + \frac{1}{2i} + \frac{1}{2i^2} + \dots + \frac{1}{2i^{i-1}}\right) + \frac{1}{2} - \left(\frac{1}{2} + \frac{1}{2i} + \frac{1}{2i^2} + \dots + \frac{1}{2i^{i-1}}\right) + \dots$$

Vbi notandum est harum progressionum harmonicarum primam $\frac{1}{2} + \frac{1}{2i} + \frac{1}{2i^2} + \dots + \frac{1}{2i^{i-1}}$ ob i numerum infinitum exprimere $1/2$ secundam $1/2$, tertiam

tertiam $1/2$, etc. ita ut habeatur per seriem fatis simplicem et regularem:

$$O = 1 - 1/2 + 1/2 - 1/2 + 1/2 - 1/2 + 1/2 - 1/2 + \dots$$

27. Eandem hanc seriem ex prima statim forma derivare licuisset, si enim ibi (24) ponatur $x = \infty$ fit $O = S - 1/x - O$ ita ut fit

$$O = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2} - 1/x$$

quia vero tam series $1 + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2}$ quam $1/x$ habet valorem infinitum, quo facilis posterior a priori auferri queat conveniet $1/x$ in totidem partes dividere, quod prior series habet terminos, quod manifesto fit hoc modo

$$1/x = 1/2 + 1/2 + 1/2 + \dots + 1/x$$

unde series inuenta conficitur. Haec series nunc pluribus modis in alias formas transmutari potest, ex quibus valorem numeri O facile quam proxime saltem colligere licebit.

Primo enim cum fit $\frac{1}{n} - 1/n^{2i} = \frac{1}{2n^2} - \frac{1}{6n^3} + \frac{1}{4n^4} - \frac{1}{5n^5} + \dots$ habebimus

$$O = \frac{1}{2} \left(1 + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots \right) - \frac{1}{6} \left(1 + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \dots \right) + \frac{1}{4} \left(1 + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \dots \right) - \frac{1}{5} \left(1 + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \dots \right) + \dots$$

Deinde ob $\frac{1}{n-1} - 1/n = \frac{1}{2n^2} + \frac{1}{3n^3} + \frac{1}{4n^4} + \frac{1}{5n^5} + \dots$ erit

ita ut hinc nihil novi eliciatur, cum adiecta parte priori $\frac{1}{2} - \frac{1}{2}a$ oriatur ut per se constat

$$O = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{i} - \frac{1}{i}$$

Manet ergo quaestio magni momenti, cuiusnam indolis sit numerus iste O , et ad quodnam genus quantitarum sit referendus.

Si terminus generalis $X = 1/x$.

30. Hic ergo erit $\int X dx = x/x - x$ et ob $\frac{dx}{dx} = \frac{1}{x}$ fiet porro $\frac{d^2x}{dx^2} = -\frac{1}{x^2}$; $\frac{d^3x}{dx^3} = \frac{1}{x^3}$; $\frac{d^4x}{dx^4} = -\frac{1}{x^4}$; $\frac{d^5x}{dx^5} = \frac{1}{x^5}$; etc. Vnde si ponamus indefinite:

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{x}$$

habebimus hanc summationem

$$\frac{A}{2^2} - \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 \cdot 16 \cdot 17 \cdot 18 \cdot 19 \cdot 20 \cdot 21 \cdot 22 \cdot 23 \cdot 24 \cdot 25 \cdot 26 \cdot 27 \cdot 28 \cdot 29 \cdot 30}{2^2 \cdot 3^2 \cdot 4^2 \cdot 5^2 \cdot 6^2 \cdot 7^2 \cdot 8^2 \cdot 9^2 \cdot 10^2 \cdot 11^2 \cdot 12^2 \cdot 13^2 \cdot 14^2 \cdot 15^2 \cdot 16^2 \cdot 17^2 \cdot 18^2 \cdot 19^2 \cdot 20^2 \cdot 21^2 \cdot 22^2 \cdot 23^2 \cdot 24^2 \cdot 25^2 \cdot 26^2 \cdot 27^2 \cdot 28^2 \cdot 29^2 \cdot 30^2} + \text{etc.} = S - \frac{1}{2} / x - x/x + x - O$$

