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Considerationes de theoria motus Lunae perficienda et imprimis de eius variatione

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CONSIDERATIONES DE THEORIA MOTVS LVNAE PERFICIENDA ET IMPRIMIS DE EIVS VARIATIONE.

Auctore

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I.

Etſi Theoria motuum Lunae a praeſtantiſſimis Geometris ſummo ſtudio eſt inueſtigata, atque adeo a Celeb. Profeſſore Göttingenſi *Mayero* Tabulae Lunares obſeruationibus apprime ſatisfacientes ſunt in medium allatae, plurimum tamen adhuc abeſt, quo minus ipſa Theoria penitus exculta exiſtimari poſſit. Quanquam enim forma iſtarum Tabularum ex Theoria eſt deriuata; quae etiam plures inaequalitates in motu Lunae accurate ſuppeditauit nonnullae tamen maximi momenti occurrunt, quarum quantitas ex ſolis obſeruationibus eſt definita cum earum determinatio per ſolam Theoriam nimis incerta relinqueretur. Quin etiam nullum eſt dubium quin verus Lunae motus multo pluribus inaequalitatibus, quam quae in his Tabulis aſſignantur, perturbetur quae etſi in vſu practico ob paruitatem facile praetermitti poſſunt, tamen in Theoria minime contemnendae videntur neque Theoria ante ſatis exculta

culta censeri poterit, quam omnes prorsus motus inaequalitates, ne minimis quidem exceptis, accurate assignare valuerimus.

II. Ad Theoriam autem motuum Lunae feliciter inuestigandam, non statim ab eius motu vero exordiendum videtur, quemadmodum ab iis, qui hoc opus susceperunt, est factum; verus enim motus, quatenus non solum secundum longitudinem, sed etiam secundum latitudinem continuo perturbatur, tot tantisque difficultatibus implicatur, et penitus obruitur, vt singulis expediendis neque nostrae neque Analyseos vires sufficiant. Quam ob causam in hoc tam difficili negotio methodum ab Astronomis praecipue felicissimo cum successu vsitatam adhiberi conueniet, vt ante quam veros Lunae motus inuestigemus, casus nobis fingamus simpliciores, multo paucioribus difficultatibus obnoxios, quos si expedire licuerit, tum demum studia nostra continuo propius ad veritatem applicare licebit.

III. Primo igitur motus Lunae in latitudinem prorsus remouendus videtur, ita vt non huius, sed alius cuiusdam Lunae, quae in ipso eclipticae plano moueatur, motus sit inuestigandus; quandoquidem hoc modo calculus a grauissimis illis difficultatibus, quibus motus nodorum et inclinatio ad eclipticam premitur, liberatur. Deinde ne ipse solis motus quatenus non est vniformis molestiam facessat, hoc quoque obstaculum in principio tollatur,

 et

et motus solis quasi esset vniformis spectetur. Hac ratione aliae inaequalitates inuestigandae non supererunt, nisi quae partim ab excentricitate orbitae lunaris, partim ab elongatione Lunae a Sole pendent. Ac si simplicitas adhuc maior desideretur, etiam excentricitas abiiciatur, et eiusmodi Lunae motus indagetur, quae sine vlla excentricitate in plano eclipticae moueretur sole cursum suum vniformiter absoluente. Hunc tantum casum adeo simplicem qui accurate et ad computum accommodate euoluere potuerit, is certe iam plurimum in Theoria praestitisse esset censendus.

Tab. I. Fig. 2. IV. Remota ergo inclinatione orbitae Lunaris, centrum terrae vt quiescens spectetur in T, et tabula referente planum eclipticae, sit tempore quodam t elapso centrum Lunae in L et Solis in S. Assumta iam recta fixa TA ad principium scilicet arietis ducta vocentur distantiae: $TL = v$, $TS = u$ et $LS = z$, et anguli $ATL = \Phi$, $ATS = \theta$, sitque breuitatis gratia $STL = \Phi - \theta = \eta$, erit $z = \sqrt{(uu - 2vu \operatorname{cos}.\eta + vv)}$ vbi quidem distantia v est valde parua prae u. Porro demisso ab L in rectam TA perpendiculo LV sit $TV = x$ et $VL = y$, eritque $x = v \operatorname{cos}.\Phi$ et $y = v \operatorname{sin}.\Phi$. Hinc $x \operatorname{cos}.\Phi + y \operatorname{sin}.\Phi = v$ et $x \operatorname{sin}.\Phi - y \operatorname{cos}.\Phi = 0$: Ergo differentiando $dx \operatorname{cos}.\Phi + dy \operatorname{sin}.\Phi - d\Phi(x \operatorname{sin}.\Phi - y \operatorname{cos}.\Phi) = dv$ seu $dx \operatorname{cos}.\Phi + dy \operatorname{sin}.\Phi = dv$ et $dx \operatorname{sin}.\Phi - dy \operatorname{cos}.\Phi + d\Phi(x \operatorname{cos}.\Phi + y \operatorname{sin}.\Phi) = 0$ seu $dx \operatorname{sin}.\Phi - dy \operatorname{cos}.\Phi = -v d\Phi$. Porro denuo differentiando:

ddx

$$ddx\,\text{cof.}\,\Phi + ddy\,\text{fin.}\,\Phi - d\Phi(dx\,\text{fin.}\,\Phi - dy\,\text{cof.}\,\Phi) = ddv$$ feu
$$ddx\,\text{cof.}\,\Phi + ddy\,\text{fin.}\,\Phi = ddv - vd\Phi^2$$
$$ddx\,\text{fin.}\,\Phi - ddy\,\text{cof.}\,\Phi + d\Phi(dx\,\text{cof.}\,\Phi + dy\,\text{fin.}\,\Phi) = -dvd\Phi - vdd\Phi$$ feu
$$ddy\,\text{cof.}\,\Phi - ddx\,\text{fin.}\,\Phi = 2dvd\Phi + vdd\Phi.$$

V. Iam maffae Solis, terrae ac Lunae defignentur litteris S, T et L, ita vt fint vires acceleratrices, quibus Luna vrgetur ad terram fecundum $LT = \frac{T}{vv}$; et ad folem fecundum $LS = \frac{S}{zz}$, quae ducta recta sLt ipfi TS parallela refoluitur in has binas vires:

1°. Secundum $LT = \frac{Sv}{z^3}$ et 2°. fecundum $Ls = \frac{Su}{z^3}$. Quia deinde terra ad folem vrgetur vi fecundum $TS = \frac{S}{uu}$, et ad Lunam vi fecundum $TL = \frac{L}{vv}$, hae vires contrarie in Lunam translatae dant vim fecundum $Lt = \frac{S}{uu}$, et fecund. $LT = \frac{L}{vv}$ ita vt iam Luna his viribus vrgeri cenfenda fit;

1°. Sec. $LT = \frac{T+L}{vv} + \frac{Sv}{z^3}$; 2°. fec. $Lt = \frac{S}{uu} - \frac{Su}{z^3}$ quae porro fecundum directiones coordinatarum TV et VL, feu ducta LR ipfi TA parallela fecundum LR et VL, refolutae dant

fecundum LR vim $= \frac{T+L}{vv}\,\text{cof.}\,\Phi + \frac{Sv}{z^3}\,\text{cof.}\,\Phi + \frac{S}{uu}\,\text{cof.}\,\theta - \frac{Su}{z^3}\,\text{cof.}\,\theta$

fecundum LV vim $= \frac{T+L}{vv}\,\text{fin.}\,\Phi + \frac{Sv}{z^3}\,\text{fin.}\,\Phi + \frac{S}{uu}\,\text{fin.}\,\theta - \frac{Su}{z^3}\,\text{fin.}\,\theta.$

VI. His viribus inuentis sumendo temporis elemento dt constante, principia motus praebent has aequationes

$$\frac{ddx}{dt^2} = -\frac{(T+L)\,\text{cos.}\,\Phi}{vv} - \frac{S\,v\,\text{cos.}\,\Phi}{z^3} - \frac{S\,\text{cos.}\,\theta}{uu} + \frac{S\,u\,\text{cos.}\,\theta}{z^3}$$

$$\frac{ddy}{dt^2} = -\frac{(T+L)\,\text{sin.}\,\Phi}{vv} - \frac{S\,v\,\text{sin.}\,\Phi}{z^3} - \frac{S\,\text{sin.}\,\theta}{uu} + \frac{S\,u\,\text{sin.}\,\theta}{z^3}$$

vnde ob $ddy\,\text{cos.}\,\Phi - ddx\,\text{sin.}\,\Phi = 2dvd\Phi + vdd\Phi$

et $ddx\,\text{cos.}\,\Phi + ddy\,\text{sin.}\,\Phi = ddv - vd\Phi^2$

nanciscimur has binas aequationes principales:

1°. $\frac{2dvd\Phi + vdd\Phi}{dt^2} = \frac{S\,\text{sin.}\,\eta}{uu} - \frac{S\,u\,\text{sin.}\,\eta}{z^3}$

2°. $\frac{ddv - vd\Phi^2}{dt^2} = -\frac{(T+L)}{vv} - \frac{Sv}{z^3} - \frac{S\,\text{cos.}\,\eta}{uu} + \frac{S\,u\,\text{cos.}\,\eta}{z^3}$.

Vt iam pro dt^2 valorem determinatum introducamus, consideremus motum Solis, qui cum ad terram sollicitari censendus sit vi $\frac{S+T}{uu}$, habebitur simili modo:

$$\frac{2dud\theta + udd\theta}{dt^2} = 0 \text{ et } \frac{ddu - ud\theta^2}{dt^2} = -\frac{(S+T)}{uu}$$

sumamus iam Solis distantiam a terra mediam $= a$, et motum medium tempori t conuenientem $= \zeta$, erit ex posteriori aequatione $\frac{ad\zeta^2}{dt^2} = \frac{S+T}{aa}$, vnde colligimus $\frac{1}{dt^2} = \frac{T+S}{a^3 d\zeta^2}$

sicque loco elementi dt introducimus elementum cognitum pariter constans $d\zeta$, et has formulas adipiscimur:

1°. $2dvd\Phi + vdd\Phi = \frac{S\,a^3\,d\zeta^2\,\text{sin.}\,\eta}{S+T}\left(\frac{1}{uu} - \frac{u}{z^3}\right)$

2°. $ddv - vd\Phi^2 = -\frac{(T+L)a^3 d\zeta^2}{(T+S)vv} - \frac{Sa^3 d\zeta^2}{S+T}\left(\frac{v}{z^3} + \frac{\text{cos.}\,\eta}{uu} - \frac{u\,\text{cos.}\,\eta}{z^3}\right)$

vbi

vbi notandum eſt loco $\frac{S}{S+T}$ vnitatem ſcribi licere cum maſſa T prae S euaneſcat.

