1768

Methodus facilis motus corporum coelestium utcunque perturbatos ad rationem calculi astronomici revocandi

Leonhard Euler

METHODVS FACILIS MOTVS CORPORVM COELESTIVM VTCVNQVE PERTVRBATOS AD RATIONEM CALCVLI ASTRONOMICI REVOCANDI.

Auctore

L. EVLERO.

I.

Cum hanc inueſtigationem, quomodo motus corporum coeleſtium ob actionem mutuam perturbentur, iam ſaepius eſſem aggreſſus, in calculos plerumque nimis prolixos et operoſos ſum delapſus, quos vero denique poſt multas ambages ad formulas multo ſimpliciores reducere licuit. Cauſa autem huius prolixitatis manifeſto in multitudine elementorum, quae in calculum introduci oportet, eſt ſita: neque enim ſolum ad cunctas determinationes, quibus motus corporis perturbantis continetur, eſt reſpiciendum, ſed etiam ipſa motus perturbatio, quatenus non fit in eodem plano, plurima poſtulat elementa, quibus ea pro more apud Aſtronomos recepto ad variationes inde in linea nodorum et orbitae inclinatione oriundas referatur. Quodſi omnes iſtae conſiderationes ſimul in calcu-

calculum ingerantur; mirum sane non est, maximam hinc molestiam et confusionem oriri debere, cui euitandae aliud remedium non superesse videtur, nisi vt omnia elementa sollicite distinguantur, et singulae operationes ita instituantur, vt in eas non plura admittantur elementa, quam in eam necessario ingrediuntur. Ita enim commodissime cauebitur, ne attentio nostra nimia elementorum multitudine obruatur.

II.

Praecipua quidem huius inuestigationis pars ad mechanicam est referenda, cum motus perturbatio ex viribus corporis perturbantis sit definienda; principia autem mechanica ita sunt comparata, vt ex iis locus corporis, cuius motus quaeritur, ad quoduis tempus per ternas coordinatas inuicem normales commodissime determinetur; verum altera pars haud leuiorem euolutionem requirit, qua locus in priori parte definitus ad morem in Astronomia receptum reduci debet, quo scilicet diuersa coeli loca per longitudinem et latitudinem exprimi solent. Atque hanc posteriorem partem, quam geometricam appellare licet, probe distingui conuenit a priori, quae tota mechanicae est tribuenda. Obseruaui autem, has duas partes non solum commode a se inuicem separari, sed etiam vtramque tum multo faciliori negotio pertractari posse, quam si ambas coniunctim expedire vellemus. Quamuis autem

autem inuestigatio mechanica geometricam praecedere debere videatur, tamen satis concinne a parte geometrica exordiri licet, cum nihil impediat, quominus locum corporis, cuius motum quaerimus, tanquam cognitum, ac per ternas coordinatas definitum spectemus. Hanc ordinis inuersionem ideo sequi est visum, quod euolutio partis geometricae plura insignia suppeditet subsidia, quibus deinceps in parte mechanica calculi labor non mediocriter subleuabitur. Hoc scilicet modo id potissimum lucri consequimur, quod dum partem geometricam tractamus, nullae quantitates ad vires perturbantes relatae in calculum ingrediantur.

Pars Geometrica.

III.

Motum igitur corporis Z iam ita determinatum assumo, vti per principia mechanica immediate definiri solet. Primum scilicet motus ad certum quoddam punctum A, quod vt fixum spectatur, etiamsi forte ipsum versetur in motu referri solet, tum vero consideratur planum quoddam per id punctum transiens pariter vti fixum, quod ipso tabulae plano repraesentetur, in quo accepta linea fixa AB, ad quoduis tempus locus corporis Z per ternas coordinatas inter se normales AX, XY et YZ ita definitur, vt primo ex loco Z in planum illud fixum demittatur perpendiculum ZY, tum vero Tab. II. Fig. 1.

 ro

to ex Y ad rectam AB ducatur normalis YX. Vocemus ergo has coordinatas:

$$AX = X, \quad XY = Y \quad \text{et} \quad YZ = Z$$

quarum valores ad quoduis tempus elapsum $= t$ vt cogniti spectantur. Hinc igitur statim habetur distantia corporis Z a puncto fixo A quae si breuitatis gratia dicatur $AZ = v$ erit $vv = XX + YY + ZZ$.

Tum vero momento temporis dt corpus ex Z in z progredi concipiamus, vt sit $Az = v + dv$, et angulus elementaris $ZAz = d\Phi$, quem corpus Z interea ex A in orbita sua conficere cernitur, eritque vti constat, $Zz = \sqrt{(dv^2 + vv\,d\Phi^2)}$, at per coordinatarum elementa est etiam $Zz = \sqrt{(dX^2 + dY^2 + dZ^2)}$, vnde fit

$$dv^2 + vv\,d\Phi^2 = dX^2 + dY^2 + dZ^2$$

sicque patet, quomodo angulum elementarem $d\Phi$ per coordinatas exprimi conueniat, quod vero mox succinctius ostendetur.

IV.

Per elementum Zz cum puncto fixo A certum planum determinatur, in quo nunc quidem corpus Z moueri censetur: hoc planum alicubi secabit planum tabulae fixum; fiat ergo haec intersectio secundum rectam AN, quae in Astronomia linea nodorum appellatur, et cuius variatio ob perturbationem motus corporis Z potissimum est inuestiganda; deinde etiam angulum, quo planum NAZ ad pla-

planum fixum inclinatur, notari conuenit, qui simpliciter in Astronomia inclinatio vocatur, et ob motus perturbationem insignes mutationes subire potest. Pro his ergo nouis elementis ponamus:

Longitudinem lineae nodorum seu angulum $BAN = \psi$
Inclinationem orbitae ZAN ad planum fixum $= \omega$
et argumentum latitudinis seu angulum $NAZ = \sigma$

quae vt ad coordinatas reuocemus, tam ex Y quam ex Z ad lineam nodorum AN agamus normales YO et ZO, sicque angulus YOZ ipsam inclinationem ω metietur. At ob angulum $NAZ = \sigma$ et distantiam $AZ = v$ habebimus:

$$AO = v \operatorname{cos}.\sigma \quad \text{et} \quad ZO = v \operatorname{sin}.\sigma$$

hincque porro

$$ZY = v \operatorname{sin}.\sigma \operatorname{sin}.\omega \quad \text{et} \quad OY = v \operatorname{sin}.\sigma \operatorname{cos}.\omega$$

vnde ob angulum $BAN = \psi = XYO$, concludimus:

$$AX = v \operatorname{cos}.\sigma \operatorname{cos}.\psi - v \operatorname{sin}.\sigma \operatorname{cos}.\omega \operatorname{sin}.\psi$$
$$\text{et } XY = v \operatorname{cos}.\sigma \operatorname{sin}.\psi + v \operatorname{sin}.\sigma \operatorname{cos}.\omega \operatorname{cos}.\psi.$$

Quare ternae nostrae coordinatae ita hinc definiuntur, vt sit

$$X = v(\operatorname{cos}.\sigma \operatorname{cos}.\psi - \operatorname{sin}.\sigma \operatorname{cos}.\omega \operatorname{sin}.\psi)$$
$$Y = v(\operatorname{cos}.\sigma \operatorname{sin}.\psi + \operatorname{sin}.\sigma \operatorname{cos}.\omega \operatorname{cos}.\psi)$$
$$\text{et } Z = v \operatorname{sin}.\sigma \operatorname{sin}.\omega$$

 vbi

vbi praeterea notetur eſſe tang. anguli $NAY = \text{tang.}\,\sigma$ $\text{coſ.}\,\omega$, qui angulus dicitur longitudo puncti Z a nodo.

V.

Cum nunc angulus BAY in Aſtronomia exhibeat longitudinem, angulus vero ZAY latitudinem puncti Z, ſiquidem planum tabulae eclipticam referat, et recta AB ad principium arietis ſit porrecta; quas denominationes autem quoque in latiori ſenſu accipere licet: habebimus

longitudinem puncti Z ſeu angulum $BAY = \psi + NAY$

exiſtente $\text{tang.}\,NAY = \text{tang.}\,\sigma\ \text{coſ.}\,\omega$

pro latitudine vero ſeu angulo ZAY erit

$$\text{ſin.}\,ZAY = \frac{ZY}{AZ} = \text{ſin.}\,\sigma\ \text{ſin.}\,\omega$$

Tab. II. Fig. 2. quae eaedem formulae vulgo ex trigonometria ſphaerica elici ſolent. In ſuperficie ſcilicet ſphaerica centro A deſcripta circulus maximus BNY repraeſentet planum fixum, et punctum B ſit initium, a quo longitudo computatur. Porro ſit N nodus et NZ orbita, ad quam nunc motus corporis Z refertur, tum ex Z ad circulum BNY ducatur arcus normalis ZY; quo facto arcus BNY praebet longitudinem, arcus vero ZY latitudinem puncti Z, ad quas inueniendas primo habemus:

arcum BN, ſeu longitudinem nodi $= \psi$

angulum ZNY ſeu inclinationem $= \omega$

et arcum NZ ſeu argumentum latitudinis $= \sigma$.

Ex

Ex his reſolutio trianguli ſphaerici rectanguli NYZ dat

$$\text{ſin.}\,ZY = \text{ſin.}\,\sigma\ \text{ſin.}\,\omega \quad \text{et} \quad \text{tang.}\,NY = \text{tang.}\,\sigma\ \text{coſ.}\,\omega$$

prorſus vt ante.

IV.