quae constans ita esse debet comparata, ut vni ipsius x valori satisfaciatur. Sit ergo $x = 1$ et cum sit $S = 1 = 0$ erit

$$-O + 1 = \frac{A}{2} - \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 \cdot 16 \cdot 17 \cdot 18 \cdot 19 \cdot 20 \cdot 21 \cdot 22 \cdot 23 \cdot 24 \cdot 25 \cdot 26 \cdot 27 \cdot 28 \cdot 29 \cdot 30}{2^2 \cdot 3^2 \cdot 4^2 \cdot 5^2 \cdot 6^2 \cdot 7^2 \cdot 8^2 \cdot 9^2 \cdot 10^2 \cdot 11^2 \cdot 12^2 \cdot 13^2 \cdot 14^2 \cdot 15^2 \cdot 16^2 \cdot 17^2 \cdot 18^2 \cdot 19^2 \cdot 20^2 \cdot 21^2 \cdot 22^2 \cdot 23^2 \cdot 24^2 \cdot 25^2 \cdot 26^2 \cdot 27^2 \cdot 28^2 \cdot 29^2 \cdot 30^2} + \text{etc.}$$

Cum igitur sit

$$A 4y - B a^2 y^3 + C a^4 y^5 - D a^6 y^7 + \text{etc.} = \frac{1 + e^{-2ay}}{1 - e^{-2ay}} - \frac{1}{2} \frac{dy}{dy}$$

ob $f e^{-2y} y^n dy = 1 \cdot 2 \dots n$ multiplicemus per $e^{-2y} \frac{dy}{y}$ et integratio suppedietur

$$A a - 1 \cdot 2 B a^2 + 1 \cdot 2 \cdot 3 \cdot 4 C a^4 - \text{etc.} = \frac{1}{2} \int \frac{1 + e^{-2ay}}{1 - e^{-2ay}} \frac{dy}{y} - \frac{1}{2} \int \frac{e^{-2y} dy}{y}$$

$$= \int \frac{e^{-y} dy}{y(1 - e^{-2ay})} - \frac{1}{2} \int \frac{e^{-y} dy}{y} - \frac{1}{2a} \int \frac{e^{-y}}{y} \frac{dy}{y}$$

inte-

integralibus his a termino $y = 0$ vsque ad $y = \infty$ extentis.

31. Statuamus nunc $a = \frac{1}{2x}$, et obtinebimus hanc aequationem:

$$-O + S + x - \frac{1}{2}x - x/x = \int \frac{e^{-y} dy}{y(1 - e^{-y/x})} - \frac{1}{2} \int \frac{e^{-y} dy}{y} - x \int \frac{e^{-y}}{y^2}$$

in quibus integrationibus quantitatem x ut constantem spectari oportet. Quare summo $x = 1$ fiet

$$-O + 1 = \int \frac{e^{-y} dy}{y(1 - e^{-y})} - \frac{1}{2} \int \frac{e^{-y} dy}{y} - \int \frac{e^{-y} dy}{y^2}$$

et quoniam est $-\int \frac{e^{-y} dy}{y^2} = \frac{e^{-y}}{y} + \int \frac{e^{-y} dy}{y}$ erit

$$-O + 1 = \int \frac{e^{-y} dy}{y(1 - e^{-y})} + \frac{1}{2} \int \frac{e^{-y} dy}{y} + \frac{e^{-y}}{y} - \frac{e^{-y}}{y}$$

Hic si ponatur $e^{-y} = z$ et integralia a termino $z = 0$ vsque ad $z = 1$ extendantur, reperitur:

$$-O + 1 = \frac{1}{2} \int \frac{z^{1/2}}{1-z} - \int \frac{dz}{(1-z)^2} - \int \frac{dz}{(1-z)^2}$$

neque vero hinc natura huius numeri O cognosci potest cum tamen aliunde constat eum esse $= \frac{1}{2} \pi$, sicque partim per logarithmos partim per circuli peripheriam π determinari. Quemadmodum ergo iste valor eruatur operae pretium erit accuratius perpendisse.

32. Quoniam a Wallisso inventa est haec aequalitas

$$\frac{\pi}{2} = \frac{1 \cdot 2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11} \text{ etc.}$$

Tom. XIV. Nou. Comm.