VII. Vt litteras maiusculas S, T, L ex calculo exterminemus, contemplemur etiam motum Lunae medium, qui quidem eſſet futurus, ſi vires perturbantes a Sole oriundae abeſſent; hoc caſu ſtatuatur diſtantia Lunae media a terra $= c$, et ratio eius motus medii ad motum medium Solis $= n : 1$; cum igitur ſit $v = c$ et $d\Phi = n\,d\zeta$, poſterior aequatio praebet $-cnnd\zeta^2 = -\frac{(T+L)a^3 d\zeta^2}{(T+S)cc}$ vnde ſit $\frac{T+L}{T+S} = \frac{nnc^3}{a^3}$; ex quo noſtrae aequationes principales ſequentes induent formas:

$$1^\circ.\ 2\,dv\,d\Phi + v\,dd\Phi = a^3 d\zeta^2 \,\text{ſin.}\,\eta\left(\frac{1}{uu} - \frac{u}{z^3}\right)$$

$$2^\circ.\ ddv - v\,d\Phi^2 = -\frac{nnc^3}{vv} d\zeta^2 - \frac{a^3 v}{z^3} d\zeta^2 - a^3 d\zeta^2 \,\text{coſ.}\,\eta\left(\frac{1}{uu} - \frac{u}{z^3}\right).$$

Totum ergo negotium huc redit, vt iſtae aequationes commode tractentur, ac ſi fieri queat ad integrationem perducantur: vbi quidem notaſſe iuuabit, membra poſteriora quantitates u et z inuoluentia prae reliquis eſſe valde parua, indeque rationem approximandi eſſe petendam.

VIII. Ponamus autem breuitatis gratia:

$$\frac{1}{uu} - \frac{u}{z^3} = -M \text{ et } \frac{v}{z^3} + \text{coſ.}\,\eta\left(\frac{1}{uu} - \frac{u}{z^3}\right) = N$$

vt aequationes noſtrae fiant

$$1^\circ.\ 2\,dv\,d\Phi + v\,dd\Phi = -a^3 M\,d\zeta^2 \,\text{ſin.}\,\eta \text{ et}$$

$$2^\circ.\ ddv - v\,d\Phi^2 = -\frac{nnc^3}{vv} d\zeta^2 - a^3 N\,d\zeta^2$$

 vbi

vbi ob v prae u valde paruum et $z = \sqrt{(uu - 2uv\,\mathrm{coſ.}\,\eta + vv)}$ erit per approximationem

$$\frac{1}{z^3} = \frac{1}{u^3} + \frac{3v}{u^4}\mathrm{coſ.}\,\eta - \frac{3vv}{2u^5}(1 - 5\,\mathrm{coſ.}\,\eta^2) - \frac{5v^3}{2u^6}(3\,\mathrm{coſ.}\,\eta - 7\,\mathrm{coſ.}\,\eta^3) + \frac{15v^4}{8u^7}(1 - 14\,\mathrm{coſ.}\,\eta^2 + 21\,\mathrm{coſ.}\,\eta^4)\ \mathrm{etc.}$$

ideoque litterarum M et N valores prodibunt

$$M = \frac{3v}{u^3}\mathrm{coſ.}\,\eta - \frac{3vv}{2u^4}(1 - 5\,\mathrm{coſ.}\,\eta^2) - \frac{5v^3}{2u^5}(3\,\mathrm{coſ.}\,\eta - 7\,\mathrm{coſ.}\,\eta^3) + \frac{15v^4}{8u^6}(1 - 14\,\mathrm{coſ.}\,\eta^2 + 21\,\mathrm{coſ.}\,\eta^4)$$

$$N = \frac{v}{u^3}(1 - 3\,\mathrm{coſ.}\,\eta^2) + \frac{3vv}{2u^4}(3\,\mathrm{coſ.}\,\eta - 5\,\mathrm{coſ.}\,\eta^3) - \frac{v^3}{2u^5}(3 - 30\,\mathrm{coſ.}\,\eta^2 + 35\,\mathrm{coſ.}\,\eta^4) - \frac{5v^4}{8u^6}(15\,\mathrm{coſ.}\,\eta - 70\,\mathrm{coſ.}\,\eta^3 + 63\,\mathrm{coſ.}\,\eta^5)$$

vbi ſingula membra ſequentia prae antecedentibus ſunt vehementer exigua.

IX. Prima aequationum noſtrarum ad integrabilitatem perducitur multiplicando eam per v tum vero etiam per $2v^3 d\Phi$, poſteriori modo prodit $v^4 d\Phi^2 = -2a^3 d\zeta^2 \int M v^3 d\Phi\, \mathrm{ſin.}\,\eta$.

Deinde prior multiplicetur per $2v\,d\Phi$ et poſterior per $2dv$ ac ſumma dabit:

$$2dv\,ddv + 2v\,dv\,d\Phi^2 + 2vv\,d\Phi\,dd\Phi = -2a^3 M v\,d\zeta^2 d\Phi\, \mathrm{ſin.}\,\eta - \frac{2nnc^3 dv}{vv} d\zeta^2 - 2a^3 N d\zeta^2 dv$$

vnde per integrationem eruitur:

$$dv^2 + vv\,d\Phi^2 = \frac{2nnc^3 d\zeta^2}{v} - 2a^3 d\zeta^2 \int M v\,d\Phi\, \mathrm{ſin.}\,\eta - 2a^3 d\zeta^2 \int N dv.$$

Statuamus breuitatis gratia:

$$a^3 \int M v^3 d\Phi\, \mathrm{ſin.}\,\eta = -c^4 P \quad \text{et} \quad a^3 \int M v\,d\Phi\, \mathrm{ſin.}\,\eta + a^3 \int N dv = -ccQ$$

vt

vt obtineamus has formas:

1°. $v^4 d\Phi^2 = +2c^4 P d\zeta^2$ et 2°. $dv^2 + vv d\Phi^2 = \frac{2nnc^3 d\zeta^2}{v} + 2cc Q d\zeta^2$

quae facto $v = cx$ fiunt

1°. $x^4 d\Phi^2 = +2P d\zeta^2$ et 2°. $dx^2 + xx d\Phi^2 = \frac{2nn d\zeta^2}{x} + 2Q d\zeta^2$

eritque:

$dP = \frac{-a^3}{c} M x^3 d\Phi \text{ fin.}\eta$ et $dQ = \frac{-a^3}{c}(Mx d\Phi \text{ fin.}\eta + N dx)$

existente

$$M = \frac{3cx}{u^3}\text{cof.}\eta - \frac{3ccxx}{2u^4}(1 - 5\text{ cof.}\eta^2) - \frac{5c^3x^3}{2u^5}(3\text{ cof.}\eta - 7\text{ cof.}\eta^3)$$

$$N = \frac{cx}{u^3}(1 - 3\text{cof.}\eta^2) + \frac{3ccxx}{2u^4}(3\text{cof.}\eta - 5\text{cof.}\eta^3) - \frac{c^3x^3}{2u^5}(3 - 30\text{cof.}\eta^2 + 35\text{ cof.}\eta^4).$$

X. Ex priore aequatione iam est $d\Phi = \frac{d\zeta \sqrt{2P}}{xx}$, qui valor in altera substitutus dat:

$$dx^2 + \frac{2P d\zeta^2}{xx} = \frac{2nn d\zeta^2}{x} + 2Q d\zeta^2$$

vnde elicitur:

$$dx = d\zeta \sqrt{(2Q + \frac{2nn}{x} - \frac{2P}{xx})} \text{ vel etiam}$$

$$\frac{dx\sqrt{2P}}{xx} = d\Phi \sqrt{(2Q + \frac{2nn}{x} - \frac{2P}{xx})}$$

hincque discimus quantitatem $2Q + \frac{2nn}{x} - \frac{2P}{xx}$ nunquam fieri posse negatiuam; euanescere autem potest, quod fit dum Luna vel in apogeo versatur vel in perigeo, quandoquidem vtroque casu fit $dx = 0$. Ceterum si vires perturbantes abessent pro motu medio,

medio, quo $x=1$ et $d\Phi=nd\zeta$ foret $n=\sqrt{2P}$, seu $P=\frac{1}{2}nn$ et $nn=2nn+2Q$ seu $Q=-\frac{1}{2}nn$, qui ergo valores his litteris proxime conueniunt.

XI. Nisi excentricitas orbitae euanescat vel sit quam minima, eius introductio in calculum satis commode ad formulas differentiales primi gradus manuducit, quae ad computum astronomicum maxime videntur accommodatae. Duplici imprimis modo haec reductio institui potest, vnde deinceps alias latius patentes eiusmodi resolutiones deriuare licet. Alterum quidem modum iam alibi fusius sum persecutus, sed dignitas materiae omnino requirere videtur, vt vtrumque hic dilucide exponam, simulque cognationem ostendam, quo facilius intelligi possit, quanta emolumenta inde expectare liceat.

Reductio prior formularum inuentarum ope excentricitatis facta.

XII. Ordiamur a formula posteriori, quae per $\sqrt{2}$ diuisa est:

$$\frac{dx}{xx}\sqrt{P}=d\Phi\sqrt{(Q+\frac{nn}{x}-\frac{P}{xx})}$$

ac statuamus $\frac{1}{x}=\frac{1-q\cos.\omega}{p}$, seu $x=\frac{p}{1-q\cos.\omega}$, vbi sequentia sunt obseruanda:

1°. Quantitas p in a ducta ob $v=cx$ exprimit semiparametrum orbitae, quatenus ea cum ellipsi comparatur; foretque p quantitas constans, si vires per-

perturbantes abessent, nunc autem erit quantitas variabilis.

2°. Quantitas q in eadem comparatione denotat excentricitatem, quae ob vires perturbantes pariter vt variabilis est spectanda.

3°. Angulus ω designat anomaliam veram ab apogeo computatam, et ob $v = cx$, erit distantia apogei $= \frac{cp}{1-q}$ et distantia perigei $= \frac{cp}{1+q}$, vnde semiaxis transuersus orbitae $= \frac{cp}{1-qq}$.

4°. Loco vnius variabilis x introduximus tres nouas p, q et ω, inter quas autem iam vnam determinationem stabiliuimus qua dx euanescere debet si $\sin.\omega = 0$; alteram determinationem consideratio formulae irrationalis suppeditabit.

XIII. In formula $Q + \frac{nn}{x} - \frac{P}{xx}$ loco $\frac{1}{x}$ substituamus valorem assumtum $\frac{1 - q\cos.\omega}{p}$, et prodibit

$$Q + \frac{nn}{p} - \frac{P}{pp} - \frac{nnq}{p}\cos.\omega + \frac{2Pq}{pp}\cos.\omega - \frac{Pqq}{pp}\cos.\omega^2$$

cuius radix quadrata quia factorem habere debet $\sin.\omega$ oportet vt sit 1°. $P = \frac{1}{2}nnp$, et 2°. $Q + \frac{nn}{p} - \frac{P}{pp} = \frac{Pqq}{pp}$ sicque prodeat $Q + \frac{nn}{x} - \frac{P}{xx} = \frac{Pqq}{pp}\sin.\omega^2$. Quo facto erit $\frac{dx}{xx}\sqrt{P} = -\frac{qd\Phi\sin.\omega}{p}\sqrt{P}$, seu $\frac{dx}{xx} = -\frac{qd\Phi}{p}\sin.\omega$. Cum autem sit $\frac{dx}{xx} = \frac{dp}{pp} + \cos.\omega\, d.\frac{q}{p} - \frac{q}{p}d\omega\sin.\omega$, habebimus $\frac{q}{p}(d\Phi - d\omega)\sin.\omega = -\frac{dp}{pp} - \cos.\omega,\, d.\frac{q}{p}$. Ex factis autem binis hypothesibus erit primo $p = \frac{2P}{nn}$ et ob $nn = \frac{2P}{p}$ altera dat $Q + \frac{P(1-qq)}{pp} = 0$,

 seu

ſeu $Q + \frac{nn}{2p}(1 - qq) = 0$ hincque $\frac{1-qq}{p} = -\frac{2Q}{nn}$. Denique prima aequatio $d\Phi = \frac{d\zeta\sqrt{2P}}{xx}$ dat $\frac{d\Phi}{d\zeta} = \frac{n(1-q\cos.\omega)^2}{p\sqrt{p}}$, ſeu $d\zeta = \frac{p\,d\Phi\sqrt{p}}{n(1-q\cos.\omega)^2}$.

