1769

De curva hypergeometrica hac aequatione expressa $y = 1.2.3.....x$

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DE CVRVA
HYPERGEOMETRICA
HAC AEQVATIONE EXPRESSA
$y = 1 . 2 . 3 . \ldots . x$.

Auctore

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I.

Denotante hic littera x abſciſſam et y applicatam, aequatio haec immediate nonniſi pro iis abſciſſis, quae numeris integris exprimuntur, applicatarum quantitatem indicat; hinc enim ſi fuerint

abſciſſae x . . 0, 1, 2, 3, 4, 5, 6 etc.
erunt
applicatae y . . 1, 1, 2, 6, 24, 120, 720 etc.

ita, vt dum abſciſſae ſecundum numeros naturales capiuntur, applicatae ſecundum progreſſionem hypergeometricam *Walliſii* progrediantur; quam ob cauſam etiam hanc curuam hypergeometricam appellari conueniet. Etſi autem per hanc aequationem innumerabilia quidem iſtius curuae puncta, ſed inter ſe diſcreta aſſignantur; vniuerſa tamen huius curuae

 indo-

indoles per eam aequationem definiri eſt cenſenda, ita vt cuique abſciſſae certa ac vi iſtius ipſius aequationis determinata reſpondeat applicata. Ratio enim iſtius aequationis omnino poſtulat, vt ſi abſciſſae cuicunque $x = p$ conueniat applicata $y = q$, tum abſciſſae $x = p + 1$ reſpondeat applicata $y = q(p + 1)$ abſciſſae vero $x = p - 1$ haec applicata $y = \frac{q}{p}$. Quam ob rem neutiquam arbitrio noſtro relinquitur per infinita illa puncta data curuam quandam parabolici generis ducere, cum omnia plane eius puncta ex ipſa aequatione determinentur.

II.

Praeter has autem applicatas, quae abſciſſis per numeros integros expreſſis reſpondent, imprimis notari merentur, quae inter eas ex aequo interiacent; et omnes per eam, quam abſciſſae $x = \frac{1}{2}$ reſpondere et quantitati $\frac{1}{2}\sqrt{\pi}$ aequari olim oſtendi, determinantur. Cum igitur ſit $\sqrt{\pi} = 1,77245385090548$; hae applicatae coniunctim tam pro abſciſſis poſitiuis quam negatiuis ſequenti modo ſe habebunt:

pro

pro abſciſſis poſitiuis		pro abſciſſis negatiuis	
x	eſt applicata y	x	eſt applicata y
0	1	0	$+1$
$\frac{1}{2}$	0,8862269	$-\frac{1}{2}$	$+1,7724538$
1	1	-1	$\pm\infty$
$1\frac{1}{2}$	1,3293404	$-1\frac{1}{2}$	$-3,5449077$
2	2	-2	$\mp\infty$
$2\frac{1}{2}$	3,3233509	$-2\frac{1}{2}$	$+2,3632718$
3	6	-3	$\pm\infty$
$3\frac{1}{2}$	11,6317284	$-3\frac{1}{2}$	$-0,9453087$
4	24	-4	$\mp\infty$
$4\frac{1}{2}$	52,3427777	$-4\frac{1}{2}$	$+0,2700882$
5	120	-5	$\pm\infty$
$5\frac{1}{2}$	287,8852775	$-5\frac{1}{2}$	$-0,0600196$
6	720	-6	$\mp\infty$
$6\frac{1}{2}$	1871,2543038	$-6\frac{1}{2}$	$+0,0109126$
7	5040	-7	$\pm\infty$

Hinc delineaui iſtam curuam in fig. 1. expreſſam Tab. I. quae ab abſciſſa negatiua $x=-1$, vbi applicata fit Fig. 1. infinita vsque ad $x=3$, vbi fit $y=6$ porrigitur, hinc vero continuo in infinitam aſcendere eſt intelligenda; ſiniſtrorſum vero, vbi pro ſingulis abſciſſarum valoribus integris applicatae abeunt in aſymtotas, vltra $x=-1$ non expreſſi.

III.

Conſideratio huius curuae plures ſuppeditat quaeſtiones haud parum curioſas, eius naturae accuratius cognoſcendae inſeruientes, quarum euolutio eo maio-

maiori attentione digna videtur, quod aequatio pro curua more ſolito explicari nequit. Huiusmodi quaeſtiones primo circa determinationem reliquorum curuae punctorum praeter ea, quae facile aſſignare licet, verſantur. Deinde in ſingulis punctis poſitio tangentis inſignem inueſtigationem requirit, quo facilius tractus totius curuae definiri queat. Tum vero ex inſpectione figurae perſpicuum eſt inter abſciſſas $x=0$ et $x=1$, alicubi applicatam omnium minimam eſſe debere; cuius adeo tam locum quam ipſam quantitatem aſſignari operae erit pretium.

Praeterea vero inter binas abſciſſas negatiuas $-1, -2, -3, -4, -5$ etc. vbi applicatae in infinitum extenduntur, neceſſe eſt dari quoque applicatas minimas, quae quo magis ſiniſtrorſum progrediamur, continuo minores euadunt, donec tandem prorſus euaneſcant. Denique etiam quaeſtio de radio curvaminis in ſingulis curuae punctis attentionem noſtram meretur, isque imprimis curuae locus notatu dignus videtur, vbi curuatura eſt maxima, ſiquidem manifeſtum eſt, in elongatione ab axe curuae ramos continuo propius ad lineam rectam accedere. Has igitur quaeſtiones reſoluere inſtitui.

Quaeſtio prima.

Pro curua hypergeometrica inuenire aequationem continuam inter abſciſſam x *et applicatam* y, *quae aeque locum habeat, ſiue pro* x *capiatur numerus integer, ſiue fractus quicunque.*

4. Cum

4. Cum aequatio propoſita $y = 1.2.3.\ldots x$ locum proprie habere nequeat, niſi x ſit numerus integer; eam in aliam formam transfundi oportet, quae ab hac conditione ſit liberata; quod pluribus modis per expreſſiones in infinitum excurrentes fieri poteſt, inter quas primum occurrit iſta:

$$y = \frac{1}{1+x}\left(\frac{2}{1}\right)^x \cdot \frac{2}{2+x}\left(\frac{3}{2}\right)^x \cdot \frac{3}{3+x}\left(\frac{4}{3}\right)^x \cdot \frac{4}{4+x}\left(\frac{5}{4}\right)^x. \text{ etc.}$$

qui factores in infinitum continuari debent. Ratio huius expreſſionis inde eſt manifeſta, quod quo plures capiantur factores, veritas eo propius, ſumtis autem infinitis, accurate obtineatur: ſi enim factorum numerus ſit $= n$, habetur

$$y = \frac{1}{1+x} \cdot \frac{2}{2+x} \cdot \frac{3}{3+x} \cdot \ldots\ldots \frac{n}{n+x}(n+1)^x$$

cuius numerator ſi ita repraeſentetur:

$$1.2.3.\ldots\ldots x(x+1)(x+2)(x+3)\ldots n$$

denominator vero ita

$$(1+x)(2+x)(3+x)\ldots n(n+1)(n+2)\ldots(n+x)$$

deletis factoribus communibus prouenit

$$y = \frac{1.2.3.\ldots\ldots\ldots\ldots\ldots\ldots x}{(n+1)(n+2)(n+3)\ldots\ldots(n+x)}(n+1)^x.$$

Quare ſi n ſit numerus infinitus, ob denominatoris ſingulos factores $= n+1$ eorumque numerorum $= x$, totus denominator per multiplicatorem $(n+1)^x$ tollitur, proditque aequatio propoſita $y = 1.2.3.\ldots x$.

5. Haec forma aliquanto generalior reddi poteſt; cum enim totum negotium eo redeat, vt multi-

multiplicator $(n+1)^x$ postremo denominatori $(n+1)(n+2)(n+3) \ldots (n+x)$ aequiualeat, casu quo numerus n est infinitus, euidens est huic conditioni quoque satisfieri, si multiplicator ille in genere statuatur $(n+a)^x$ existente a numero quocunque finito; maxime vero hanc formulam ad institutum fore accommodatam, si litterae a medius quidam valor inter 1 et x veluti $a=\frac{1+x}{2}$ seu $a=\sqrt{x}$ tribuatur. Nunc vero necesse est hunc multiplicatarem $(n+a)^x$ in tot factores, quot numerus n continet vnitates, resolui, quod commode hac resolutione praestatur:

$$(n+a)^x = a^x . \left(\frac{a+1}{a}\right)^x . \left(\frac{a+2}{a+1}\right)^x . \left(\frac{a+3}{a+2}\right)^x \ldots \left(\frac{a+n}{a+n-1}\right)^x .$$

Quocirca pro abscissa quacunque x habebimus applicatam:

$$y = a^x . \frac{1}{1+x}\left(\frac{a+1}{a}\right)^x . \frac{1}{2+x}\left(\frac{a+2}{a+1}\right)^x . \frac{3}{3+x}\left(\frac{a+3}{a+2}\right)^x . \text{ etc. in infinitum}$$

quae expressio semper veritati est consentanea, quicunque numerus pro a accipiatur, promtissime autem ad veritatem perducet, si sumatur $a=\frac{1+x}{2}$, vnde fiet:

$$y = \left(\frac{1+x}{2}\right)^x . \frac{1}{1+x}\left(\frac{3+x}{1+x}\right)^x . \frac{2}{2+x}\left(\frac{5+x}{3+x}\right)^x . \frac{3}{3+x}\left(\frac{7+x}{5+x}\right)^x . \text{ etc.}$$

quae expressio ex infinitis factoribus formae $\frac{m}{m+x}\left(\frac{a+m}{a+m-1}\right)^x$ praeter primum a^x constat, et quo plures quouis casu inuicem multiplicantur, eo propius ad ad veritatem accedetur. Hinc autem naſcitur prima expreſſio, ſi ſumatur $a = 1$.

6. Eo magis autem haec expreſſio ad vſum eſt accommodata, quo promtius factores ad vnitatem conuergunt, id quod euenit ſumendo $a = \frac{1+x}{2}$, tum vero calculus eo facilius expedietur, quo minores numeri loco x ſubſtituuntur, ſemper autem ſufficit applicatas pro abſciſſis x vnitate vel adeo nihilo minoribus inueſtigaſſe, quoniam inde facili negotio applicatae per abſciſſis $x+1$, $x+2$, $x+3$, $x+4$ etc. deriuantur. Sit igitur $x = \frac{\alpha}{\beta}$ exiſtente $\alpha < \beta$, eritque

$$y = \left(\frac{\alpha+\beta}{2\beta}\right)^{\frac{\alpha}{\beta}} \cdot \frac{\beta}{\alpha+\beta}\left(\frac{3\beta+\alpha}{\beta+\alpha}\right)^{\frac{\alpha}{\beta}} \cdot \frac{2\beta}{\alpha+2\beta}\left(\frac{5\beta+\alpha}{3\beta+\alpha}\right)^{\frac{\alpha}{\beta}} \cdot \frac{3\beta}{\alpha+3\beta}\left(\frac{7\beta+\alpha}{5\beta+\alpha}\right)^{\frac{\alpha}{\beta}} \text{. etc.}$$

vnde applicatae poteſtas y^β ita prodit expreſſa:

$$y^\beta = \left(\frac{\alpha+\beta}{2\beta}\right)^\alpha \cdot \frac{\beta^\beta(3\beta+\alpha)^\alpha}{(\beta+\alpha)^\beta(\beta+\alpha)^\alpha} \cdot \frac{(2\beta)^\beta(5\beta+\alpha)^\alpha}{(2\beta+\alpha)^\beta(3\beta+\alpha)^\alpha} \cdot \frac{(3\beta)^\beta(7\beta+\alpha)^\alpha}{(3\beta+\alpha)^\beta(5\beta+\alpha)^\alpha} \text{ etc.}$$

Pro abſciſſa autem $x = -\frac{\alpha}{\beta}$ applicata y hinc colligetur

$$y^\beta = \left(\frac{2\beta}{\beta-\alpha}\right)^\alpha \cdot \frac{\beta^\beta(\beta-\alpha)^\alpha}{(\beta-\alpha)^\beta(3\beta-\alpha)^\alpha} \cdot \frac{(2\beta)^\beta(3\beta-\alpha)^\alpha}{(2\beta-\alpha)^\beta(5\beta-\alpha)^\alpha} \cdot \frac{(3\beta)^\beta(5\beta-\alpha)^\alpha}{(3\beta-\alpha)^\beta(7\beta-\alpha)^\alpha} \text{ etc.}$$

Sumamus exempli gratia $x = \frac{1}{2}$ et impetrabimus:

$$y^2 = \frac{3}{4} \cdot \frac{2 \cdot 2 \cdot 7}{3 \cdot 3 \cdot 3} \cdot \frac{4 \cdot 4 \cdot 11}{5 \cdot 5 \cdot 7} \cdot \frac{6 \cdot 6 \cdot 15}{7 \cdot 7 \cdot 11} \cdot \frac{8 \cdot 8 \cdot 19}{9 \cdot 9 \cdot 15} \text{ etc.}$$

cuius factor in genere cum fit $\frac{2n \cdot 2n \cdot (4n+3)}{(2n+1)(2n+1)(4n-1)} = \frac{16n^3+12nn}{16n^3+12nn-1} = 1 + \frac{1}{(2n+1)^2(4n-1)}$, hinc intelligitur, quam promte hi factores ad vnitatem accedunt, erit igitur:

$$y^2 = \tfrac{3}{4}\left(1+\tfrac{1}{3^2 \cdot 3}\right)\left(1+\tfrac{1}{5^2 \cdot 7}\right)\left(1+\tfrac{1}{7^2 \cdot 11}\right)\left(1+\tfrac{1}{9^2 \cdot 15}\right)\left(1+\tfrac{1}{11^2 \cdot 19}\right) \text{ etc.}$$

vbi quidem nouimus esse $y^2 = \frac{\pi}{4}$. Sin autem statuamus $x = -\frac{1}{2}$, cui conuenit $y = \sqrt{\pi}$ erit ex altera expressione

$$\pi = 4 \cdot \tfrac{2 \cdot 2 \cdot 1}{1 \cdot 1 \cdot 5} \cdot \tfrac{4 \cdot 4 \cdot 5}{3 \cdot 3 \cdot 9} \cdot \tfrac{6 \cdot 6 \cdot 9}{5 \cdot 5 \cdot 13} \cdot \tfrac{8 \cdot 8 \cdot 13}{7 \cdot 7 \cdot 17} \text{ etc.}$$

seu $\pi = 4\left(1-\frac{1}{1^2 \cdot 5}\right)\left(1-\frac{1}{3^2 \cdot 9}\right)\left(1-\frac{1}{5^2 \cdot 13}\right)\left(1-\frac{1}{7^2 \cdot 17}\right)$ etc. inde vero est $\pi = 3\left(1+\frac{1}{3^2 \cdot 3}\right)\left(1+\frac{1}{5^2 \cdot 7}\right)\left(1+\frac{1}{7^2 \cdot 11}\right)\left(1+\frac{1}{9^2 \cdot 15}\right)$ etc.

ita vt altera crescendo, altera decrescendo ad veritatem appropinquet.

7. Commodius autem calculus instituetur, si expressio nostra in singulis factoribus abrumpatur, tum enim sequentes formulae prodibunt continuo propius ad veritatem accedentes:

$$y = \frac{1}{1+x}\left(\frac{3+x}{2}\right)^x$$

$$y = \frac{1}{1+x} \cdot \frac{2}{2+x}\left(\frac{5+x}{2}\right)^x$$

$$y = \frac{1}{1+x} \cdot \frac{2}{2+x} \cdot \frac{3}{3+x}\left(\frac{7+x}{2}\right)^x$$

$$y = \frac{1}{1+x} \cdot \frac{2}{2+x} \cdot \frac{3}{3+x} \cdot \frac{4}{4+x}\left(\frac{9+x}{2}\right)^x$$

$$y = \frac{1}{1+x} \cdot \frac{2}{2+x} \cdot \frac{3}{3+x} \cdot \frac{4}{4+x} \cdot \frac{5}{5+x}\left(\frac{11+x}{2}\right)^x.$$

Quia si loco x scribatur $-x$ prodit applicata $=\frac{y}{x}$. erit per similes formulas:

$$y=\left(\frac{2+x}{2}\right)^{x-1}$$

$$y=\frac{2}{1+x}\left(\frac{4+x}{2}\right)^{x-1}$$

$$y=\frac{2}{1+x}\cdot\frac{3}{2+x}\left(\frac{6+x}{2}\right)^{x-1}$$

$$y=\frac{2}{1+x}\cdot\frac{3}{2+x}\cdot\frac{4}{3+x}\left(\frac{8+x}{2}\right)^{x-1}$$

$$y=\frac{2}{1+x}\cdot\frac{3}{2+x}\cdot\frac{4}{3+x}\cdot\frac{5}{4+x}\left(\frac{10+x}{2}\right)^{x-1}.$$

Quare posito $x=\frac{1}{2}$ pro applicata $y=\frac{1}{2}\sqrt{\pi}$ duplex series formularum eo conuergentium resultat:

$$\frac{1}{2}\sqrt{\pi}=\frac{2}{3}\sqrt{\frac{7}{4}}$$
$$\frac{1}{2}\sqrt{\pi}=\frac{2\cdot 4}{3\cdot 5}\sqrt{\frac{11}{4}}$$
$$\frac{1}{2}\sqrt{\pi}=\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\sqrt{\frac{15}{4}}$$
$$\frac{1}{2}\sqrt{\pi}=\frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9}\sqrt{\frac{19}{4}}$$
etc.

$$\frac{1}{2}\sqrt{\pi}=\sqrt{\frac{4}{5}}$$
$$\frac{1}{2}\sqrt{\pi}=\frac{4}{3}\sqrt{\frac{4}{9}}$$
$$\frac{1}{2}\sqrt{\pi}=\frac{4\cdot 6}{3\cdot 5}\sqrt{\frac{4}{13}}$$
$$\frac{1}{2}\sqrt{\pi}=\frac{4\cdot 6\cdot 8}{3\cdot 5\cdot 7}\sqrt{\frac{4}{17}}$$
$$\frac{1}{2}\sqrt{\pi}=\frac{4\cdot 6\cdot 8\cdot 10}{3\cdot 5\cdot 7\cdot 9}\sqrt{\frac{4}{21}}$$
etc.