X

erit

erit logarithimis fuerendis

$$\frac{1}{2}\pi = 1/2 - 1/3 + 1/4 - 1/5 + 1/6 - 1/7 + 1/8 - 1/9 + \text{etc.}$$

feu hoc modo per duplicem seriem

$$\frac{1}{2}\pi = 1/2 + 1/4 + 1/6 + 1/8 + 1/10 + 1/12 + \text{etc.} \dots + 1/2x + \frac{1}{2}(x+1)$$

$$- 1/3 - 1/5 - 1/7 - 1/9 - 1/11 - 1/13 - \text{etc.} - 1/(2x+1)$$

siquidem utraque series in infinitum quidem sed tamen parem terminorum numerum continetur, seu ipsa x utriusque idem valor tribuitur: quae duplex series etiam hoc modo exhiberi potest

$$\frac{1}{2}\pi = 1/2 + 1/4 + 1/6 + 1/8 + \dots + 1/2x - \frac{1}{2}x - 1/2x - 1/3 - 1/5 - 1/7 - \dots - 1/(2x-1)$$

At ex ipsa nostra serie summo x infinito habemus

$$1x + 1/2 + 1/3 + 1/4 + \dots + 1/x = 0 - x^2 + (x + \frac{1}{2})/x$$

vnde si x 1/2 seu ad quemlibet terminum 1/2, ad datur fit

$$1x + 1/4 + 1/6 + 1/8 + \dots + 1/2x = 0 - x + x/2 + (x + \frac{1}{2})/x$$

Deinde si ibi loco x scribamus 2x prodit

$$1x + 1/2 + 1/3 + 1/4 + \dots + 1/2x = 0 - 2x + (2x + \frac{1}{2})/2 + (2x + \frac{1}{2})/x$$

2 qua si illa auferatur relinquitur:

$$1x + 1/3 + 1/5 + \dots + 1/(2x-1) = -x + (x + \frac{1}{2})/2 + x/x$$

quae summae si in illa forma loco utriusque seriei substituantur orietur haec aequatio:

$$\frac{1}{2}\pi + \frac{1}{2}x = 0 - x + x/2 + (x + \frac{1}{2})/x \quad \left\{ \begin{array}{l} = 0 - \frac{1}{2}x + \frac{1}{2}x \\ + x - (x + \frac{1}{2})/2 - x/x \end{array} \right.$$

vnde

vnde concluditur $0 = \frac{1}{2}\pi + \frac{1}{2}x + \frac{1}{2}x + \frac{1}{2}x - \frac{1}{2}x = \frac{1}{2}x$ seu $0 = 0, 9189385332046727417803297$.

33. Cum ergo sit $0 = \frac{1}{2}x$ hinc vicissim colligimus fore

$$\int \frac{dx}{1-x} = -\int \frac{dx}{(1-x)^2} = 1 - \frac{1}{2}x$$

siquae patet has tres integrationes, siquidem a termino $x = 0$ ad terminum $x = 1$ extendantur perducunt ad quantitatem $1/2\pi$ quod quomodo per calculum ostendi possit, haud liquet, vnde haec investigatio eo maiori attentione digna videtur. Facile quidem perspicitur esse

$$-\int \frac{dx}{(1-x)^2} = \frac{x}{1-x} - \int \frac{dx}{1-x} = \frac{x}{1-x} - \ln(1-x) = 1 - \frac{1}{2}x$$

Parum quoque lucramur ponendo $x = v^i$ et $1-x = v^j$ existente i numero infinito consequimur autem hanc aequationem.

$$\frac{-v^i}{i(1-v)} + \int \frac{v^{i-1}dv}{1-v} + \int \frac{v^{i-1}dv}{(1-v)(1-v^i)} = 1 - \frac{1}{2}x$$

quae integralia pariter ab $v = 0$ vsque ad $v = 1$ extendi debent.