XIV. Quia nunc P et Q ſunt quantitates, quarum differentialia ſaltem vt cognita ſpectantur, variationes momentaneae elementorum motus ſequenti modo ſe habebunt.

1°. Pro quantitate p erit $dp = \frac{2dP}{nn}$, ideoque

$$dp = \frac{-2a^3}{nnc} M x^3 d\Phi \sin.\eta \text{ vbi } x = \frac{p}{1-\cos.\omega}.$$

2°. Pro ſemiaxe orbitae $\frac{cp}{1-qq}$ habemus $d.\frac{1-qq}{p} = -\frac{2dQ}{nn}$

ideoque $d.\frac{1-qq}{p} = \frac{2a^3}{nnc}(Mx\,d\Phi\sin.\eta + N\,dx)$

quia vero eſt $dx = \frac{-qxx\,d\Phi}{p}\sin.\omega$ erit

$$d.\frac{1-qq}{p} = \frac{2a^3}{nnc}x\,d\Phi\left(M\sin.\eta - \frac{Nq\sin.\omega}{1-q\cos.\omega}\right).$$

3°. Inuento differentiali quantitatis $\frac{1-qq}{p}$, quam tantiſper vocabo R, erit $qq = 1 - pR$ et $\frac{qq}{pp} = \frac{1}{pp} - \frac{R}{p}$, vnde fit

$$d\frac{qq}{pp} = \frac{2q}{p}d.\frac{q}{p} = -\frac{2dp}{p^3} + \frac{R\,dp}{pp} - \frac{1}{p}.d.R = -\frac{(1+qq)dp}{p^3} - \frac{1}{p}dR$$

vbi ſi loco dp et dR valores inuenti ſubſtituantur, reperitur

$$\frac{2q}{p}d.\frac{q}{p} = \frac{2a^3qx\,d\Phi}{nncp}\left(\frac{M(2\cos.\omega + q\sin.\omega^2)\sin.\eta}{(1-q\cos.\omega)^2} + \frac{N\sin.\omega}{1-q\cos.\omega}\right) \text{ ideoque}$$

$$d.\frac{q}{p} = \frac{a^3x\,d\Phi}{nnc}\left(\frac{M(2\cos.\omega + q\sin.\omega^2)\sin.\eta}{(1-q\cos.\omega)^2} + \frac{N\sin.\omega}{1-q\cos.\omega}\right)$$

vnde

vnde concluditur :

$$\frac{q}{p}(d\Phi - d\omega)\,\text{ſin.}\,\omega = \frac{2\,a^3}{n\,n\,c}\,x\,d\Phi . \frac{M\,\text{ſin.}\,\eta}{(1-q\,\text{coſ.}\,\omega)^2} - \text{coſ.}\,\omega\, d.\frac{q}{p} \quad \text{ſeu}$$

$$\frac{q}{p}(d\Phi - d\omega)\,\text{ſin.}\,\omega = \frac{a^3 x\, d\Phi}{n\,n\,c}\left(\frac{M(2\,\text{ſin.}\,\omega^2 - q\,\text{ſin.}\,\omega^2\,\text{coſ.}\,\omega)\,\text{ſin.}\,\eta}{(1-q\,\text{coſ.}\,\omega)^2} - \frac{N\,\text{ſin.}\,\omega\,\text{coſ.}\,\omega}{1-q\,\text{coſ.}\,\omega}\right)$$

ſicque habebimus :

$$d\Phi - d\omega = \frac{a^3\,x\,x\,d\Phi}{n\,n.c\,q}\left(\frac{M(2-q\,\text{coſ.}\,\omega)\,\text{ſin.}\,\eta\,\text{ſin.}\,\omega}{1-q\,\text{coſ.}\,\omega} - N\,\text{coſ.}\,\omega\right)$$

vnde motus lineae abſidum definitur.

4°. His variationibus definitis erit tandem

$$x = \frac{p}{1-q\,\text{coſ.}\,\omega} \quad \text{et} \quad d\zeta = \frac{p\,d\Phi\sqrt{p}}{n(1-q\,\text{coſ.}\,\omega)^2}$$

qua poſteriori formula ratio inter $d\Phi$ et $d\zeta$, illinc vero ratio inter $d\Phi$ et $d\omega$ exprimitur.

Reductio altera formularum inuentarum ad differentialia primi gradus.

XV. Aequationi poſteriori haec inducatur forma :

$$\frac{dx}{x}\sqrt{P} = -d\Phi\sqrt{(Qxx + nnx - P)}$$

priore exiſtente $xx\,d\Phi = d\zeta\sqrt{2P}$, et excentricitas ita introducatur vt ponatur $x = p + q\,\text{coſ.}\,\omega$, ſicque diſtantia maxima ſit $= p + q$ et minima $= p - q$, vbi autem quantitates p et q ſunt variabiles. Cum nunc ſit :

$\frac{dx}{x} = \frac{dp + dq\,\text{coſ.}\,\omega - q\,d\omega\,\text{ſin.}\,\omega}{p + q\,\text{coſ.}\,\omega}$, quae expreſſio euaneſcere debet ſi ſin. $\omega = 0$, valor ipſius x in altera parte ſubſtitutus dabit

$$Qxx + nnx - P = Qpp + 2Qpq\,\text{cof.}\,\omega + Qqq\,\text{cof}\,\omega^2 + nnp + nnq\,\text{cof.}\,\omega - P.$$

Hic ergo ponatur $2Qp + nn = 0$ et $Qqq = -Qpp - nnp + P$ vt fiat $\sqrt{(Qxx + nnx - P)} = \text{fin.}\,\omega\sqrt{(Qpp + nnp - P)} = +q\,\text{fin.}\,\omega\sqrt{-Q}$. At ob $nnp = -2Qpp$, habemus $Qqq = Qpp + P$, feu

$Q = \frac{-P}{pp-qq} = \frac{-nn}{2p}$, vnde fit $\frac{nn(pp-qq)}{p} = 2P$, et $\frac{nn}{p} = -2Q$.

Quare altera aequatio hanc induit formam:

$$\frac{dx}{x}\sqrt{P} = -qd\Phi\,\text{fin.}\,\omega\sqrt{-Q}, \text{ feu } \frac{dx}{x}\sqrt{(pp-qq)} = -qd\Phi\,\text{fin.}\,\omega$$

vnde colligimus:

$$dp + dq\,\text{cof.}\,\omega - qd\omega\,\text{fin.}\,\omega = \frac{-q(p+q\,\text{cof.}\,\omega)\,d\Phi\,\text{fin.}\,\omega}{\sqrt{(pp-qq)}}.$$

Hinc fingularum quantitatum variationes momentaneas ex differentialibus cognitis dP et dQ affignare poterimus.

1°. Aequatio $\frac{nn}{p} = -2Q$ dat $\frac{nndp}{pp} = 2dQ$, ideoque $dp = \frac{2ppdQ}{nn}$.

2°. Ex aequatione $\frac{nn(pp-qq)}{p} = 2P$ feu $p - \frac{qq}{p} = \frac{2P}{nn}$, fequitur

$$dp + \frac{qqdp}{pp} - \frac{2qdq}{p} = \frac{2dP}{nn}, \text{ feu } qdq = \frac{dp(pp+qq)}{2p} - \frac{pdP}{nn}$$

vnde fit $qdq = \frac{p(pp+qq)dQ - pdP}{nn}$.

3°. Hi valores in vltima aequatione fubftituti dabunt:

$$\frac{2ppdQ}{nn} + \frac{p(pp+qq)dQ\,\text{cof.}\,\omega}{nnq} - \frac{pdP\,\text{cof.}\,\omega}{nnq} - qd\omega\,\text{fin.}\,\omega = \frac{-q(p+q\,\text{cof.}\,\omega)d\Phi\,\text{fin.}\,\omega}{\sqrt{(pp-qq)}}$$

vnde

vnde fit:

$$qq\,d\omega\,\text{fin.}\omega = \frac{p\,dQ}{nn}(2pq + (pp+qq)\text{cof.}\,\omega) - \frac{p\,dP}{nn}\,\text{cof.}\,\omega + \frac{qq(p+q\,\text{cof.}\,\omega)\,d\Phi\,\text{fin.}\,\omega}{\sqrt{(pp-qq)}}.$$

4°. Cum autem fit $dx = -\frac{qx\,d\Phi\,\text{fin.}\,\omega}{\sqrt{(pp-qq)}}$ erit $dP = \frac{-a^3}{c}Mx^3 d\Phi\,\text{fin.}\,\eta$ et $dQ = \frac{-a^3}{c}d\Phi(Mx\,\text{fin.}\,\eta - \frac{Nq\,x\,\text{fin.}\,\omega}{\sqrt{(pp-qq)}})$, qui valores in formulis inuentis fubftituti praebent:

$$dp = \frac{-2a^3ppx\,d\Phi}{nnc}(M\,\text{fin.}\,\eta - \frac{Nq\,\text{fin.}\,\omega}{\sqrt{(pp-qq)}})$$

$$dq = \frac{a^3px\,d\Phi}{nnc}(M\,\text{fin.}\,\eta(2p\,\text{cof.}\,\omega - q\,\text{fin.}\,\omega^2) + \frac{N(pp+qq)\,\text{fin.}\,\omega}{\sqrt{(pp-qq)}})$$

$$d\omega = \frac{x\,d\Phi}{\sqrt{(pp-qq)}} - \frac{a^3px\,d\Phi}{nncq}(M\,\text{fin.}\,\eta\,(2p+q\,\text{cof.}\,\omega)\,\text{fin.}\,\omega - \frac{N(2pq+(pp+qq)\text{cof.}\,\omega)}{\sqrt{(pp-qq)}}).$$

Denique ob $2P = \frac{nn(pp-qq)}{p}$ eft $d\zeta = \frac{xx\,d\Phi\sqrt{p}}{n\sqrt{(pp-qq)}}$ exiftente $x = p + q\,\text{cof.}\,\omega$.

Reductio generalior binas praecedentes in fe complectens.