8. Huiusmodi autem producta commodissime per logarithmos euoluuntur; ac primo quidem ex forma generali numerum quemcunque a implicante nanciscimur:

$$ly=x\,la+x\,l\frac{a+1}{a}+x\,l\frac{a+2}{a+1}+x\,l\frac{a+3}{a+2}+x\,l\frac{a+4}{a+3}\text{ etc.}$$
$$-l(1+x)-l(1+\tfrac{x}{2})-l(1+\tfrac{x}{3})-l(1+\tfrac{x}{4})\text{ etc.}$$

et sumto $a=\frac{1+x}{2}$, vt haec series maxime conuergens reddatur:

$$ly = xl\frac{1+x}{2} + xl\frac{x+3}{x+1} + xl\frac{x+5}{x+3} + xl\frac{x+7}{x+5} + xl\frac{x+9}{x+7} \text{ etc.}$$
$$-l(1+x) - l(1+\tfrac{x}{2}) - l(1+\tfrac{x}{3}) - l(1+\tfrac{x}{4}) \text{ etc.}$$

Sumtis igitur his logarithmis naturalibus, cum ſit in genere:

$$xl\frac{x+2m+1}{x+2m-1} = \frac{2x}{x+2m} + \frac{2x}{3(x+2m)^3} + \frac{2x}{5(x+2m)^5} + \frac{2x}{7(x+2m)^7} + \text{etc.}$$
$$\text{et } l(1+\tfrac{x}{m}) = \frac{2x}{x+2m} + \frac{2x^3}{3(x+2m)^3} + \frac{2x^5}{5(x+4m)^5} + \frac{2x^7}{7(x+2m)^7} + \text{etc.}$$

ſequentem formam infinitis ſeriebus conſtantem adipiſcimur:

$$ly = xl\frac{1+x}{2} + \tfrac{2}{3}x(1-xx)\left(\frac{1}{(x+2)^3} + \frac{1}{(x+4)^3} + \frac{1}{(x+6)^3} + \frac{1}{(x+8)^3} + \text{etc.}\right)$$
$$+ \tfrac{2}{5}x(1-x^4)\left(\frac{1}{(x+2)^5} + \frac{1}{(x+4)^5} + \frac{1}{(x+6)^5} + \frac{1}{(x+8)^5} + \text{etc.}\right)$$
$$+ \tfrac{2}{7}x(1-x^6)\left(\frac{1}{(x+2)^7} + \frac{1}{(x+4)^7} + \frac{1}{(x+6)^7} + \frac{1}{(x+8)^7} + \text{etc.}\right)$$
$$+ \tfrac{2}{9}x(1-x^8)\left(\frac{1}{(x+2)^9} + \frac{1}{(x+4)^9} + \frac{1}{(x+6)^9} + \frac{1}{(x+8)^9} + \text{etc.}\right)$$
$$\text{etc.}$$

9. Primae ſeriei ſumamus definitum terminorum numerum qui ſit $= n$, et cum ſuperior pars ad vnicum membrum $xl(a+n)$ redigatur, erit

$$ly = xl(a+n) - l(1+x) - l(1+\tfrac{1}{2}x) - l(1+\tfrac{1}{3}x) \ldots - l(1+\tfrac{1}{n}x)$$

quae expreſſio eo propius ad veritatem accedit, quo maior capiatur numerus n. Sit igitur n numerus praemagnus ac primo quidem habebimus $l(n+a) = ln + \frac{a}{n} - \frac{aa}{2n^2} + \frac{a^3}{3n^3} - \text{etc.}$ vbi loco a ſumi $\frac{1+x}{2}$ conueniet; tum vero poſita breuitatis gratia fractione $0,5772156649015325 = \Delta$, nouimus eſſe ſummam progreſſionis harmonicae:

$1 + \frac{1}{2}$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\ldots\ldots+\frac{1}{n}=\Delta+ln+\frac{1}{2n}-\frac{1}{12nn}+\frac{1}{120n^4}-\text{etc.}$$

vnde cum ſit:

$$l(n+a)=1+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}\ldots\ldots+\frac{1}{n}-\Delta-\frac{1}{2n}+\frac{1}{12nn}-\frac{1}{120n^4}$$
$$+\frac{a}{n}-\frac{aa}{2nn}+\frac{a^3}{3n^3}$$

colligimus ſumto $a=\frac{a+x}{2}$

$$ly=-\Delta x+x+\frac{1}{2}x+\frac{1}{3}x\ \ldots\ldots+\frac{1}{n}x+\frac{xx}{2n}$$
$$-l(1+x)-l(1+\tfrac{1}{2}x)-l(1+\tfrac{1}{3}x)-l(1+\tfrac{1}{n}x)-\frac{x-6xx-3x^3}{24nn}.$$

Reuera ergo augendo numerum n in infinitum erit:

$$ly=-\Delta x+x+\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x+\text{etc.}$$
$$-l(1+x)-l(1+\tfrac{1}{2}x)-l(1+\tfrac{1}{3}x)-l(1+\tfrac{1}{4}x)-\text{etc.}$$

et ſingulis logarithmis per ſeries euolutis:

$$ly=-\Delta x+\frac{1}{2}xx(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{etc.})$$
$$-\frac{1}{3}x^3(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\text{etc.})$$
$$+\frac{1}{4}x^4(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.})$$
$$-\frac{1}{5}x^5(1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\text{etc.})$$
$$\text{etc.}$$

10. Praeter has autem formulas, quibus cuique abſciſſae x conueniens applicata y aſſignatur, methodus mea progreſſiones indefinite ſummandi ſingularem ſuppeditat expreſſionem ad eundem ſcopum accommodatam.

Cum enim ſit $ly=l1+l2+l3+l4\ldots+lx$ hanc progreſſionem indefinite ſummari oportet; introducendo autem valores numericos:

$A = \frac{1}{6}$, $B = \frac{1}{90}$, $C = \frac{1}{945}$, $D = \frac{1}{9450}$, $E = \frac{1}{93555}$, $F = \frac{691}{1.\,3.\,5.\,\ldots\ldots\,15.\,315}$ etc. quorum progreſſio ita eſt comparata vt ſit

$$5B = 2AA; \quad 7C = 4AB; \quad 9D = 4AC + 2BB; \quad 11E = 4AD + 4BC \text{ etc.}$$

oſtendi alibi fore

$$ly = \tfrac{1}{2} l 2\pi + (x + \tfrac{1}{2}) l x - x + \frac{A}{2x} - \frac{1.\,2\,B}{2^3 x^3} + \frac{1.\,2.\,3.\,4\,C}{2^5 x^5} - \frac{1.\,2.\,3.\,4.\,5.\,6\,D}{2^7 x^7} + \text{etc.}$$

quae ſeries prae ſuperioribus hunc praeſtat vſum, vt quo maiores capiantur abſciſſae x eo promtius verum valorem applicatae y exhibeat. Cum igitur ſi abſciſſae x conueniat applicata y, abſciſſae maiori $x + n$ conueniat applicata $y(x+1)(x+2)(x+3)\ldots.(x+n)$, habebimus ſemper per ſeriem valde conuergentem:

$$ly = \tfrac{1}{2} l 2\pi - l(x+1) - l(x+2) - l(x+3) \ldots\ldots - l(x+n) + (x + n + \tfrac{1}{2}) l(x+n) - x - n + \frac{A}{2(x+n)} - \frac{1.\,2\,B}{2^3 (x+n)^3} + \frac{1.\,2.\,3.\,4\,C}{2^5 (x+n)^5} - \frac{1.\,2.\,3.\,4.\,5.\,6\,D}{2^7 (x+n)^7} + \text{etc.}$$

Quodſi ergo e denotet numerum, cuius logarithmus naturalis $= 1$, breuitatis gratia ponatur:

$$\frac{A}{2(x+n)} - \frac{1.\,2\,B}{2^3 (x+n)^3} + \frac{1.\,2.\,3.\,4\,C}{2^5 (x+n)^5} - \text{etc.} = s$$

concludimus a logarithmus ad numeros regrediendo

$$y = \frac{\sqrt{2\pi(x+n)}}{(x+1)(x+2)(x+3)\ldots.(x+n)} \left(\frac{x+n}{e}\right)^{x+n} e^s$$

vbi numerus integer n arbitrio noſtro relinquitur, quo

quo maior is autem accipiatur, eo facilius verum valorem ipſius s inuenire licet.

11. Denique etiam applicatam y per formulam integralem exhibere licet, poſita enim abſciſſa $x = p$, nouaque introducta variabili u, prae qua quantitas p vt conſtans tractetur, erit applicata $y = \int du (l\frac{1}{u})^p$ ſiquidem integratio a valore $u = 0$ vsque ad valorem $u = 1$ extendatur. Vel ſi forma exponentiali vti malimus, erit quoque

$$y = \int e^{-v} v^p \, dv$$

integrationem a valore $v = 0$ ad $v = \infty$ extendendo. Ex his quidem formulis, quoties abſciſſa p eſt numerus integer, integratio ſtatim praebet $y = 1 . 2 . 3 \ldots p$. at ſi p fuerit numerus fractus, hinc ſimul intelligitur ad quodnam genus quantitatum tranſcendentium valor ipſius y referri debeat. Alio autem loco oſtendi, quomodo tum integrale per quadraturas curuarum algebraicarum exprimi queat.

12. En ergo plurimas ſolutiones quaeſtionis noſtrae primae, qua pro qualibet abſciſſa x etiamſi numero non integro exprimatur, valor applicatae y reperiebatur: quarum praecipuas ſimul aſpectui expoſuiſſe iuuabit, vt inde quouis caſu ea, quae maxime ad vſum accommodata videatur, eligi queat:

I. $y = \frac{1}{1+x}\left(\frac{2}{1}\right)^x \cdot \frac{2}{2+x}\left(\frac{3}{2}\right)^x \cdot \frac{3}{3+x}\left(\frac{4}{3}\right)^x \cdot \frac{4}{4+x}\left(\frac{5}{4}\right)^x \cdot$ etc.

II. $y = \left(\frac{1+x}{2}\right)^x \cdot \frac{1}{1+x}\left(\frac{3+x}{2+x}\right)^x \cdot \frac{2}{2+x}\left(\frac{5+x}{3+x}\right)^x \cdot \frac{3}{3+x}\left(\frac{7+x}{5+x}\right)^x \cdot$ etc.

III.

$$\text{III.}\ ly = xl\tfrac{2}{1} + xl\tfrac{3}{2} + xl\tfrac{4}{3} + xl\tfrac{5}{4} + \text{etc.}$$
$$- l(1+x) - l(1+\tfrac{1}{2}x) - l(1+\tfrac{1}{3}x) - l(1+\tfrac{1}{4}x) - \text{etc.}$$

$$\text{IV.}\ ly = xl\frac{1+x}{2} + xl\frac{x+3}{x+1} + xl\frac{x+5}{x+3} + xl\frac{x+7}{x+5} + xl\frac{x+9}{x+7} + \text{etc.}$$
$$- l(1+x) - l(1+\tfrac{1}{2}x) - l(1+\tfrac{1}{3}x) - l(1+\tfrac{1}{4}x) - \text{etc.}$$

$$\text{V.}\ ly = -\Delta x + x + \tfrac{1}{2}x + \tfrac{1}{3}x + \tfrac{1}{4}x + \text{etc.}$$
$$- l(1+x) - l(1+\tfrac{1}{2}x) - l(1+\tfrac{1}{3}x) - l(1+\tfrac{1}{4}x) - \text{etc.}$$

$$\text{VI.}\ ly = -\Delta x + \tfrac{1}{2}xx\left(1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \text{etc.}\right)$$
$$+ \tfrac{1}{3}x^3\left(1 + \tfrac{1}{2^3} + \tfrac{1}{3^3} + \tfrac{1}{4^3} + \text{etc.}\right)$$
$$+ \tfrac{1}{4}x^4\left(1 + \tfrac{1}{2^4} + \tfrac{1}{3^4} + \tfrac{1}{4^4} + \text{etc.}\right)$$
$$+ \tfrac{1}{5}x^5\left(1 + \tfrac{1}{2^5} + \tfrac{1}{3^5} + \tfrac{1}{4^5} + \text{etc.}\right)$$
$$+ \text{etc.}$$

$$\text{VII.}\ ly = \tfrac{1}{2}l2\pi + (x+\tfrac{1}{2})lx - x + \frac{A}{2x} - \frac{1.2\,B}{2^3\,x^3} + \frac{1.2.3.4\,C}{2^5\,x^7} - \frac{1.2\ldots6\,D}{2^7\,x^7} + \text{etc.}$$

existente $\Delta = 0,57721566490014225$ et

$$A = \tfrac{1}{6},\ B = \tfrac{1}{90},\ C = \tfrac{1}{945},\ D = \tfrac{1}{9450},\ E = \tfrac{1}{93555}\ \text{etc.}$$

Tum in tribus postremis formis logarithmos naturales accipi oportet.

Quaestio secunda.

In curua hypergeometrica ad quoduis eius punctum directionem tangentis definire.

13. Hic igitur assumimus pro quauis abscissa x valorem applicatae y iam esse inuentum; et cum directio tangentis ratione differentialium $\frac{dy}{dx}$ definiatur, quippe qua fractione tangens anguli, quo curvae

vae tangens in loco propoſito ad axem inclinatur, exprimi ſolet, tantum opus eſt, vt quandam formularum inuentarum differentiemus. In hunc autem finem formula V maxime videtur idonea, ex qua colligimus:

$$\frac{dy}{ydx} = -\Delta + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.}$$
$$- \frac{1}{1+x} - \frac{1}{2+x} - \frac{1}{3+x} - \frac{1}{4+x} - \text{etc.}$$

quae expreſſio in hanc concinniorem contrahitur:

$$\frac{dy}{ydx} = -\Delta + \frac{x}{1+x} + \frac{x}{2(2+x)} + \frac{x}{3(3+x)} + \frac{x}{4(4+x)} + \text{etc.}$$

vnde ſtatim patet, ſi x ſit numerus integer negatiuus, fieri non ſolum applicatam y, ſed etiam formulam $\frac{dy}{dx}$ infinitam, ita vt in his locis ipſae applicatae, vtpote aſymtotae fiant tangentes. Ponamus autem in genere angulum, quem tangens cum axe conſtituit $= \Phi$ vt ſit $\frac{dy}{dx} = \text{tang.}\, \Phi$.

14. Primum ergo hinc definiamus tangentes pro abſciſſis x, quae numeris poſitiuis exprimuntur, ſiquidem applicatae y ſponte dantur.

I. Sit ergo $x = 0$, et ob $y = 1$ fit

$$\frac{dy}{dx} = -\Delta = -0,5772156649 = \text{tang.}\, \Phi$$

vnde fit angulus $\Phi = -29^\circ, 59', 39''$, vbi ſignum $-$ indicat, tangentem dextrorſum in axem incidere, cum eoque angulum tantum non 30° conſtituere.

II. Sit $x = 1$ et ob $y = 1$ fit $\frac{dy}{dx} = 1 - \Delta = 0,422784335 = \text{tang.}\, \Phi$, hincque angulus $\Phi = 22^\circ, 55'$.

III. Sit $x=2$ et ob $y=2$ fit $\frac{dy}{dx}=2(1+\frac{1}{2}-\Delta)=1,845568670$ $=\text{tang.}\,\Phi$ hincque angulus $\Phi=61^\circ,33'$.

IV. Sit $x=3$ et ob $y=6$ fit $\frac{dy}{dx}=6(1+\frac{1}{2}+\frac{1}{3}-\Delta)=\text{tang.}\,\Phi$ seu $\text{tang.}\,\Phi=7,536706010$ et $\Phi=82^\circ,26'$.

V. Sit $x=4$ et ob $y=24$ fit $\frac{dy}{dx}=24(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\Delta)$ hincque $\text{tang.}\,\Phi=36,146824040$ et $\Phi=88^\circ,25'$.

In genere igitur si abscissa x aequetur numero integro cuicunque n, ob $y=1.\,2\ldots\ldots n$ erit

$\frac{dy}{dx}=\text{tang.}\,\Phi=1.\,2.\,3\ldots\ldots n(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}-\Delta)$.

15. Definiamus hinc etiam tangentes pro locis intermediis, ac primo quidem ad abscissas positiuas relatis:

I. Sit $x=\frac{1}{2}$, erit $y=\frac{1}{2}\sqrt{\pi}$ atque

$\frac{dy}{ydx}=-\Delta+1-\frac{2}{3}+\frac{1}{2}-\frac{2}{5}+\frac{1}{3}-\frac{2}{7}$ etc.

seu $\frac{dy}{ydx}=-\Delta+2(\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\text{etc.})=-\Delta+2(1-l2)$

hincque $\frac{dy}{dx}=\text{tang.}\,\Phi=y(2(1-l2)-\Delta)=0,036489973\,y$

II. Sit $x=\frac{3}{2}$ erit $y=\frac{1.\,3}{2.\,2}\sqrt{\pi}$ atque

$$\frac{dy}{ydx}=-\Delta+2(1+\frac{1}{3}-l2)\text{ vnde fit}$$

$\frac{dy}{dx}=\text{tang.}\,\Phi=y(2(1+\frac{1}{3}-l2)-\Delta)=0,7031566405\,y$

III. Sit $x=\frac{5}{2}$ erit $y=\frac{1.\,3.\,5}{2.\,2.\,2}\sqrt{\pi}$ et $\frac{dy}{ydx}=-\Delta+2(1+\frac{1}{3}+\frac{1}{5}-l2)$

hinc $\text{tang.}\,\Phi=y(2(1+\frac{1}{3}+\frac{1}{5}-l2)-\Delta)=1,1031566405\,y$.

Cum

Cum nunc fit $\frac{1}{2}\sqrt{\pi}.(2(1-l2)-\Delta)=0,0323383973$ erit pro his cafibus:

$$x=\tfrac{1}{2};\ y=0,8862269\quad \text{tang.}\Phi=0,0323384$$
$$x=\tfrac{3}{2};\ y=1,3293404\quad \text{tang.}\Phi=0,9347345$$
$$x=\tfrac{5}{2};\ y=3,3233509\quad \text{tang.}\Phi=3,6661767$$
$$x=\tfrac{7}{2};\ y=11,6317284\quad \text{tang.}\Phi=16,1549694$$
$$x=\tfrac{9}{2};\ y=52,3427777\quad \text{tang.}\Phi=84,3290907$$

etc.

16. Antequam vlterius progrediar, obferuo fi fuerit pro abfciffa quacunque

$$x=p;\quad y=q;\quad \text{tang.}\Phi=r$$

tum pro abfciffa fequente fore

$$x=p+1;\ y=q(p+1)\ \text{et tang.}\Phi=r(p+1)+q$$

pro abfciffa autem antecedente

$$x=p-1;\ y=\frac{q}{p};\quad \text{et tang.}\Phi=\frac{r}{p}-\frac{q}{pp}$$

vnde fuperiores valores facile retro continuare poterimus:

$$x=\tfrac{1}{2};\ y=0,8862269;\ \text{tang.}\Phi=0,0323384$$
$$x=-\tfrac{1}{2};\ y=1,7724538;\ \text{tang.}\Phi=-3,4802308$$
$$x=-\tfrac{3}{2};\ y=-3,5449077;\ \text{tang.}\Phi=-0,1293538$$
$$x=-\tfrac{5}{2};\ y=+2,3632718;\ \text{tang.}\Phi=+1,6617504$$
$$x=-\tfrac{7}{2};\ y=-0,9453087;\ \text{tang.}\Phi=-1,0428236$$
$$x=-\tfrac{9}{2};\ y=+0,2700882;\ \text{tang.}\Phi=+0,3751176$$
$$x=-\tfrac{11}{2};\ y=-0,0600196;\ \text{tang.}\Phi=-0,0966971$$
$$x=-\tfrac{13}{2};\ y=+0,0109126;\ \text{tang.}\Phi=+0,0195654$$

etc.

17. Eadem aequatio differentialis ei curuae puncto μ inueniendo inferuit, vbi applicata est minima feu tangens axi parallela. Pofito igitur $\frac{dy}{dx} = 0$, abfciffa refpondens x ex hac aequatione quaeri debet:

$$\Delta = \frac{x}{1+x} + \frac{x}{2(2+x)} + \frac{x}{3(3+x)} + \frac{x}{4(4+x)} + \frac{x}{5(5+x)} + \text{etc.}$$

quae euoluitur in hanc:

$$\begin{aligned} \Delta = &+ x \left(1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \text{etc.}\right) \\ &- x^2 \left(1 + \tfrac{1}{2^3} + \tfrac{1}{3^3} + \tfrac{1}{4^3} + \text{etc.}\right) \\ &+ x^3 \left(1 + \tfrac{1}{2^4} + \tfrac{1}{3^4} + \tfrac{1}{4^4} + \text{etc.}\right) \\ &- x^4 \left(1 + \tfrac{1}{2^5} + \tfrac{1}{3^5} + \tfrac{1}{5^5} + \text{etc.}\right) \\ &\text{etc.} \end{aligned}$$

Summis autem harum ferierum proximis fubftitutis erit

$$\begin{aligned} 0 = &+ 0,5772156649 - 1,6449340668\, x \\ &+ 1,2020569032\, x^2 - 1,0823232337\, x^3 \\ &+ 1,0369277551\, x^4 - 1,0173430620\, x^5 \\ &+ 1,0083492774\, x^6 - 1,0040773562\, x^7 \\ &+ 1,0020083928\, x^8 - 1,0009945751\, x^9 \\ &+ 1,0004941886\, x^{10} - 1,0002460866\, x^{11} \\ &+ 1,0001227233\, x^{12} - 1,0000612481\, x^{13} \\ &+ 1,0000305882\, x^{14} - 1,0000152823\, x^{15} \\ &\text{etc.} \end{aligned}$$

Sin autem duae primae fractiones retineantur, fequens feries multo magis conuergens emergit

0 =

$$0 = +0{,}5772156649 - \frac{x}{1+x} - \frac{x}{2(2+x)}$$
$$+0{,}0770569032\,x^2 - 0{,}3949340668\,x$$
$$+0{,}0056777551\,x^4 - 0{,}0198232337\,x^3$$
$$+0{,}00[illegible]367774\,x^6 - 0{,}0017180620\,x^5$$
$$+0{,}0000552678\,x^8 - 0{,}0001711062\,x^7$$
$$+0{,}0000059074\,x^{10} - 0{,}0000180126\,x^9$$
$$+0{,}0000006530\,x^{12} - 0{,}0000019460\,x^{11}$$
$$+0{,}0000000706\,x^{14} - 0{,}0000002130\,x^{13}$$
$$+0{,}0000000078\,x^{16} - 0{,}0000000235\,x^{15}.$$

Hinc proxime reperitur $x = \frac{1}{2}$, verum haec applicata minima facilius ope sequentis quaeſtionis definietur.