34. Evolutio harum formularum nihil aliud suppeditat nisi quod statim ex prima aequatione sumendo numerum x infinitum concludi potest, quia enim tum series litteras A. B. C. D etc. complectens evanescit, habebimus

$$0 = \frac{1}{2}x = 1x + 1/2 + 1/3 + \dots + 1/x - (x + \frac{1}{2})/x + x$$

X 2

vbi

vbi cum series $1x + 1/2 \dots + 1/x$ contet x terminis quaelibet reliquarum partium $(x + \frac{1}{2})/x$ et x in feriem totidem terminorum conueratur. Ac posterior quidem x totidem terminos unitati aequalis praebet, prior vero $(x + \frac{1}{2})/x$ sequenti modo euoluitur:

$$\frac{1}{2} \pi = 1 + 1/2 + 1/3 \dots + 1/(x-1) + 1/x$$

$$+ 1 + 1 + 1 \dots + 1 \quad + 1$$

$$\frac{1}{2} / 1 - \frac{1}{2} / 2 - \frac{1}{3} \dots - (x - \frac{1}{2}) / (x - 1) - (x + \frac{1}{2}) / x$$

$$+ \frac{1}{2} / 1 + \frac{1}{2} / 2 \dots + (x - \frac{1}{2}) / (x - 2) + (x - \frac{1}{2}) / (x - 1)$$

vnde colligitur haec series satis concinna:

$$\frac{1}{2} \pi = 1 - (\frac{1}{2} / \frac{1}{2} - 1) - (\frac{1}{2} / \frac{1}{2} - 1) - (\frac{1}{2} / \frac{1}{2} - 1) - (\frac{1}{2} / \frac{1}{2} - 1) \text{ etc.}$$

quae commodius hac forma exhibetur:

$$1 - \frac{1}{2} \pi = \frac{1}{2} / \frac{1}{2} - 1 + \frac{1}{2} / \frac{1}{2} - 1 + \frac{1}{2} / \frac{1}{2} - 1 + \frac{1}{2} / \frac{1}{2} - 1 \text{ etc.}$$

Terminus generalis huius ferier est $\frac{x}{x} + \frac{1}{x} - 1$, qui in hanc feriem euoluitur: $\frac{1}{3x^2} + \frac{1}{5x^4} + \frac{1}{7x^6} + \frac{1}{9x^8} + \dots$ etc. ex quo per infinitas series habebimus:

$$1 - \frac{1}{2} \pi = \frac{1}{3 \cdot 3^2} + \frac{1}{5 \cdot 5^2} + \frac{1}{7 \cdot 7^2} + \frac{1}{9 \cdot 9^2} + \dots \text{ etc.}$$

$$\frac{1}{2 \cdot 5^2} + \frac{1}{5 \cdot 5^4} + \frac{1}{7 \cdot 7^6} + \frac{1}{9 \cdot 9^8} + \dots \text{ etc.}$$

$$\frac{1}{3 \cdot 7^2} + \frac{1}{5 \cdot 7^4} + \frac{1}{7 \cdot 7^6} + \frac{1}{9 \cdot 7^8} + \dots \text{ etc.}$$

$$\frac{1}{4 \cdot 9^2} + \frac{1}{6 \cdot 9^4} + \frac{1}{8 \cdot 9^6} + \frac{1}{10 \cdot 9^8} + \dots \text{ etc.}$$

etc.

vb

vbi cum series potestatum reciprocarum primo termino truncatarum occurrant, erit

$$1 - \frac{1}{2} \pi = \frac{1}{2} (\frac{1}{2}^2 - 1) A \pi^{-1} + \frac{1}{2} (\frac{1}{2}^4 - 1) B \pi^{-1} + \frac{1}{2} (\frac{1}{2}^6 - 1) C \pi^{-1} + \dots$$

supra autem inuenimus:

$$1 - \frac{1}{2} \pi = \frac{1}{2} A - \frac{1}{2} B + \frac{1}{2} C - \frac{1}{2} D + \dots \text{ etc.}$$

35. Huiusmodi relationes eo maiorem attentionem merentur, quo magis sunt absconditae, vnde operae erit pretium feriem modo inuentam accuratius euoluere. Hunc in finem eam generatiori forma complectar, statuens:

$$P = \frac{1}{2} A \pi^{-2} u^2 + \frac{1}{2} B \pi^{-4} u^4 + \frac{1}{2} C \pi^{-6} u^6 + \frac{1}{2} D \pi^{-8} u^8 + \dots$$

$$Q = \frac{1}{2} A \frac{\pi^{-2} u^2}{2} + \frac{1}{2} B \frac{\pi^{-4} u^4}{2} + \frac{1}{2} C \frac{\pi^{-6} u^6}{2} + \frac{1}{2} D \frac{\pi^{-8} u^8}{2} + \dots$$