XVI. Statuamus $x = \frac{p+q\,\text{cof.}\,\omega}{1-r\,\text{cof.}\,\omega}$, vbi angulus ω ita fe habeat vt cafu fin. $\omega = 0$ euanefcat dx; feu vt diftantia fiat maxima cafu $\omega = 0$, minima vero cafu $\omega = 180°$.

Erit ergo $\frac{dx}{x} = \frac{dp + dq\,\text{cof.}\,\omega - q\,d\omega\,\text{fin.}\,\omega}{p+q\,\text{cof.}\,\omega} + \frac{dr\,\text{cof.}\,\omega - r\,d\omega\,\text{fin.}\,\omega}{1-r\,\text{cof.}\,\omega}$ feu

$$\frac{dx}{x} = \frac{dp + dq\,\text{cof.}\,\omega}{p+q\,\text{cof.}\,\omega} + \frac{dr\,\text{cof.}\,\omega}{1-r\,\text{cof.}\,\omega} - \frac{(pr+q)\,d\omega\,\text{fin.}\,\omega}{(p+q\,\text{cof.}\,\omega)(1-r\,\text{cof.}\,\omega)}.$$

Nunc fiat fubftitutio in expreffione $Qxx + nnx - P$ quae abibit in hanc formam:

$$\left\{\begin{array}{lll} +Qpp & +2Qpq\,\text{cof.}\,\omega & +Qqq\,\text{cof.}\,\omega^2 \\ +nnP & +nnq\,\text{cof.}\,\omega & -nnqr\,\text{cof.}\,\omega^2 \\ -P & -nnpr\,\text{cof.}\,\omega & -Prr\,\text{cof.}\,\omega^2 \\ & +2Pr\,\text{cof.}\,\omega & \end{array}\right\} : (1-r\,\text{cof.}\,\omega)^2$$

Hic primo ſtatuatur $nn(pr-q)=2Pr+2Qpq$, deinde ſit $Qpp+nnp-P=-Qqq+nnqr+Prr$; vt fiat

$$\sqrt{(Qxx+nnx-P)}=\frac{\sqrt{(Qpp+nnp-P)}}{1-r\,\text{cof.}\,\omega}\,\text{ſin.}\,\omega \text{ ideoque}$$

$$\frac{dx}{x}\sqrt{P}=\frac{-d\Phi\sqrt{(Qpp+nnp-P)}}{1-r\,\text{cof.}\,\omega}\,\text{ſin.}\,\omega,$$

vnde ſequentes determinationes deducentur.

XVII. Quaeramus primo rationem inter P et Q quae ob $nn=\frac{2Pr+2Qpq}{pr-q}$ ex aequatione

$$Qpp+nnp-P=-Qqq+nnqr+Prr$$

ita reperitur:

$$Q(p^3r-pqqr+ppq-q^3)=P(-pr+pr^3-q+qrr)$$

quae per $pr+q$ diuiſa dat

$$Q(pp-qq)=-P(1-rr) \text{ ſeu } Q=\frac{-P(1-rr)}{pp-qq}.$$

Hinc prior determinatio $nn(pr-q)=2Pr+2Qpq$ praebet

$$nn(pr-q)=2Pr-\frac{2Ppq(1-rr)}{pp-qq}=\frac{2P(ppr-qqr-pq+pqrr)}{pp-qq}$$

et per $pr-q$ diuidendo $nn=\frac{2P(p+qr)}{pp-qq}$, ita vt ſit

$$\frac{pp-qq}{p+qr}=\frac{2P}{nn} \quad \text{et} \quad \frac{1-rr}{p+qr}=\frac{-2Q}{nn}.$$

Deinde

Deinde loco nn iterum ſcribendo $\frac{2Pr+2Qpq}{pr-q}$ fit

$$Qpp+nnp-P=\frac{(pr+q)(P+Qpp)}{pr-q}=\frac{P(pr+q)^2}{pp-qq}$$

vnde concludimus:

$$\frac{dx}{x}=\frac{-(pr+q)\,d\Phi\,\sin.\omega}{(1-r\cos.\omega)\sqrt{(pp-qq)}}.$$

At ob $x=\frac{p+q\cos.\omega}{1-r\cos.\omega}$, forma differentialis $\frac{dx}{x}$ ita exhiberi poteſt vt ſit

$$\frac{dx}{x}=\frac{dp+dq\cos.\omega}{x(1-r\cos.\omega)}+\frac{dr\cos.\omega}{1-r\cos.\omega}-\frac{(pr+q)d\omega\sin.\omega}{x(1-r\cos.\omega)^2}=\frac{-(pr+q)\,d\Phi\,\sin.\omega}{(1-r\cos.\omega)\sqrt{(pp-qq)}}.$$

Quocirca erit

$$\frac{(pr+q)d\omega\sin.\omega}{1-r\cos.\omega}=\frac{(pr+q)x\,d\Phi\sin.\omega}{\sqrt{(pp-qq)}}+dp+dq\cos.\omega+xdr\cos.\omega.$$

XVIII. Quodſi iam formulas ſuperiores ad P et Q reductas differentiemus, ad ſequentes expreſſiones perueniemus:

$$+dp(pp+2pqr+qq)-dq(2pq+ppr+qqr)-qdr(pp-qq)=\frac{2(p+qr)^2dP}{nn}.$$

$$dp(1-rr)+rdq(1-rr)+qdr(1+rr)+2prdr=\frac{2(p+qr)^2dQ}{nn}.$$

vnde cum differentialia dP et dQ dentur, bina tantum trium elementorum dp, dq et dr definiuntur, tertio quaſi arbitrio noſtro relicto. Verum ob $dx=\frac{-(pr+q)x\,d\Phi\sin.\omega}{(1-r\cos.\omega)\sqrt{(pp-qq)}}$

erit: $dP=\frac{-a^3x\,d\Phi}{c}.\,Mxx\,\text{ſin.}\,\eta$ et

$$dQ=\frac{-a^3x\,d\Phi}{c}\left(M\,\text{ſin.}\,\eta-\frac{N(pr+q)\sin.\omega}{(1-r\cos.\omega)\sqrt{(pp-qq)}}\right).$$

Vel

Vel etiam angulum ω pro lubitu aſſumere licet, ac tum binis illis aequationibus hanc tertiam iungendo $dp + dq \operatorname{cof.} \omega + x\, dr \operatorname{cof.} \omega = \frac{(pr+q)\, d\omega \operatorname{fin.} \omega}{1 - r \operatorname{cof.} \omega} - \frac{(pr+q) x\, d\Phi \operatorname{fin.} \omega}{\surd(pp-qq)}$ omnia tria elementa dp, dq et dr definiri poterunt. Denique ob $2\,P = \frac{nn(pp-qq)}{p+qr}$ erit $d\zeta = \frac{xx\, d\Phi \surd(p+qr)}{n\surd(pp-qq)}$.

XIX. Mirum videbitur, quod in hac reductione angulus ω arbitrio noſtro relinquatur, cum certe poſitio et motus lineae abſidum minime a noſtra voluntate pendeant. Verum hic perpendi oportet, eatenus tantum diſtantiam $v = cx$ fieri maximam vel minimam facto $\operatorname{fin.} \omega = 0$, quatenus idem angulus ω non in reliquas quantitates ita ingreditur, vt in valore pro $\frac{dx}{x}$ inuento factor $\operatorname{fin.} \omega$ iterum tollatur. Quodſi exempli cauſa reperiretur $\surd(pp-qq) = s \operatorname{fin.} \omega$, minime amplius concludere liceret poſito $\operatorname{fin.} \omega = 0$, formulam $\frac{dx}{x\, d\Phi}$ eſſe euanituram. Quocirca angulus ω neutiquam inter quantitates aſſumtas admitti poteſt, niſi forte conſtet a cuiusmodi angulo poſitio lineae abſidum pendeat.

XX. Antequam hunc caſum deſeram, binas illas aequationes differentiales pro elementis dp, dq et dr inuentas diligentius examinaſſe iuuabit. Ac ſi inde primo elementum dp elidatur reperitur:

$$\frac{dq(1-rr)(pr+q)}{p+qr} + dr(pr+q) = \frac{-(1-rr)\,dP + (pp+qq+2pqr)\,dQ}{nn}$$

ſin autem inde elementum dq exterminetur, prodit

$$\frac{dp(1-rr)(pr+q)}{p+qr} + \frac{dr(pr+q)^2}{p+qr} = \frac{r(1-rr)\,dP + (2pq+ppr+qqr)\,dQ}{nn}$$

Eiecto

Eiecto autem elemento dr obtinetur

$$dp(pr+q)-\frac{dq(pr+q)^2}{p+qr}=\frac{(2pr+q+qrr)dP+q(pp-qq)dQ}{nn}.$$

Quod ſi iam harum binas quasque in locum illarum ſubſtituamus, calculus haud parum fiet ſimplicior hae vero videntur commodiſſimae:

$$dp-\frac{dq(pr+q)}{p+qr}=\frac{(2pr+q+qrr)dP+q(pp-qq)dQ}{nn(pr+q)}$$

$$dr+\frac{dq(1-rr)}{p+qr}=-\frac{(1-rr)dP+(pp+qq+2pqr)dQ}{nn(pr+q)}.$$

Vnde aſſumto q reliqua elementa facile determinantur ſin autem angulus ω vt cognitus ſpectetur, hinc valores pro dp et dr in poſtrema aequatione differentiali ſupra data (XVIII.) ſubſtituti determinationem elementi dq ſuppeditabunt. Peruenitur autem ad hanc aequationem:

$$\begin{aligned}\frac{dq(pr+q)\sin.\omega^2}{p+qr}=&(pr+q)\sin.\omega\left(d\omega-\frac{d\Phi(p+q\cos.\omega)}{\surd(pp-qq)}\right)\\ &-\frac{dP}{nn(pr+q)}(2pr+q+qrr-(p+qr)(1+rr)\cos.\omega\\ &\qquad -q(1-rr)\cos.\omega^2)\\ &-\frac{dQ}{nn(pr+q)}(q(pp-qq)+(p+qr)(pp+qq)\cos.\omega\\ &\qquad +q(pp+qq+2pqr)\cos.\omega^2).\end{aligned}$$

XXI. Subſtituendo denique hic pro dP et dQ valores ſupra indicatos (XVIII.)

$$\begin{aligned}\frac{dq\sin.\omega}{p+qr}=&d\omega-\frac{d\Phi(p+q\cos.\omega)}{\surd(pp-qq)}\\ &+\frac{a^3Mx\,d\Phi\sin.\eta\sin.\omega}{nnc(pr+q)(1-r\cos.\omega)^2}(2pp-qq+pqr+(p+qr)(3q-pr)\cos.\omega\\ &\qquad -q(pr-q+2qrr)\cos.\omega^2)\\ &-\frac{a^3N\,x\,d\Phi}{nnc(pr+q)(1-r\cos.\omega)}\left(\frac{q(pp-qq)+(p+qr)(pp+qq)\cos.\omega+q(pp+qq+2pqr)\cos.\omega^2}{\surd(pp-qq)}\right)\end{aligned}$$

ſi quidem nunc totam aequationem per $(pr+q)\,\text{ſin.}\,\omega$ diuidere licuit; commode enim vſu venit, vt membrum elemento $M\,\text{ſin.}\,\eta$ affectum factorem $1-\text{coſ.}\,\omega^2 = \text{ſin.}\,\omega^2$ ſortiretur.