Quamuis autem tam linea nodorum, quam inclinatio ſit variabilis; tamen quia ambo puncta Z et z ad idem planum NAZ pertinent, per differentiationem a puncto Z ad z perueniri debet, etiamſi angulus $BAN = \psi$ et inclinatio ω pro conſtantibus habeantur, dum ſcilicet angulus $NAZ = \sigma$ angulo elementari $ZAz = d\Phi$ creſcere ſumatur vt ſit $d\sigma = d\Phi$. Deinde vero etiam per differentiationem ad idem punctum z perueniri neceſſe eſt, ſi tam linea nodorum, quam inclinatio variabiles ſtatuantur, quoniam punctum z quoque ad orbitam variatam pertinere debet, hic vero non amplius differentiale $d\sigma$ ipſi $d\Phi$ aequale eſt ponendum, ſed ipſi proprius valor eſt tribuendus, qui ſimul ab orbitae mutatione pendet. Cum igitur haec duplex differentiatio eodem perducere debeat, aequationes hinc adipiſcemur, quibus certae relationes inter variationes in orbita ortas definiuntur, quae in ſequenti calculo maximum praeſtabunt vſum. Neque vero hoc ſolum in differentialibus ipſarum coordinatarum locum habet, ſed etiam quantitatum inde deriuatarum, cuiusmodi ſunt:

$$\frac{X}{Z} = \frac{\text{coſ.}\,\sigma\ \text{coſ.}\,\psi}{\text{ſin.}\,\sigma\ \text{ſin.}\,\omega} - \frac{\text{coſ.}\,\omega\ \text{ſin.}\,\psi}{\text{ſin.}\,\omega} \quad \text{et} \quad \frac{Y}{Z} = \frac{\text{coſ.}\,\sigma\ \text{ſin.}\,\psi}{\text{ſin.}\,\sigma\ \text{ſin.}\,\omega} + \frac{\text{coſ.}\,\omega\ \text{coſ.}\,\psi}{\text{ſin.}\,\omega}$$

qua-

quarum ergo differentialia duobus illis modis ſumta eosdem valores praebere debent.

VII.

Prima igitur differentiatio ſumtis angulis ψ et ω conſtantibus et $d\sigma = d\Phi$ dat:

$$d.\frac{X}{Z} = \frac{-d\Phi \,\mathrm{cos.}\psi}{\mathrm{sin.}\sigma^2 \,\mathrm{sin.}\omega} \text{ et } d.\frac{Y}{Z} = \frac{-d\Phi \,\mathrm{sin.}\psi}{\mathrm{sin.}\sigma^2 \,\mathrm{sin.}\omega}.$$

Pro altera differentiatione notentur primo hae formulae:

$$\frac{X}{Z}\mathrm{cos.}\psi + \frac{Y}{Z}\mathrm{sin.}\psi = \frac{\mathrm{cos.}\sigma}{\mathrm{sin.}\sigma \,\mathrm{sin.}\omega} \text{ et } \frac{Y}{Z}\mathrm{cos.}\psi - \frac{X}{Z}\mathrm{sin.}\psi = \frac{\mathrm{cos.}\omega}{\mathrm{sin.}\omega}$$

quae more conſueto differentiatae praebent:

$$\mathrm{cos.}\psi\, d.\frac{X}{Z} + \mathrm{sin.}\psi\, d.\frac{Y}{Z} - d\psi\left(\frac{Y}{Z}\mathrm{cos.}\psi - \frac{X}{Z}\mathrm{sin.}\psi\right) = \frac{-d\sigma}{\mathrm{sin.}\sigma^2 \,\mathrm{sin.}\omega} - \frac{d\omega \,\mathrm{cos.}\sigma \,\mathrm{cos.}\omega}{\mathrm{sin.}\sigma \,\mathrm{sin.}\omega^2}$$

$$\mathrm{cos.}\psi\, d.\frac{Y}{Z} - \mathrm{sin.}\psi\, d.\frac{X}{Z} - d\psi\left(\frac{X}{Z}\mathrm{cos.}\psi + \frac{Y}{Z}\mathrm{sin.}\psi\right) = \frac{-d\omega}{\mathrm{sin.}\omega^2}$$

vnde prioribus valoribus ſubſtitutis colligitur:

$$-\frac{d\Phi \,\mathrm{cos.}\psi^2}{\mathrm{sin.}\sigma^2 \,\mathrm{sin.}\omega} - \frac{d\Phi \,\mathrm{sin.}\psi^2}{\mathrm{sin.}\sigma^2 \,\mathrm{sin.}\omega} + \frac{d\psi \,\mathrm{cos.}\omega}{\mathrm{sin.}\omega} = \frac{-d\sigma}{\mathrm{sin.}\sigma^2 \,\mathrm{sin.}\omega} - \frac{d\omega \,\mathrm{cos.}\sigma \,\mathrm{cos.}\omega}{\mathrm{sin.}\sigma \,\mathrm{sin.}\omega^2}$$

$$-\frac{d\Phi \,\mathrm{sin.}\psi \,\mathrm{cos.}\psi}{\mathrm{sin.}\sigma^2 \,\mathrm{sin.}\omega} + \frac{d\Phi \,\mathrm{sin.}\psi \,\mathrm{cos.}\psi}{\mathrm{sin.}\sigma^2 \,\mathrm{sin.}\omega} - \frac{d\psi \,\mathrm{cos.}\sigma}{\mathrm{sin.}\sigma \,\mathrm{sin.}\omega} = \frac{-d\omega}{\mathrm{sin.}\omega^2}$$

quae formae contrahuntur in has:

$$-\frac{d\Phi}{\mathrm{sin.}\sigma^2 \,\mathrm{sin.}\omega} + \frac{d\psi \,\mathrm{cos.}\omega}{\mathrm{sin.}\omega} = \frac{-d\sigma}{\mathrm{sin.}\sigma^2 \,\mathrm{sin.}\omega} - \frac{d\omega \,\mathrm{cos.}\sigma \,\mathrm{cos.}\omega}{\mathrm{sin.}\sigma \,\mathrm{sin.}\omega^2}$$

$$\text{et } -\frac{d\psi \,\mathrm{cos.}\sigma}{\mathrm{sin.}\sigma \,\mathrm{sin.}\omega} = \frac{-d\omega}{\mathrm{sin.}\omega^2} \text{ ſeu } \frac{d\omega}{\mathrm{sin.}\omega} = \frac{d\psi}{\mathrm{tang.}\sigma}$$

qui poſterior valor in illa ſubſtitutus ſuppeditat

$$\frac{d\sigma - d\Phi}{\mathrm{sin.}\sigma^2 \,\mathrm{sin.}\omega} = \frac{-d\psi \,\mathrm{cos.}\omega}{\mathrm{sin.}\omega} - \frac{d\psi \,\mathrm{cos.}\sigma^2 \,\mathrm{cos.}\omega}{\mathrm{sin.}\sigma^2 \,\mathrm{sin.}\omega} = \frac{-d\psi \,\mathrm{cos.}\omega}{\mathrm{sin.}\sigma^2 \,\mathrm{sin.}\omega}$$

ſeu $d\psi \,\mathrm{cos.}\omega = d\Phi - d\sigma$.

VIII.

Hinc igitur primo diſcimus variationem in inclinatione orbitae ortam $d\omega$ ſemper ita pendere a variatione lineae nodorum $d\psi$ vt ſit $d\omega = \frac{d\psi \,\mathrm{sin.}\omega}{\mathrm{tang.}\sigma}$; ſeu incrementum inclinationis ſe habebit ad promotionem lineae nodorum, vt ſinus inclinationis ad tangentem argumenti latitudinis; vnde ſequentia conſectaria deducuntur:

1°. Si argumentum latitudinis σ ſit vel nullum vel 6ˢ vbi latitudo eſt nulla, lineam nodorum quieſcere, quantumuis interea varietur inclinatio.

2°. Si argumentum latitudinis σ ſit vel 3ˢ vel 9ˢ, ſeu tang.$\sigma = \infty$ vbi latitudo eſt maxima, tum inclinationem nullam mutationem pati; quantumuis interea linea nodorum vel progrediatur vel regrediatur.

3°. Si argumentum latitudinis σ vel intra limites 0ˢ et 3ˢ vel intra 6ˢ et 9ˢ continetur, hoc eſt dum latitudo creſcit, tum inclinationem ω creſcere, ſiquidem linea nodorum progrediatur, ſin autem regrediatur, inclinationem decreſcere.

4°. Si argumentum latitudinis σ vel intra limites 3ˢ et 6ˢ vel intra 9ˢ et 12ˢ continetur, hoc eſt dum latitudo decreſcit, tum progrediente linea nodorum inclinationem imminui, ea vero regrediente augeri.

IX.

Deinde obſeruandum eſt, augmentum argumenti latitudinis σ promotioni in propria orbita ſeu elemento $d\Phi$ non eſſe aequale, niſi linea nodorum maneat immota; cum inuenerimus $d\sigma = d\Phi - d\psi \cos\omega$, ſolo excepto caſu, quo inclinatio ω foret angulus rectus. Haec vero phaenomena per trigonometriam ſphaericam magis perſpicua redduntur. Si enim circulus BNY vt ante repraeſentet planum fixum, ad quod motus puncti Z refertur, eiusque motus praeſens fiat ſecundum circulum NZ, vt ſit $BN = \psi$; $YNZ = \omega$ et arcus $NZ = \sigma$; at poſtquam punctum Z per elementum $Zz = d\Phi$ fuerit progreſſum, eius motus fiat ſecundum circulum nz, erit promotio lineae nodorum $Nn = d\psi$ inclinatio variata $Ynz = \omega + d\omega$, et argumentum latitudinis $nz = \sigma + d\sigma$. Ducto ergo arculo $n\nu$ ad NZ normali erit $N\nu = d\psi \cos\omega$ et $n\nu = d\psi \sin\omega$; inde autem colligitur $Zn = \sigma - d\psi \cos\omega$, ideoque $nz = \sigma - d\psi \cos\omega + d\Phi = \sigma + d\sigma$ et conſequenter $d\sigma = d\Phi - d\psi \cos\omega$ vt ante; ſimul autem intelligimus ob $d\psi \cos\omega = d\Phi - d\sigma = N\nu$, formulam $d\Phi - d\sigma$ exhibere promotionem lineae nodorum in propria orbita, quia cum nodus fuiſſet in orbitae NZ puncto N; is iam in eius punctum n ſeu ν eſſet translatus. Praeterea ex triangulo ſphaerico NnZ colligimus:

Tab. II. Fig. 3.

$$\sin\omega : \sin(\omega + d\omega) = \sin(\sigma - d\psi \cos\omega) : \sin\sigma$$

ſeu

ſeu $\sin\omega : \sin\omega + d\omega \cos\omega = \sin\sigma - d\psi \cos\sigma \cos\omega : \sin\sigma$ et diuidendo $\sin\omega : d\omega \cos\omega = \sin\sigma - d\psi \cos\sigma \cos\omega : d\psi \cos\sigma \cos\omega$ vnde fit $d\omega \sin\sigma = d\psi \sin\omega \cos\sigma$ ſeu $\frac{d\omega}{\sin\omega} = \frac{d\psi}{\tan\sigma}$ prorſus vt ante.