Quaeſtio tertia.

Pro dato quouis curuae hypergeometricae puncto, indolem portionis minimae iſtius curuae circa id punctum ſitae inueſtigare.

18. Pro abſciſſa ergo data $x = p$ inuenta ſit applicata $y = q$; et nunc quaeri oportet applicatam, quae abſciſſae parumper ab illa diſcrepanti $p + \omega$ reſpondeat; quae applicata ſtatuatur $= q + \psi$. Cum igitur ſit ſecundum formulam V

$$lq = -\Delta p + p + \tfrac{1}{2}p + \tfrac{1}{3}p + \tfrac{1}{4}p + \text{etc.}$$
$$- l(1+p) - l(1+\tfrac{1}{2}p) - l(1+\tfrac{1}{3}p) - l(1+\tfrac{1}{4}p) - \text{etc.}$$

ſi hic loco p ſcribatur $p + \omega$, loco lq prodibit valor ipſius $l(q + \psi)$, quo ipſo quaeſtio reſoluetur. At ſi ponamus $lq = P$, ſcribendo $p + \omega$ loco p notum eſt prodire

$$l(q+\psi) = P + \frac{\omega\, dP}{1\, dp} + \frac{\omega^2\, ddP}{1 \cdot 2\, dp^2} + \frac{\omega^3\, d^3P}{1 \cdot 2 \cdot 3\, dp^3} + \frac{\omega^4\, d^4P}{1 \cdot 2 \cdot 3 \cdot 4\, dp^4} + \text{etc.}$$

Eſt vero vt vidimus:

$$\frac{dP}{dp} = -\Delta + \frac{p}{1+p} + \frac{p}{2(2+p)} + \frac{p}{3(3+p)} + \frac{p}{4(4+p)} + \text{etc.}$$

hincque porro:

$$\frac{ddP}{1 \cdot dp^2} = \frac{1}{(1+p)^2} + \frac{1}{(2+p)^2} + \frac{1}{(3+p)^2} + \frac{1}{(4+p)^2} + \text{etc.}$$

$$\frac{d^3 P}{1 \cdot 2\, dp^3} = -\frac{1}{(1+p)^3} - \frac{1}{(2+p)^3} - \frac{1}{(3+p)^3} - \frac{1}{(4+p)^3} - \text{etc.}$$

$$\frac{d^4 P}{1 \cdot 2 \cdot 3\, dp^4} = \frac{1}{(1+p)^4} + \frac{1}{(2+p)^4} + \frac{1}{(3+p)^4} + \frac{1}{(4+p)^4} + \text{etc.}$$

etc.

vnde ob $P = lq$ colligimus:

$$l\left(1 + \frac{\psi}{q}\right) = -\Delta\omega + \omega\left(\frac{p}{1+p} + \frac{p}{2(2+p)} + \frac{p}{3(3+p)} + \text{etc.}\right)$$

$$+ \tfrac{1}{2}\omega^2\left(\frac{1}{(1+p)^2} + \frac{1}{(2+p)^2} + \frac{1}{(3+p)^2} + \text{etc.}\right)$$

$$- \tfrac{1}{3}\omega^3\left(\frac{1}{(1+p)^3} + \frac{1}{(2+p)^3} + \frac{1}{(3+p)^3} + \text{etc.}\right)$$

$$+ \tfrac{1}{4}\omega^4\left(\frac{1}{(1+p)^4} + \frac{1}{(2+p)^4} + \frac{1}{(3+p)^4} + \text{etc.}\right)$$

$$- \tfrac{1}{5}\omega^5\left(\frac{1}{(1+p)^5} + \frac{1}{(2+p)^5} + \frac{1}{(3+p)^5} + \text{etc.}\right)$$

etc.

19. Hic iam coordinatae p et q vt conſtantes ſpectari poſſunt, quoniam litterae ω et ψ nouas coordinatas a dato curuae puncto ſumtas atque illis parallelas referunt; ex quarum relatione hic definita indoles curuae circa id punctum verſantis facile inveſtigatur. Quare cum iam innumerabilia curuae puncta aſſignauerimus, hinc tractus ſingularum curvae portionum inter bina illorum punctorum interiacentium vero proxime definiri poterit. Primo ſcilicet

ſcilicet ex illa aequatione differentiata colligitur vt ante inclinatio tangentis ad axem Φ, ſitque

$$\frac{d\psi}{d\omega} = \text{tang.}\,\Phi = q\left(-\Delta + \frac{p}{1+p} + \frac{p}{2(2+p)} + \frac{p}{3(3+p)} + \text{etc.}\right).$$

Deinde ſi pro aequatione differentiali breuitatis gratia ponamus $d\psi = A\,d\omega + B\,\omega\,d\omega + C\,\omega^2\,d\omega + \text{etc.}$ erit radius curuaturae in dato curuae puncto $= \frac{(1+AA)^{\frac{3}{2}}}{B} = B\frac{1}{\cos.\Phi^3}$ ob $A = \text{tang.}\,\Phi$. Eſt vero

$$B = \text{tang.}\,\Phi\left(-\Delta + \frac{p}{1+p} + \frac{p}{2(2+p)} + \frac{p}{3(3+p)} + \text{etc.}\right)$$
$$+ q\left(\frac{1}{(1+p)^2} + \frac{1}{(2+p)^2} + \frac{1}{(3+p)^2} + \frac{1}{(4+p)^2} + \text{etc.}\right)$$

Vnde ſi radius curuaturae ponatur $= r$ erit

$$\frac{1}{r} = \frac{\sin.\Phi^2\cos.\Phi}{q} + q\left(\frac{1}{(1+p)^2} + \frac{1}{(2+p)^2} + \frac{1}{(3+p)^2} + \text{etc.}\right)$$

20. Quo autem inueſtigationem directionis et curuaturae ad curuae puncta a puncto principali coordinatis p et q definito extendere queamus, ponamus breuitatis cauſa

$$-\Delta + \frac{p}{1+p} + \frac{p}{2(2+p)} + \frac{p}{3(3+p)} + \frac{p}{4(4+p)} + \text{etc} = P$$
$$\frac{1}{(1+p)^2} + \frac{1}{(2+p)^2} + \frac{1}{(3+p)^2} + \frac{1}{(4+p)^2} + \text{etc.} = Q$$
$$\frac{1}{(1+p)^3} + \frac{1}{(2+p)^3} + \frac{1}{(3+p)^3} + \frac{1}{(4+p)^3} + \text{etc.} = R$$
$$\frac{1}{(1+p)^4} + \frac{1}{(2+p)^4} + \frac{1}{(3+p)^4} + \frac{1}{(4+p)^4} + \text{etc.} = S$$
etc.

vt ſit $l\left(1 + \frac{\psi}{q}\right) = P\omega + \frac{1}{2}Q\omega^2 - \frac{1}{3}R\omega^3 + \frac{1}{4}S\omega^4 - \frac{1}{5}T\omega^5 + \text{etc.}$

Iam

Iam hinc differentiando elicimus:

$$\frac{d\psi}{d\omega}=(q+\psi)(P+Q\omega-R\omega^2+S\omega^3-T\omega^4+\text{etc.})$$

atque vlterius differentiando

$$\frac{dd\psi}{d\omega^2}=(q+\psi)(P+Q\omega-R\omega^2+S\omega^3-T\omega^4+\text{etc.})^2$$
$$+(q+\psi)(Q-2R\omega+3S\omega^2-4T\omega^3+\text{etc.})$$

$$\frac{d^3\psi}{d\omega^3}=3(q+\psi)(Q-2R\omega+3S\omega^2-4T\omega^3+\text{etc.})(P+Q\omega-R\omega^2+S\omega^3-\text{etc.})$$
$$+(q+\psi)(P+Q\omega-R\omega^2+S\omega^3-T\omega^4+\text{etc.})^3$$
$$-(q+\psi)(2R-6S\omega+12T\omega^2-\text{etc.}).$$

His expeditis pro curuae puncto, quod conuenit abscissae $x=p+\omega$ et applicatae $y=q+\psi$ directio tangentis ita se habebit vt sit

$$\text{tang.}\,\Phi=\frac{d\psi}{d\omega}=(q+\psi)(P+Q\omega-R\omega^2+S\omega^3-T\omega^4+\text{etc.}).$$

Tum vero posito radio curuaturae $=r$, nouimus fore

$$r=\left(1+\frac{d\psi^2}{d\omega^2}\right)^{\frac{3}{2}}:\frac{dd\psi}{d\omega^2}=1:\frac{dd\psi}{d\omega^2}\,\text{cof.}\,\Phi^3$$

seu $\frac{1}{r}=\frac{dd\psi}{d\omega^2}\,\text{cof.}\,\Phi^3$, vnde pro variabilitate curuaturae elicimus:

$$-\frac{dr}{rr\,d\omega}=\frac{d^3\psi}{d\omega^3}\,\text{cof.}\,\Phi^3-\frac{3\,dd\psi}{d\omega^2}\cdot\frac{d\Phi}{d\omega}\,\text{fin.}\,\Phi\,\text{cof.}\,\Phi^2.$$

Eft vero $\frac{d\Phi}{\text{cof.}\,\Phi^2}=\frac{dd\psi}{d\omega}$ vnde conficitur:

$$-\frac{dr}{rr\,d\omega}=\frac{d^3\psi}{d\omega^3}\,\text{cof.}\,\Phi^3-3\left(\frac{dd\psi}{d\omega^2}\right)^2\text{fin.}\,\Phi\,\text{cof.}\,\Phi^4.$$

Quae-

Quaestio quarta.

Naturam curuae hypergeometricae circa punctum eius infimum μ, *vbi applicata est minima, inuestigare.*

21. Quoniam hoc punctum parum distat a loco, cui respondet abscissa $=\frac{1}{2}$ et applicata $=\frac{1}{2}\sqrt{\pi}$ statuamus $p=\frac{1}{2}$ vt sit $q=\frac{1}{2}\sqrt{\pi}$, hincque primo quaeramus valores litterarum P, Q, R, S etc. qui prodibunt

$$P=-\Delta+\frac{1}{3}+\frac{1}{2.5}+\frac{1}{3.7}+\text{etc.}=2(1-l2)-\Delta=0,0364899739785\overline{7}$$

$$Q=\frac{4}{3^2}+\frac{4}{5^2}+\frac{4}{7^2}+\frac{4}{9^2}+\text{etc.}=0,93480220054468$$

$$R=\frac{8}{3^3}+\frac{8}{5^3}+\frac{8}{7^3}+\frac{8}{9^3}+\text{etc.}=0,41439832211716$$

$$S=\frac{16}{3^4}+\frac{16}{5^4}+\frac{16}{7^4}+\frac{16}{9^4}+\text{etc.}=0,23484850566707$$

$$T=\frac{32}{3^5}+\frac{32}{5^5}+\frac{32}{7^5}+\frac{32}{9^5}+\text{etc.}=0,14476040831276$$

$$V=\frac{64}{3^6}+\frac{64}{5^6}+\frac{64}{7^6}+\frac{64}{9^6}+\text{etc.}=0,09261290502029$$

$$W=\frac{128}{3^7}+\frac{128}{5^7}+\frac{128}{7^7}+\frac{128}{9^7}+\text{etc.}=0,06035822809843$$

Deinde vero est $q=\frac{1}{2}\sqrt{\pi}=0,886226925452\overline{74}$.

22. Hinc iam ante omnia definiamus locum μ, vbi applicata est omnium minima, quem cum leues approximationes ostendant respondere abscissae $x=0,4616$, posito $p+\omega=\frac{1}{2}+\omega=0,4616$, colligitur proxime $\omega=-0,0383$, qui iam valor ex aequatione $\frac{d\psi}{d\omega}=0$ seu

$$P+Q\omega-R\omega^2+S\omega^3-T\omega^4+\text{etc.}=0$$

accuratius inuestigari debet. Cum igitur sit prope $\omega = -\frac{1}{26}$ statuatur $\omega = -\frac{1}{26} - z$, et facta substitutione necesse est fiat . . . $0{,}0364899739785 7 =$

$$
\begin{aligned}
&+0{,}0359539307901 8 + 0{,}934802200\, z\\
&+0{,}0006130152694 0 + 0{,}031876794\, z + 0{,}414398\, zz\\
&+0{,}0000133618858 5 + 0{,}001042227\, z + 0{,}027097\, zz\\
&+0{,}0000003167790 0 + 0{,}000032945\, z + 0{,}001285\, zz\\
&+0{,}0000000077947 9 + 0{,}000001013\, z + 0{,}000053\, zz\\
&+0{,}0000000001953 8 + 0{,}000000030\, z + 0{,}000002\, zz\\
&+0{,}0000000000049 6\\
&\qquad 13\\
\hline
&\quad 0{,}0365806327197 0 + 0{,}967755211\, z + 0{,}442835\, zz\\
&\quad 0{,}0364899739785 7\\
\hline
&0 = 0{,}0000906587411 3 + 0{,}967755211\, z + 0{,}442835\, zz
\end{aligned}
$$

vnde reperitur $z = -0{,}000093683 23$ hincque $\omega = -0{,}038367855 23$.

Quocirca minima applicata $m\mu$ respondet, abscissae $Om = 0{,}46163214477$. Pro applicata vero $m\mu = q + \psi$ euolui oportet aequationem

$$l\left(1 + \frac{\psi}{q}\right) = P\omega + \frac{1}{2}Q\omega^2 - \frac{1}{3}R\omega^3 + \frac{1}{4}S\omega^4 - \frac{1}{5}T\omega^5 + \text{etc.}$$

ex qua colligitur $l\left(1 + \frac{\psi}{q}\right) = -0{,}000704053$ porroque $1 + \frac{\psi}{q} = 1 - 0{,}000703805$, ita vt fiat applicata minima $m\mu = q + \psi = 0{,}8856031945$.

23. Definiamus iam in genere ex aequatione logarithmica valorem ipsius ψ ac calculo subducto obtinebimus:

$\frac{\psi}{q} =$

$$\frac{\psi}{q} = + 0,0364899740\,\omega + 0,468066860\,\omega^2$$
$$- 0,121069221\,\omega^3 + 0,16321479\,\omega^4$$
$$- 0,09360753\,\omega^5 \text{ etc.}$$

qui termini si quidem ω valde paruum accipiatur sufficiunt. Ponamus autem breuitatis gratia

$$\frac{\psi}{q} = \mathfrak{P}\omega + \mathfrak{Q}\omega^2 - \mathfrak{R}\omega^3 + \mathfrak{S}\omega^4 - \mathfrak{T}\omega^5 \text{ vt fit}$$

$\mathfrak{P} = 0,0364899740$; $\mathfrak{Q} = 0,468066860$

$\mathfrak{R} = 0,121069221$; $\mathfrak{S} = 0,16321479$

$\mathfrak{T} = 0,09360753$

atque hinc habebimus:

$$\frac{d\psi}{d\omega} = q(\mathfrak{P} + 2\mathfrak{Q}\omega - 3\mathfrak{R}\omega^2 + 4\mathfrak{S}\omega^3 - 5\mathfrak{T}\omega^4),$$
$$\frac{dd\psi}{d\omega^2} = q(2\mathfrak{Q} - 6\mathfrak{R}\omega + 12\mathfrak{S}\omega^2 - 20\mathfrak{T}\omega^3).$$

Quodsi iam hinc radium curuaturae in loco infimo μ vbi est $\omega = -0,0383678523$ indagare velimus, quoniam ibi est $\frac{d\psi}{d\omega} = 0$, erit is $= \frac{d\omega^2}{dd\psi}$. Ponatur in hoc loco radius curuaturae $= r$ et cum sit

$$\frac{1}{r} = 2q(\mathfrak{Q} - 3\mathfrak{R}\omega + 6\mathfrak{S}\omega^2 - 10\mathfrak{T}\omega^3) = 0,9669949$$

prodit pro puncto μ radius curuaturae $r = 1,'66893$.

24. Determinationes has puncti curuae infimi μ ideo omni studio inuestigaui, quod non sine ratione suspicari licebat quemadmodum hoc punctum singulari praerogatiua est praeditum, ita numeros eius indolem exhibentes elegantiam quandam in se esse complexuros, ac nisi satis simpliciter siue rationaliter

naliter siue irrationaliter exprimantur, ad simplicius saltem genus quoddam transcendentium quantitatum relatum iri. Praeter expectationem autem vsu venit, vt tale criterium elegantiae neque in abscissa $Om = 0,4616321447$ neque in applicata $m\mu = 0,8856031945$ neque in radio curuaturae ibidem $= 1,166893$ appareat; nulla enim affinitas neque cum numeris rationalibus neque irrationalibus duntaxat simplicioribus, neque cum quadratura circuli, nec logarithmis vel exponentialibus deprehenditur. Cum etiam si abscissa Om vt logarithmus consideretur, numerus ei conueniens aliquid promittere videri posset, hunc numerum quaesiui et inueni $= 1,5866616$, in quo autem nulla affinitas cum quantitatibus cognitis cernitur.

25. Antequam huic speculationi finem imponam, obseruasse iuuabit formulam $1 . 2 . 3 \ldots\ldots x$ etiam per sequentem seriem indefinite exprimi posse

$$x^x - x(x-1)^x + \frac{x(x-1)}{1 . 2}(x-2)^x - \frac{x(x-1)(x-2)}{1 . 2 . 3}(x-3)^x + \text{etc.}$$

quippe quae quoties x est numerus integer positiuus sponte dat illud productum $1 . 2 . 3 \ldots\ldots x$. Hoc vero etiam praestat ista expressio latius patens:

$$a^x - x(a-1)^x + \frac{x(x-1)}{1 . 2}(a-2)^x - \frac{x(x-1)(x-2)}{1 . 2 . 3}(a-3)^x + \text{etc.}$$

erit enim si loco x successiue substituantur numeri 0, 1, 2, 3 etc. vt sequitur:

$a^0 = 1$

$$a^0 = 1$$
$$a^1-(a-1)^1 = 1$$
$$a^2-2(a-1)^2+(a-2)^2 = 1.1$$
$$a^3-3(a-1)^3+3(a-2)^3-(a-3)^3 = 1.2.3$$
$$a^4-4(a-1)^4+6(a-2)^4-4(a-3)^4+(a-4)^4 = 1.2.3.4$$
$$a^5-5(a-1)^5+10(a-2)^5-10(a-3)^5+5(a-4)^5-(a-5)^5 = 1.2.3.4.5.$$
etc.

26. Manifesta haec quidem sunt ex iis, quae de differentiis cuiusque ordinis progressionum algebraicarum sunt demonstrata, verumtamen ex ipsa harum serierum natura veritas haud facile euincitur; vnde sequens demonstratio non superflua videtur. Cum pro exponentibus minoribus x res per se sit perspicua, ratiocinium ita instruo vt concessa pro casu $x=n$ veritate, eam quoque pro casu $x=n+1$ locum habere sim ostensurus.