Quodſi iam hic ponatur $q=0$, reductio reſultat prior ſcribendo q' loco r, ſin autem ponatur $r=0$, reductio habetur poſterior, vnde intelligitur quanto latius pateat haec reductio generalior ambas praecedentes in ſe complectens. Loco dP et dQ etiam in praecedentibus formulis ſubſtituantur valores ac reperietur:

$$dp = \frac{dq(pr+q)}{p+qr} - \frac{a^3 M x\, d\Phi\, \text{ſin.}\,\eta}{nnc(1-r\,\text{coſ.}\,\omega)^2}\big(2pp-qq+pqr+2q(p+qr)\,\text{coſ.}\,\omega + q(pr+q)\,\text{coſ.}\,\omega^2\big) + \frac{a^3 N x\, d\Phi\, \text{ſin.}\,\omega}{nnc(1-r\,\text{coſ.}\,\omega)} \cdot q\sqrt{(pp-qq)}$$

$$dr = \frac{-dq(1-rr)}{p+qr} - \frac{a^3 M x\, d\Phi\, \text{ſin.}\,\eta}{nnc(1-r\,\text{coſ.}\,\omega)^2}\big(pr+q-2(p+qr)\,\text{coſ.}\,\omega + (pr-q+2qrr)\,\text{coſ.}\,\omega^2\big) + \frac{a^3 N x\, d\Phi\, \text{ſin.}\,\omega}{nnc(1-r\,\text{coſ.}\,\omega)}\left(\frac{pp+qq+2pqr}{\sqrt{(pp-qq)}}\right)$$

et loco dq valorem ſuperiorem ſubſtituendo:

$$\frac{dp\,\text{ſin.}\,\omega}{p+qr} = \frac{pr+q}{p+qr}\left(d\omega - \frac{d\Phi(p+q\,\text{coſ.}\,\omega)}{\sqrt{(pp-qq)}}\right) - \frac{a^3 M x\, d\Phi\, \text{ſin.}\,\eta\, \text{ſin.}\,\omega\, \text{coſ.}\,\omega}{nnc(1-r\,\text{coſ.}\,\omega)^2}(pr-q+2qr\,\text{coſ.}\,\omega) - \frac{a^3 N x\, d\Phi\, \text{coſ.}\,\omega}{nnc(1-r\,\text{coſ.}\,\omega)}\left(\frac{pp+qq+2pq\,\text{coſ.}\,\omega}{\sqrt{(pp-qq)}}\right)$$

$$\frac{dr(pr+q)\,\text{ſin.}\,\omega}{p+qr} = -\frac{(pr+q)(1-rr)}{p+qr}\left(d\omega - \frac{d\Phi(p+q\,\text{coſ.}\,\omega)}{\sqrt{(pp-qq)}}\right) - \frac{a^3 M x\, d\Phi\, \text{ſin.}\,\eta\, \text{ſin.}\,\omega}{nnc(1-r\,\text{coſ.}\,\omega)^2}\big(2p+qr-prr+(q-3pr-3qrr+pr^3)\,\text{coſ.}\,\omega + r(pr-q+2qrr)\,\text{coſ.}\,\omega^2\big)$$

+

$$+\frac{a^3 N x d\Phi}{nnc(1-r\cos.\omega)\sqrt{(pp-qq)}}(ppr+2pq+qqr+(1-rr)(pp+qq)\cos.\omega$$
$$-r(pp+qq+2pqr)\cos.\omega^2).$$

XXII. Si excentricitas orbitae ſatis fuerit notabilis, commodiſſime reductione prima vtemur, quia ibi aberrationes a motu regulari in ellipſi facto definiuntur. Sin autem excentricitas fuerit quam minima vel adeo nulla, neque primam reductionem neque ſecundam in vſum vocare licebit, quandoquidem anomaliae ω tum ne locus quidem relinquitur; ac ſpectata quantitate q ſaltem vt minima, quia ea denominatorem formulae pro $d\Phi - d\omega$ inventae afficit, motus lineae abſidum nimis fit vagus et incertus. Neque etiam adhuc perſpicio, quomodo poſtrema reductio ſumendo $\omega = \eta$ in hac inveſtigatione vtilitatem afferre poſſet, tam propter multitudinem, quam complicationem formularum, quas reſolui oporteret. Nihilo tamen minus caſus quo excentricitas plane euaneſceret ſine dubio pro ſimpliciſſimo eſſet habendus; ex quo in eius reſolutione merito omne ſtudium collocandum videtur quo his difficultatibus ſuperatis deinceps veri motus lunaris inueſtigatio feliciori ſucceſſu ſuſcipi, neque tantum ad vſum practicum ſatis conuenienter, ſed etiam multo accuratius abſolui queat. Neque autem ad hunc caſum euoluendum alia via aptior videtur, quam vt ad ipſas aequationes differentio-differentiales reuertamur indeque approximationes idoneas petamus.

Inueſtigatio motus ſi Luna in ecliptica ſine vlla excentricitate ſol autem vniformiter moueretur.

XXIII. Ponamus in ipſis aequationibus differentio-differentialibus $v = cx$, et habebimus.

$$1^\circ.\ 2dx\,d\Phi + x\,dd\Phi + \frac{a^3}{c} M\,d\zeta^2 \,\text{ſin.}\,\eta = 0$$

$$2^\circ.\ ddx - x\,d\Phi^2 + \frac{nn}{xx} d\zeta^2 + \frac{a^3}{c} N\,d\zeta^2 = 0$$

et quia motus ſolis aſſumitur vniformis erit $u = a$ et $\theta = \zeta$ ideoque $\Phi = \zeta + \eta$, hinc

$$\frac{a^3}{c} M = 3x\,\text{coſ.}\,\eta - \frac{3cxx}{2a}(1 - 5\,\text{coſ.}\,\eta^2) - \frac{5ccx^3}{2aa}(3\,\text{coſ.}\,\eta - 7\,\text{coſ.}\,\eta^3)$$

$$\frac{a^3}{c} N = x(1 - 3\,\text{coſ.}\,\eta^2) + \frac{3cxx}{2a}(3\,\text{coſ.}\,\eta - 5\,\text{coſ.}\,\eta^3) - \frac{ccx^3}{2aa}(3 - 30\,\text{coſ.}\,\eta^2 + 35\,\text{coſ.}\,\eta^4)$$

vnde binae noſtrae aequationes erunt

$$1^\circ. \left\{ \begin{array}{l} \frac{2dx\,d\Phi}{d\zeta^2} + \frac{x\,dd\Phi}{d\zeta^2} \\ + 3x\,\text{ſin.}\,\eta\,\text{coſ.}\,\eta - \frac{3c}{2a}xx\,\text{ſin.}\,\eta(1 - 5\,\text{coſ.}\,\eta^2) - \frac{5cc}{2aa}x^3\,\text{ſin.}\,\eta(3\,\text{coſ.}\,\eta - 7\,\text{coſ.}\,\eta^3) \end{array} \right\} = 0$$

$$2^\circ. \left\{ \begin{array}{l} \frac{ddx}{d\zeta^2} - \frac{x\,d\Phi^2}{d\zeta^2} + \frac{nn}{xx} \\ + x(1 - 3\,\text{coſ.}\,\eta^2) + \frac{3cxx}{2a}(3\,\text{coſ.}\,\eta - 5\,\text{coſ.}\,\eta^3) - \frac{cc}{2aa}x^3(3 - 30\,\text{coſ.}\,\eta^2 + 35\,\text{coſ.}\,\eta^4) \end{array} \right\} = 0$$

vbi cum $\frac{c}{a}$ ſit quantitas quam minima, has aequationes in partes ſectas concipere licet, quae ſequentibus multo ſint maiores, ad quem ordinem etiam approximationem accommodari conuenit.

XXIV.

XXIV. Si omnis perturbatio abesset, foret ob excentricitatem euanescentem, vti vidimus, $x = 1$ et $\frac{d\Phi}{d\zeta} = n$ hincque $\frac{d\eta}{d\zeta} = n - 1$. Nunc perturbatione accedente statuamus:

$$x = 1 + P + Q + R \text{ et } \frac{d\Phi}{d\zeta} = n + p + q + r$$

hincque $\frac{d\eta}{d\zeta} = n - 1 + p + q + r$, vbi P, Q, R et p, q, r series maxime decrescentes referant, cum seriebus superioribus ex perturbatione natis comparandas ac has ipsas quantitates tanquam functiones anguli η spectemus, siquidem nouimus omnes inaequalitates ab hoc solo angulo pendere. Erit ergo $dx = dP + dQ + dR$ et per $\frac{d\eta}{d\zeta} = (n-1) + p + q + r$ multiplicando:

$$\frac{dx}{d\zeta} = \left\{ \begin{matrix} (n-1)dP + (n-1)dQ + (n-1)dR \\ + p\,dP \quad + \quad p\,dQ \\ + \; q\,dP \end{matrix} \right\} : d\eta$$

quae forma differentiata sumto iam elemento $d\eta$ constante dabit

$$\frac{ddx}{d\zeta} = \left\{ \begin{matrix} (n-1)ddP + (n-1)ddQ + (n-1)ddR \\ + p\,ddP \;+\; p\,ddQ \\ + dP\,dp + dp\,dQ \\ + \; q\,ddP \\ + \; dq\,dP \end{matrix} \right\} : d\eta$$

multiplicetur denuo per $\frac{d\eta}{d\zeta}$, prodibitque

$$\frac{ddx}{d\zeta^2} = \left\{\begin{array}{r}(n-1)^2 ddP + (n-1)^2 ddQ + (n-1)^2 ddR \\ +2(n-1)pddP + 2(n-1)pddQ \\ +(n-1)dpdP + (n-1)dpdQ \\ +2(n-1)qddP \\ +(n-1)dqdP \\ +ppddP \\ +pdpdP\end{array}\right\} : d\eta^2$$

simili modo cum sit

$\frac{dd\Phi}{d\zeta} = dp + dq + dr$ per $\frac{d\eta}{d\zeta}$ multiplicando erit

$$\frac{dd\Phi}{d\zeta^2} = \left\{\begin{array}{r}(n-1)dp + (n-1)dq + (n-1)dr \\ +pdp \quad +pdq \\ +qdp\end{array}\right\} : d\eta$$

et $\frac{d\Phi^2}{d\zeta^2} = \begin{array}{l}nn + 2np + 2nq + 2nr \\ + pp + 2pq\end{array}$ ac tandem

$$\frac{1}{xx} = \begin{array}{l}1 - 2P - 2Q - 2R \\ + 3PP + 6PQ \\ - 4P^3.\end{array}$$

XXV. Hos igitur valores in aequationes nostras introductos secundum ordines stabilitos distribuamus, vbi quidem elementum $d\eta$, quippe quod sponte intelligitur, omittamus.