X.

Formulae autem differentiales, ante §. 7. inventae, ſi euoluantur praebent:

$$\frac{X\,dZ - Z\,dX}{ZZ} = \frac{d\Phi \cos\psi}{\sin\sigma^2 \sin\omega} \quad \text{et} \quad \frac{Y\,dZ - Z\,dY}{ZZ} = \frac{d\Phi \sin\psi}{\sin\sigma^2 \sin\omega}$$

vnde ob $Z = v \sin\sigma \sin\omega$ nanciscimur has formulas notatu dignas:

$$X\,dZ - Z\,dX = vv\,d\Phi \sin\omega \cos\psi \quad \text{et}$$
$$Y\,dZ - Z\,dY = vv\,d\Phi \sin\omega \sin\psi.$$

Hinc vt dZ eliminemus, ſi priorem multiplicemus per Y poſteriorem vero per X, hoc productum ab illo ablatum relinquet:

$$Z(X\,dY - Y\,dX) = vv\,d\Phi \sin\omega (Y\cos\psi - X\sin\psi).$$

Cum autem ſit vt vidimus §. 7. $Y\cos\psi - X\sin\psi = \frac{Z \cos\omega}{\sin\omega}$ haec formula ad inſignem ſimplicitatem contrahitur:

$$X\,dY - Y\,dX = vv\,d\Phi \cos\omega.$$

Hac formulae cum illis, quas ſtatim ab initio invenimus, coniunctae ſcilicet hiſce:

$$XX + YY + ZZ = vv \quad \text{et} \quad dX^2 + dY^2 + dZ^2 = dv^2 + vv\,d\Phi^2$$

in

in parte mechanica maximum praeſtabunt vſum, ad coordinatas ex calculo eiiciendas, vt is deinceps eiusmodi quantitatibus, quae in Aſtronomia vſu ſunt receptae, contineatur.

XI.

Tab. II. Fig. 1. Iſtae autem reductiones Geometricae latiſſime patent nihilque intereſt, ad quodnam punctum A, planumque fixum BAY, motum puncti Z referre velimus. Verum ſi ad vſum aſtronomicum ſpectemus, plurimum refert, quomodo tam illud punctum A, quod quaſi eſt centrum motus, quam illud planum, ad quod motus puncti Z per longitudinem et latitudinem refertur, accipiatur, quoniam hinc potiſſimum ſimplicitas determinationis pendet. Ad quam electionem inſtituendam ante omnia notari oportet, cuncta artificia, quae adhuc ſunt excogitata, tum ſolum cum aliquo ſucceſſu adhiberi poſſe, cum motus corporis, qui quaeritur, non multum a legibus *Keplerianis* diſcrepet, ideoque perturbationes admodum fuerint exiguae. Quando autem motus ita eſt comparatus, vt areae circa quodpiam punctum deſcriptae ſint ſatis prope tempori proportionales, in hoc puncto aptiſſime punctum illud fixum A ſtatuitur. Quod cum eueniat, ſi inter vires corpus ſollicitantes, vna reliquas multum ſuperet, in eo puncto, ad quod haec vis dirigitur, punctum A accipi conueniet: Ita ſi quaeſtio fuerit de perturbationibus planetae cuiusdam principalis ſeu cometae, punctum A commodiſſime in centro ſolis capietur: ſin autem perturbationes in motu lunae, vel alius planetae ſecundarii factae definiri debeant; tum punctum A in centro terrae vel planetae primarii accipi oportet, ita vt vis corpus propoſitum Z ad A pellens reliquas vires, quibus hoc corpus ſimul vrgetur, multum ſuperet.

XII.

Si corpus Z hac ſola vi principali ſollicitaretur, corpus omnino regulariter circa punctum A in ſectione conica reuolueretur, idque perpetuo in eodem plano, ita vt quomodocunque planum fixum BAY acciperetur, neque in linea nodorum neque in inclinatione vnquam vlla mutatio oriretur; interim tamen calculus ſine dubio ſimpliciſſimus euaderet, ſi planum fixum in ipſo plano motus acciperetur. Verum ſi motus ab alio corpore coeleſti perturbetur, cuiusquidem motum in hac inueſtigatione tanquam cognitum aſſumi oportet, planum fixum conuenientiſſime cum orbita illius corporis perturbantis congruens ſumetur. Ita ſi perturbationes lunae a ſole oriundae quaerantur, planum eclipticae, in quo ſol ex terra tanquam centro motus A moueri cernitur, dabit planum fixum BAY; et a quocunque alio corpore perturbatio efficiatur, id planum, in quo hoc corpus ex centro motus A moueri cernitur, erit eligendum. At ſi hoc corpus ipſum non in eodem plano moueatur, tum planum

num aliquod medium, commodiſſime aſſumetur; vix autem opus videtur, calculum ad iſtum caſum accommodari, quod quidem ſi vſus poſtulauerit facile praeſtabitur.

Pars Mechanica.

XIII.

Tab. II. Fig. 1. In mechanica tractatione tria corpora veniunt conſideranda. Primum eſt id, quod in centro motus A poſitum vim praecipuam exerit in corpus Z, cuius motum inueſtigamus, qualis ſpectatori in ipſo puncto A conſtituto apparet, huius igitur corporis in A ſiti maſſam vocemus $= A$.

Alterum corpus, a cuius actione motus corporis Z perturbatur, in ipſo plano fixo BAY vtcunque moueri aſſumimus, ita vt eius locus ad quoduis tempus aſſignari poſſit. Sit maſſa huius corporis $= B$, idque nunc quidem verſetur in S, ita vt ſit eius diſtantia a corpore centrali $AS = u$, et longitudo ſeu angulus $BAS = \theta$, vnde ex S in rectam fixam AB demiſſo perpendiculo SP fit $AP = u \cos. \theta$ et $PS = u \sin. \theta$.

Tertium corpus eſt id ipſum in Z, in cuius motum inquirimus, ſit eius maſſa $= C$, et vt ante poſuimus, diſtantia a centro motus $AZ = v$, vocatis ternis coordinatis orthogonalibus $AX = X$, $XY = Y$

$XY = Y$ et $YZ = Z$, quas vero ex calculo exuimus introducendo ſequentia elementa:

1°. longitudinem lineae nodorum $BAN = \psi$

2°. Inclinationem orbitae praeſentem $YOZ = \omega$

3°. argumentum latitudinis ſeu $NAZ = \sigma$.

Denique tempuſculo dt a corpore Z angulum elementarem $ZAz = d\Phi$ abſolui ſtatuimus. Horum autem elementorum relationes ex parte geometrica ſunt repetendae.

XIV.

Quoniam corpus Z ad A vrgetur vi $= \frac{A}{vv}$, contra vero A ad Z trahitur vi $= \frac{C}{vv}$, vt punctum A tanquam quieſcens conſiderari poſſit, corpus Z ad A attrahi cenſendum eſt vi $= \frac{A+C}{vv}$, quae ſecundum directiones ternarum coordinatarum reſoluta dat vires:

$$\text{ſec. } XA = \frac{A+C}{v^3}.X;\ \text{ſec. } YX = \frac{A+C}{v^3}.Y;\ \text{ſec. } ZY = \frac{A+C}{v^3}.Z.$$

Deinde pro vi, qua corpus Z ad S ſollicitatur, vocemus breuitatis gratia diſtantiam $SZ = w$; vt vis ZS ſit $= \frac{B}{ww}$, quae reſoluitur ſtatim in vires ſec. $ZY = \frac{B}{w^3}.Z$ et ſec. $YS = \frac{B}{w^3}.YS$, haec vero porro ob $XP = u \cos. \theta - X$ et $PS - XY = u \sin. \theta - Y$ in vires:

$$\text{ſec. } XP = \frac{B}{w^3}(u \cos. \theta - X) \text{ et ſec. } XY = \frac{B}{w^3}(u \sin. \theta - Y).$$

Deni-

Denique quia corpus A quoque ad S vrgetur vi $=\frac{B}{uu}$ haec contrarie in Z translata dabit insuper vires

$$\text{secundum } XA = \frac{B}{uu}\cos.\theta \text{ et secundum } YX = \frac{B}{uu}\sin.\theta$$

quae vires in corpus Z agentes collectae praebent:

1°. Vim sec. $XA = \frac{A+C}{v^3}X + \frac{B}{w^3}(X - u\cos.\theta) + \frac{B}{uu}\cos.\theta$

2°. Vim sec. $XY = \frac{A+C}{v^3}Y + \frac{B}{w^3}(Y - u\sin.\theta) + \frac{B}{uu}\sin.\theta$

3°. Vim sec. $ZY = \frac{A+C}{v^3}Z + \frac{B}{w^3}Z$,

quibus cum accelerationes corporis Z secundum easdem directiones sint proportionales, statuamus pro temporis elemento dt constanti sumto:

$$ddX = -\alpha dt^2\left(\frac{A+C}{v^3}X + \frac{B}{w^3}X - Bu\cos.\theta\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\right)$$

$$ddY = -\alpha dt^2\left(\frac{A+C}{v^3}Y + \frac{B}{w^3}Y - Bu\sin.\theta\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\right)$$

$$ddZ = -\alpha dt^2\left(\frac{A+C}{v^3}Z + \frac{B}{w^3}Z\right)$$

vbi constans α ex vnico motu huius generis cognito, veluti ex motu solis apparente, definiri potest.

XV.