$$R = \frac{1}{2} u^2 + \frac{1}{2} u^4 + \frac{1}{2} u^6 + \frac{1}{2} u^8 + \dots = \frac{1}{2} / \frac{1}{2} \pm \frac{1}{2} - 1$$

vt posito $u = 1$ fit $1 - \frac{1}{2} \pi = P - Q - R$

Iam ad valores litterarum P et Q definiendos sumo aequationem supra §. 6 datam

$$A x^2 + B x^4 + C x^6 + D x^8 + \dots = \frac{1}{2} - \frac{1}{2} x \cot x$$

vnde per integrationem fit

$$\frac{1}{2} A x^3 + \frac{1}{2} B x^5 + \frac{1}{2} C x^7 + \dots = \frac{1}{2} x + \frac{1}{2} \int \frac{x dx \cot x}{\int \pi x}$$

$$\text{feu } \frac{1}{2} A x^3 + \frac{1}{2} B x^5 + \frac{1}{2} C x^7 + \dots = \frac{1}{2} - \frac{1}{2} \int \frac{x dx \cot x}{\int \pi x}$$

Hinc posito primo $x = \pi u$ tum vero $x = \frac{\pi}{2}$ deducimus

$$P = \frac{1}{2} - \frac{1}{2} \int \frac{\pi u du \cot \pi u}{\int \pi u} = \frac{1}{2} - \frac{1}{2} \int \frac{u du \cot \pi u}{\int \pi u} \text{ Ob } u = 1$$

$$X = 3 \quad Q = \frac{1}{2}$$

$$Q = 1 - \frac{1}{\pi u} \int \frac{\pi u du \cos \frac{1}{2} \pi u}{4 \sin \frac{1}{2} \pi u} = \frac{1}{2} - \frac{\pi}{4} \int \frac{u du \cos \frac{1}{2} \pi u}{\sin \frac{1}{2} \pi u}$$

et ob $\sin \pi u = 2 \sin \frac{1}{2} \pi u \cos \frac{1}{2} \pi u$ et $\cos \pi u = \cos \frac{1}{2} \pi u^2 - \sin \frac{1}{2} \pi u^2$ fiet

$$P - Q = \frac{\pi}{4} \int \frac{u du \cos \frac{1}{2} \pi u^2 - \cos \frac{1}{2} \pi u^2 \sin \frac{1}{2} \pi u^2}{\sin \frac{1}{2} \pi u \cos \frac{1}{2} \pi u} = \frac{\pi}{4} \int \frac{u du \sin \frac{1}{2} \pi u}{\cos \frac{1}{2} \pi u}$$

ita ut sit $1 - \frac{1}{2} \int_0^{\pi} \frac{u du \sin \frac{1}{2} \pi u}{\cos \frac{1}{2} \pi u} = \frac{1}{2} \int_0^{\pi} \frac{1 + u}{1 - u} + 1$

$$\text{seu } \int_0^{\pi} \frac{1 + u}{1 - u} du = \int_0^{\pi} \frac{1 + u}{1 - u} du = \int_0^{\pi} \frac{1 + u}{1 - u} du = \int_0^{\pi} \frac{1 + u}{1 - u} du$$

siquidem integratione absoluta ponatur $u = 1$.

36. Statuamus nunc angulum $\frac{1}{2} \pi u = \Phi$, seu $u = \frac{2\Phi}{\pi}$, ut integrale a termino $\Phi = 0$ vsque ad $\Phi = \frac{\pi}{2} = 90^\circ$ extendi oporteat, ac praecedens aequatio abibit in hanc formam

$$\int_0^{\pi/2} \frac{1 + \frac{2\Phi}{\pi}}{1 - \frac{2\Phi}{\pi}} d\Phi = \int_0^{\pi/2} \frac{1 + \frac{2\Phi}{\pi}}{1 - \frac{2\Phi}{\pi}} d\Phi = \int_0^{\pi/2} \frac{1 + \frac{2\Phi}{\pi}}{1 - \frac{2\Phi}{\pi}} d\Phi$$

$$\int_0^{\pi/2} \frac{1 + \frac{2\Phi}{\pi}}{1 - \frac{2\Phi}{\pi}} d\Phi = \int_0^{\pi/2} \frac{1 + \frac{2\Phi}{\pi}}{1 - \frac{2\Phi}{\pi}} d\Phi = \int_0^{\pi/2} \frac{1 + \frac{2\Phi}{\pi}}{1 - \frac{2\Phi}{\pi}} d\Phi$$