* Aequa-

* Aequatio Prima

II.	III.	IV.
$+2n(n-1)dP$	$+2n(n-1)dQ$	$+2n(n-1)dR$
$+(n-1)dp$	$+2npdP$	$+2npdQ$
$+3\sin.\eta\cos.\eta$	$+2(n-1)pdP$	$+2nqdP$
	$+(n-1)dq$	$+2(n-1)pdQ$
	$+pdp$	$+2ppdP$
	$+(n-1)Pdp$	$+2(n-1)qdP$
	$+3P\sin.\eta\cos.\eta$	$+(n-1)dr$
	$-\frac{3c}{2a}\sin.\eta(1-5\cos.\eta^2)$	$+pdq$
		$+qdp$
		$+(n-1)Pdq$
		$+Ppdp$
		$+(n-1)Qdp$
		$+3Q\sin.\eta\cos.\eta$
		$-\frac{3c}{a}P\sin.\eta(1-5\cos.\eta^2)$
		$-\frac{5cc}{2aa}\sin.\eta(3\cos.\eta-7\cos.\eta^3)$

Hic ſcilicet ordo primus deeſt, quia ſublata perturbatione primae aequationis omnia membra ſponte euaneſcunt.

Pro

* Huius aequationis nonniſi integrale particulare hic quaeritur, quod ſcilicet hypotheſi aſſumtae, qua excentricitas euaneſcit, conueniat, et manifeſto huiusmodi habet formam $P = A + B\cos.\eta^2$. Integrale autem completum foret

$$P = A + B\cos.\eta^2 + M\sin.\tfrac{n}{n-1}\eta + N\cos.\tfrac{n}{n-1}\eta$$

vbi M et N ſunt conſtantes arbitrariae, quibus conditio excentricitatis continetur. Id quod peculiarem euolutionem meretur.

Pro ſequentibus autem ordinibus terminos ad quemuis pertinentes ſeorſim nihilo aequari oportet.

XXVI. Aequatio altera ſequenti modo in membra diſtribuitur.

Aequatio ſecunda.

I.	II.	III.	IV.
$-nn$	$(n-1)^2 ddP$	$+(n-1)^2 ddQ$	$+(n-1)^2 ddR$
$+nn$	$-2np$	$+2(n-1)pddP$	$+2(n-1)pddQ$
	$-3nnP$	$+(n-1)dpdP$	$+2(n-1)qddP$
	$+(1-3\cos.\eta^2)$	$-2nq$	$+(n-1)dpdQ$
		$-pp$	$+(n-1)dqdP$
		$-2nPp$	$+ppddP$
		$-3nnQ$	$+pdpdP$
		$+3nnPP$	$-2nr$
		$+P(1-3\cos.\eta^2)$	$-2pq$
		$+\frac{3c}{2a}(3\cos.\eta-5\cos.\eta^3)$	$-2nPq$
			$-Ppp$
			$-2nQq$
			$-3nnR$
			$+6nnPQ$
			$-4nnP^3$
			$+Q(1-3\cos.\eta^2)$
			$+\frac{3c}{a}P(3\cos.\eta-5\cos.\eta^3)$
			$-\frac{cc}{2aa}(3-30\cos.\eta^2+35\cos.\eta^4)$

vbi membrum primum ſponte ſe tollit.

XXVII. Secundus ordo ex vtraque aequatione quantitatibus ſecundo loco aſſumtis definiendis inſer-

vit,

vit, quae sunt P et p, ideoque ex his duabus aequationibus determinandae.

1°. $2n(n-1)dP + (n-1)dp + 3d\eta \sin.\eta \cos.\eta = 0$

2°. $(n-1)^2 ddP - 2npd\eta^2 - 3nnPd\eta^2 + d\eta^2(1-3\cos.\eta^2) = 0.$

Prior autem integrata dat $2n(n-1)P + (n-1)p = \Delta + \frac{3}{2}\cos.\eta^2$ seu $p = -2nP + \frac{\Delta}{n-1} + \frac{3\cos.\eta^2}{2(n-1)}$, qui valor in altera substitutus praebet:

$(n-1)^2 ddP + nnPd\eta^2 - \frac{2n}{n-1}\Delta d\eta^2 - \frac{3nd\eta^2 \cos.\eta^2}{n-1} + d\eta^2(1-3\cos.\eta^2) = 0$

seu $(n-1)^2 ddP + nnPd\eta^2 - \frac{2n}{n-1}\Delta d\eta^2 - \frac{3(2n-1)}{n-1}d\eta^2\cos.\eta^2 + d\eta^2 = 0.$

Statuamus, quandoquidem forma integralis sponte patet $P = A + B\cos.\eta^2$, si esset $nn = 4(n-1)^2$ poni deberet $P = A + B\cos.\eta^2 + C\eta \sin.\eta \cos.\eta$, erit $\frac{dP}{d\eta} = -2B\sin.\eta\cos.\eta$ et $\frac{ddP}{d\eta^2} = -2B\cos.\eta^2 + 2B\sin.\eta^2 = 2B - 4B\cos.\eta^2$ et facta substitutione oritur:

$$\left.\begin{aligned} &+2(n-1)^2 B + nnA - \tfrac{2n}{n-1}\Delta + 1 \\ &-4(n-1)^2 B\cos.\eta^2 + nnB\cos.\eta^2 - \tfrac{3(2n-1)}{n-1}\cos.\eta^2 \end{aligned}\right\} = 0$$

hincque $B = \frac{-3(2n-1)}{(n-1)(n-2)(3n-2)}$ et $\frac{2n}{n-1}\Delta = 1 + nnA + 2(n-1)^2 B$

et $p = -2nA \quad -2nB\cos.\eta^2$

$\quad + \frac{1}{2n} \quad + \frac{3}{2(n-1)}\cos.\eta^2.$

$\quad + \frac{1}{2}nA$

$\quad + \frac{(n-1)^2}{n}B.$

Quare si ponamus:

$P = A + B\cos.\eta^2$ et $p = \mathfrak{A} + \mathfrak{B}\cos.\eta^2$

 quan-

quantitas A arbitrio noſtro relinquitur. eritque

$$\mathrm{B} = \frac{-3(2n-1)}{(n-1)(n-2)(3n-2)} \text{ atque}$$

$$\mathfrak{A} = \frac{1}{2n} - \frac{3}{2}n\mathrm{A} + \frac{(n-1)^2}{n}\mathrm{B} \text{ et } \mathfrak{B} = \frac{3}{2(n-1)} - 2n\mathrm{B}.$$

Quantitas A ideo manet indefinita, vt vel diſtantia media vel motus medius ad veritatem definiri posſit, ob perturbationem enim, ſi c conueniat cum diſtantia media n non amplius cum ratione $\frac{d\Phi}{d\zeta}$ congruit et viciſſim.

XXVIII. Ad quantitates tertii ordinis Q et q determinandas, has habemus aequationes:

$$2n(n-1)d\mathrm{Q} + (n-1)dq + 2(2n-1)pd\mathrm{P} + pdp + (n-1)\mathrm{P}dp$$
$$+ 3\mathrm{P}\,\text{ſin.}\eta\,\text{coſ.}\eta - \frac{3c}{2a}\,\text{ſin.}\eta(1 - 5\,\text{coſ.}\eta^2) = 0$$

$$(n-1)^2 dd\mathrm{Q} - 2nq - 3nn\mathrm{Q} + 2(n-1)pdd\mathrm{P} + (n-1)dpd\mathrm{P} - pp - 2n\mathrm{P}p$$
$$+ 3nn\mathrm{PP} + \mathrm{P}(1 - 3\,\text{coſ.}\eta^2) + \frac{3c}{2a}(3\,\text{coſ.}\eta - 5\,\text{coſ.}\eta^3) = 0.$$

Cum autem ſit $\mathrm{P} = \mathrm{A} + \mathrm{B}\,\text{coſ.}\eta^2$ et $p = \mathfrak{A} + \mathfrak{B}\,\text{coſ.}\eta^2$ erit $d\mathrm{P} = -2\mathrm{B}\,\text{ſin.}\eta\,\text{coſ.}\eta$ et $dp = -2\mathfrak{B}\,\text{ſin.}\eta\,\text{coſ.}\eta$

hi valores in prima aequatione ſubſtituti dant

$$\frac{2n(n-1)d\mathrm{Q} + (n-1)dq}{d\eta} \begin{array}{l} -4(2n-1)\mathfrak{A}\mathrm{B} \\ -2\mathfrak{A}\mathfrak{B} \\ -2(n-1)\mathfrak{B}\mathrm{A} \\ +3\mathrm{A} \end{array} \text{ſin.}\eta\,\text{coſ.}\eta \begin{array}{l} -4(2n-1)\mathfrak{B}\mathrm{B} \\ -2\mathfrak{B}\mathfrak{B} \\ -2(n-1)\mathfrak{B}\mathrm{B} \\ +3\mathrm{B} \end{array} \text{ſin.}\eta\,\text{coſ.}\eta^3 - \frac{3c}{2a}\,\text{ſin}\,\eta(1 - 5\,\text{coſ.}\eta^2) = 0$$

vnde

vnde per integrationem elicitur

$$2n(n-1)Q+(n-1)q+(2(2n-1)\mathfrak{A}B+\mathfrak{A}B+(n-1)\mathfrak{B}A-\tfrac{3}{2}A)\cos.\eta^2 + \tfrac{3c}{2a}\cos.\eta - \tfrac{5c}{2a}\cos.\eta^3 = \Delta$$
$$+((2n-1)\mathfrak{B}B+\tfrac{1}{2}\mathfrak{B}\mathfrak{B}+\tfrac{1}{2}(n-1)\mathfrak{B}B-\tfrac{3}{4}B)\cos.\eta^4$$

fit breuitatis gratia

$$2(2n-1)\mathfrak{A}B+\mathfrak{A}\mathfrak{B}+(n-1)\mathfrak{B}A-\tfrac{3}{2}A=\alpha$$
$$\tfrac{1}{2}(5n-3)\mathfrak{B}B+\tfrac{1}{2}\mathfrak{B}\mathfrak{B}-\tfrac{3}{4}B=\beta$$

erit $q=-2nQ+\frac{\Delta}{n-1}-\frac{\alpha}{n-1}\cos.\eta^2-\frac{\beta}{n-1}\cos.\eta^4-\frac{3c}{2a(n-1)}\cos.\eta + \frac{5c}{2a(n-1)}\cos.\eta^3.$

Tum pro altera aequatione ob $ddP=2B-4B\cos.\eta^2$ est

$$\begin{array}{l|lll|l}
2(n-1)p\,ddP & 4(n-1)\mathfrak{A}B & -8(n-1)\mathfrak{A}B\cos.\eta^2 & -8(n-1)\mathfrak{B}B\cos.\eta^4 & \\
+(n-1)dp\,dP & & +4(n-1)\mathfrak{B}B & & \\
 & & +4(n-1)\mathfrak{B}B & -4(n-1)\mathfrak{B}B & \\
-pp & -\mathfrak{A}\mathfrak{A} & -2\mathfrak{A}\mathfrak{B} & -\mathfrak{B}\mathfrak{B} & \\
-2nPp & -2n\mathfrak{A}A & -2n\mathfrak{A}B & -2nB\mathfrak{B} & \\
+3nnPP & & -2n\mathfrak{B}A & & \\
 & +3nnAA & +6nnAB & +3nnBB & \\
+P(1-3\cos.\eta^2) & +A & -3A & -3B & \\
 & & +B & & +\frac{9c}{2a}\cos.\eta-\frac{15c}{2a}\cos.\eta^3 \\
\hline
+\frac{(n-1)^2\,ddQ}{a\eta^2} & -\frac{2n\Delta}{n-1} & +\frac{2n\alpha}{n-1} & +\frac{2n\beta}{n-1} & +\frac{3nc}{a(n-1)}\cos.\eta-\frac{5nc}{a(n-1)}\cos.\eta^3 \\
+nnQ & & & &
\end{array}$$