Antequam autem has formulas euoluamus, distantiam $SZ = w$ in calculum de nouo introductam accuratius definire debemus. Cum autem manifesto sit:

$$SZ^2 = XZ^2 + XP^2 + (PS - XY)^2$$

habebimus:

$$ww = ZZ + XX + YY + uu - 2uX\cos.\theta - 2uY\sin.\theta$$

quae

quae forma ob $XX + YY + ZZ = vv$ reducitur ad hanc:

$$ww = vv + uu - 2u(X\cos.\theta + Y\sin.\theta).$$

Ex valoribus autem pro X et Y supra §. 4. inuentis colligimus:

$$X\cos.\theta + Y\sin.\theta = v(\cos.\sigma\cos.(\theta - \psi) + \sin.\sigma\cos.\omega\sin.(\theta - \psi))$$

vbi angulus $\theta - \psi$ exprimit distantiam corporis perturbantis S a linea nodorum seu angulum $NAS = \theta - \psi$, ita vt sit

$$ww = vv + uu - 2vu\cos.\sigma\cos.(\theta - \psi) + \sin.\sigma\cos.\omega\sin.(\theta - \psi))$$

Verum si iam breuitatis gratia vocemus ang. $SAZ = \mu$ quo distantia corporis Z a corpore perturbante S ex A visa designatur, ob $AZ = v$ et $AS = u$ constat fore

$$ww = vv + uu - 2vu\cos.\mu$$

vnde concluditur esse:

$$\cos.\sigma\cos.(\theta - \psi) + \sin.\sigma\cos.\omega\sin.(\theta - \psi) = \cos.\mu$$

id quod facillime per trigonometriam sphaericam probatur. Cum enim in fig. 2. sit $BN = \psi$, $NZ = \sigma$ et ang. $YNZ = \omega$ si capiatur $BS = \theta$, erit $NS = \theta - \psi$, et in triangulo sphaerico latus $SZ = \mu$ ex lateribus $NZ = \sigma$, $NS = \theta - \psi$ cum angulo intercepto $ZNS = \omega$ hoc ipso modo determinatur. Fig. 2.

XVI.

Cum tres aequationes ex principiis mechanicis deductae totidem determinationes suppeditent, totum artificium in hoc constat, quemadmodum eas inde commodissime deriuemus. Ac primo quidem statim se offert haec ratio, qua prima per $2dX$ secunda per $2dY$ et tertia per $2dZ$ multiplicatae in vnam summam colliguntur; quia enim vt supra vidimus est

$$dX^2+dY^2+dZ^2=dv^2+vvd\Phi^2$$

et $XX+YY+ZZ=vv$

erit $2dXddX+2dYddY+2dZddZ=d.(dv^2+vvd\Phi^2)$

et $XdX+YdY+ZdZ=vdv.$

Quare memorata ratione peruenietur ad hanc aequationem:

$$d.(dv^2+vvd\Phi^2)=-2\alpha dt^2\left(\tfrac{A+C}{vv}dv+\tfrac{B}{w^3}vdv-Bu(dX\text{cos}.\theta+dY\text{sin}.\theta)\left(\tfrac{1}{w^3}-\tfrac{1}{u^3}\right)\right)$$

vbi formulam $dX\text{cos}.\theta+dY\text{sin}.\theta$ euolui conuenit. Verum ex formulis supra §. 10. erutis colligimus:

$$dX=\tfrac{XdZ}{Z}-\tfrac{vd\Phi\text{cos}.\psi}{\text{sin}.\sigma} \text{ et } dY=\tfrac{YdZ}{Z}-\tfrac{vd\Phi\text{sin}.\psi}{\text{sin}.\sigma}$$

ob $Z=v\,\text{sin}.\sigma\,\text{sin}.\omega$ vnde fit

$$dX\text{cos}.\theta+dY\text{sin}.\theta=\tfrac{dZ}{Z}(X\text{cos}.\theta+Y\text{sin}.\theta)-\tfrac{vd\Phi\text{cos}.(\theta-\psi)}{\text{sin}.\sigma}$$

modo autem vidimus esse

$$X\text{cos}.\theta+Y\text{sin}.\theta=v(\text{cos}.\sigma\text{cos}.(\theta-\psi)+\text{sin}.\sigma\text{cos}.\omega\,\text{sin}.(\theta-\Phi))=v\,\text{cos}.\mu$$

tum

tum vero est $\frac{dZ}{Z}=\frac{dv}{v}+\frac{d\sigma\,\text{cos}.\sigma}{\text{sin}.\sigma}+\frac{d\omega\,\text{cos}.\omega}{\text{sin}.\omega}$, vel

$$\tfrac{dZ}{Z}=\tfrac{dv}{v}+\tfrac{d\sigma\,\text{cos}.\sigma}{\text{sin}.\sigma}+\tfrac{d\psi\text{cos}.\omega}{\text{tang}.\sigma} \text{ ob } d\omega=\tfrac{d\psi\,\text{sin}.\omega}{\text{tang}.\sigma}.$$

Cum igitur sit $d\Phi=d\sigma+d\psi\,\text{cos}.\omega$, adipiscimur

$$\tfrac{dZ}{Z}=\tfrac{dv}{v}+\tfrac{d\Phi}{\text{tang}.\sigma}, \text{ ita vt sit}$$

$$dX\text{cos}.\theta+dY\text{sin}.\theta=dv\,\text{cos}.\mu+\tfrac{vd\Phi\text{cos}.\mu}{\text{tang}.\sigma}-\tfrac{vd\Phi\text{cos}.(\theta-\psi)}{\text{sin}.\sigma}=$$
$$dv\text{cos}.\mu+\tfrac{vd\Phi}{\text{sin}.\sigma}(\text{sin}.\sigma\text{cos}.\sigma\text{cos}.\omega\text{sin}.(\theta-\psi)+\text{cos}.\sigma^2\text{cos}.(\theta-\psi)-\text{cos}.(\theta-\psi))$$

ideoque

$$dX\text{cos}.\theta+dY\text{sin}.\theta=dv\,\text{cos}.\mu-vd\Phi(\text{sin}.\sigma\,\text{cos}.(\theta-\psi)-\text{cos}.\sigma\,\text{cos}.\omega\,\text{sin}.(\theta-\psi)).$$

Quocirca aequatio nostra inuenta erit: $d.(dv^2+vvd\Phi^2)$

$$=-2\alpha dt^2dv\left(\tfrac{A+C}{vv}+\tfrac{Bv}{w^3}-Bu\,\text{cos}.\mu\left(\tfrac{1}{w^3}-\tfrac{1}{u^3}\right)\right)$$
$$-2\alpha Bdt^2d\Phi.\,uv(\text{sin}.\sigma\,\text{cos}.(\theta-\psi)-\text{cos}.\sigma\,\text{cos}.\omega\,\text{sin}.(\theta-\psi))\left(\tfrac{1}{w^3}-\tfrac{1}{u^3}\right).$$

XVII.

Hanc euolutionem formulae $dX\text{cos}.\theta+dY\text{sin}.\theta$ nimis prolixam multo concinnius ex ipsis valoribus pro X et Y inuentis conficere licet. Cum enim eorum differentialia rite prodeant, si anguli ψ et ω vt constantes tractentur et pro $d\sigma$ scribatur $d\Phi$ haec differentiatio praebet:

$$dX=dv(\text{cos}.\sigma\text{cos}.\psi-\text{sin}.\sigma\text{cos}.\omega\text{sin}.\psi)-vd\Phi(\text{sin}.\sigma\text{cos}.\psi+\text{cos}.\sigma\text{cos}.\omega\text{sin}.\psi)$$

$$dY=dv(\text{cos}.\sigma\text{sin}.\psi+\text{sin}.\sigma\text{cos}.\omega\text{cos}.\psi)-vd\Phi(\text{sin}.\sigma\text{sin}.\psi-\text{cos}.\sigma\text{cos}.\omega\,\text{cos}.\psi)$$

vnde

vnde statim colligitur:

$$dX\cos.\theta + dY\sin.\theta = dv(\cos.\sigma\cos.(\theta-\psi) + \sin.\sigma\cos.\omega\sin(\theta-\psi)) - vd\Phi(\sin.\sigma\cos.(\theta-\psi) - \cos.\sigma\cos.\omega\sin(\theta-\psi))$$

ad quam formam magis contrahendam obseruo in
Fig. 2. fig. 2. vbi $NS = \theta-\psi$; $NZ = \sigma$ $SNZ = \omega$ et $SZ = \mu$ fore primo vt supra $\cos.\sigma\cos.(\theta-\psi) + \sin.\sigma\cos.\omega\sin.(\theta-\psi) = \cos.\mu$, deinde vero si ponatur angulus $NZS = \xi$, reperiri

$$\cot.\xi = \frac{\sin.\sigma\cos.(\theta-\psi) - \cos.\sigma\cos.\omega\sin.(\theta-\psi)}{\sin.\omega\sin.(\theta-\psi)}$$

vnde concluditur:

$$\sin.\sigma\cos.(\theta-\psi) - \cos.\sigma\cos.\omega\sin.(\theta-\psi) = \frac{\sin.\omega\sin.(\theta-\psi)\cos.\xi}{\sin.\xi} = \sin.\mu\cos.\xi$$

ob $\sin.\xi : \sin.(\theta-\psi) = \sin.\omega : \sin.\mu$. Ex his ergo impetramus

$$dX\cos.\theta + dY\sin.\theta = dv\cos.\mu - vd\Phi\sin.\mu\cos.\xi.$$

Vel si in Z ad arcum NZ alium arcum normalem ducamus, in eumque ex S perpendiculum in superficie sphaerica demittamus, quod vocemus $= \nu$ erit $\sin.\nu = \sin.\mu\cos.\xi$ seu

$$\sin.\sigma\cos.(\theta-\psi) - \cos.\sigma\cos.\omega\sin.(\theta-\psi) = \sin.\nu$$

aequatio primam determinationem continens ita se habebit

$$d.(dv^2 + vvd\Phi^2) = -2\alpha dt^2\left(\frac{A+C}{vv}dv + \frac{Bvdv}{w^3} - Bu(dv\cos.\mu - vd\Phi\sin.\nu)\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\right).$$

XVIII.

XVIII.