Sit ergo

I. $a^n-n(a-1)^n+\frac{n(n-1)}{1.2}(a-2)^n-\text{etc.} = N = 1.2.3..n$

et quia summa N non ab a pendet erit etiam

II. $(a-1)^n-n(a-2)^n+\frac{n(n-1)}{1.2}(a-3)^n-\text{etc.} = N$

quae ab illa subtracta relinquit

III. $a^n-\frac{(n+1)}{1}(a-1)^n+\frac{(n+1)n}{1.2}(a-2)^n-\frac{(n+1)n(n-1)}{1.2.3}(a-3)^n+\text{etc.} = 0$

haec multiplicetur per a vt prodeat

IV. $a^{n+1}-\frac{(n+1)}{1}a(a-1)^n+\frac{(n+1)n}{1.2}a(a-2)^n-\frac{(n+1)n(n-1)}{1.2.3}a(a-3)^n+\text{etc.} = 0$

huic addatur aequatio II in $n+1$ ducta, nempe:

V. $+(n+1)1(a-1)^n-\frac{(n+1)n}{1.\ 2}2(a-2)^n+\frac{(n+1)n(n-1)}{1.\ 2.\ 3}3(a-3)^n-\text{etc}=(n+1)N$

atque aggregatum IV + V dabit

VI. $a^{n+1}-\frac{(n+1)}{1}(a-1)^{n+1}+\frac{(n+1)n}{1.\ 2}(a-2)^{n+1}-\frac{(n+1)n(n-1)}{1.\ 2.\ 3}(a-3)^{n+1}+\text{etc.}=(n+1)N$

vbi ob $N=1.2.3\ldots n$ erit $(n+1)N=1.2.3\ldots(n+1)$.

Euictum ergo eſt, quod ſi propoſitio noſtra

$$a^x-x(a-1)^x+\frac{x(x-1)}{1.\ 2}(a-2)^x-\frac{x(x-1)(x-2)}{1.\ 2.\ 3}(a-3)^x+\text{etc.}=1.\ 2.\ 3\ldots x$$

vera fuerit caſu $x=n$, eam quoque veram eſſe caſu $x=n+1$. Quoniam igitur ea manifeſto vera eſt caſu $x=1$, hinc ſequitur eam quoque veram eſſe pro omnibus numeris integris poſitiuis loco x aſſumtis.

27. Quanquam autem haec expreſſio ſatis eſt elegans et omni attentione digna, tamen ad noſtrum inſtitutum, cui curua hypergeometrica eſt propoſita, minus eſt accommodata quoniam pro caſibus quibus x eſt numerus fractus, haec ſeries non ſolum in infinitum excurrit, ſed etiam ſi denominator eſt numerus par, terminos imaginarios complectitur, ita vt eius valorem ne appropinquando quidem colligere liceat. Ita poſito $x=\frac{1}{2}$ prodit haec ſeries infinita:

$$\sqrt{a}-\tfrac{1}{2}\sqrt{(a-1)}-\tfrac{1}{2}.\tfrac{1}{4}\sqrt{(a-2)}-\tfrac{1}{2}.\tfrac{1}{4}.\tfrac{3}{6}\sqrt{(a-3)}-\tfrac{1}{2}.\tfrac{1}{4}.\tfrac{3}{6}.\tfrac{5}{8}\sqrt{(a-4)}-\text{etc.}$$

cuius valorem eſſe $=\frac{1}{2}\sqrt{\pi}$, vix quisquam oſtendere poterit. Pari modo ſumendo $x=-\frac{1}{2}$ ex ſuperioribus quidem iam nouimus eſſe,

$\sqrt{\pi}=$

$$\sqrt{\pi} = \frac{1}{\sqrt{a}} + \frac{1}{2\sqrt{(a-1)}} + \frac{1 \cdot 3}{2 \cdot 4 \sqrt{(a-2)}} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \sqrt{(a-3)}} + \text{etc.}$$

Nihilo vero minus vberior huius ſeriei inueſtigatio geometris merito commendatur, imprimis ſi ei amplior extenſio inducatur, atque hac forma repraeſentetur:

$$s = x^n - m(x-1)^n + \frac{m(m-1)}{1 \cdot 2}(x-2)^n - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}(x-3)^n + \text{etc.}$$

ſeuſ enim ſtudio adhibito, mox admodum inſignes proprietates deprehenduntur, quarum euolutio omnem attentionem noſtram mereri videtur. Equidem quae mihi circa eam obſeruare contigit phaenomena prorſus ſingularia hic in medium afferam.

Obſeruationes circa hanc ſeriem.

$$s = x^n - m(x-1)^n + \frac{m(m-1)}{1 \cdot 2}(x-2)^n - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}(x-3)^n + \text{etc.}$$

I. In praecedentibus igitur iam demonſtraui, ſi fuerit exponens $n = m$, fore huius ſeriei ſummam

$$s = 1 \cdot 2 \cdot 3 \ldots\ldots m$$

ita vt ea hoc caſu non a numero x pendeat. Hinc autem primo colligo ſi fuerit $n = m-1$, tum fore $s = 0$. Cum enim ſumto $n = m$ ſit

$$♄ \ldots 1 \cdot 2 \cdot 3 \ldots\ldots m = x^m - m(x-1)^m + \frac{m(m-1)}{1 \cdot 2}(x-2)^m - \text{etc.}$$

erit ſcribendo $x-1$ loco x et $m-1$ loco m ſimili modo:

$$♃ \ldots 1 \cdot 2 \cdot 3 \ldots (m-1) = (x-1)^{m-1} - (m-1)(x-2)^{m-1} + \frac{(m-1)(m-2)}{1 \cdot 2}(x-2)^{m-1} - \text{etc.}$$

Iam

Iam illa aequatio hoc modo referatur:

$$♂ \ldots 1.2.3\ldots m = x.x^{m-1} - mx(x-1)^{m-1} + \frac{m(m-1)}{1.2}x(x-2)^{m-1} - \text{etc.}$$
$$+ m(x-1)^{m-1} - \frac{m(m-1)}{1}(x-2)^{m-1} + \text{etc.}$$

aequatio autem ♃ per m multiplicata dat:

$$☉\ 1.2.3\ldots m = m(x-1)^{m-1} - \frac{m(m-1)}{1}(x-2)^{m-1} + \frac{m(m-1)(m-2)}{1.2}(x-2)^{m-1} - \text{etc.}$$

quae ab illa ♂ subtracta et diuisione per x facta praebet

$$♀\ 0 = x^{m-1} - \frac{m}{1}(x-1)^{m-1} + \frac{m(m-1)}{1.2}(x-2)^{m-1} - \text{etc.}$$

quae est ipsa aequatio proposita pro casu $n = m-1$, cuius idcirco valor est $= 0$.

II. Eodem modo ostenditur seriei propositae summam s quoque euanescere casu $n = m-2$. Series enim illa ♀ hoc modo repraesentetur:

$$☿ = 0 = x.x^{m-2} - \frac{m}{1}x(x-1)^{m-2} + \frac{m(m-1)}{1.2}x(x-2)^{m-2} - \text{etc.}$$
$$+ m(x-1)^{m-2} - \frac{m(m-1)}{1}(x-2)^{m-2} + \text{etc.}$$

et si in eadem serie ♀ scribatur $x-1$ loco x et $m-1$ loco m, tota vero series per m multiplicetur, fit

$$☽ \ldots 0 = m(x-1)^{m-2} - \frac{m(m-1)}{1}(x-2)^{m-2} + \text{etc.}$$

Hac ab illa subtracta residuum per x diuidatur, prodibitque:

$$0 = x^{m-2} - \frac{m}{1}(x-1)^{m-2} + \frac{m(m-1)}{1.2}(x-2)^{m-2} - \text{etc.}$$

Sicque

Sicque feriei propofitae fumma s etiam euanefcit cafu $n = m - 2$, parique modo oftendi poteft eam quoque euanefcere cafibus $n = m - 3$, $n = m - 4$, et in genere $n = m - i$, exiftente i numero quocunque integro pofitiuo. Teneatur ergo feriei propofitae fummam effe $s = 1 . 2 . 3 \ldots m$ cafu $n = m$, cafibus autem quibus exponens n minor eft numero m fummam in nihilum abire, fiquidem numeri m et n fint integri, feu faltem $n - m$ numerus integer pofitiuus.

III. Quo igitur indolem reliquorum cafuum perfcrutemur fingulos terminos noftrae feriei euoluamus et fecundum poteftates ipfius x difponamus, quo pacto confequemur:

$$s = x^n\left(1 - m + \frac{m(m-1)}{1 . 2} - \frac{m(m-1)(m-2)}{1 . 2 . 3} + \text{etc.}\right)$$
$$+ n x^{n-1}\left(m - \frac{2m(m-1)}{1 . 2} + \frac{3m(m-1)(m-2)}{1 . 2 . 3} - \text{etc.}\right)$$
$$- \frac{n(n-1)}{1 . 2} x^{n-2}\left(m - \frac{4m(m-1)}{1 . 2} + \frac{9m(m-1)(m-2)}{1 . 2 . 3} - \text{etc.}\right)$$
$$+ \frac{n(n-1)(n-2)}{1 . 2 . 3} x^{n-3}\left(m - \frac{8m(m-1)}{1 . 2} + \frac{27m(m-1)(m-2)}{1 . 2 . 3} - \text{etc.}\right)$$
etc.

quarum fingularum ferierum fummas fequenti modo inueniemus; prima aliquanto generalius exhibeatur, et cum eius fumma fit cognita:

$$1 - mu + \frac{m(m-1)}{1 . 2} u^2 - \frac{m(m-1)(m-2)}{1 . 2 . 3} u^3 + \text{etc.} = (1 - u)^m$$

continuo eam differentiemus, et perpetuo loco du reftituamus u, fietque fignis mutatis:

$$mu - \frac{2m(m-1)}{1 \cdot 2}u^2 + \frac{3m(m-1)(m-2)}{1 \cdot 2 \cdot 3}u^3 - \text{etc.} = mu(1-u)^{m-1}$$

$$mu - \frac{2^2 m(m-1)}{1 \cdot 2}u^2 + \text{etc.} = mu(1-u)^{m-1} - m(m-1)u^2(1-u)^{m-2}$$

$$mu - \frac{2^3 m(m-1)}{1 \cdot 2}u^2 + \text{etc.} = mu(1-u)^{m-1} - 3m(m-1)u^2(1-u)^{m-2} + m(m-1)(m-2)u^3(1-u)^{m-3}$$

$$mu - \frac{2^4 m(m-1)}{1 \cdot 2}u^2 + \text{etc.} = mu(1-u)^{m-1} - 7m(m-1)uu(1-u)^{m-2} + 6m(m-1)(m-2)u^3(1-u)^{m-3} - m(m-1)(m-2)(m-3)u^4(1-u)^{m-4}$$

etc.

Hic ergo iam ſcribi oportet $u = 1$, quo facto omnes termini in quouis ordine euaneſcunt praeter eos, vbi exponens ipſius $1-u$ fit $= 0$.

IV. Tribuantur nunc ſucceſſiue ipſi m valores $1, 2, 3, 4, 5$ etc. et loco coefficientis in genere $\frac{n(n-1)(n-2) \dots (n-i)}{1 \cdot 2 \cdot 3 \dots (i+1)}$ ſcribatur breuitatis gratia $\left(\frac{n-i}{i+1}\right)$, quo facto nanciſcimur valores ſequentes:

ſi	erit
$m = 1$	$\frac{s}{1} = \left(\frac{n}{1}\right)x^{n-1} - \left(\frac{n-1}{2}\right)x^{n-2} + \left(\frac{n-2}{3}\right)x^{n-3} - \left(\frac{n-3}{4}\right)x^{n-4} + \left(\frac{n-4}{5}\right)x^{n-5} - \text{etc.}$
$m = 2$	$\frac{s}{1 \cdot 2} = \left(\frac{n-1}{2}\right)x^{n-2} - 3\left(\frac{n-2}{3}\right)x^{n-3} + 7\left(\frac{n-3}{4}\right)x^{n-4} - 15\left(\frac{n-4}{5}\right)x^{n-5} + 31\left(\frac{n-5}{6}\right)x^{n-6} - \text{etc.}$
$m = 3$	$\frac{s}{1 \cdot 2 \cdot 3} = \left(\frac{n-2}{3}\right)x^{n-3} - 6\left(\frac{n-3}{4}\right)x^{n-4} + 25\left(\frac{n-4}{5}\right)x^{n-5} - 90\left(\frac{n-5}{6}\right)x^{n-6} + 301\left(\frac{n-6}{7}\right)x^{n-7} + \text{etc.}$
$m = 4$	$\frac{s}{1 \cdot 2 \dots 4} = \left(\frac{n-3}{4}\right)x^{n-4} - 10\left(\frac{n-4}{5}\right)x^{n-5} + 65\left(\frac{n-5}{6}\right)x^{n-6} - 350\left(\frac{n-6}{7}\right)x^{n-7} + 1701\left(\frac{n-7}{8}\right)x^{n-8} + \text{etc.}$
$m = 5$	$\frac{s}{1 \cdot 2 \dots 5} = \left(\frac{n-4}{5}\right)x^{n-5} - 15\left(\frac{n-5}{6}\right)x^{n-6} + 140\left(\frac{n-6}{7}\right)x^{n-7} - 1050\left(\frac{n-7}{8}\right)x^{n-8} + 6951\left(\frac{n-8}{9}\right)x^{n-9} + \text{etc.}$
$m = 6$	$\frac{s}{1 \cdot 2 \dots 6} = \left(\frac{n-5}{6}\right)x^{n-6} - 21\left(\frac{n-6}{7}\right)x^{n-7} + 266\left(\frac{n-7}{8}\right)x^{n-8} - 2646\left(\frac{n-8}{9}\right)x^{n-9} + 22827\left(\frac{n-9}{10}\right)x^{n-10} + \text{etc.}$

vbi

vbi formatio coefficientium numericorum ex antecedentibus est manifesta, est nempe pro postrema sexta serie:

$$21=6.1+15;\ 266=6.21+140;\ 2646=6.266+1050 \text{ etc.}$$

Atque hinc statim perspicitur, si fuerit $n<m$ valorem ipsius s euanescere, in postrema enim serie si $n<6$ ideoque vel 5 vel 4 vel 3 etc. erit $(\frac{n-5}{6})=0$, $(\frac{n-6}{7})=0$ etc.

Tum vero etiam si sit $n=m$, euidens est fore $\frac{s}{1.2\ldots m}=1$, est enim in infima serie:

$$(\tfrac{6-5}{6})=1,\ (\tfrac{6-6}{7})=0,\ (\tfrac{6-7}{8})=0,\ (\tfrac{6-8}{9})=0 \text{ etc.}$$

Euolutio casuum $n=m+1$.

V. Hinc primo casus euoluamus, quibus est $n=m+1$, et forma postrema praebet

si	has summas
$m=1, n=2$	$\frac{s}{1}=2x-1$
$m=2, n=3$	$\frac{s}{1.2}=3x-3$
$m=3, n=4$	$\frac{s}{1.2.3}=4x-6$
$m=4, n=5$	$\frac{s}{1.2.3.4}=5x-10$
$m=5, n=6$	$\frac{s}{1.2\ldots5}=6x-15$
	etc

vbi priores coefficientes ipsius x ipsi n, numeri absoluti autem trigonalibus ipsius n aequentur, habebimus in genere

si sit	hanc aequationem
$n = m + 1$	$\frac{s}{1 \cdot 2 \ldots m} = (m+1)x - \frac{m(m+1)}{1 \cdot 2} = (m+1)(x - \frac{m}{2})$

ita vt sit

$$x^{m+1} - m(x-1)^{m+1} + \frac{m(m-1)}{1 \cdot 2}(x-2)^{m+1} - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}(x-3)^{m+1} + \text{etc.}$$
$$= 1 \cdot 2 \cdot 3 \ldots \ldots (m+1)(x - \tfrac{m}{2}).$$

Euolutio casuum $n = m + 2$.

VI. Pro his ergo casibus habebimus

si fuerit	has aequationes
$m = 1, n = 3$	$\frac{s}{1} = 3x^2 - 3 \cdot 1 x + 1 \cdot 1 = 3(xx - x + \frac{2}{6})$
$m = 2, n = 4$	$\frac{s}{1 \cdot 2} = 6x^2 - 4 \cdot 3x + 1 \cdot 7 = 6(xx - 2x + \frac{7}{6})$
$m = 3, n = 5$	$\frac{s}{1 \cdot 2 \cdot 3} = 10x^2 - 5 \cdot 6x + 1 \cdot 25 = 10(xx - 3x + \frac{15}{6})$
$m = 4, n = 6$	$\frac{s}{1 \cdot 2 \cdot 3 \cdot 4} = 15x^2 - 6 \cdot 10x + 1 \cdot 65 = 15(xx - 4x + \frac{25}{6})$
$m = 5, n = 7$	$\frac{s}{1 \cdot 2 \ldots 5} = 21x^2 - 7 \cdot 15x + 1 \cdot 140 = 21(xx - 5x + \frac{40}{6})$
$m = 6, n = 8$	$\frac{s}{1 \cdot 2 \ldots 6} = 28x^2 - 8 \cdot 21x + 1 \cdot 266 = 28(xx - 6x + \frac{57}{6})$

etc.

quae formae ita repraesentari possunt

si fuerit	erit
$m = 1; n = 3$	$\frac{s}{1} = \frac{2 \cdot 3}{1 \cdot 2}(xx - x + \frac{1 \cdot 4}{12})$
$m = 2; n = 4$	$\frac{s}{1 \cdot 2} = \frac{3 \cdot 4}{1 \cdot 2}(xx - 2x + \frac{2 \cdot 7}{12})$
$m = 3; n = 5$	$\frac{s}{1 \cdot 2 \cdot 3} = \frac{4 \cdot 5}{1 \cdot 2}(xx - 3x + \frac{3 \cdot 10}{12})$
$m = 4; n = 6$	$\frac{s}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{5 \cdot 6}{1 \cdot 2}(xx - 4x + \frac{4 \cdot 13}{12})$
$m = 5; n = 7$	$\frac{s}{1 \cdot 2 \cdot 3 \ldots 5} = \frac{6 \cdot 7}{1 \cdot 2}(xx - 5x + \frac{5 \cdot 16}{12})$
$m = 6; n = 8$	$\frac{s}{1 \cdot 2 \ldots 6} = \frac{7 \cdot 8}{1 \cdot 2}(xx - 6x + \frac{6 \cdot 19}{12})$

vnde

vnde manifesto sequitur, si in genere sit $n = m + 2$

fore $\frac{s}{1 \cdot 2 \cdot \ldots \cdot m} = \frac{m+1}{1} \cdot \frac{m+2}{2} \left(xx - mx + \frac{m(3m+1)}{12} \right)$

seu $\frac{s}{1 \cdot 2 \cdot \ldots \cdot m} = \frac{m+1}{1} \cdot \frac{m+2}{2} \left(\left(x - \frac{m}{2}\right)^2 + \frac{m}{12} \right)$.

Ergo hinc obtinetur ista summatio

$$x^{m+2} - m(x-1)^{m+2} + \frac{m(m-1)}{1 \cdot 2}(x-2)^{m+2} - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}(x-3)^{m+2} + \text{etc.}$$
$$= 1 \cdot 2 \cdot 3 \ldots \ldots (m+2) \left(\tfrac{1}{2}\left(x - \tfrac{m}{2}\right)^2 + \tfrac{m}{24} \right).$$

Euolutio casuum $n = m + 3$.