XXIX. Pro resolutione huius aequationis poni oportere manifestum est:

$$Q=C+D\cos.\eta^2+E\cos.\eta^4+F\cos.\eta+G\cos.\eta^3$$

eritque
$$\frac{ddQ}{d\eta^2}=\begin{array}{llll}2D & -4D\cos.\eta^2-16E\cos.\eta^4 & -F\cos.\eta & -9G\cos.\eta^3\\ & +12E & +6G & \end{array}$$

 Hinc

Hinc istae nascuntur aequationes:

$$0 = nnC + 2(n-1)^2 D + 4(n-1)\mathfrak{A}B - \mathfrak{A}\mathfrak{A} - 2n\mathfrak{A}A + 3nnAA + A - \frac{2n\Delta}{n-1}$$

$$0 = 12(n-1)^2 E - (n-2)(3n-2)D - 2(5n-4)\mathfrak{A}B + 8(n-1)\mathfrak{B}B - 2\mathfrak{A}\mathfrak{B} - 2n\mathfrak{B}A + 6nnAB - 3A + B + \frac{2n\alpha}{n-1}$$

$$0 = -(3n-4)(5n-4)E - 12(n-1)\mathfrak{B}B - \mathfrak{B}\mathfrak{B} - 2n\mathfrak{B}B + 3nnBB - 3B + \frac{2n\mathfrak{c}}{n-1}$$

$$0 = 6(n-1)^2 G + (2n-1)F + \frac{3(5n-3)c}{2(n-1)a}$$

$$0 = -(2n-3)(4n-3)G - \frac{5(5n-3)c}{2(n-1)a}$$

vnde ordine retrogado litterae G, F, E et D determinantur tum vero ex prima valor ipsius Δ quaeratur, vt C maneat quantitas indefinita, ac tum etiam valor litterae q innotescet per quantitates iam definitas A, B, $\mathfrak{A}$ et $\mathfrak{B}$, ex quibus α et $\mathfrak{c}$ resultant. Si enim ponatur $q = \mathfrak{C} + \mathfrak{D}\cos.\eta^2 + \mathfrak{E}\cos.\eta^4 + \mathfrak{F}\cos.\eta + \mathfrak{G}\cos.\eta^3$ erit

$$\mathfrak{C} = -2nC + \frac{\Delta}{n-1}$$

$$\mathfrak{D} = -2nD - \frac{\alpha}{n-1}$$

$$\mathfrak{E} = -2nE - \frac{\mathfrak{c}}{n-1}$$

$$\mathfrak{F} = -2nF - \frac{3c}{2(n-1)a}$$

$$\mathfrak{G} = -2nG + \frac{5c}{2(n-1)a}.$$

XXX. Simili modo perturbationes sequentium ordinum ex aequationibus supra datis colligi posse per

per ſe eſt manifeſtum. Calculus quidem haud parum ſit moleſtus ac taedioſus; verum ſufficit methodum eum euoluendi hic dilucide expoſuiſſe, ita vt nulla difficultas ſit metuenda praeter calculi prolixitatem. Interim ſequentia membra ita fiunt parva, vt pro vſu aſtronomico facile reiici queant. Inventis autem his omnibus litteris binae aequationes quibus iam motus lunae continetur ita ſe habent:

$$x = 1 + A + C + (B + D)\cos.\eta^2 + E\cos.\eta^4 + F\cos.\eta + G\cos.\eta^5$$

$$\frac{d\Phi}{d\zeta} = n + \mathfrak{A} + \mathfrak{C} + (\mathfrak{B} + \mathfrak{D})\cos.\eta^2 + \mathfrak{E}\cos.\eta^4 + \mathfrak{F}\cos.\eta + \mathfrak{G}\cos.\eta^5$$

existente $d\Phi = d\eta + d\zeta$, et $\frac{d\eta}{d\zeta} = \frac{d\Phi}{d\zeta} - 1$; vbi manifeſtum eſt conſtantem A nihilo aequalem poni poſſe, dummodo C in calculo retineatur, vt ratio media $d\Phi : d\zeta$ ob ſequentes terminos aliquantillum a vero valore n depulſa corrigi et ad veritatem reduci poſſit modo infra exponendo.

Operae autem pretium erit hos ſingulos terminos euoluere ſumendo pro n valorem per obſervationes definitum, quia eaedem inaequalitates; etiamſi ad caſum hunc maxime particularem pertinentes, tamen in vero quoque lunae motu locum inueniunt.

XXXI. Sumamus ergo $A = 0$, et cum motum lunae medium cum ſolis motu comparando ſit $n = 13,25586$, calculus pro determinatione harum inaequalitatum ita ſe habebit:

$n = 13,25586$	$l(2n-1) = 1,4067394$
$n-1 = 12,25586$	$l\,3 = 0,4771213$
$2n-1 = 25,51172$	$l\,3(2n-1) = 1,8838607$
$3n-2 = 37,76758$	$l(n-1) = 1,0883440$
$n-2 = 11,25586$	$l(n-2) = 1,0513787$
$B = -\ 0,0146899$	$l(3n-2) = 1,5771191$
$\frac{3}{2(n-1)} = +0,1223904$	$3,7168418$
$2nB = -\ 0,3894546$	$l - B = 8,1670189$
$\mathfrak{B} = +\ 0,5118450$	$l\frac{3}{2} = 0,1760913$
$\frac{1}{2n} = +\ 0,0377191$	$l(n-1) = 1,0883440$
$\frac{(n-1)^2}{n}B = -0,1664559$	$l\frac{3}{2(n-1)} = 9,0877473$
$\mathfrak{A} = -\ 0,1287368$	$l\,2 = 0,3010300$
	$l\,n = 1,1224079$
	$l\,2n = 1,4234379$
	$l - 2nB = 9,5904568$
	$l(n-1)^2 = 2,1766880$
	$l\frac{(n-1)^2}{n} = 1,0542801$
	$l - \frac{(n-1)^2}{n}B = 9,2212990$

XXXII. Nunc pro litteris α et β calculus ita ſe habebit:

	$l\mathfrak{A} = -\ 9,1097027$
$2(n-1)\mathfrak{A}B = +0,096492$	$lB = -\ 8,1670189$
$\mathfrak{A}\mathfrak{B} = -\ 0,065893$	$l\,2 = 0,3010300$
$\alpha = +\ 0,030599$	$l(2n-1) = 1,4067394$
$\frac{1}{2}(5n-3) = 31,63965$	$+\ 8,9844910$
$\frac{1}{2}(5n-3)\mathfrak{B}B = -0,237897$	$l\mathfrak{A} = -\ 9,1097027$
$\frac{1}{2}\mathfrak{B}\mathfrak{B} = +0,130993$	$l\mathfrak{B} = +\ 9,7091385$
$-\ 0,106904$	$-\ 8,8188412$
	$-\frac{3}{4}B$

$-\frac{3}{4}B = +0,011018$	$l\frac{1}{2}(5n-3) = 1,5002316$
$\mathfrak{C} = -0,095886$	$l\mathfrak{B} = +9,7091385$
	$lB = -8,1670189$
	$-9,3763890$
	$l\mathfrak{B}\mathfrak{B} = 9,4182770$
	$l2 = 0,3010300$
	$9,1172470$

Hinc primo quaeratur littera E.

$-12(n-1)\mathfrak{B}B = +1,105820$	$l\mathfrak{B}B = -7,8761574$
$-2n\mathfrak{B}B = +0,199340$	$l(n-1) = 1,0883440$
$+1,305160$	$l12 = 1,0791812$
$-\mathfrak{B}\mathfrak{B} = -0,261986$	$-0,0436826$
$+1,043174$	$l2n = 1,4234379$
$+3nnBB = +0,113756$	$-9,2995953$
$-3B = +0,044069$	$lBB = 6,3340378$
$+1,200999$	$lnn = 2,2448158$
$+\frac{2n\mathfrak{C}}{n-1} = -0,207419$	$l3 = 0,4771213$
$(3n-4)(5n-4)E = +0,993580$	$+9,0559749$
ergo $E = +0,00044603$	$l\mathfrak{C} = -8,9817552$
	$l2n = 1,4234379$
	$-0,4051931$
	$l(n-1) = 1,0883440$
	$-9,3168491$
	$l+0,99358 = 9,9972028$
	$l(3n-4) = 1,5534895$
	$l(5n-4) = 1,7943437$
	$3,3478332$
	$lE = +6,6493696$

XXXIII.

XXXIII. Porro pro littera D.

	$lE = +6,6493696$
$12(n-1)^2 E = +0,803968$	$l(n-1)^2 = 2,1766880$
$-2(5n-4)\mathfrak{A}B = -0,235557$	$l\,12 = 1,0791812$
$+0,568411$	$+9,9052388$
$+8(n-1)\mathfrak{B}B = -0,737223$	$l\,2\mathfrak{A}B = +7,5777516$
$-0,168812$	$l(5n-4) = 1,7943437$
$-2\mathfrak{A}\mathfrak{B} = +0,131786$	$+9,3720953$
$-0,037026$	$l\alpha = +8,4857072$
$B = -0,014690$	$l\frac{2n}{n-1} = 0,3350939$
$-0,051716$	$+8,8208011$
$+\frac{3n\alpha}{n-1} = +0,066192$	$l\,0,014476 = 8,1606486$
$(n-2)(3n-2)D = +0,014476$	$l(n-2)(3n-2) = 2,6284978$
Ergo $D = +0,000034052$	$lD = 5,5321508$
$2(n-1)^2 D = +0,010230$	$l(n-1)^2 = 2,1766880$
$+4(n-1)\mathfrak{A}B = +0,192984$	$l\,2 = 0,3010300$
$+0,203214$	$8,0098688$
$-\mathfrak{A}\mathfrak{A} = -0,016573$	$l\mathfrak{A}\mathfrak{A} = +8,2194054$
$+0,186641$	$l\ldots = 9,2710070$
$nnC - \frac{2n\Delta}{n-1} = +0,186641$	$l\,2n = 1,4234379$
$\frac{\Delta}{n-1} = 0,007040 + \frac{1}{2}nC$	$7,8475691$
Ergo $\mathfrak{C} = 0,007040 - \frac{3}{2}nC$	$lD = 5,5321508$
$-2nD = -0,0009028$	$l\,2nD = +6,9555887$
$-\frac{\alpha}{n-1} = -0,0024970$	$l\alpha = +8,4857072$
Ergo $\mathfrak{D} = -0,0033998$	$l(n-1) = 1,0883440$
$-2nE = -0,0118250$	$+7,3973632$
	$lE = +6,6493696$
$-\frac{E}{n-1} = +0,0078237$	$l\,2n = 1,4234379$
	$+8,0728075$