ubi fractio $\frac{1 + \frac{2\Phi}{\pi}}{1 - \frac{2\Phi}{\pi}}$ casu $\Phi = \frac{\pi}{2}$ abit in $\frac{2}{0} = \infty$, ita ut sit $\int_0^{\pi/2} \frac{1 + \frac{2\Phi}{\pi}}{1 - \frac{2\Phi}{\pi}} d\Phi = \int_0^{\pi/2} \frac{1 + \frac{2\Phi}{\pi}}{1 - \frac{2\Phi}{\pi}} d\Phi$

Quod si ergo demonstrari posset formulae integralis $\int d\Phi \cos \Phi$ valorem a termino $\Phi = 0$ ad $\Phi = 90^\circ = \frac{\pi}{2}$ extensum, fevera quantitati $\frac{\pi}{2} = \frac{\pi}{2}$ aequari, omnino per aliam viam id assequeremur quod ante per ambages circa valorem litterae $Q = \frac{1}{2} \int_0^{\pi/2} \pi$ concludimus. Quoniam vero ante de hoc valore

sumus

sumus certi, insigne consecuti sumus hoc theorema quod sit integrali a valore $\Phi = 0$ vsque ad $\Phi = \frac{\pi}{2} = 90^\circ$ extensum $\int d\Phi \cos \Phi = -\frac{\pi}{2}$

vel si ponamus $\cos \Phi = v$, ut termini integrationis sint $v = 1$ et $v = 0$ ostendendum est fore

$$\int \frac{dv v}{\sqrt{1-v^2}} = -\frac{\pi}{2}$$

vnde hoc integrali in seriem euoluto erit

$$\frac{\pi}{2} = 1 + \frac{1}{2} + \frac{1^2 \cdot 1^2}{2 \cdot 4 \cdot 2^2} + \frac{1^2 \cdot 3^2 \cdot 1^2}{2 \cdot 4 \cdot 6 \cdot 2^2} + \dots$$

quae series vicissim reducitur ad $\int_0^{\pi/2} \frac{1}{\cos s} ds$. Ang. $\sin s$ ponendo post integrationem $s = 1$: haec vero porro posito $s = \sin \Phi$ ad hanc

$$\int \frac{d\Phi \cos \Phi}{\sin \Phi} = \int \frac{d\Phi}{\sin \Phi} = \int \frac{d\Phi}{\sin \Phi} = \int \frac{d\Phi}{\sin \Phi}$$

quae cum superiori congruit.

36. Quod autem sit $\int d\Phi \cos \Phi = -\frac{\pi}{2}$ hoc modo demonstratur. Cum sit $\frac{d\Phi}{\sin \Phi} = 2 \sin \frac{1}{2} \Phi + 2$

$$\sin \frac{1}{2} \Phi + 2 \sin \frac{1}{2} \Phi + 2 \sin \frac{1}{2} \Phi + 2 \sin \frac{1}{2} \Phi + 2$$

$\int \sin \frac{1}{2} \Phi = -\cos \frac{1}{2} \Phi = 2 \cos \frac{1}{2} \Phi = 2 \cos \frac{1}{2} \Phi = 2 \cos \frac{1}{2} \Phi$ etc. $-\int_0^{\pi/2} \frac{1}{\sin \frac{1}{2} \Phi} d\Phi = -\int_0^{\pi/2} \frac{1}{\sin \frac{1}{2} \Phi} d\Phi = -\int_0^{\pi/2} \frac{1}{\sin \frac{1}{2} \Phi} d\Phi$

ergo $\int_0^{\pi/2} \frac{1}{\sin \frac{1}{2} \Phi} d\Phi = -\int_0^{\pi/2} \frac{1}{\sin \frac{1}{2} \Phi} d\Phi = -\int_0^{\pi/2} \frac{1}{\sin \frac{1}{2} \Phi} d\Phi$

iam facto $\Phi = \frac{\pi}{2}$ fit $\int_0^{\pi/2} \frac{1}{\sin \frac{1}{2} \Phi} d\Phi = -\frac{\pi}{2}$.