VII. Pro his casibus habebimus

si fuerit	has aequationes
$m = 1; n = 4$	$\frac{s}{1} = 4x^3 - 6 \cdot 1x^2 + 4 \cdot 1x - 1 \cdot 1$
$m = 2; n = 5$	$\frac{s}{1 \cdot 2} = 10x^3 - 10 \cdot 3x^2 + 5 \cdot 7x - 1 \cdot 15$
$m = 3; n = 6$	$\frac{s}{1 \cdot 2 \cdot 3} = 20x^3 - 15 \cdot 6x^2 + 6 \cdot 25x - 1 \cdot 90$
$m = 4; n = 7$	$\frac{s}{1 \cdot 2 \cdot \cdot 4} = 35x^3 - 21 \cdot 10x^2 + 7 \cdot 65x - 1 \cdot 350$
$m = 5; n = 8$	$\frac{s}{1 \cdot 2 \cdot \cdot \cdot 5} = 56x^3 - 28 \cdot 15x^2 + 8 \cdot 140x - 1 \cdot 1050$

quae hoc modo repraesententur:

$$\frac{s}{1} = \frac{2 \cdot 3 \cdot 4}{1 \cdot 2 \cdot 3}\left(x^3 - \frac{3}{2}x^2 + \frac{1 \cdot 4}{4}x - \frac{1 \cdot 1 \cdot 2}{8}\right)$$
$$\frac{s}{1 \cdot 2} = \frac{3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3}\left(x^3 - \frac{6}{2}x^2 + \frac{2 \cdot 7}{4}x - \frac{2 \cdot 2 \cdot 3}{8}\right)$$
$$\frac{s}{1 \cdot 2 \cdot 3} = \frac{4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3}\left(x^3 - \frac{9}{2}x^2 + \frac{3 \cdot 10}{4}x - \frac{3 \cdot 3 \cdot 4}{8}\right)$$
$$\frac{s}{1 \cdot 2 \cdot \cdot 4} = \frac{5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3}\left(x^3 - \frac{12}{2}x^2 + \frac{4 \cdot 13}{4}x - \frac{4 \cdot 4 \cdot 5}{8}\right)$$
$$\frac{s}{1 \cdot 2 \cdot \cdot 5} = \frac{6 \cdot 7 \cdot 8}{1 \cdot 2 \cdot 3}\left(x^3 - \frac{15}{2}x^2 + \frac{5 \cdot 16}{4}x - \frac{5 \cdot 5 \cdot 6}{8}\right)$$

etc.

vnde in genere concluditur pro caſu $n = m + 3$

$$\frac{s}{1.2\ldots m} = \frac{m+1}{1}.\frac{m+2}{2}.\frac{m+3}{3}\left(x^3 - \frac{3m}{2}x^2 + \frac{m(3m+1)}{4}x - \frac{mm(m+1)}{8}\right)$$

$$= \frac{m+1}{1}.\frac{m+2}{2}.\frac{m+3}{3}\left(\left(x-\frac{m}{2}\right)^3 + \frac{m}{4}\left(x-\frac{m}{2}\right)\right)$$

ita vt iam conſequamur:

$$x^{m+3} - m(x-1)^{m+3} + \frac{m(m-1)}{1.\ 2.}(x-2)^{m+3} - \frac{m(m-1)(m-2)}{1.\ 2.\ 3}(x-3)^{m+3}\ \text{etc.}$$

$$= 1.\ 2.\ 3\ \ldots\ldots(m+3)\left(\frac{1}{6}\left(x-\frac{m}{2}\right)^3 + \frac{m}{24}\left(x-\frac{m}{2}\right)\right).$$

Praeparatio ad caſus ſequentes.

VIII. Cum §. 4. formulas tantum ad caſum $m = 6$ produxerimus, conemur pro iis formam generalem eruere. In hunc finem ſtatuamus $n = m + \lambda$, et ad abbreuiandum loco talis expreſſionis $\frac{k(k-1)(k-2)(k-3)\ldots\ldots(k-i+1)}{1.\ 2.\ 3.\ 4\ \ldots\ldots\ i}$ ſcribamus $\left(\frac{k}{i}\right)$ ita vt k denotet primum factorem numeratoris, i vero vltimum denominatoris. Ponamus igitur eſſe pro caſu

$$m-1 \quad \left| \frac{s}{1.2\ldots(m-1)} = \left(\frac{m+\lambda}{m-1}\right)x^{\lambda+1} - A\left(\frac{m+\lambda}{m}\right)x^{\lambda} + B\left(\frac{m+\lambda}{m+1}\right)x^{\lambda-1} - C\left(\frac{m+\lambda}{m+2}\right)x^{\lambda-2}\ \text{etc.}\right.$$

$$m \quad \left| \frac{s}{1.\ 2.\ 3\ \ldots\ m} = \left(\frac{m+\lambda}{m}\right)x^{\lambda} - A^{1}\left(\frac{m+\lambda}{m+1}\right)x^{\lambda-1} + B^{1}\left(\frac{m+\lambda}{m+2}\right)x^{\lambda-2} - C^{1}\left(\frac{m+\lambda}{m+3}\right)x^{\lambda-3}\ \text{etc.}\right.$$

ita vt A^1, B^1, C^1, D^1 etc. ſint ii coefficientes, quos inueſtigari oportet. Ex lege autem iſtarum formularum vidimus eſſe; $A^1 = m.1 + A$; $B^1 = mA^1 + B$; $C^1 = mB^1 + C$; $D^1 = mC^1 + D$ etc. vbi euidens eſt eſſe $A = \frac{m(m-1)}{1.\ 2}$ et $A^1 = \frac{(m+1)m}{1.\ 2}$ ſeu noſtro ſignando modo $A = \left(\frac{m}{2}\right)$ et $A^1 = \left(\frac{m+1}{2}\right)$. Iam pro ſequentibus operationibus obſeruo eſſe:

$$\left(\frac{m+\mu+1}{\nu}\right) - \left(\frac{m+\mu}{\nu}\right) = \left(\frac{m+\mu}{\nu-1}\right)$$

quod

quod facile inde patet, quod fit euoluendo:

$$\left(\frac{m+\mu+1}{\nu}\right)=\frac{(m+\mu+1)(m+\mu)(m+\mu-1)\ldots\ldots(m+\mu+2-\nu)}{1.\quad 2.\quad 3\quad .\quad .\quad \nu}$$

$$\left(\frac{m+\mu}{\nu}\right)=\frac{(m+\mu)(m+\mu-1)\ldots\ldots(m+\mu+2-\nu)(m+\mu+1-\nu)}{1.\quad 2\quad \ldots\ldots\quad (\nu-1).\quad \nu}$$

vnde perspicitur esse $\left(\frac{m+1}{2}\right)-\left(\frac{m}{2}\right)=\left(\frac{m}{1}\right)=m.$

IX. Iam vt fiat

$$B^{I}-B=mA^{I}=\left(\frac{m+1}{2}\right)m=3\left(\frac{m+1}{3}\right)+\left(\frac{m+1}{2}\right)$$

statuamus $B=\alpha\left(\frac{m+1}{4}\right)+\beta\left(\frac{m+1}{3}\right)$

hincque $B^{I}=\alpha\left(\frac{m+2}{4}\right)+\beta\left(\frac{m+2}{3}\right)$

prodibitque

$$B^{I}-B=\alpha\left(\frac{m+1}{3}\right)+\beta\left(\frac{m+1}{2}\right)$$

vnde fit $\alpha=3$ et $\beta=1$ ita vt fit

$$B^{I}=3\left(\frac{m+2}{4}\right)+\left(\frac{m+2}{3}\right).$$

Pro sequentibus operationibus autem notetur esse in genere:

$$\left(\frac{m+\mu}{\nu}\right)m=(\nu+1)\left(\frac{m+\mu}{\nu+1}\right)+(\nu-\mu)\left(\frac{m+\mu}{\nu}\right)$$

quippe quae forma prodit, si valor ipsius $\left(\frac{m+\mu}{\nu}\right)$ supra euolutus multiplicetur per

$$m=m+\mu-\nu+\nu-\mu=(\nu+1).\frac{m+\mu-\nu}{\nu+1}+(\nu-\mu).$$

X. His obseruatis cum esse debeat $C^{I}-C=mB^{I}$, ob

$$\left(\frac{m+2}{4}\right)m=5\left(\frac{m+2}{5}\right)+2\left(\frac{m+2}{4}\right)$$

et

et $\left(\frac{m+2}{3}\right) = 4\left(\frac{m+2}{4}\right) + 1\left(\frac{m+2}{3}\right)$ erit

$$mB' = 15\left(\frac{m+2}{6}\right) + 10\left(\frac{m+2}{4}\right) + 1\left(\frac{m+2}{3}\right)$$

ſtatuatur ergo

$$C = 15\left(\frac{m+2}{6}\right) + 10\left(\frac{m+2}{5}\right) + 1\left(\frac{m+2}{4}\right)$$

hinc $C' = 15\left(\frac{m+3}{6}\right) + 10\left(\frac{m+3}{5}\right) + 1\left(\frac{m+3}{4}\right)$.

XI. Simili modo cum eſſe debeat $D' - D = mC'$ quia eſt

$$m\left(\frac{m+3}{6}\right) = 7\left(\frac{m+3}{7}\right) + 3\left(\frac{m+3}{6}\right)$$

$$m\left(\frac{m+3}{5}\right) = 6\left(\frac{m+3}{6}\right) + 2\left(\frac{m+3}{5}\right)$$

$$m\left(\frac{m+3}{4}\right) = 5\left(\frac{m+3}{5}\right) + 1\left(\frac{m+3}{4}\right)$$

erit

$$mC' = 105\left(\frac{m+3}{7}\right) + 105\left(\frac{m+3}{6}\right) + 25\left(\frac{m+3}{5}\right) + \left(\frac{m+3}{4}\right)$$

vnde colligimus

$$D' = 105\left(\frac{m+4}{8}\right) + 105\left(\frac{m+4}{7}\right) + 25\left(\frac{m+4}{6}\right) + 1\left(\frac{m+4}{5}\right)$$

XII. Porro ob $E' - E = mD'$ quia eſt

$$m\left(\frac{m+4}{8}\right) = 9\left(\frac{m+4}{9}\right) + 4\left(\frac{m+4}{8}\right)$$

$$m\left(\frac{m+4}{7}\right) = 8\left(\frac{m+4}{8}\right) + 3\left(\frac{m+4}{7}\right)$$

$$m\left(\frac{m+4}{6}\right) = 7\left(\frac{m+4}{7}\right) + 2\left(\frac{m+4}{6}\right)$$

$$m\left(\frac{m+4}{5}\right) = 6\left(\frac{m+4}{6}\right) + 1\left(\frac{m+4}{5}\right)$$

colli-

colligimus

$$mD^{I} = 945\binom{m+4}{9} + 1260\binom{m+4}{8} + 490\binom{m+4}{7} + 56\binom{m+4}{6} + 1\binom{m+4}{5}$$

hincque

$$E^{I} = 945\binom{m+5}{10} + 1260\binom{m+5}{9} + 490\binom{m+5}{8} + 56\binom{m+5}{7} + 1\binom{m+5}{6}$$

et vlterius progrediendo

$$F^{I} = 10395\binom{m+6}{12} + 17325\binom{m+6}{11} + 9450\binom{m+6}{10} + 1918\binom{m+6}{9} + 119\binom{m+6}{8} + \binom{m+6}{7}.$$

Euolutio casus $n = m + \lambda$.

XIII. Pro serie ergo nostra casu quo $n = m + \lambda$

$$s = x^{m+\lambda} - \frac{m}{1}(x-1)^{m+\lambda} + \frac{m(m-1)}{1.\ 2.}(x-2)^{m+\lambda} - \frac{m(m-1)(m-2)}{1.\ 2.\ 3}(x-3)^{m+\lambda} + \text{etc.}$$

si aequationem generalem supra §. VIII. exhibitam diuidamus per $\binom{m+\lambda}{m} = \frac{(m+\lambda)(m+\lambda-1)(m+\lambda-2)\ldots\ldots(\lambda+1)}{1.\ 2.\ 3\ \ldots\ldots\ m}$

perueniemus ad hanc expressionem

$$\frac{s}{(\lambda+1)(\lambda+2)\ldots(\lambda+m)} = x^{\lambda} - \frac{\lambda}{m+1}A^{I}x^{\lambda-1} + \frac{\lambda(\lambda-1)}{(m+1)(m+2)}B^{I}x^{\lambda-2} - \frac{\lambda(\lambda-1)(\lambda-2)}{(m+1)(m+2)(m+3)}C^{I}x^{\lambda-3} + \text{etc.}$$

vbi loco litterarum A^{I}, B^{I}, C^{I}, D^{I} etc. sequentes valores substitui oportet:

$$A^{I} = \binom{m+1}{2} = \frac{(m+1)m}{1.\ 2}$$

$$B^{I} = 3\binom{m+2}{4} + \binom{m+2}{3}$$

$$C^{I} = 15\binom{m+3}{6} + 10\binom{m+3}{5} + \binom{m+3}{4}$$

$$D^{I} = 105\binom{m+4}{8} + 105\binom{m+4}{7} + 25\binom{m+4}{6} + \binom{m+4}{5}$$

$$E^{I} = 945\binom{m+5}{10} + 1260\binom{m+5}{9} + 490\binom{m+5}{8} + 56\binom{m+5}{7} + \binom{m+5}{6}$$

$$F^{I} = 10395\binom{m+6}{12} + 17325\binom{m+6}{11} + 9450\binom{m+6}{10} + 1918\binom{m+6}{9} + 119\binom{m+6}{8} + \binom{m+6}{7}$$

 vbi

vbi est $10395 = 11.945$; $17325 = 10.1260 + 5.945$

$$\begin{aligned}
9450 &= 9.490 + 4.1260\\
1918 &= 8.56 + 3.490\\
119 &= 7.1 + 2.56\\
1 &= 6.0 + 1.1
\end{aligned}$$

hinc si pro valore sequente ponatur:

$$G^{I} = \alpha\left(\tfrac{m+7}{14}\right) + \beta\left(\tfrac{m+7}{13}\right) + \gamma\left(\tfrac{m+7}{12}\right) + \delta\left(\tfrac{m+7}{11}\right) + \varepsilon\left(\tfrac{m+7}{10}\right) + \zeta\left(\tfrac{m+7}{9}\right) + \eta\left(\tfrac{m+7}{8}\right)$$

hi coefficientes ita determinabuntur:

$$\begin{array}{l|l}
\alpha = 13.10395 & \varepsilon = 9.119 + 3.1918\\
\beta = 12.17325 + 6.10395 & \zeta = 8.1 + 2.119\\
\gamma = 11.9450 + 5.17325 & \eta = 7.0 + 1.1\\
\delta = 10.1918 + 4.9450 &
\end{array}$$

XIV. Iidem autem valores commodius ita exprimentur:

$$A^{I} = \left(\tfrac{m+1}{2}\right).1$$

$$B^{I} = \left(\tfrac{m+2}{3}\right)\left(1 + 3.\tfrac{m-1}{4}\right)$$

$$C^{I} = \left(\tfrac{m+3}{4}\right)\left(1 + 10.\tfrac{m-1}{5} + 15.\tfrac{m-1}{5}.\tfrac{m-2}{6}\right)$$

$$D^{I} = \left(\tfrac{m+4}{5}\right)\left(1 + 25.\tfrac{m-1}{6} + 105.\tfrac{m-1}{6}.\tfrac{m-2}{7} + 105.\tfrac{m-1}{6}.\tfrac{m-2}{7}.\tfrac{m-3}{8}\right)$$

$$E^{I} = \left(\tfrac{m+5}{6}\right)\left(1 + 56.\tfrac{m-1}{7} + 490.\tfrac{m-1}{7}.\tfrac{m-2}{8} + 1260.\tfrac{m-1}{7}.\tfrac{m-2}{8}.\tfrac{m-3}{9} + 945.\tfrac{m-1}{7}.\tfrac{m-2}{8}.\tfrac{m-3}{9}.\tfrac{m-4}{10}\right)$$

$$F^{I} = \left(\tfrac{m+6}{7}\right)\left(1 + 119.\tfrac{m-1}{8} + 1918.\tfrac{m-1}{8}.\tfrac{m-2}{9} + 9450.\tfrac{m-1}{8}.\tfrac{m-2}{9}.\tfrac{m-3}{10} + 17325.\tfrac{m-1}{8}.\tfrac{m-2}{9}.\tfrac{m-3}{10}.\tfrac{m-4}{11} + 10395.\tfrac{m-1}{8}.\tfrac{m-2}{9}.\tfrac{m-3}{10}.\tfrac{m-4}{11}.\tfrac{m-5}{12}\right)$$

cuius

cuius progreſſionis lex quo facilius perſpiciatur, ponamus in genere

$$M^{\text{I}} = \left(\frac{m+\mu-1}{\mu}\right)\left(1+\alpha.\frac{m-1}{\mu+1}+\beta.\frac{m-1}{\mu+1}.\frac{m-2}{\mu+2}+\gamma.\frac{m-1}{\mu+1}.\frac{m-2}{\mu+2}.\frac{m-3}{\mu+3}+\text{etc.}\right.$$

et ſequentem

$$N^{\text{I}} = \left(\frac{m+\mu}{\mu+1}\right)\left(1+\alpha^{\text{I}}.\frac{m-1}{\mu+2}+\beta^{\text{I}}.\frac{m-1}{\mu+2}.\frac{m-2}{\mu+3}+\gamma^{\text{I}}.\frac{m-1}{\mu+2}.\frac{m-2}{\mu+3}.\frac{m-3}{\mu+4}+\text{etc.}\right.$$

atque hi coefficientes hoc modo per praecedentes determinantur

$$\alpha^{\text{I}} = 2\alpha+\mu+1;$$
$$\beta^{\text{I}} = 3\beta+(\mu+2)\alpha$$
$$\gamma^{\text{I}} = 4\gamma+(\mu+3)\beta$$
$$\delta^{\text{I}} = 5\delta+(\mu+4)\gamma$$
$$\varepsilon^{\text{I}} = 6\varepsilon+(\mu+5)\delta$$

vnde has formulas facile quousque libuerit continuare licet.

XV. Subſtituamus iam hos valores, ac pro ſumma s ſeriei propoſitae quando $n = m+\lambda$ obtinebimus ſequentem expreſſionem:

$$\frac{s}{(\lambda+1)(\lambda+2)\ldots(\lambda+m)} =$$

$$x^{\lambda} - \frac{\lambda m}{1.2}x^{\lambda-1} + \frac{\lambda(\lambda-1)m}{1.2.3}x^{\lambda-2}\left(1+\frac{3(m-1)}{4}\right)$$

$$- \frac{\lambda(\lambda-1)(\lambda-2)m}{1.2.3.4}x^{\lambda-3}\left(1+10.\frac{m-1}{5}+15.\frac{m-1}{5}.\frac{m-2}{6}\right)$$

$$+ \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)m}{1.2.3.4.5}x^{\lambda-4}\left(1+25.\frac{m-1}{6}+105.\frac{m-1}{6}.\frac{m-2}{7}+105.\frac{m-1}{6}.\frac{m-2}{7}.\frac{m-3}{8}\right)$$

$$- \frac{\lambda\ldots(\lambda-4)m}{1\ldots5.6}x^{\lambda-5}\left(1+56.\frac{m-1}{7}+490.\frac{m-1}{7}.\frac{m-2}{8}+1260.\frac{m-1}{7}\ldots\frac{m-3}{9}\right.$$
$$\left.+945\frac{m-1}{7}\ldots\frac{m-4}{10}\right)$$

$$+ \frac{\lambda\ldots(\lambda-5)m}{1\ldots6.7}x^{\lambda-6}\left(1+119.\frac{m-1}{8}+1918.\frac{m-1}{8}.\frac{m-2}{9}+9450.\frac{m-1}{8}\ldots\frac{m-3}{10}\right.$$
$$\left.+17325.\frac{m-1}{8}\ldots\frac{m-4}{11}+10395.\frac{m-1}{8}\ldots\frac{m-5}{12}\right\}$$

etc.