Binas reliquas determinationes, quas aequationes differentio-differentiales ex principiis motus deductae suppeditant, commodissime per sequentes combinationes obtinebimus: Primo ergo harum aequationum §. 14 inuentarum prima per Y multiplicata a secunda per X multiplicata subtrahatur vt prodeat:

$$XddY - YddX = -\alpha dt^2 Bu(Y\cos.\theta - X\sin.\theta)\left(\frac{1}{w^3} - \frac{1}{u^3}\right)$$ seu valoribus pro x et y substitutis

$$XddY - YddX = \alpha Bvudt^2(\cos.\sigma\sin(\theta-\psi) - \sin.\sigma\cos.\omega\cos.(\theta-\psi))\left(\frac{1}{w^3} - \frac{1}{u^3}\right).$$

Cum igitur sit $XddY - YddX$ differentiale ipsius $XdY - YdX$ hanc habebimus aequationem:

$$d.vvd\Phi\cos.\omega = \alpha Bvudt^2(\cos.\sigma\sin.(\theta-\psi) - \sin.\sigma\cos.\omega\cos.(\theta-\psi))\left(\frac{1}{w^3} - \frac{1}{u^3}\right).$$

Simili modo ex prima ac tertia colligimus

$$XddZ - ZddX = -\alpha BuZdt^2\cos.\theta\left(\frac{1}{w^3} - \frac{1}{u^3}\right)$$ seu

$$XddZ - ZddX = -\alpha Bvudt^2\cos.\theta\sin.\sigma\sin.\omega\left(\frac{1}{w^3} - \frac{1}{u^3}\right)$$

sicque habebitur:

$$d.vvd\Phi\sin.\omega\cos.\psi = -\alpha Bvudt^2\cos.\theta\sin.\sigma\sin\omega\left(\frac{1}{w^3} - \frac{1}{u^3}\right).$$

Pari modo secunda aequatio cum tertia coniuncta dat:

$$YddZ - ZddY = -\alpha Bvudt^2\sin.\theta\sin\sigma\sin.\omega\left(\frac{1}{w^3} - \frac{1}{u^3}\right)$$ seu

$$d.vvd\Phi\sin.\omega\sin.\psi = -\alpha Bvudt^2\sin.\theta\sin.\sigma\sin.\omega\left(\frac{1}{w^3} - \frac{1}{u^3}\right)$$

 verum

verum probe eſt notandum, in his tribus aequationibus tantum duas determinationes contineri, cum in binis tertia iam ſponte includatur.

XIX.

Binas poſtremas ita tractemus, vt cum membra priora differentiari debeant, pars communis $vvd\Phi \,\text{ſin.}\,\omega$ tanquam vnica quantitas ſpectetur, ſicque fiet:

$$\text{coſ.}\,\psi\, d.vvd\Phi\,\text{ſin.}\,\omega - d\psi\,\text{ſin.}\,\psi.\, vvd\Phi\,\text{ſin.}\,\omega = -\alpha B v u dt^2 \,\text{coſ.}\,\theta\,\text{ſin.}\,\sigma\,\text{ſin.}\,\omega\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$$

$$\text{ſin.}\,\psi\, d.vvd\Phi\,\text{ſin.}\,\omega + d\psi\,\text{coſ.}\,\psi.\, vvd\Phi\,\text{ſin.}\,\omega = -\alpha B v u dt^2 \,\text{ſin.}\,\theta\,\text{ſin.}\,\sigma\,\text{ſin.}\,\omega\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$$

vnde d. $vvd\Phi\,\text{ſin.}\,\omega$ eliminando colligitur

$$d\psi.\, vvd\Phi\,\text{ſin.}\,\omega = -\alpha B v u dt^2\,\text{ſin.}\,\sigma\,\text{ſin.}\,\omega\,\text{ſin.}\,(\theta-\psi)\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$$

ſicque variatio lineae nodorum ita definitur vt ſit

$$d\psi = -\frac{-\alpha B u dt^2\,\text{ſin.}\,\sigma\,\text{ſin.}\,(\theta-\psi)}{v d\Phi}\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$$

vnde ſimul variatio inclinationis innoteſcit ob $\frac{d\omega}{\text{ſin.}\,\omega} = \frac{d\psi}{\text{tang.}\,\sigma}$ ſin autem ex illis binis formis membrum $vvd\Phi\,\text{ſin.}\,\omega$ eliminetur, obtinetur

$$d.vvd\Phi\,\text{ſin.}\,\omega = -\alpha B v u dt^2\,\text{ſin.}\,\sigma\,\text{ſin.}\,\omega\,\text{coſ.}\,(\theta-\psi)\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$$

quae iam cum prima exuendo angulum ω comparata praebet

$$\text{coſ.}\,\omega\, d.vvd\Phi - d\omega\,\text{ſin.}\,\omega vvd\Phi = \alpha B v u dt^2(\text{coſ.}\,\sigma\,\text{ſin.}\,(\theta-\psi) - \text{ſin.}\,\sigma\,\text{coſ.}\,\omega\,\text{coſ.}\,(\theta-\psi))\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$$

ſin.

$$\text{ſin.}\,\omega\, d.vvd\Phi + d\omega\,\text{coſ.}\,\omega.\, vvd\Phi = -\alpha B v u dt^2(\text{ſin.}\,\sigma\,\text{ſin.}\,\omega\,\text{coſ.}\,(\theta-\psi))\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$$

vnde concludimus:

$$d.vvd\Phi = -\alpha B v u dt^2(\text{ſin.}\,\sigma\,\text{coſ.}\,(\theta-\psi) - \text{coſ}\,\sigma\,\text{coſ.}\,\omega\,\text{ſin.}\,(\theta-\psi))\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$$

ſeu $d.\,vvd\Phi = -\alpha B v u dt^2\,\text{ſin.}\,\nu\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$

quae eſt altera determinatio, quam quaeri oportebat.

XX.

Poſtremam hanc aequationem multiplicemus per $2vvd\Phi$ et, integrali ſaltem indicato, fiet

$$v^4 d\Phi^2 = -2\alpha B dt^2 \int v^3 u d\Phi\,\text{ſin.}\,\nu\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$$

qua aequatione relatio inter angulum elementarem $d\Phi$ et tempuſculum dt continetur, vbi quidem manifeſtum eſt, ſi maſſa corporis perturbantis B euaneſceret, futurum eſſe $v^2 d\Phi$ tempori dt proportionale, ſeu areas circa A deſcriptas tempori proportionales. Ad hanc aequationem ſi adiungatur primo §. 17. inuenta, pariter integrata quatenus fieri poteſt erit

$$dv^2 + vvd\Phi^2 = 2\alpha dt^2(A+C)\left(\frac{1}{v}-\frac{1}{f}\right) - 2\alpha B dt^2 \int \frac{v dv}{w^3} + 2\alpha B dt^2 \int u(dv\,\text{coſ.}\,\mu - v d\Phi\,\text{ſin.}\,\nu)\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$$

quae aequatio inſuper variationem diſtantiae v cum elemento $d\Phi$ vel tempuſculo dt comparat, quae duae res proprie ad motum corporis Z in ſua orbita

bita ſpectant. Praeterea vero pro ipſius orbitae variatione, habemus:

$$d\psi = \frac{-\alpha B u\, dt^2\, \text{ſin.}\,\sigma\, \text{ſin.}\,(\theta-\psi)}{v\, d\Phi}\left(\frac{1}{w^3} - \frac{1}{u^3}\right)$$

$$\frac{d\omega}{\text{ſin.}\,\omega} = \frac{-\alpha B u\, dt^2\, \text{coſ.}\,\sigma\, \text{ſin.}\,(\theta-\psi)}{v\, d\Phi}\left(\frac{1}{w^3} - \frac{1}{u^3}\right) = \frac{d\psi}{\text{tang.}\,\sigma}.$$

Ac denique argumentum latitudinis σ ad eadem elementa reuocatur ope huius aequationis $d\sigma = d\Phi - d\psi\, \text{coſ.}\,\omega$.

XXI.

Elementum temporis dt cum quantitate conſtante α commodiſſime ex calculo tolletur, ſi motus quidam regularis et cognitus introducatur, veluti motus medius ſolis, vel alius corporis, quod circa centrum virium in circulo vniformiter reuoluitur. Ponamus ergo circa corpus in A poſitum cuius maſſa ſit $= \mathfrak{A}$ aliud corpus, cuius maſſa $= \mathfrak{C}$ ad diſtantiam $= a$ in circulo ita circumferri vt tempore t angulum ipſi proportionalem τ abſoluat, atque noſtrae formulae ad hunc caſum accommodabuntur ſtatuendo $A = \mathfrak{A}$, $C = \mathfrak{C}$ et $B = 0$, ita vt tum fiat $v = a$ et $d\Phi = d\tau$. Motus igitur, quem cognitum aſſumimus, his duabus aequationibus continetur:

$$v^4 d\Phi^2 = 2\alpha D dt^2 \text{ et } dv^2 + vv\, d\Phi^2 = 2\alpha dt^2 (\mathfrak{A}+\mathfrak{C})\left(\frac{1}{v} - \frac{2}{f}\right)$$

vbi primum conſtantes D et f huic caſui conuenienter definiri oportet. Hunc in finem ex priori valor $2\alpha dt^2 = \frac{v^4 d\Phi^2}{D}$ in altera ſubſtitutus dat: $dv^2 + vv\, d\Phi^2 = \frac{(\mathfrak{A}+\mathfrak{C}) v^4 d\Phi^2}{D}\left(\frac{1}{v} - \frac{2}{f}\right)$

ſeu

ſeu $D f dv^2 + D f vv\, d\Phi^2 = (\mathfrak{A}+\mathfrak{C}) v^3 d\Phi^2 (f - v)$ vnde colligitur

$$d\Phi = \frac{dv \sqrt{Df}}{v\sqrt{((\mathfrak{A}+\mathfrak{C}) v (f-v) - Df)}}$$

cui aequationi differentiali ſatis fit tribuendo ipſi v eiusmodi valorem conſtantem, quo denominator euaneſcat, verum alio loco oſtendi, hanc ſolutionem in integrali admitti non poſſe, niſi iſte denominationis factor euaneſcens ad minimum vnius ſit dimenſionis, vnde neceſſe eſt, vt poſt ſignum radicale idem factor occurrat geminatus ſeu quadratus, vel quod eodem redit, vt etiam differentiale quantitatis poſt ſignum poſitae eundem inuoluat factorem. Poſito ergo hoc differentiali $= 0$, fit $v = \frac{1}{2} f$ quare cum per hypotheſin eſſe debeat $v = a$, erit $f = 2a$, quo caſu ipſe denominator fit $(\mathfrak{A}+\mathfrak{C}) aa - 2 D a$ nihilo aequandus, ita vt ſit $D = \frac{1}{2}(\mathfrak{A}+\mathfrak{C}) a$. Iam in alterutra aequatione ſtatuatur $v = a$ et $d\Phi = d\tau$, eritque $a^4 d\tau^2 = \alpha(\mathfrak{A}+\mathfrak{C}) a dt^2$ ſeu $\alpha dt^2 = \frac{a^3 d\tau^2}{\mathfrak{A}+\mathfrak{C}}$.