Ergo

Ergo $\mathfrak{E} = -0,0040013$

$(2n-3)(4n-3)G = -12,90796 \frac{c}{a}$

$G = -0,010975 . \frac{c}{a}$

$+6(n-1)^2 G = -9,89097 . \frac{c}{a}$

$\frac{3(5n-3)c}{2(n-1)a} = +7,74478 . \frac{c}{a}$

$(2n-1)F = +2,14619 . \frac{c}{a}$

Ergo $F = +0,084126 . \frac{c}{a}$

$-2nF = -2,23032 . \frac{c}{a}$

$-\frac{3c}{2(n-1)a} = -0,122390 . \frac{c}{a}$

$\mathfrak{F} = -2,35271 . \frac{c}{a}$

$-2nG = +0,29096 . \frac{c}{a}$

$+\frac{5c}{2(n-1)a} = +0,20399 . \frac{c}{a}$

$\mathfrak{G} = +0,49495 . \frac{c}{a}$

$l\mathfrak{E} = -8,9817552$
$l(n-1) = 1,0883440$
$-7,8934112$
$l\frac{5n-3}{2} = 1,5002316$
$l(n-1) = 1,0883440$
$l\frac{5n-3}{2(n-1)} = 0,4118876$
$l5 = 0,6989700$
$1,1108576$

$1,1108576 = l12,90796$
$1,3712844 = l(2n-3)$
$1,6991735 = l(4n-3)$
$3,0704579$
$-8,0403997 = lG$
$2,1766880 = l(n-1)^2$
$0,7781513 = l6$
$-0,9952390$

$l\frac{5n-3}{2(n-1)} = 0,4118876$
$l3 = 0,4771213$
$0,8890089$
$l2,14619 = 0,3316683$
$l(2n-1) = 1,4067394$
$lF = +8,9249289$
$lG = -8,0403997$
$l2n = 1,4234379$
$l2nF = +0,3483668$
$l2nG = -9,4638376.$

XXXIV. Ex his igitur valoribus colligimus:

$$x = 1 + C - 0{,}014656 \cos.\eta^2 + 0{,}000446 \cos.\eta^4$$
$$+ 0{,}084126.\tfrac{c}{a}\cos.\eta - 0{,}010975.\tfrac{c}{a}\cos.\eta^3$$
$$\tfrac{d\Phi}{d\zeta} = 13{,}134163 - \tfrac{3}{2}nC + 0{,}508445\cos.\eta^2 - 0{,}004001\cos.\eta^4$$
$$-2{,}3527\tfrac{c}{a}.\cos.\eta + 0{,}4949.\tfrac{c}{a}\cos.\eta^3$$

vbi conſtans C ita definiri debet, vt motus medius ex poſteriori forma erutus praeciſe conueniat cum motu medio ex obſeruationibus deducto. In hunc autem finem poteſtates $\cos.\eta^2$ et $\cos.\eta^4$ ad coſinus angulorum ſimplicium reduci debent, quia inde partes conſtantes emergunt cum principali coniungendae. Scilicet ob $\cos.\eta^2 = \frac{1}{2} + \frac{1}{2}\cos.2\eta$ et $\cos.\eta^4 = \frac{3}{8} + \frac{1}{2}\cos.2\eta + \frac{1}{8}\cos.4\eta$, fit pars conſtans:

$13{,}386885 - \frac{3}{2}nC$ ipſi $n = 13{,}25586$ aequanda, vnde fit $\frac{3}{2}nC = 0{,}131025$, ideoque $C = 0{,}0065895$.

Euolutis autem poteſtatibus $\cos.\eta$ reperitur

$$x = 0{,}999428 - 0{,}007105\cos.2\eta + 0{,}000056\cos.4\eta$$
$$+ 0{,}07589\tfrac{c}{a}\cos.\eta - 0{,}00274\tfrac{c}{a}\cos.3\eta$$
$$\tfrac{d\Phi}{d\zeta} = 13{,}25586 + 0{,}252222\cos.2\eta - 0{,}000500\cos.4\eta$$
$$-1{,}9815\tfrac{c}{a}\cos.\eta + 0{,}1237\tfrac{c}{a}\cos.3\eta.$$

XXXV. Quo hinc facilius ipſum angulum Φ definire queamus, ponamus breuitatis gratia $\frac{d\Phi}{d\zeta} = n + r$ erit

$$\frac{d\Phi}{d\eta} = \frac{n+r}{n-1+r} = \frac{n}{n-1} - \frac{r}{(n-1)^2} + \frac{rr}{(n-1)^3}$$

fit

sit $r = \alpha \cos. 2\eta + \beta \cos. 4\eta + \gamma \cos. \eta + \delta \cos. \eta^3$ erit

$rr = \frac{1}{2}\alpha\alpha + \frac{1}{2}\alpha\alpha \cos. 4\eta$ omissis reliquis terminis, qui ad ordines sequentes deuoluerentur, et ob paruitatem facile negliguntur. Integratione ergo instituta prodit

$$\Phi = \Delta + \frac{n}{n-1}\eta - \frac{\alpha \sin. 2\eta}{2(n-1)^2} - \frac{\beta \sin. 4\eta}{4(n-1)^2} - \frac{\gamma \sin. \eta}{(n-1)^2} - \frac{\delta \sin. 3\eta}{3(n-1)^2}$$
$$+ \frac{\alpha\alpha}{2(n-1)^3}\eta \qquad + \frac{\alpha\alpha \sin. 4\eta}{8(n-1)^3}$$

vbi est:

$\alpha = +0,252222$; $\beta = -0,000500$; $\gamma = -1,9815.\frac{c}{a}$;
$\delta = +0,1237.\frac{c}{a}$

atque iam ante quidem C ita definiri debuisset, vt et hic particula $\frac{\alpha\alpha}{2(n-1)^3}\eta$ tolleretur, prodiretque secundum motum medium $\Phi = \Delta + \frac{n}{n-1}\eta = \Delta + 1,081593\eta$. Singulis igitur terminis euolutis et in minuta secunda conuersis habebitur:

$$\Phi = \Delta + 1.081593\eta - 173'',177 \sin. 2\eta + 1'',063 \sin. 4\eta$$
$$+ 2721''.\frac{c}{a} \sin. \eta - 57''.\frac{c}{a} \sin. 3\eta.$$

XXXVI. Sed cum η ex motu medio non innotescat, relatio primo inter ζ et η est stabilienda, quae ob $\Phi = \zeta + \eta$ elicitur:

$$\zeta = \Delta + \frac{1}{n-1}\eta - \frac{\alpha \sin. 2\eta}{2(n-1)^2} - \frac{\beta \sin. 4\eta}{4(n-1)^2} - \frac{\gamma \sin. \eta}{(n-1)^2} - \frac{\delta \sin. 3\eta}{3(n-1)^2}$$
$$+ \frac{\alpha\alpha \sin. 4\eta}{8(n-1)^3}$$

 hinc-

hincque colligitur:

$$\eta = \text{Const.} + 12{,}25586\zeta + 2122'',43 \,\text{sin.}\, 2\eta - 13'',023 \,\text{sin.}\, 4\eta - 33348'' \tfrac{c}{a} \,\text{sin.}\, \eta + 694'' \tfrac{c}{a} \,\text{sin.}\, 3\eta$$

vnde haud difficulter ad datam solis longitudinem mediam angulus η colligitur, tum vero erit $\Phi = \eta + \zeta$. Denique distantia lunae a terra habebitur:

$$v = c\left(0{,}999428 - 0{,}007105 \cos. 2\eta + 0{,}000056 \cos. 4\eta + 0{,}07589 \tfrac{c}{a} \cos. \eta - 0{,}00274 \tfrac{c}{a} \cos. 3\eta\right)$$

At ex massis Solis, lunae, et terrae quantitas c ita definitur vt sit $\frac{nnc^3}{a^3} = \frac{T+L}{T+S} = \frac{T}{S}$, vnde patet ob actionem solis distantiam lunae mediam aliquantillum imminui.

XXXVII. In vero lunae motu eaedem istae inaequalitates quoque occurrunt, vnde haud inutile erat eas omni cura determinasse; ab Astronomis autem nomine variationis lunae designantur, quia omnes in vna tabula comprehendi possunt, argumentum distantiae solis a luna prae se ferente. Patet autem eius partem posteriorem a parallaxi solis pendere, seu a fractione $\frac{c}{a}$, dum prior absolute datur. Quare si quantitas huius inaequalitatis pro variis angulis per obseruationes innotesceret, inde vicissim parallaxis solis concludi posset. Cum igitur Tabulae *Mayerianae* cum coelo ita exacte conueniant,

niant, vt inaequalitates tanquam ex obseruationibus conclusae spectari queant; comparatio nostrae formulae inuentae cum his Tabulis, parallaxin solis nobis exhibere poterit. Consideremus solum casum, quo angulus $\eta = 90°$, quia tum pars variationis absoluta euanescit, eritque per formulam nostram variatio $= -34042''.\frac{c}{a}$ tabulae autem *Mayerianae* habent $-1', 57'' = -117''$ vnde sequitur $\frac{a}{c} = \frac{34042}{117} = 291$, cui rationi cum ratio parallaxium sit aequalis, parallaxis autem lunae media sit $57', 15'' = 3435''$, erit parallaxis solis $= \frac{3435}{291} = 11\frac{4}{5}''$. Haec fortasse methodus parallaxin solis definiendi reliquis excepto veneris transitu, longe anteferenda videtur, si quidem tabulae *Mayerianae* nunquam vltra minutum a coelo diffident; quia enim haec variationis portio ad $117''$ assurgit, leuis mutatio in parallaxi solis assumta sensibilem aberrationem a veritate produceret vt scilicet parallaxis solis prodiret $= 8\frac{1}{2}''$ tabulae *Mayerianae* loco $-117''$ habere deberent $-84''$, ex hac autem solis parallaxi foret $\frac{a}{c} = 400$. Considerari potest quoque maxima variatio angulo $\eta = 135°$ fere respondens, quae ex nostra forma est $= -2122'' - 23050''.\frac{c}{a}$; at ex Tabulis *Mayerianis* $= -41', 41'' = -2501''$, vnde sequitur $\frac{a}{c} = \frac{23050}{379}$, sed haec conclusio minus est certa, ob effectum a parallaxi solis ortum multo minorem. Contra vero maxima variatio hinc potius oriri videtur $= -2202''$ seu $36', 42''$. Verum hic probe animaduerti oportet, ex ex excentricitate partem quoque ipſi ſin. 2η proportionalem naſci, quae in his tabulis cum vera variatione eſt coniuncta. Haecque eſt cauſa, cur parallaxin ſolis ex variatione vbi ſin. $2\eta = 0$ et ſin. $4\eta = 0$ feliciter determinare licuerit minime vero ex variatione maxima.