ſubtra-

ſubtrahatur hinc primo poteſtas

$$\left(x-\tfrac{m}{2}\right)^{\lambda} = x^{\lambda} - \frac{\lambda m}{2} x^{\lambda-1} + \frac{\lambda(\lambda-1)m^2}{1.\ 2.\ 4} x^{\lambda-2} - \frac{\lambda(\lambda-1)(\lambda-2)m^3}{1.\ 2.\ 3.\ 8} x^{\lambda-3}$$
$$+ \frac{\lambda \ldots (\lambda-3)m^4}{1 \ldots 4.\ 16} x^{\lambda-4} - \frac{\lambda \ldots (\lambda-4)m^5}{1. \ldots 5.\ 32} x^{\lambda-5}$$
$$+ \frac{\lambda \ldots (\lambda-5)m^6}{1 \ldots 6.\ 64} x^{\lambda-6} - \text{etc.}$$

Commode autem hic euenit vt ſit

$$\frac{15}{5.\ 6} = \tfrac{4}{8};\ \frac{105}{6.7.8} = \tfrac{5}{16};\ \frac{945}{7.8.9.10} = \tfrac{6}{32};\ \frac{10395}{8.9.10.11.12} = \tfrac{7}{64}$$

cuius quidem rei ratio per ſe eſt perſpicua; quamobrem expreſſio ſuperior euoluta ſequentem induit formam

$$\left(x-\tfrac{m}{2}\right)^{\lambda} + \frac{\lambda(\lambda-1)m}{1.\ 2.\ 3} x^{\lambda-2}.\tfrac{1}{4} - \frac{\lambda(\lambda-1)(\lambda-2)m}{1.\ 2.\ 3.\ 4} x^{\lambda-3}.\tfrac{m}{2} + \frac{\lambda \ldots (\lambda-3)m}{1 \ldots 4.\ 5} x^{\lambda-4}\left(\tfrac{5}{8}m^2 + \tfrac{5}{48}m - \tfrac{1}{24}\right)$$
$$- \frac{\lambda \ldots (\lambda-4)m}{1 \ldots 5.\ 6} x^{\lambda-5}\left(\tfrac{5}{8}m^3 + \tfrac{5}{16}m^2 - \tfrac{1}{8}m\right)$$
$$+ \frac{\lambda \ldots (\lambda-5)m}{1 \ldots 6.\ 7} x^{\lambda-6}\left(\tfrac{35}{64}m^4 + \tfrac{35}{64}m^3 - \tfrac{91}{576}m^2 - \tfrac{7}{96}m + \tfrac{1}{36}\right).\ \text{etc.}$$

XVI. In hac expreſſione denuo poteſtas ipſius $x-\frac{m}{2}$ ſcilicet $\frac{\lambda(\lambda-1)m}{2.\ 3.\ 4}\left(x-\frac{m}{2}\right)^{\lambda-2}$ contineri deprehenditur qua inde ſeparata expreſſio noſtra erit:

$$\left(x-\tfrac{m}{2}\right)^{\lambda} + \frac{\lambda(\lambda-1)m}{2.\ 3.\ 4}\left(x-\tfrac{m}{2}\right)^{\lambda-2} + \frac{\lambda \ldots (\lambda-3)m}{1 \ldots 4.\ 5} x^{\lambda-4}\left(\tfrac{5}{48}m - \tfrac{1}{24}\right)$$
$$- \frac{\lambda \ldots (\lambda-4)m}{1 \ldots 5.\ 6} x^{\lambda-5}\left(\tfrac{5}{16}m^2 - \tfrac{1}{8}m\right)$$
$$+ \frac{\lambda \ldots (\lambda-5)m}{1 \ldots 6.\ 7} x^{\lambda-6}\left(\tfrac{35}{64}m^3 - \tfrac{91}{576}m^2 - \tfrac{7}{96}m + \tfrac{1}{36}\right)\ \text{etc.}$$

in qua adhuc continetur $\frac{\lambda \ldots (\lambda-3)m}{1 \ldots 4.\ 5}\left(\frac{5}{48}m - \frac{1}{24}\right)\left(x-\frac{m}{2}\right)^{\lambda-4}$ ac praeterea ſupereſt

$$\frac{\lambda \ldots (\lambda-5)m}{1 \ldots 6.\ 7} x^{\lambda-6}\left(\tfrac{35}{576}m^2 - \tfrac{7}{96}m + \tfrac{1}{36}\right)$$

vnde

vnde ſine dubio inſuper haec poteſtas accedit:

$$+\frac{\lambda \ldots\ldots (\lambda-5)m}{1 \ldots\ldots 6 \cdot 7} \cdot \frac{35m^2-42m+16}{576}\left(x-\frac{m}{2}\right)^{\lambda-6}.$$

Quocirca aequatio noſtra ita erit comparata:

$$\frac{s}{(\lambda+1)(\lambda+2)\ldots(\lambda+m)} = \left(x-\frac{m}{2}\right)^{\lambda} + \frac{\lambda(\lambda-1)}{1 \cdot 2} \cdot \frac{m}{12}\left(x-\frac{m}{2}\right)^{\lambda-2}$$
$$+\frac{\lambda(\lambda-1)\ldots\ldots(\lambda-3)}{1 \cdot 2 \ldots\ldots 4} \cdot \frac{m(5m-2)}{240}\left(x-\frac{m}{2}\right)^{\lambda-4}$$
$$+\frac{\lambda(\lambda-1)\ldots\ldots(\lambda-5)}{1 \cdot 2 \ldots\ldots 6} \cdot \frac{m(35m^2-42m+16)}{4032}\left(x-\frac{m}{2}\right)^{\lambda-6} \text{ etc.}$$

XVII. En ergo feriei noſtrae propoſitae generalis:

$$s = x^{m+\lambda} - \frac{m}{1}(x-1)^{m+\lambda} + \frac{m(m-1)}{1 \cdot 2}(x-2)^{m+\lambda}$$
$$-\frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}(x-3)^{m+\lambda} + \text{etc.}$$

eximiam transformationem, quae cum per plures ambages ſit eruta, ac pluribus operationibus admodum intricatis innixa, tantopere abſtruſa videtur, vt eius inueſtigatio directa ingentia ſubſidia in Analyſin ſit allatura. Quo autem facilius hanc transformationem perſcrutari liceat eam hoc modo repraeſentabo, vt ſit

$$\frac{s}{(\lambda+1)(\lambda+2)\ldots\ldots(\lambda+m)} = \left(x-\frac{m}{2}\right)^{\lambda} + \frac{\lambda(\lambda-1)}{1 \cdot 2} P\left(x-\frac{m}{2}\right)^{\lambda-2}$$
$$+\frac{\lambda(\lambda-1)\ldots(\lambda-3)}{1 \cdot 2 \ldots 4} Q\left(x-\frac{m}{2}\right)^{\lambda-4}$$
$$+\frac{\lambda(\lambda-1)\ldots(\lambda-5)}{1 \cdot 2 \ldots 6} R\left(x-\frac{m}{2}\right)^{\lambda-6}$$
$$+\frac{\lambda(\lambda-1)\ldots(\lambda-7)}{1 \cdot 2 \ldots 8} S\left(x-\frac{m}{2}\right)^{\lambda-8}$$
$$+\frac{\lambda(\lambda-1)\ldots(\lambda-9)}{1 \cdot 2 \ldots 10} T\left(x-\frac{m}{2}\right)^{\lambda-10}$$
$$\text{etc.}$$

pro qua expreſſione hactenus quidem inueni

$$P = \frac{m}{3 \cdot 4}$$

$$Q = \frac{m(5m - 2)}{5 \cdot 6 \cdot 8}$$

$$R = \frac{m(35mm - 42m + 16)}{6 \cdot 7 \cdot 96}$$

$$S = \frac{m(175m^3 - 420m^2 + 404m - 144)}{34560}$$

ſed methodus deſideratur harum litterarum valores expedite inueniendi.

XVIII. Imprimis autem hic notaſſe iuuabit, ſeriem noſtram in aliam eſſe transmutatam, quae ſecundum poteſtates formulae $x - \frac{m}{2}$ ita progrediatur, vt earum exponentes ſint λ, $\lambda - 2$, $\lambda - 4$ etc. binario continuo decreſcentes; tum vero litteras P, Q, R, etc. a ſolo numero m pendere, ita vt neque exponens λ neque quantitas x in eas ingrediatur; praeterea vero coefficientes praefixos ſolum numerum λ implicare, ac legem progreſſionis ex euolutione binomii ortae ſeruare. Hac forma probe obſeruata manifeſtum eſt valores litterarum P, Q, R, S etc. ſeorſim ex ipſa ſerie propoſita vel ex eius transformata §. XV. cuius lex progreſſionis itidem eſt cognita elici poſſe, ſiquidem ponatur $x = \frac{m}{2}$ ſi enim tum capiatur $\lambda = 2$ fit $P = \frac{s}{(\lambda+1)(\lambda+2)\ldots(\lambda+m)}$ poſito autem $\lambda = 4$ fit $Q = \frac{s}{(\lambda+1)(\lambda+2)\ldots(\lambda+m)}$ at poſito $\lambda = 6$ fit $R = \frac{s}{(\lambda+1)(\lambda+2)\ldots(\lambda+m)}$,

etc.

XIX.

XIX. Quodſi ergo hic loco $\frac{s}{(\lambda+1)(\lambda+2)\ldots\ldots(\lambda+m)}$ ſeries ſupra §. XV. inuenta ſubſtituatur, atque in hunc finem breuitatis gratia ponatur:

$\mathfrak{A} = 1$

$\mathfrak{B} = 1 + 3 . \frac{m-1}{4}$

$\mathfrak{C} = 1 + 10 . \frac{m-1}{5} + 15 . \frac{m-1}{5} . \frac{m-2}{6}$

$\mathfrak{D} = 1 + 25 . \frac{m-1}{6} + 105 . \frac{m-1}{6} . \frac{m-2}{7} + 105 . \frac{m-1}{6} . \frac{m-2}{7} . \frac{m-3}{8}$

$\mathfrak{E} = 1 + 56 . \frac{m-1}{7} + 490 . \frac{m-1}{7} . \frac{m-2}{8} + 1260 . \frac{m-1}{7} \ldots \frac{m-3}{9}$
$+ 945 . \frac{m-1}{7} \ldots \frac{m-4}{10}$

$\mathfrak{F} = 1 + 119 . \frac{m-1}{8} + 1918 . \frac{m-1}{8} . \frac{m-2}{9} + 9450 . \frac{m-1}{8} \ldots \frac{m-3}{10}$
$+ 17325 . \frac{m-1}{8} \ldots \frac{m-4}{11} + 10395 . \frac{m-1}{8} \ldots \frac{m-5}{12}$

etc.

adipiſcimur ſequentes valores

$P = \frac{m^2}{2^2} - 2\,\mathfrak{A}\frac{m}{2} . \frac{m}{2} + \mathfrak{B}\frac{m}{3}$

$Q = \frac{m^4}{2^4} - 4\,\mathfrak{A} . \frac{m}{1} . \frac{m^3}{2^3} + 6\,\mathfrak{B} . \frac{m}{3} . \frac{m^2}{2^2} - 4\,\mathfrak{C} . \frac{m}{4} . \frac{m}{2} + \mathfrak{D}\frac{m}{5}$

$R = \frac{m^6}{2^6} - 6\,\mathfrak{A} . \frac{m}{2} . \frac{m^5}{2^5} + 15\,\mathfrak{B} . \frac{m}{3} . \frac{m^4}{2^4} - 20\,\mathfrak{C} . \frac{m}{4} . \frac{m^3}{2^3} + 15\,\mathfrak{D} . \frac{m}{5} . \frac{m^2}{2^2}$
$- 6\,\mathfrak{E}\frac{m}{6} . \frac{m}{2} + \mathfrak{F} . \frac{m}{7}$

quem in finem valores illos litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. euolui conueniet, vnde prodit

$\mathfrak{A} = 1$

$\mathfrak{B} = \frac{3}{4}m + \frac{1}{4} = \frac{3}{4}(m + \frac{1}{3})$

$\mathfrak{C} = \frac{1}{2}m^2 + \frac{1}{2}m = \frac{1}{2}(mm + m)$

$\mathfrak{D} =$

$$\mathfrak{D} = \tfrac{5}{16}m^3 + \tfrac{5}{8}m^2 + \tfrac{5}{48}m - \tfrac{1}{24} = \tfrac{5}{16}(m^3 + 2m^2 + \tfrac{1}{3}m - \tfrac{2}{15})$$

$$\mathfrak{E} = \tfrac{5}{16}m^4 + \tfrac{5}{8}m^3 + \tfrac{5}{16}m^2 - \tfrac{1}{8}m = \tfrac{6}{32}(m^4 + \tfrac{10}{3}m^3 + \tfrac{5}{3}m^2 - \tfrac{2}{3}m)$$

$$\mathfrak{F} = \tfrac{7}{64}m^5 + \tfrac{35}{64}m^4 + \tfrac{35}{64}m^3 - \tfrac{91}{576}m^2 - \tfrac{7}{96}m + \tfrac{1}{3}$$

seu . . . $\mathfrak{F} = \tfrac{7}{64}(m^5 + 5m^4 + 5m^3 - \tfrac{13}{9}m^2 - \tfrac{2}{3}m + \tfrac{16}{63})$

hic autem praeterquam in primis terminis nullus ordo perſpicitur

XX. Quod vero ſeries transformata ſecundum poteſtates quantitatis $x = \frac{m}{2}$ progrediatur, id quidem per ſolam inductionem agnouimus, verumtamen hoc neceſſario euenire ita oſtendi poteſt. Quoniam progreſſio propoſita ſimili modo deſinit, quo incipit, ita vt vltimi bini termini futuri ſint $\pm m (x - m + 1)^{m+\lambda} \mp (x-m)^{m+\lambda}$, vbi ſigna ſuperiora valent ſi m ſit numerus impar, inferiora vero ſi par; ſumamus m eſſe numerum parem, (eadem enim concluſio producitur ſi fuerit impar) et ponamus $x - \frac{m}{2} = y$, eritque

$$2s = +(y + \tfrac{1}{2}m)^{m+\lambda} - \tfrac{m}{1}(y + \tfrac{1}{2}m - 1)^{m+\lambda} + \tfrac{m(m-1)}{1.\,2}(y + \tfrac{1}{2}m - 2)^{m+\lambda}$$
$$+ (y - \tfrac{1}{2}m)^{m+\lambda} - \tfrac{m}{1}(y - \tfrac{1}{2}m + 1)^{m+\lambda} + \tfrac{m(m-1)}{1.\,2}(y - \tfrac{1}{2}m + 2)^{m+\lambda}$$

et facta euolutione ſecundum poteſtates ipſius $y = x - \frac{m}{2}$ reperitur;

$$s = y^{m+\lambda}\left(1 - \tfrac{m}{1} + \tfrac{m(m-1)}{1.\,2} - \text{etc.}\right)$$
$$+ \left(\tfrac{m+\lambda}{2}\right) y^{m+\lambda-2}\left(\left(\tfrac{m}{2}\right)^2 - \tfrac{m}{1}\left(\tfrac{m}{2} - 1\right)^2 + \tfrac{m(m-1)}{1.\,2}\left(\tfrac{m}{2} - 2\right)^2 - \text{etc.}\right)$$
$$+ \left(\tfrac{m+\lambda}{4}\right) y^{m+\lambda-4}\left(\left(\tfrac{m}{2}\right)^4 - \tfrac{m}{1}\left(\tfrac{m}{2} - 1\right)^4 + \tfrac{m(m-1)}{1.\,2}\left(\tfrac{m}{2} - 2\right)^4 - \text{etc.}\right)$$
$$\text{etc.}$$

Hae

Hae ſeries autem omnes euaneſcunt, donec perueniatur ad eam in qua exponentes ſunt m, eiusque ſummam nouimus eſſe $= 1 . 2 . 3 \ldots . m$, omiſſis ergo prioribus, quarum ſumma ad nihilum reducitur, obtinebimus:

$$s = \left(\frac{m+\lambda}{m}\right) y^{\lambda} \left(\left(\frac{m}{2}\right)^m - \frac{m}{1}\left(\frac{m}{2}-1\right)^m + \frac{m(m-1)}{1 . 2}\left(\frac{m}{2}-2\right)^m - \text{etc.}\right)$$
$$+ \left(\frac{m+\lambda}{m+2}\right) y^{\lambda-2} \left(\left(\frac{m}{2}\right)^{m+2} - \frac{m}{1}\left(\frac{m}{2}-1\right)^{m+2} + \frac{m(m-1)}{1 . 2}\left(\frac{m}{2}-2\right)^{m+2} - \text{etc.}\right)$$
$$\text{etc.}$$

ſicque manifeſtum eſt, quod demonſtrare ſuſcepi, ſcilicet hanc ſeriem ſecundum poteſtates $y^{\lambda}, y^{\lambda-2}, y^{\lambda-4}$ etc. deſcendere.

XXI. Tribuamus huic ſeriei ſimilem formam ei quam §. XVII. habuimus, fietque

$$\frac{s}{(\lambda+1)(\lambda+2)\ldots(\lambda+m)} = \frac{y^{\lambda}}{1 . 2 \ldots m}\left(\left(\frac{m}{2}\right)^m - \frac{m}{1}\left(\frac{m}{2}-1\right)^m + \text{etc}\right)$$
$$+ \frac{1 . 2\, y^{\lambda-2}}{1 . 2 \ldots (m+2)} \cdot \frac{\lambda(\lambda-1)}{1 . 2}\left(\left(\frac{m}{2}\right)^{m+2} - \frac{m}{1}\left(\frac{m}{2}-1\right)^{m+2} + \text{etc.}\right)$$
$$+ \frac{1 . 2 . 3 . 4\, y^{\lambda-4}}{1 . 2 \ldots (m+4)} \cdot \frac{\lambda(\lambda-1)\ldots(\lambda-3)}{1 . 2 \ldots 4}\left(\left(\frac{m}{2}\right)^{m+4} - \text{etc.}\right)$$
$$\text{etc.}$$

vnde valores litterarum P, Q, R etc. nouo modo ita determinare licet

$$P = \frac{1}{3 . 4 \ldots (m+2)}\left(\left(\frac{m}{2}\right)^{m+2} - \frac{m}{1}\left(\frac{m}{2}-1\right)^{m+2} + \text{etc.}\right)$$
$$Q = \frac{1}{5 . 6 \ldots (m+4)}\left(\left(\frac{m}{2}\right)^{m+4} - \frac{m}{1}\left(\frac{m}{2}-1\right)^{m+4} + \text{etc.}\right)$$
$$R = \frac{1}{7 . 8 \ldots (m+6)}\left(\left(\frac{m}{2}\right)^{m+6} - \frac{m}{1}\left(\frac{m}{2}-1\right)^{m+6} + \text{etc.}\right)$$
$$S = \frac{1}{9 . 10 \ldots (m+8)}\left(\left(\frac{m}{2}\right)^{m+8} - \frac{m}{1}\left(\frac{m}{2}-1\right)^{m+8} + \text{etc.}\right)$$
$$\text{etc.}$$

Hic quidem similium serierum summatione opus est quoniam vero istae series solum numerum m inuoluunt, nostra inuestigatio ad casum simpliciorem perducta est censenda. Ceterum nunc demum certo agnoscimus has litteras tantum a numero m pendere.

XXII. Quodsi autem hic litterae m successiue tribuamus valores definitos 1. 2. 3. 4. 5. 6 etc. totidem inde valores pro litteris P, Q, R, S etc. consequemur, quibus cognitis, facile earum formas generales colligere licebit.

Ita pro littera P inuenienda reperiemus

si $m =$	0,	1,	2,	3,	4	etc.
$3 . 2^2 P =$	0,	1,	2,	3,	4	etc.
diff.		1,	1,	1,	1,	

ita vt hinc fit $3 . 2^2 P = m$ et $P = \frac{m}{2^2 . 3}$ vt ante. Porro pro littera Q

si $m =$	0,	1,	2,	3,	4,	5,	6
$2^4 . 3 . 5 Q =$	0,	3,	16,	39,	72,	115,	168
Diff. I.		3,	13,	23,	33,	43,	53
Diff. II.			10,	10,	10,	10,	10

erit ergo $2^4 . 3 . 5 Q = 3m + 10 \frac{m(m-1)}{1 . 2} = m(5m-2)$, hincque $Q = \frac{m(5m-2)}{2^4 . 3 . 5}$.