XXII

Cum igitur ob motum iſtum cognitum ad datum quoduis tempus t innoteſcat motus medius τ, hic loco temporis in noſtrum calculum introducetur, ſi modo vbique loco αdt^2 ſcribatur valor modo inuentus $\frac{a^3 d\tau^2}{\mathfrak{A}+\mathfrak{C}}$. Statuamus ergo ad noſtras formulas ſimpliciores reddendas primo $\frac{A+C}{\mathfrak{A}+\mathfrak{C}} = m$, deinde $B = n(A+C)$, vt fiat $\alpha dt^2 (A+C) = m a^3 d\tau^2$ et $\alpha B dt^2 = m n a^3 d\tau^2$ vbi notandum eſt, perturba-

tiones

tiones fore minimas, ſi termini numero n affecti fuerint minimi. Noſtrae ergo aequationes ſequentes induent formas:

$$1^\circ.\ v^4 d\Phi^2 = -2mna^3 d\tau^2 \int v^3 u\, d\Phi \sin.\nu\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right)$$

$$2^\circ.\ dv^2 + vv\,d\Phi^2 = 2ma^3 d\tau^2\left(\tfrac{1}{v} - \tfrac{1}{f}\right) - 2mna^3 d\tau^2 \int \tfrac{v\,dv}{w^3}$$
$$+ 2mna^3 d\tau^2 \int u(dv \cos.\mu - v\,d\Phi \sin.\nu)\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right)$$

$$3^\circ.\ d\psi = -mna^3 d\tau^2 . \tfrac{u \sin.\sigma \sin.(\theta-\psi)}{v\,d\Phi}\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right)$$

$$4^\circ.\ \tfrac{d\omega}{\sin.\omega} = -mna^3 d\tau^2 . \tfrac{u \cos.\sigma \sin.(\theta-\psi)}{v\,d\Phi}\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right) = \tfrac{d\psi}{\tang.\sigma}$$

$$5^\circ.\ d\sigma = d\Phi + mna^3 d\tau^2 \cos.\omega . \tfrac{u \sin.\sigma \sin.(\theta-\psi)}{v\,d\Phi}\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right) = d\Phi - d\psi \cos.\omega$$

Hisque aequationibus totus corporis Z motus cum omnibus perturbationibus ab actione corporis S oriundis determinatur: vbi imprimis obſeruetur formulas integrales, quibus binae priores aequationes ſunt affectae, ad perturbationes tantum pertinere ideoque ſufficere ſi earum valores proxime veri colligantur, ex quo his integrationibus negotium approximationis vix impediri eſt cenſendum. Mox autem methodum exponam calculum adeo ab his integralibus liberandi.

XXIII.

Statuamus tantiſper ad abbreuiandum:

$$\int v^3 u\, d\Phi \sin.\nu\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right) = \mathrm{P}$$
$$\int \tfrac{v\,dv}{w^3} = \mathrm{Q}$$
$$\int u(dv \cos.\mu - v\,d\Phi \sin.\nu)\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right) = \mathrm{R}$$

vt binae aequationes priores contrahantur in has formas:

$$1^\circ.\ v^4 d\Phi^2 = 2ma^3 d\tau^2(\mathrm{D} - n\mathrm{P})$$
$$2^\circ.\ dv^2 + v^2 d\Phi^2 = 2ma^3 d\tau^2\left(\tfrac{1}{v} - \tfrac{1}{f} - n(\mathrm{Q}-\mathrm{R})\right)$$

quae ſolae totum negotium conficerent, ſi corpus Z in eodem plano moueretur, in quo corpus perturbans S circumferri aſſumimus; reliquae aequationes ad motum, vt dicitur, latitudinis pertinent, earumque reſolutio multo minoribus laborat difficultatibus, vnde omne ſtudium in binis prioribus eſt conſumendum. Inde autem eliminato elemento $d\tau$ haec naſcitur aequatio

$$(\mathrm{D} - n\mathrm{P})(dv^2 + vv\,d\Phi^2) = v^4 d\Phi^2\left(\tfrac{1}{v} - \tfrac{1}{f}\right) - n(\mathrm{Q}-\mathrm{R})$$

vnde elicitur:

$$d\Phi = \frac{dv \surd(\mathrm{D} - n\mathrm{P})}{v \surd\left(v - \frac{vv}{f} - nvv(\mathrm{Q}-\mathrm{R}) - \mathrm{D} + n\mathrm{P}\right)}$$

hincque porro $2ma^3 d\tau^2 = \dfrac{vv\,dv^2}{v - \frac{vv}{f} - nvv(\mathrm{Q}-\mathrm{R}) - \mathrm{D} + n\mathrm{P}}$

ſeu $a\,d\tau \surd 2ma = \dfrac{v\,dv}{\surd\left(-\mathrm{D} + n\mathrm{P} + v - vv\left(\frac{1}{f} + n\mathrm{Q} - n\mathrm{R}\right)\right)}$

quarum formularum integratio foret in promtu, ſi fractio n vel perturbationes in nihilum abirent.

XXIV.

Aequationem illam hac forma repraeſentemus:

$$\tfrac{dv}{vv}\surd(\mathrm{D} - n\mathrm{P}) = d\Phi \surd\left(-\tfrac{1}{f} + n(\mathrm{R}-\mathrm{Q}) + \tfrac{1}{v} - \tfrac{\mathrm{D} + n\mathrm{P}}{vv}\right)$$

ex qua discimus distantiam $AZ = v$ tum fieri maximam vel minimam, quando quantitas posteriori signo radicali inuoluta euanescit. Haec autem loca non solum in Astronomia maximi sunt momenti, quia tum corpus Z in absidibus versari dicitur, sed etiam inde eiusmodi egregia subsidia petere licet, quibus motus perturbatus admodum concinne cum motu regulari comparari, eiusque aberrationes ab eo assignari queant. Commodissime hoc praestabitur introducendo in calculum nouum angulum ω, qui in Astronomia anomalia vera appellatur, et ita est comparatus, vt eo vel euanescente vel ad duos rectos excrescente distantia v fiat vel minima vel maxima. Quo igitur motus propius ad similitudinem motus regularis in ellipsi facti reducatur, statuamus $v = \frac{p}{1 + q\cos.\omega}$, ita vt nunc motus conformis sit motui regulari in eiusmodi ellipsi, cuius semiparameter sit $= p$, excentricitas $= q$, ideoque semi-axis transuersus $= \frac{p}{1 - qq}$, anomalia vera seu angulo ab axe existente $= \omega$. Facile autem perspicitur ob perturbationes hanc ellipsis speciem continuo mutari, vnde non solum anomalia ω sed etiam litterae p et q vt variabiles sunt spectandae, quarum variationes iam sum inuestigaturus.

XXV.

Quo haec inuestigatio facilior reddatur, ponamus breuitatis gratia:

$$\tfrac{1}{2} - n(R - Q) = M \text{ et } D - nP = N$$

vt

vt habeamus hanc formam euoluendam:

$$\frac{dv}{vv}\sqrt{N} = d\Phi\sqrt{\left(-M + \frac{1}{v} - \frac{N}{vv}\right)}.$$

Quoniam igitur nunc ponimus $v = \frac{p}{1 + q\cos.\omega}$ seu $\frac{1}{v} = \frac{1 + q\cos.\omega}{p}$ per hypothesin tam casu $\omega = 0$, quo fit $\frac{1}{v} = \frac{1+q}{p}$, quam casu $\omega = 180°$, quo casu fit $\frac{1}{v} = \frac{1-q}{p}$ quantitas $-M + \frac{1}{v} - \frac{N}{vv}$ in nihilum abire debet, ex quo hae duae nascuntur aequationes:

$$-M + \frac{1+q}{p} - \frac{N(1+q)^2}{pp} = 0 \text{ et}$$

$$-M + \frac{1-q}{p} - \frac{N(1-q)^2}{pp} = 0$$

quarum differentia dat $\frac{2q}{p} - \frac{4Nq}{pp} = 0$, ita vt sit $p = 2N$, seu $N = \frac{1}{2}p$, vnde fit $M = \frac{1+q}{p} - \frac{(1+q)^2}{2p} = \frac{1-qq}{2p}$.

Quodsi ergo nostrae ellipsis semiaxis transuersus ponatur $= r$ vt sit $r = \frac{p}{1-qq}$, erit $M = \frac{1}{2r}$, ideoque:

$$\tfrac{1}{2} - n(R - Q) = \frac{1-qq}{2p} = \frac{1}{2r} \text{ et } D - nP = \tfrac{1}{2}p.$$

XXVI.

His valoribus in nostra aequatione substitutis habebimus:

$$\frac{dv}{vv}\sqrt{\tfrac{1}{2}p} = d\Phi\sqrt{\left(-\frac{1-qq}{2p} + \frac{1}{v} - \frac{p}{2vv}\right)}$$

in cuius posteriori membro primum pro $\frac{1}{v}$ valorem $\frac{1+q\cos.\omega}{p}$ scribamus, vt fiat:

$$\frac{dv}{vv}\sqrt{\tfrac{1}{2}p} = d\Phi\sqrt{\left(\frac{-1+qq}{2p} + \frac{1+q\cos.\omega}{p} - \frac{1+2q\cos.\omega+qq\cos.\omega^2}{2p}\right)}$$ ideoque

$$\frac{dv}{vv}\sqrt{\tfrac{1}{2}p} = d\Phi\sqrt{\frac{qq-qq\cos.\omega^2}{2p}} = \frac{qd\Phi\sin.\omega}{\sqrt{2p}}$$

ita

ita vt ſit $\frac{dv}{vv} = \frac{q\,d\Phi}{p}\,\text{ſin.}\,\vartheta$; vnde vtique quod nobis erat propoſitum, agnoſcimus, ſcilicet quoties anomaliae ϑ ſinus euaneſcit, ſimul diſtantiae v differentiale in nihilum abire, eamque propterea vel maximam vel minimam euadere. Tum vero hinc in genere incrementum diſtantiae v ad elementum $d\Phi$ reducitur, quod ipſum iam cum elemento cognito $d\tau$ ita comparatur, vt ob $D - nP = \frac{1}{2}p$ ſit

$$v^4 d\Phi^2 = m a^3 p\, d\tau^2,\ \text{ſeu}\ \ vv\,d\Phi = a\,d\tau \sqrt{map}.$$

Cum autem ſit $\frac{1}{v} = \frac{1 + q\,\text{coſ.}\,\vartheta}{p}$ erit

$$\frac{dv}{vv} = \frac{dp(1 + q\,\text{coſ.}\,\vartheta)}{pp} - \frac{dq\,\text{coſ.}\,\vartheta + q\,d\vartheta\,\text{ſin.}\,\vartheta}{p}$$

quae forma ipſi $\frac{q\,d\Phi}{p}\,\text{ſin.}\,\vartheta$ aequalis facta, praebet

$$q(d\Phi - d\vartheta)\,\text{ſin.}\,\vartheta = \frac{dp}{p}(1 + q\,\text{coſ.}\,\vartheta) - dq\,\text{coſ.}\,\vartheta = \frac{dp}{v} - dq\,\text{coſ.}\,\vartheta$$

qua noua differentialium relatio continetur.