Eodem

Eodem modo pro littera R

si $m =$	0,	1,	2,	3	4,	5	6
$2^6.3.7\,R =$	0,	3,	48,	205,	544,	1135,	2048
Diff. I.		3,	45,	157,	339,	591,	913
Diff. II.			42,	112,	182,	252,	322
Diff. III.				70,	70,	70	

vnde concluditur $2^6.3.7\,R = 3m + 21m(m-1)$
$+ \frac{35}{3}m(m-1)(m-2)$

atque $R = \frac{m(35m^2 - 42m + 16)}{2^6.3^2.7}$

quos eosdem valores iam supra sumus nacti, hinc igitur eandem operationem ad litteras sequentes accommodemus.

XXIII. Pro littera igitur S habebimus:

si fuerit $m =$	0,	1,	2,	3,	4,	5,	6
$2^8.5.9\,S =$	0,	5,	256,	2013,	7936,	22085,	49920
Diff. I.		5,	251,	1757,	5923,	14149,	27835
Diff. II.			246,	1506,	4166,	8226,	13686
III.				1260,	2660,	4060,	5460
IV.					1400,	1400,	1400

vnde fit $2^8.5.9\,S = 5m + 123m(m-1) + 210m(m-1)(m-2)$
$+ \frac{175}{3}m(m-1)(m-2)(m-3)$

et $S = \frac{m(175m^3 - 420m^2 + 404m - 144)}{2^8.5.5.9}$.

Nunc porro pro littera T habebimus:

si fuerit $m =$	0,	1,	2,	3,	4,	5,	6
$2^{10}.3.11\,T =$	0,	3,	512,	7665,	46080,	174255,	499968
Diff. I.		3,	509,	7153,	38415,	128175,	325713
II.			506,	6644,	31262,	89760,	197538
III.				6138,	24618,	58498,	107778
IV.					18480,	33880,	49280
V.						15400,	15400

vnde fit
$$2^{10}.3.11\,T = 3m + 253m(m-1) + 1023m(m-1)(m-2)$$
$$+ 770m(m-1)(m-2)(m-3)$$
$$+ \tfrac{385}{3}m(m-1)(m-2)(m-3)(m-4)$$
et
$$T = \frac{m(385m^4 - 1540m^3 + 2684m^2 - 2288m + 768)}{2^{10}.9.11}.$$

XXIV. Hos nunc valores ita repraeſentemus, quo facilius lex progreſſionis explorari poſſit:

$$P = \frac{1\,m}{12}$$
$$Q = \frac{1.3\,m}{12^2}\left(m - \tfrac{2}{5}\right)$$
$$R = \frac{1.3.5\,m}{12^3}\left(m^2 - \tfrac{6}{5}m + \tfrac{16}{35}\right)$$
$$S = \frac{1.3.5.7\,m}{12^4}\left(m^3 - \tfrac{12}{5}m^2 + \tfrac{404}{175}m - \tfrac{144}{175}\right)$$
$$T = \frac{1.3.5.7.9\,m}{12^5}\left(m^4 - \tfrac{20}{5}m^3 + \tfrac{244}{35}m^2 - \tfrac{208}{35}m + \tfrac{768}{385}\right)$$

atque hic in primis et ſecundis terminis lex progreſſionis ita eſt manifeſta, vt iidem pro omnibus ſequentibus litteris tuto aſſignari poſſint, in reliquis autem terminis nullam plane legem etiamnum obſeruare licet.

XXV.

XXV. Pro valore ergo litterae V inueniendo ſtatuamus

$$V = \frac{1.\,3.\,5.\,7.\,9.\,11\,m}{12^6}\left(m^5 - \tfrac{30}{5}m^4 + \alpha m^3 - \beta m^2 + \gamma m - \delta\right).$$

Ex forma autem generali

$$V = \frac{1}{13.\,14\ldots(m+12)}\left(\left(\tfrac{m}{2}\right)^{m+12} - m\left(\tfrac{m}{2}-1\right)^{m+12} + \frac{m(m-1)}{1.\,2}\left(\tfrac{m}{2}-2\right)^{m+12} - \text{etc.}\right.$$

colligimus

ſi ſit fore

$$m=1;\ V = \frac{1}{2^{12}.\,13} = \frac{5.\,7.\,11}{2^{12}.\,3^3}(-5+\alpha-\beta+\gamma-\delta)$$

$$m=2;\ V = \frac{1}{7.\,13} = \frac{5.\,7.\,11}{2^{11}.\,3^3}(-64+8\alpha-4\beta+2\gamma-\delta)$$

$$m=3;\ V = \frac{597871}{2^{12}.\,5.\,7.\,13} = \frac{5.\,7.\,11}{2^{12}.\,3^2}(-243+27\alpha-9\beta+3\gamma-\delta)$$

$$m=4;\ V = \frac{5461}{2^2.\,5.\,7.\,13} = \frac{5.\,7.\,11}{2^{10}.\,3^3}(-512+64\alpha-16\beta+4\gamma-\delta)$$

$$m=5;\ V = \frac{5838647}{2^{12}.\,7.\,13} = \frac{25.\,7.\,11}{2^{12}.\,3^3}(-625+125\alpha-25\beta+5\gamma-\delta)$$

$$m=6;\ V = \frac{6\ 047}{2^2.\,3^4.\,13} = \frac{5.\,7.\,11}{2^{11}.\,3^2}(-0+216\alpha-36\beta+6\gamma-\delta).$$

Hinc ergo ſequentes formemus aequationes:

$$\alpha - \beta + \gamma - \delta = \frac{27}{5.\,7.\,11.\,13} + 5$$

$$8\alpha - 4\beta + 2\gamma - \delta = \frac{27.\,2048}{5.\,7^2.\,11.\,13} + 64.$$

$$27\alpha - 9\beta + 3\gamma - \delta = \frac{9.\,597871}{5^2.\,7^2.\,11.\,13} + 243$$

$$64\alpha - 16\beta + 4\gamma - \delta = \frac{27.\,256.\,5461}{5^2.\,7^2.\,11.\,13} + 512$$

$$125\alpha - 25\beta + 5\gamma - \delta = \frac{27.\,5838647}{5^2.\,7^2.\,11.\,13} + 625$$

$$216\alpha - 36\beta + 6\gamma - \delta = \frac{3.\,512.\,63047}{5.\,7^2.\,11.\,13} + 0.$$

Differentiae nunc primae ita ſe habebunt

$$7\alpha - 3\beta + \gamma = \frac{27\cdot 157}{5\cdot 7^2\cdot 11} + 59$$

$$19\alpha - 5\beta + \gamma = \frac{9\cdot 43627}{5^2\cdot 7^2\cdot 11} + 179$$

$$37\alpha - 7\beta + \gamma = \frac{9\cdot 276629}{5^2\cdot 7^2\cdot 11} + 269$$

$$61\alpha - 9\beta + \gamma = \frac{27\cdot 341587}{5^2\cdot 7^2\cdot 11} + 113$$

$$91\alpha - 11\beta + \gamma = \frac{3\cdot 8373269}{5^2\cdot 7^2\cdot 11} - 625$$

ſecundae vero per 2 diuiſae dant

$$6\alpha - \beta = \frac{9\cdot 268}{5^2\cdot 7} + 60$$

$$9\alpha - \beta = \frac{9\cdot 1513}{5^2\cdot 7} + 45$$

$$12\alpha - \beta = \frac{9\cdot 4858}{5^2\cdot 7} - 78$$

$$15\alpha - \beta = \frac{3\cdot 34409}{5^2\cdot 7} - 369.$$

Tertiae tandem differentiae per 3 diuiſae praebent

$$\alpha = \frac{3\cdot 249}{5\cdot 7} - 5 = \frac{3\cdot 669}{5\cdot 7} - 41 = \frac{3967}{5\cdot 7} - 97$$

quae tres aequationes praebent eundem valorem

$$\alpha = \frac{572}{5\cdot 7} = \frac{4\cdot 11\cdot 13}{5\cdot 7}.$$

ex quo valore iam reliqui definiuntur ſequenti modo

$$\beta = \frac{6\cdot 572}{5\cdot 7} - \frac{9\cdot 268}{5^2\cdot 7} - 60 = \frac{12\cdot 1229}{5^2\cdot 7} - 60 = \frac{4248}{175} = \frac{8\cdot 9\cdot 59}{175}$$

$$\gamma = 3\beta - 7\alpha + \frac{27\cdot 157}{5\cdot 7^2\cdot 11} + 59 = \frac{255968}{5^2\cdot 7^2\cdot 11}$$

$$\delta = \alpha - \beta + \gamma - \frac{27}{5\cdot 7\cdot 11\cdot 13} - 5 = \frac{1061376}{5^2\cdot 7^2\cdot 11\cdot 13}.$$

XXVI.

XXVI. Confpectui ergo fimul exponamus litterarum P, Q, R, etc. valores hactenus inuentos

$P = \frac{1\,m}{12}$

$Q = \frac{1.\,3\,m}{12^2}(m - \frac{2}{5})$

$R = \frac{1.\,3.\,5\,m}{12^3}(m^2 - \frac{6}{5}m + \frac{15}{35})$

$S = \frac{1.\,3.\,5.\,7\,m}{12^4}(m^3 - \frac{12}{5}m^2 + \frac{404}{175}m - \frac{144}{175})$

$T = \frac{1.\,3.\,5.\,7.\,9\,m}{12^5}(m^4 - \frac{20}{5}m^3 + \frac{244}{35}m^2 - \frac{208}{35}m + \frac{768}{385})$

$V = \frac{1.\,3.\,5.\,7.\,9.\,11\,m}{12^6}(m^5 - \frac{30}{5}m^4 + \frac{572}{35}m^3 - \frac{4248}{175}m^2 + \frac{255968}{13475}m - \frac{1061376}{175175})$

Ex prioribus terminis concludo poteftates hic occurrere, quibus feorfim pofitis ordo facilius perfpici poffe videtur:

$P = \frac{1\,m}{12}$

$Q = \frac{1.\,3\,m}{12^2}(m - \frac{2}{5})^1$

$R = \frac{1.\,3.\,5\,m}{12^3}((m - \frac{3}{5})^2 + \frac{17}{175})$

$S = \frac{1.\,3.\,5.\,7\,m}{12^4}((m - \frac{4}{5})^3 + \frac{4.\,17}{175}m - \frac{16.\,17}{5.\,175})$

$T = \frac{1.\,3.\,5.\,7.\,9\,m}{12^5}((m - \frac{5}{5})^4 + \frac{2.\,17}{35}m^2 - \frac{4.\,17}{35}m + \frac{383}{385})$

$V = \frac{1.\,3.\,5.\,7.\,9.\,11\,m}{12^6}((m - \frac{6}{5})^5 + \frac{4.\,17}{35}m^3 - \frac{72.\,17}{175}m^2 + \frac{581296}{5^3.\,7^2.\,11}m - \frac{7118556}{5^5.\,7^2.\,11.\,13}$

quin etiam proxime fequentes termini hoc modo contrahi poffint vt prodeat

$P = \frac{1\,m}{12}.\,1$

$Q = \frac{1.\,3\,m}{12^2}(m - \frac{2}{5})$

$R = \frac{1.\,3.\,5\,m}{12^3}((m - \frac{3}{5})^2 + \frac{17}{175})$

S =

$$S = \frac{1.3.5.7\,m}{12^4}\left(\left(m-\frac{4}{5}\right)^3 + \frac{4.17}{175}\left(m-\frac{4}{5}\right)\right)$$

$$T = \frac{1.3.5.7.9\,m}{12^5}\left(\left(m-\frac{5}{5}\right)^4 + \frac{10.17}{175}\left(m-\frac{5}{5}\right)^2 + \frac{9}{385}\right)$$

$$V = \frac{1.3.5.7.9.11\,m}{12^6}\left(\left(m-\frac{6}{5}\right)^5 + \frac{20.17}{175}\left(m-\frac{6}{5}\right)^3 + \frac{15808}{5^3.7^2.11}m - \frac{467128}{5^5.7^2.11.13}\right).$$

Nisi vltimum valorem euoluissemus, videretur omnes has expressiones ad huiusmodi potestates reduci, quod autem nunc secus euenire agnoscimus. Quocirca ex alio fonte in legem harum litterarum inquiri oportebit.

XXVIII. Singulos igitur terminos harum formarum potius euolutos repraesentemus:

$$P = \frac{m}{4.3}$$

$$Q = \frac{mm}{16.3} - \frac{m}{8.3.5}$$

$$R = \frac{5m^3}{64.9} - \frac{mm}{32.3} + \frac{m}{4.9.7}$$

$$S = \frac{5.7m^4}{256.27} - \frac{7m^3}{64.9} + \frac{101m^2}{64.27.5} - \frac{m}{16.3.5}$$

$$T = \frac{5.7m^5}{1024.9} - \frac{5.7m^4}{256.9} + \frac{61m^3}{256.9} - \frac{13m^2}{64.9} + \frac{m}{4.3.11}$$

$$V = \frac{5.7.11m^6}{4096.27} - \frac{5.7.11m^5}{2048.9} + \frac{1573m^4}{1024.27} - \frac{649m^3}{512.3.5} + \frac{7909m^2}{128.27.5.7} - \frac{691m}{8.9.5.7.13}$$

vbi quidem inter terminos primos et secundos iam ordinem obseruauimus postremi autem omni ordine destituti videbantur, quoad valore etiam litterae V euoluto numerus 691 criterium nobis suppeditauerit, in his postremis terminis numeros *Bernoullianos* implicari.

Desi-

Deſignemus ergo numeros *Bernoullianos* litteris α, β, γ, δ etc. vt ſit

$$\alpha = \tfrac{1}{2};\ \beta = \tfrac{1}{6};\ \gamma = \tfrac{1}{6};\ \delta = \tfrac{3}{10};\ \varepsilon = \tfrac{5}{6};\ \zeta = \tfrac{691}{210} \text{ etc.}$$

et inter eos notemus hanc legem progreſſionis:

$$\alpha = \tfrac{1}{2^1}$$

$$\beta = \frac{5 \cdot 4\,\alpha}{2^2 \cdot 1 \cdot 2 \cdot 3} - \tfrac{2}{2^3}$$

$$\gamma = \frac{7 \cdot 6\,\beta}{2^2 \cdot 1 \cdot 2 \cdot 3} - \frac{7 \cdot 6 \cdot 5 \cdot 4\,\alpha}{2^4 \cdot 1 \cdot 2 \ldots 5} + \tfrac{3}{2^5}$$

$$\delta = \frac{9 \cdot 8\,\gamma}{2^2 \cdot 1 \cdot 2 \cdot 3} - \frac{9 \cdot 8 \cdot 7 \cdot 6\,\beta}{2^4 \cdot 1 \cdot 2 \ldots 5} + \frac{9 \ldots 4\,\alpha}{2^6 \cdot 1 \ldots 7} - \tfrac{4}{2^7}$$

$$\varepsilon = \frac{11 \cdot 10\,\delta}{2^2 \cdot 1 \cdot 2 \cdot 3} - \frac{11 \ldots 8\,\gamma}{2^4 \cdot 1 \ldots 5} + \frac{11 \ldots 6\,\beta}{2^6 \cdot 1 \ldots 7} - \frac{11 \ldots 4\,\alpha}{2^8 \cdot 1 \ldots 9} - \tfrac{5}{2^9}$$

$$\zeta = \frac{13 \cdot 12\,\varepsilon}{2^2 \cdot 1 \cdot 2 \cdot 3} - \frac{13 \ldots 10\,\delta}{2^4 \cdot 1 \ldots 5} + \frac{13 \ldots 8\,\gamma}{2^6 \cdot 1 \ldots 7} - \frac{13 \ldots 6\,\beta}{2^8 \cdot 1 \ldots 9} + \frac{13 \ldots 4\,\alpha}{2^{10} \cdot 1 \ldots 11} - \tfrac{6}{2^{11}}$$

etc.

Ac poſtremi litterarum P, Q, R, S etc. termini ita concinne referri poterunt

$$\frac{\alpha m}{2 \cdot 3};\ \frac{\beta m}{4 \cdot 5};\ \frac{\gamma m}{6 \cdot 7};\ \frac{\delta m}{8 \cdot 9};\ \frac{\varepsilon m}{10 \cdot 11};\ \frac{\zeta m}{12 \cdot 13}.$$

XXIX. Quo autem nunc etiam inueſtigemus quomodo ipſae litterae P, Q, R, S etc. progrediantur, a qualibet eiusmodi multiplum praecedentis auferamus, vt primi termini tollantur, et quia has litteras praecedit littera $O = 1$, habebimus:

$$P - \tfrac{m}{12} O = 0$$

$$Q - \frac{3m}{12} P = - \frac{\beta m}{4 \cdot 5}$$

$$R - \frac{5m}{12} Q = - \frac{mm}{16 \cdot 9} + \frac{\gamma m}{6 \cdot 7} = - \tfrac{m}{12} P + \frac{\gamma m}{6 \cdot 7}$$

$$S - \frac{7m}{12} R = - \frac{7m^3}{128 \cdot 9} + \frac{3m^2}{16 \cdot 5} - \frac{\delta m}{8 \cdot 9}$$

$$T-\frac{9m}{12}S=-\frac{7m^4}{128.9}+\frac{17m^3}{64.3.5}-\frac{7m^2}{8.9.5}+\frac{\varepsilon m}{10.11}$$

$$V-\frac{11m}{12}T=-\frac{5.7.11m^5}{2048.27}+\frac{451m^4}{512.27}-\frac{121m^3}{2048.27.5}+\frac{7159m^2}{128.27.5.7}-\frac{\zeta m}{12.13};$$

Quodſi iam has formas penitius perpendamus, ac breuitatis gratia ponamus:

$$\frac{\alpha m}{2.3}=\alpha^1;\ \frac{\beta m}{4.5}=\beta^1;\ \frac{\gamma m}{6.7}=\gamma^1;\ \frac{\delta m}{8.9}=\delta^1;\ \text{etc.}$$

ſequentem legem ſatis ſimplicem in noſtris litteris P, Q, R etc. deprehendemus:

$$P-\alpha^1=0$$

$$Q-\frac{3}{1}\alpha^1P+\frac{3.2.1}{1.2.3}\beta^1=0$$

$$R-\frac{5}{1}\alpha^1Q+\frac{5.4.3}{1.2.3}\beta^1P-\frac{5.4.3.2.1}{1.2.3.4.5}\gamma^1=0$$

$$S-\frac{7}{1}\alpha^1R+\frac{7.6.5}{1.2.3}\beta^1Q-\frac{7.6\ldots3}{1.2\ldots5}\gamma^1P+\frac{7.6\ldots1}{1.2\ldots7}\delta^1=0$$

$$T-\frac{9}{1}\alpha^1S+\frac{9\ldots7}{1\ldots3}\beta^1R-\frac{9\ldots5}{1\ldots5}\gamma^1Q+\frac{9\ldots3}{1\ldots7}\delta^1P-\frac{9\ldots1}{1\ldots9}\varepsilon^1=0$$

$$V-\frac{11}{1}\alpha^1T+\frac{11\ldots9}{1\ldots3}\beta^1S-\frac{11\ldots7}{1\ldots5}\gamma^1R+\frac{11\ldots5}{1\ldots7}\delta^1Q-\frac{11\ldots3}{1\ldots9}\varepsilon^1P+\frac{11\ldots1}{1\ldots11}\zeta^1=0$$

etc.

hae autem nouae litterae α^1, β^1, γ^1, δ^1 etc. ex praecedentibus hanc ſequentur legem:

$$\alpha^1-\frac{m}{2^2.3}=0$$

$$\beta^1-\frac{3.2}{2^2.1.2.3}\alpha^1+\frac{m}{2^4.5}=0$$

$$\gamma^1-\frac{5.4}{2^2.1.2.3}\beta^1+\frac{5.4.3.2}{2^4.1\ldots5}\alpha^1-\frac{m}{2^6.7}=0$$

$$\delta^1-\frac{7.6}{2^2.1\ldots3}\gamma^1+\frac{7\ldots4}{2^4.1\ldots5}\beta^1-\frac{7\ldots2}{2^6.1\ldots7}\alpha^1+\frac{m}{2^8.9}=0$$

$$\varepsilon^1-\frac{9.8}{2^2.1\ldots3}\delta^1+\frac{9\ldots6}{2^4.1\ldots5}\gamma^1-\frac{9\ldots4}{2^6.1\ldots7}\beta^1+\frac{9\ldots2}{2^8.1\ldots9}\alpha^1-\frac{m}{2^{10}.11}=0$$

etc.

Quo-

Quocirca nunc quidem quaestionem circa seriem illam singularem, quam hactenus sum contemplatus, perfecte solutam dedisse sum censendus, vnde solutionem hic succincte sum propositurus.

Problema.

Proposita hac progressione indefinita:

$$s = x^{m+\lambda} - \frac{m}{1}(x-1)^{m+\lambda} + \frac{m(m-1)}{1.\ 2}(x-2)^{m+\lambda} - \frac{m(m-1)(m-2)}{1.\ 2.\ 3}(x-3)^{m+\lambda} + \text{etc.}$$

eius summam assignare; siquidem λ fuerit numerus quicunque integer positiuus.

Solutio.

Denotent litterae $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. numeros *Bernoullianos*, ita vt sit

$$\mathfrak{A} = \tfrac{1}{2};\ \mathfrak{B} = \tfrac{1}{6};\ \mathfrak{C} = \tfrac{1}{6};\ \mathfrak{D} = \tfrac{3}{10};\ \mathfrak{E} = \tfrac{5}{6};$$

$$\mathfrak{F} = \tfrac{691}{210};\ \mathfrak{G} = \tfrac{35}{2};\ \mathfrak{H} = \tfrac{3617}{30};\ \mathfrak{I} = \tfrac{43867}{42};$$

$$\mathfrak{K} = \frac{1222277}{110};\ \mathfrak{L} = \frac{854513}{6}$$

$$\mathfrak{M} = \frac{1181820455}{546};\ \mathfrak{N} = \frac{76977927}{2}$$

$$\mathfrak{O} = \frac{23749461029}{30};\ \mathfrak{P} = \frac{8615841276005}{462}$$

$$\mathfrak{Q} = \frac{84802531453387}{170};\ \mathfrak{R} = \frac{90219075042845}{6}$$

etc.

 Quos

Quos numeros ita progredi obſeruaui vt ſit

$$\mathfrak{A} = \tfrac{1}{2}$$

$$\mathfrak{B} = \tfrac{4}{2} \cdot \tfrac{\mathfrak{A}^2}{3}$$

$$\mathfrak{C} = \tfrac{6}{2} \cdot \tfrac{2\mathfrak{A}\mathfrak{B}}{3}$$

$$\mathfrak{D} = \tfrac{8}{2} \cdot \tfrac{2\mathfrak{A}\mathfrak{C}}{3} + \tfrac{8 \cdot 7 \cdot 6}{2 \cdot 3 \cdot 4} \cdot \tfrac{\mathfrak{B}^2}{5}$$

$$\mathfrak{E} = \tfrac{10}{2} \cdot \tfrac{2\mathfrak{A}\mathfrak{D}}{3} + \tfrac{10 \cdot 9 \cdot 8}{2 \cdot 3 \cdot 4} \cdot \tfrac{2\mathfrak{B}\mathfrak{C}}{5}$$

$$\mathfrak{F} = \tfrac{12}{2} \cdot \tfrac{2\mathfrak{A}\mathfrak{E}}{3} + \tfrac{12 \cdot 11 \cdot 10}{2 \cdot 3 \cdot 4} \cdot \tfrac{2\mathfrak{B}\mathfrak{D}}{5} + \tfrac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot \tfrac{\mathfrak{C}\mathfrak{C}}{7}$$

$$\mathfrak{G} = \tfrac{14}{2} \cdot \tfrac{2\mathfrak{A}\mathfrak{F}}{3} + \tfrac{14 \cdot 13 \cdot 12}{2 \cdot 3 \cdot 4} \cdot \tfrac{2\mathfrak{B}\mathfrak{E}}{5} + \tfrac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot \tfrac{2\mathfrak{C}\mathfrak{D}}{7}$$

etc.

Hinc iam quaerantur numeri P, Q, R, S etc. vt ſit

$$P = \tfrac{1\mathfrak{A}m}{1 \cdot 2 \cdot 3}$$

$$Q = \tfrac{3\mathfrak{A}m}{1 \cdot 2 \cdot 3} P - \tfrac{3 \cdot 2 \cdot 1\, \mathfrak{B}m}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

$$R = \tfrac{5\mathfrak{A}m}{1 \cdot 2 \cdot 3} Q - \tfrac{5 \cdot 4 \cdot 3\, \mathfrak{B}m}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} P + \tfrac{5 \cdot 4 \cdots 1\, \mathfrak{C}m}{1 \cdot 2 \cdots 7}$$

$$S = \tfrac{7\mathfrak{A}m}{1 \cdot 2 \cdot 3} R - \tfrac{7 \cdot 6 \cdot 5\, \mathfrak{B}m}{1 \cdot 2 \cdots 5} Q + \tfrac{7 \cdots 3\, \mathfrak{C}m}{1 \cdot 2 \cdots 7} P - \tfrac{7 \cdots 1\, \mathfrak{D}m}{1 \cdot 2 \cdots 9}$$

vbi lex progreſſionis etiam eſt perſpicua.

Haç ſerie inuenta ſumma quaeſita s ita exprimetur:

$$\frac{s}{(\lambda+1)(\lambda+2)\cdots(\lambda+m)} = \left(x-\tfrac{m}{2}\right)^{\lambda} + \tfrac{\lambda(\lambda-1)}{1 \cdot 2} P \left(x-\tfrac{m}{2}\right)^{\lambda-2}$$

$$+ \tfrac{\lambda \cdots (\lambda-3)}{1 \cdots 4} Q \left(x-\tfrac{m}{2}\right)^{\lambda-4}$$

$$+ \tfrac{\lambda \cdots (\lambda-5)}{1 \cdots 6} R \left(x-\tfrac{m}{2}\right)^{\lambda-6}$$

$$+ \tfrac{\lambda \cdots (\lambda-7)}{1 \cdots 8} S \left(x-\tfrac{m}{2}\right)^{\lambda-8}$$

etc.

vbi

vbi notetur fi forte numerus m non fit integer valorem producti $(\lambda+1)(\lambda+2)\ldots(\lambda+m)$ per artificia alibi expofita definiri poffe.

Corollarium 1.

Si loco numerorum *Bernoullianorum* numeros iis cognatos, quibus ad poteftatum reciprocarum fummas fum vfus, introducere velimus, hosque numeros litteris A, B, C, D etc. defignemus, vt fit $A = \frac{1}{6}$, $B = \frac{1}{90}$, $C = \frac{1}{945}$, $D = \frac{1}{9450}$; $E = \frac{1}{93555}$ quoniam hi numeri a prioribus ita pendent vt fit

$$\mathfrak{A} = \frac{1.2.3}{2^1} A;\ \mathfrak{B} = \frac{1\ldots\ldots5}{2^3} B;\ \mathfrak{C} = \frac{1\ldots\ldots7}{2^5} C;\ \text{etc.}$$

inter fe autem ita connectuntur vt fit:

$$5B = 2A^2;\ 7C = 4AB;\ 9D = 4AC + 2BB;$$
$$11E = 4AD + 4BC;\ 13F = 4AE + 4BD + 2CC\ \text{etc.}$$

Tum ex his numeris litterae P, Q, R, S etc. ita determinabuntur:

$$P = \frac{1Am}{2}$$
$$Q = \frac{3Am}{2}P - \frac{3.2.1\,Bm}{2^3}$$
$$R = \frac{5Am}{2}Q - \frac{5.4.3\,Bm}{2^3}P + \frac{5.4\ldots1\,Cm}{2^5}$$
$$S = \frac{7Am}{2}R - \frac{7.6.5\,Bm}{2^3}Q + \frac{7.6\ldots3\,Dm}{2^5}P - \frac{7.6\ldots1\,Em}{2^7}$$

etc.

Corollarium 2.

Si pro variis valoribus numeri λ summam progressionis proposito hoc signandi modo $\int(\lambda)$ indicemus, ac iam loco λ successiue scribamus numeros 0, 1, 2, 3, 4 etc. pro his casibus summae $\int(0)$, $\int(1)$, $\int(2)$, $\int(3)$ etc. sequenti modo exprimentur ponendo breuitatis gratia $x - \frac{m}{2} = y$:

$$\frac{\int(0)}{1.2\ldots m} = 1$$

$$\frac{\int(1)}{2.3\ldots(m+1)} = y$$

$$\frac{\int(2)}{3.4\ldots(m+2)} = y^2 + P$$

$$\frac{\int(3)}{4.5\ldots(m+3)} = y^3 + 3Py$$

$$\frac{\int(4)}{5.6\ldots(m+4)} = y^4 + 6Py^2 + Q$$

$$\frac{\int(5)}{6.7\ldots(m+5)} = y^5 + 10Py^3 + 5Qy$$

$$\frac{\int(6)}{7.8\ldots(m+6)} = y^6 + 15Py^4 + 15Qy^2 + R$$

etc.

Corollarium 3.

Hinc ergo istae summae sequenti modo singulae ex antecedentibus definiri possunt

$$\int(1) = \frac{m+1}{1} y \int(0)$$

$$\int(2) = \frac{m+2}{2} y \int(1) + \frac{(m+2)(m+1)}{2.2} m A \int(0)$$

$$\int(3) = \frac{m+3}{3} y \int(2) + \frac{(m+3)(m+2)}{2.3} m A \int(1)$$

$\int(4)$

$$\int(4)=\frac{m+4}{4}y\int(3)+\frac{(m+4)(m+3)}{2.\ 4}mA\int(2)-\frac{(m+4)\ldots(m+1)}{2^3.\ 4}mB\int(0)$$
$$\int(5)=\frac{m+5}{5}y\int(4)+\frac{(m+5)(m+4)}{2.\ 5}mA\int(3)-\frac{(m+5)\ldots(m+2)}{2^3.\ 5}mB\int(1)$$
$$\int(6)=\frac{m+6}{6}y\int(5)+\frac{(m+6)(m+5)}{2.\ 6}mA\int(4)-\frac{(m+6)\ldots(m+3)}{2^3.\ 6}mB\int(2)+\frac{(m+6)\ldots(m+1)}{2^5.\ 6}mC\int(0)$$
$$\int(7)=\frac{m+7}{7}y\int(6)+\frac{(m+7)(m+6)}{2.\ 7}mA\int(5)-\frac{(m+7)\ldots(m+4)}{2^3.\ 7}mB\int(3)+\frac{(m+7)\ldots(m+2)}{2^5.\ 7}mC\int(1)$$
$$\int(8)=\frac{m+8}{8}y\int(7)+\frac{(m+8)(m+7)}{2.\ 8}mA\int(6)-\frac{(m+8)\ldots(m+5)}{2^3.\ 8}mB\int(4)+\frac{(m+8)\ldots(m+2)}{2^5.\ 8}mC\int(2)$$
$$-\frac{(m+8)\ldots(m+1)}{2^7.\ 8}mD\int(0)$$

quae lex progressionis inspicienti mox fit manifesta.

Conclusio.

Nunc haud multo difficilius erit hoc negotium longe generalius expedire, ita vt, si $\phi:x$ denotet functionem quamcunque ipsius x summam huius seriei

$$s=\phi:x-m\phi.(x-1)+\frac{m(m-1)}{1.\ 2}\phi:(x-2)-\frac{m(m-1)(m-2)}{1.\ 2.\ 3}\phi:(x-3)$$

assignare queamus. Perspicuum enim est hanc formam, differentiam ordinis m exhibere istius progressionis

$$\phi:x;\ \phi:(x-1);\ \phi:(x-2);\ \phi:(x-3)\ \text{etc.}$$

Ex iis enim quae in Institutionibus Calculi Differentialis pag. 343. in medium attuli, si ponamus $\phi:x=y$, colligitur differentias singulorum ordinum esse:

Δy

$$\Delta y = \frac{dy}{dx} - \frac{ddy}{2dx^2} + \frac{d^3y}{2.3.dx^3} - \frac{d^4y}{2.3.4dx^4} + \frac{d^5y}{2\ldots 5dx^5} - \text{etc.}$$

$$\Delta^2 y = \frac{d^2y}{dx^2} - \frac{3d^3y}{3dx^3} + \frac{7d^4y}{3.4dx^4} - \frac{15d^5y}{3.4.5dx^5} + \frac{31d^6y}{3\ldots 6dx^6} - \text{etc.}$$

$$\Delta^3 y = \frac{d^3y}{dx^3} - \frac{6d^4y}{4dx^4} + \frac{25d^5y}{4.5.dx^5} - \frac{90d^6y}{4.5.6dx^6} + \frac{301d^7y}{4\ldots 7dx^7} - \text{etc.}$$

$$\Delta^4 y = \frac{d^4y}{dx^4} - \frac{10d^5y}{5dx^5} + \frac{65d^6y}{5.6dx^6} - \frac{350d^7y}{5.6.7dx^7} + \frac{1701d^8y}{5\ldots 6dx^8} - \text{etc.}$$

etc.

qui coefficientes cum ſint illi ipſi, quos ſupra §. IV habuimus, eodem modo intelligemus differentiam ordinis m ſeu $\Delta^m y$, hoc eſt ipſam ſummam ſeriei propoſitae fore

$$s = \frac{d^my}{dx^m} - \frac{A^I d^{m+1}y}{(m+1)dx^{m+1}} + \frac{B^I d^{m+2}y}{(m+1)(m+2)dx^{m+2}} - \frac{C^I d^{m+3}y}{(m+1)\ldots(m+3)dx^{m+3}} + \text{etc.}$$

quos coefficientes A^I, B^I, C^I etc. ſupra §. 13. determinaui. Quocirca erit

$$\frac{A^I}{m+1} = \frac{m}{2}$$

$$\frac{B^I}{(m+1)(m+2)} = \frac{m}{1.2.3} + \frac{3m(m-1)}{1.2.3.4}$$

$$\frac{C^I}{(m+1)\ldots(m+3)} = \frac{m}{1.2.3.4} + \frac{10m(m-1)}{1.2\ldots 5} + \frac{15m(m-1)(m-2)}{1.2\ldots 6}$$

$$\frac{D^I}{(m+1)\ldots(m+4)} = \frac{m}{1.2\ldots 5} + \frac{25m(m-1)}{1.2\ldots 6} + \frac{105m(m-1)(m-2)}{1.2\ldots 7} + \frac{105m(m-1)(m-2)(m-3)}{1.2\ldots 8}$$

etc.

Quodſi iam nunc ponamus $\phi:(x-\frac{m}{2})=v$, ita vt v oriatur ex y, ſi loco x ſcribatur $x-\frac{m}{2}$, erit vtique

$$\frac{d^mv}{dx^m} = \frac{d^my}{dx^m} - \frac{md^{m+1}y}{2dx^{m+1}} + \frac{m^2d^{m+2}y}{2.4dx^{m+2}} - \text{etc.}$$

quae

quae aequatio ſi inde ſubtrahatur, calculus idem prorſus erit inſtituendus, quem ſupra expediuimus. Vnde introducendo easdem litteras P, Q, R, S etc. quas ſupra definiuimus, obtinebimus ſequentem ſummae s valorem:

$$s = \frac{d^m v}{dx^m} + \frac{P d^{m+2} v}{1.2\, dx^{m+2}} + \frac{Q d^{m+4} v}{1.2.4\, dx^{m+4}} + \frac{R d^{m+6} v}{1.2\ldots 6\, dx^{m+6}} + \frac{S d^{m+8} v}{1.2\ldots 8\, dx^{m+8}} + \text{etc.}$$

atque hinc ſi ſumatur $y = \Phi : x = x^{m+\lambda}$ et $v = (x - \frac{m}{2})^{m+\lambda}$ manifeſto eadem ſummatio ſequitur quam ante eruimus, ſicque totum negotium redit ad litteras P, Q, R, S etc. quarum indolem ex numeris *Bernoullianis* ſupra deriuaui.

Hinc ſtatim liquet, quod ante minus apparebat, ſi in functione y vel v numerus dimenſionum minor fuerit quam exponens m, quem quidem numerum integrum poſitiuum eſſe oportet, tum omnia differentialia ordinis m et ſuperiorum in nihilum abire, foreque ſummam $s = 0$.

Deinde hinc etiam planior patet via ad valores litterarum P, Q, R, S etc. inueniendos. Cum enim poſito

$$s = \frac{d^m y}{dx^m} - \frac{\alpha d^{m+1} y}{dx^{m+1}} + \frac{\mathfrak{C} d^{m+2} y}{dx^{m+2}} - \frac{\gamma d^{m+3} y}{dx^{m+3}} + \text{etc.}$$

 ſit

fit $\alpha = \frac{m}{1.2}$

$$\beta = \frac{m}{1.2.3} + \frac{3m(m-1)}{1.2.3.4}$$

$$\gamma = \frac{m}{1\ldots4} + \frac{10m(m-1)}{1\ldots5} + \frac{15m\ldots(m-2)}{1\ldots6}$$

$$\delta = \frac{m}{1\ldots5} + \frac{25m(m-1)}{1\ldots6} + \frac{105m\ldots(m-2)}{1\ldots7} + \frac{105m\ldots(m-3)}{1\ldots8}$$

etc.

functio autem y ex functione $v = \phi : (x - \frac{m}{2})$ nafcatur fi in hac loco x fcribatur $x + \frac{m}{2}$ erit in genere

$$\frac{d^n y}{dx^n} = \frac{d^n v}{dx^n} + \frac{m}{2} \cdot \frac{d^{n+1} v}{dx^{n+1}} + \frac{m^2}{2.4} \cdot \frac{d^{n+2} v}{dx^{n+2}} + \frac{m^3}{2.4.6} \cdot \frac{d^{n+3} v}{dx^{n+3}} + \text{etc.}$$

vnde fi loco differentialium ipfius y, haec differentialia ipfius v fubftituantur, fiet

$$s = \frac{d^n v}{dx^n} + \left(\frac{m}{2} - \alpha\right)\frac{d^{n+1} v}{dx^{n+1}} + \left(\frac{m^2}{2.4} - \frac{m}{2}\alpha + \beta\right)\frac{d^{n+2} v}{dx^{n+2}}$$

$$+ \left(\frac{m^3}{2.4.6} - \frac{m^2}{2.4}\alpha + \frac{m}{2}\beta - \gamma\right)\frac{d^{n+3} v}{dx^{n+3}} + \text{etc.}$$

ficque habebimus:

$$\frac{m}{2} - \alpha = 0$$

$$\frac{m^2}{2.4} - \frac{m}{2}\alpha + \beta = \frac{\mathfrak{P}}{1.2}$$

$$\frac{m^3}{2.4.6} - \frac{m^2}{2.4}\alpha + \frac{m}{2}\beta - \gamma = 0$$

$$\frac{m^4}{2.4.6.8} - \frac{m^3}{2.4.6}\alpha + \frac{m^2}{2.4}\beta - \frac{m}{2}\gamma + \delta = \frac{\mathfrak{Q}}{1.2.3.4}$$

$$\frac{m^5}{2.4.6.8.10} - \frac{m^4}{2.4.6.8}\alpha + \frac{m^3}{2.4.6}\beta - \frac{m^2}{2.4}\gamma + \frac{m}{2}\delta - \varepsilon = 0$$

etc.

facile enim perfpicitur has expreffiones alternatim euanefcere debere.

QVO-