XXVII.

Reliquas determinationes peti oportet ex formulis ſupra inuentis:

$$p = 2D - 2nP \ \text{et}\ \frac{1}{r} = \frac{1 - qq}{p} = \frac{2}{f} - 2n(R - Q)$$

quae differentiatae et loco P, Q, R valores ſupra exhibitos reſtituendo ſuppeditant,

$$dp = -2n\,dP = -2nv^3 u\,d\Phi\,\text{ſin.}\,\nu\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right)$$

$$d.\frac{1}{r} = d.\frac{1 - qq}{p} = 2n\,dQ - 2n\,dR = \frac{2n\,v\,dv}{w^3} - 2nu(dv\,\text{coſ.}\,\mu - v\,d\Phi\,\text{ſin.}\,\nu)\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right).$$

Cum

Cum autem ſit $dv = \frac{qvv\,d\Phi\,\text{ſin.}\,\vartheta}{p} = \frac{qv\,d\Phi\,\text{ſin.}\,\vartheta}{1 + q\,\text{coſ.}\,\vartheta}$ etiam hoc poſterius differentiale ad elementum $d\Phi$ reducitur fietque:

$$d\frac{1}{r} = d.\frac{1 - qq}{p} = \frac{2nqv^3 d\Phi\,\text{ſin.}\,\vartheta}{pw^3} - 2nvu\,d\Phi\left(\tfrac{q\,\text{coſ.}\,\mu\,\text{ſin.}\,\vartheta}{1 + q\,\text{coſ.}\,\vartheta} - \text{ſin.}\,\nu\right)\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right).$$

Eſt vero $d.\frac{1 - qq}{p} = \frac{-dp}{pp}(1 - qq) - \frac{2q\,dq}{p}$, ideoque

$$q\,dq = -\frac{dp}{2p}(1 - qq) - \frac{p}{2}d.\frac{1 - qq}{p} \ \text{vnde colligitur}$$

$$q\,dq = +\frac{n(1 - qq)}{p}v^3 u\,d\Phi\,\text{ſin.}\,\nu\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right) - \frac{nqv^3 d\Phi\,\text{ſin.}\,\vartheta}{w^3} + npvu\,d\Phi\left(\tfrac{q\,\text{coſ.}\,\mu\,\text{ſin.}\,\vartheta}{1 + q\,\text{coſ.}\,\vartheta} - \text{ſin.}\,\nu\right)\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right)$$

quae ob $v = \frac{p}{1 + q\,\text{coſ.}\,\vartheta}$ contrahitur in hanc

$$q\,dq = nv^2 u\,d\Phi\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right)\left(q\,\text{coſ.}\,\mu\,\text{ſin.}\,\vartheta - \frac{q\,\text{ſin.}\,\nu(q + 2\,\text{coſ.}\,\vartheta + q\,\text{coſ.}\,\vartheta^2)}{1 + q\,\text{coſ.}\,\vartheta}\right) - \frac{nqv^3 d\Phi\,\text{ſin.}\,\vartheta}{w^3}$$

quae per q diuiſa dat

$$dq = nv^2 u\,d\Phi\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right)\left(\text{coſ.}\,\mu\,\text{ſin.}\,\vartheta - \frac{\text{ſin.}\,\nu(q + 2\,\text{coſ.}\,\vartheta + q\,\text{coſ.}\,\vartheta^2)}{1 + q\,\text{coſ.}\,\vartheta}\right) - \frac{nv^3 d\Phi\,\text{ſin.}\,\vartheta}{w^3}.$$

Denique his valoribus in formula $q(d\Phi - d\vartheta)\,\text{ſin.}\,\vartheta = \frac{dp}{v} - dq\,\text{coſ.}\,\vartheta$ ſubſtitutis obtinebitur per ſin. ϑ diuiſione facta

$$q(d\Phi - d\vartheta) = \frac{nv^3 d\Phi\,\text{coſ.}\,\vartheta}{w^3} - nv^2 u\,d\Phi\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right)\left(\text{coſ.}\,\mu\,\text{coſ.}\,\vartheta + \frac{\text{ſin.}\,\nu\,\text{ſin.}\,\vartheta(2 + q\,\text{coſ.}\,\vartheta)}{1 + q\,\text{coſ.}\,\vartheta}\right).$$

XXVIII.

Nunc igitur omnium quantitatum, quae in noſtrum calculum ingrediuntur, incrementa momentanea ad idem elementum $d\Phi$, quod eodem tempuſculo

pusculo dt, quo ſecundum motum medium hic introductum angulus $d\tau$ abſoluitur, reduximus, vnde pro-quouis tempore minimo illa incrementa facile aſſignari poterunt. Primo igitur relatio inter angulum elementarem $d\Phi$ et $d\tau$ hac formula exprimitur:

$$vv\,d\Phi = a\,d\tau\sqrt{map} \text{ vnde fit } ma^2 d\tau^2 = \tfrac{1}{p}v^4 d\Phi^2$$

ſecundo ſi ſtatuamus $v = \frac{p}{1+q\,\text{coſ.}\,\vartheta}$ et $r = \frac{p}{1-qq}$ erit

$$1^\circ.\ dp = -2nv^3u\,d\Phi\,\text{ſin.}\,\nu\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right)$$

$$2^\circ.\ d.\tfrac{1}{r} = \frac{2nqv^3 d\Phi\,\text{ſin.}\,\vartheta}{p\,w^3} - \frac{2nv^2u\,d\Phi}{p}\left(q\,\text{coſ.}\,\mu\,\text{ſin.}\,\vartheta - (1+q\,\text{coſ.}\,\vartheta)\,\text{ſin.}\,\nu\right)\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right)$$

$$3^\circ.\ dq = nv^2u\,d\Phi\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right)\left(\text{coſ.}\,\mu\,\text{ſin.}\,\vartheta - \frac{(q+2\,\text{coſ.}\,\vartheta+q\,\text{coſ.}\,\vartheta^2)\,\text{ſin.}\,\nu}{1+q\,\text{coſ.}\,\vartheta}\right) - \frac{nv^3 d\Phi\,\text{ſin.}\,\vartheta}{w^3}$$

$$4^\circ.\ d\vartheta = d\Phi - \frac{nv^3 d\Phi\,\text{coſ.}\,\vartheta}{q\,w^3} + \frac{nv^2u\,d\Phi}{q}\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right)\left(\text{coſ.}\,\mu\,\text{coſ.}\,\vartheta + \frac{(2+q\,\text{coſ.}\,\vartheta)\,\text{ſin.}\,\nu\,\text{ſin.}\,\vartheta}{1+q\,\text{coſ.}\,\vartheta}\right)$$

exiſtente

$$\text{coſ.}\,\mu = \text{coſ.}\,\sigma\,\text{coſ.}(\theta-\psi) + \text{ſin.}\,\sigma\,\text{coſ.}\,\omega\,\text{ſin.}(\theta-\psi)$$

et $\text{ſin.}\,\nu = \text{ſin.}\,\sigma\,\text{coſ.}(\theta-\psi) - \text{coſ.}\,\sigma\,\text{coſ.}\,\omega\,\text{ſin.}(\theta-\psi)$
vbi notetur $d\Phi - d\vartheta$ deſignare promotionem momentaneam lineae abſidum in ipſa orbita.
Tertio pro motu in latitudinem has habebimus formulas:

$$1^\circ.\ d\psi = -\frac{nv^3u\,d\Phi\,\text{ſin.}\,\sigma\,\text{ſin.}(\theta-\psi)}{p}\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right)$$

$$2^\circ.\ \frac{d\omega}{\text{ſin.}\,\omega} = -\frac{nv^3u\,d\Phi\,\text{coſ.}\,\sigma\,\text{ſin.}(\theta-\psi)}{p}\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right) = \frac{d\psi}{\text{tan.}\,\sigma}$$

$$3^\circ.\ d\sigma = d\Phi + \frac{nv^3u\,d\Phi\,\text{coſ.}\,\omega\,\text{ſin.}\,\sigma\,\text{ſin.}(\theta-\psi)}{p}\left(\tfrac{1}{w^3} - \tfrac{1}{u^3}\right)$$

ſeu $d\sigma = d\Phi - d\psi\,\text{coſ.}\,\omega$.

XXIX.

XXIX.

Si integratio harum formularum abſolui poſſet, nihil amplius in hac inueſtigatione eſſet deſiderandum, cum exinde omnis generis perturbationes, quantumuis fuerint magnae, definire liceret. Cum autem vires Analyſeos nondum eousque increuerint, ad approximationes confugere conuenit, quae quidem eo feliciori ſucceſſu ſuſcipi poterunt, quo minores fuerint perturbationes: quia enim tunc valores quantitatum p et q pauxillulum mutantur, eas in integratione formularum littera n affectarum ſine errore tanquam conſtantes ſpectare licet, quia etiam poſtmodum methodis vſitatis correctiones neceſſariae haud difficulter elicientur. Interim tamen ſi excentricitas q fuerit valde magna, difficultates occurrunt, quas tamen certis artificiis adhibendis ſuperare licebit, quae quidem res optime ſuccedit, ſi excentricitas q parum ab vnitate discrepat, vti fit in orbitis fere parabolicis cometarum. Maiores autem difficultates ſe exerunt, quando excentricitas q eſt quam minima, quia tum variationes anomaliae ϑ maxime increſcunt, verumtamen ſi vſus veniat, et hic remedium ſperari poſſet. Maxime vero hac operationes ob partem $\frac{1}{w^3}$ impediuntur, quam niſi commode in ſeriem ſatis conuergentem conuertere liceat, de integratione omnino erit deſperandum, neque tum alia via ſupereſſe videtur, niſi vt ex ipſis his formulis differentialibus ſingulae variationes pro temporis interuallis ſatis exiguis definifiniantur, earumque fummatione integrationis negotium compenfetur, vti alia occafione fufius docui.

Applicatio huius Theoriae ad motum Lunae.

XXX.

Statuatur in A centrum terrae, cuius maffa fit $=A$, et dum tabula planum eclipticae refert fit nunc quidem fol in S cuius maffa fit $=B$, pro quo loco definiendo dirigatur recta AB ad coeli punctum fixum veluti primam ftellam arietis, ac ponatur:

longitudo folis feu angulus $BAS=\theta$

et diftantia folis a terra feu $AS=u$.

Quae elementa vt ex theoria folis definiantur, ponatur femiaxis transuerfus orbitae folis $=a$, femiparameter orbitae folis $=b$, excentricitas orbitae eius $=e$, et anomalia vera $=v$; eritque $u=\frac{b}{1-e\,\mathrm{cof}.v}$. Tum vero fi fol fecundum motum medium tempore infinite paruo angulum $d\tau$ percurrere ftatuatur erit $uud\theta=ad\tau\sqrt{ab}$, et $dv=d\theta$, nifi ad motum apogei folis refpicere velimus. Eft vero $b=a(1-ee)$; vnde fit

$u=\frac{a(1-ee)}{1-e\,\mathrm{cof}.v}$, et $d\tau=\frac{d\theta(1-ee)^{\frac{3}{2}}}{(1-e\,\mathrm{cof}.v)^2}$ atque $dv=d\theta$ proxime, vbi

vbi notetur, neglecta excentricitate orbitae folaris fore $u=a$ et $d\theta=d\tau$.

XXXI.

Sit porro in Z luna, cuius maffa $=C$, ac ponatur $\frac{B}{A+C}=n$ exiftente $\frac{A+C}{A+B}=m$, quoniam angulus elementaris $d\tau$ ex motu medio folis eft defumtus, ita, vt fit $\mathfrak{A}=A$ et $\mathfrak{C}=B$; vnde prodit $\frac{B}{A+B}=mn$ feu $m=\frac{1}{n}$, fiquidem maffa folis B prae maffa terrae A vt infinita fpectari poteft. Iam pro loco lunae ftatuatur:

longitudo nodi afcendentis feu angulus $BAN=\psi$

inclinatio orbitae lunae ad eclipticam feu $\Upsilon OZ=\omega$

et argumentum latitudinis feu $NAZ=\sigma$.

vnde fit:

longitudo lunae $=\psi+\mathrm{Ang.tang.}(\mathrm{tang.}\sigma\,\mathrm{cof.}\omega)$

et latitudo borealis $=\mathrm{Ang.fin.}(\mathrm{fin.}\sigma\,\mathrm{fin.}\omega)$.

Deinde fit diftantia lunae a terra $AZ=v$ et diftantia lunae a fole $SZ=w$, ac definitis hinc duobus angulis μ et ν vt fit

$$\mathrm{cof}\,\mu=\mathrm{cof.}\sigma\,\mathrm{cof.}(\theta-\psi)+\mathrm{fin.}\sigma\,\mathrm{cof.}\omega\,\mathrm{fin.}(\theta-\psi)$$

$$\mathrm{fin.}\,\nu=\mathrm{fin.}\sigma\,\mathrm{cof.}(\theta-\phi)-\mathrm{cof.}\sigma\,\mathrm{cof.}\omega\,\mathrm{fin.}(\theta-\psi)$$

erit $ww=vv+uu-2vu\,\mathrm{cof.}\mu$ feu $w=\sqrt{(vv+uu-2vu\,\mathrm{cof.}\mu)}$.

XXXII.

His pofitis fi ponamus pro tempore praefente $v=\frac{p}{1-q\,\mathrm{cof.}u}$ vt p denotet orbitae lunaris parame-

trum.

trum dimidiam, (q eius excentricitatem, et angulus ϑ eius anomaliam veram; ideoque iam q negatiue accipi debeat; tum vero fit $d\Phi$ angulus a luna circa terram eodem tempore deſcriptus, quo ſol conficit angulum $d\tau$, motu medio. Hinc ergo pro motu lunae definiendo hae habebuntur aequationes:

1°. $vv\,d\Phi = a\,d\tau\sqrt{map} = a\,d\tau\sqrt{\frac{ap}{n}}$

2°. $dp = -2nv^3u\,d\Phi\,\mathrm{sin.}\,\nu\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$

3°. $dq = -nv^2u\,d\Phi\left(\mathrm{cos.}\,\mu\,\mathrm{sin.}\,\vartheta + \frac{(q-2\mathrm{cos.}\,\vartheta+q\,\mathrm{cos.}\,\vartheta^2)\,\mathrm{sin.}\,\nu}{1-q\,\mathrm{cos.}\,\vartheta}\right)\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$
$+\frac{nv^3\,d\Phi\,\mathrm{sin.}\,\vartheta}{w^3}$

4°. $\frac{dv}{vv} = -\frac{q\,d\Phi\,\mathrm{sin.}\,\vartheta}{p}$ ſeu $d.\frac{1}{v} = +\frac{q\,d\Phi\,\mathrm{sin.}\,\vartheta}{p}$

5°. $d\vartheta = d\Phi + \frac{nv^3d\Phi\,\mathrm{cos.}\,\vartheta}{qw^3} - \frac{nv^2u\,d\Phi}{q}\left(\mathrm{cos.}\,\mu\,\mathrm{cos.}\,\vartheta + \frac{(2-q\,\mathrm{cos.}\,\vartheta)\,\mathrm{sin.}\,\nu\,\mathrm{sin.}\,\vartheta}{1-q\,\mathrm{cos.}\,\vartheta}\right)$
$\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$

denotante $d\Phi - d\vartheta$ promotionem momentaneam lineae abſidum ſeu apogei lunae in ſua orbita:

6°. $d\psi = \frac{-nv^3u\,d\Phi\,\mathrm{sin.}\,\sigma\,\mathrm{sin.}(\theta-\psi)}{p}\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$

7°. $\frac{d\omega}{\mathrm{sin.}\,\omega} = \frac{d\psi}{\mathrm{tang.}\,\sigma} = \frac{-nv^3u\,d\Phi\,\mathrm{cos.}\,\sigma\,\mathrm{sin.}(\theta-\psi)}{q}\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$

8°. $d\sigma = d\Phi - d\psi\,\mathrm{cos.}\,\omega = d\Phi + \frac{nv^3u\,d\Phi\,\mathrm{sin.}\,\sigma\,\mathrm{cos.}\,\omega\,\mathrm{sin.}(\theta-\psi)}{p}\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$.

XXXIII.

Incipiamus ab euolutione formulae $\frac{1}{w^3}-\frac{1}{u^3}$; quae inde facile inſtituitur, quod diſtantia u ſemper eſt praemagna prae diſtantia v; vnde fit

$$\frac{1}{w^3} = (uu-2vu\,\mathrm{cos.}\,\mu+vv)^{-\frac{3}{2}} = \frac{1}{u^3} + \frac{3v\,\mathrm{cos.}\,\mu}{u^4} + \frac{3vv(5\,\mathrm{cos.}\,\mu^2-1)}{2u^5}$$

ideoque $\frac{1}{w^3}-\frac{1}{u^3} = \frac{3v\,\mathrm{cos.}\,\mu}{u^4} + \frac{3vv(5\,\mathrm{cos.}\,\mu^2-1)}{2u^5}$.

Atque

Atque formulae $n°$. 3 et $n°$. 5 abibunt in has:

3°. $dq = \frac{nv^3d\Phi\,\mathrm{sin.}\,\vartheta}{u^3}\left(1-3\,\mathrm{cos.}\,\mu^2 + \frac{3v\,\mathrm{cos.}\,\mu}{2u}(3-5\,\mathrm{cos.}\,\mu^2)\right)$
$-\frac{nvvd\Phi\,\mathrm{sin.}\,\nu}{1-q\,\mathrm{cos.}\,\vartheta}(q-2\,\mathrm{cos.}\,\vartheta+q\,\mathrm{cos.}\,\vartheta^2)\left(\frac{3v\,\mathrm{cos.}\,\mu}{u^3}+\frac{3vv(5\,\mathrm{cos.}\,\mu^2-1)}{2u^4}\right)$

5°. $d\vartheta = d\Phi + \frac{nv^3d\Phi\,\mathrm{cos.}\,\vartheta}{qu^3}\left(1-3\,\mathrm{cos.}\,\mu^2 + \frac{3v\,\mathrm{cos.}\,\mu}{2u}(3-5\,\mathrm{cos.}\,\mu^2)\right)$
$-\frac{nvvd\Phi\,\mathrm{sin.}\,\nu\,\mathrm{sin.}\,\vartheta}{q(1-q\,\mathrm{cos.}\,\vartheta)}\left(\frac{3v\,\mathrm{cos.}\,\mu}{u^3}+\frac{3vv(5\,\mathrm{cos.}\,\mu^2-1)}{2u^4}\right)(2-q\,\mathrm{cos.}\,\vartheta)$.

Ita vt hinc fit

$dq\,\mathrm{cos.}\,\vartheta + q(d\Phi - d\vartheta)\,\mathrm{sin.}\,\vartheta = 2nvvud\Phi\,\mathrm{sin.}\,\nu\left(\frac{3v\,\mathrm{cos.}\,\mu}{u^3}+\frac{3vv(5\,\mathrm{cos.}\,\mu^2-1)}{2u^4}\right)$

quae expreſſio ſatis ſimplex deinceps vſum habere poterit.

XXXIV.

Deinde etiam notetur, cum inclinatio ω fit ſatis exigua ideoque $\mathrm{cos.}\,\omega = 1-\frac{1}{2}\omega\omega$ fore proxime:

$\mathrm{cos.}\,\mu = \mathrm{cos.}(\sigma-\theta+\psi) - \frac{1}{2}\omega\omega\,\mathrm{sin.}\,\sigma\,\mathrm{sin.}(\theta-\psi)$

$\mathrm{sin.}\,\nu = \mathrm{sin.}(\sigma-\theta+\psi) + \frac{1}{2}\omega\omega\,\mathrm{cos.}\,\sigma\,\mathrm{sin.}(\theta-\psi)$.