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Evolutio generalior formularum comparationi curvarum inservientium

Leonhard Euler

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EVOLVTIO GENERALIOR

FORMVLARVM COMPARATIONI CVRVA-
RVM INSERVIENTIVM.

Auctore

L. EYLERO.

r.

Quae de comparatione arcuum circularium et elementis sunt cognita, et quae Illustr. Comes Fagnanus de simili comparatione arcuum curvae Lemniscatae mira sagacitate elicit, ea, vitiam aliquoties offendi, ita generalius enunciaripotest, ut si cuiuspiam lineae curvae arcus indefinite per hanc formulam integram exprimatursit $\sqrt{\frac{Mdz}{A+cz^2+Ez^4}}$, tum in ea curva, sumto arcu quocunque, ab alio quouis puncto arcum geometricè abscindi posse illi arcui aequalem. Atque hinc etiam proposito arcu quocunque ab alio quouis puncto arcus abscindi poterit, qui illius arcus sit duplus seu triplus, seu qui in genere ad eum rationem quamcunque rationalem teneat. Vnde conficitur omnium curvarum, quarum quidem rectificatio ista formula continetur, arcus perinde atque arcus circulares inter se comparari posse.

2. Deinde quae de comparatione arcuum parabolicorum iam pridem sunt inuenta, et quae similli

DE COMPARATIONE CVRVARVM. 43

milli modo Ill. Comes Fagnanus circa arcus ellipticos et hyperbolicos summo acurmine praestitit, ea deinceps tam late patere demonstravi, ut pari successu ad omnes curvas, quarum arcus indefinite per hanc formulam integram: $\int \frac{dx(B+Dz^2+Ex^4+Dz^6+Ex^8)}{\sqrt{(A+cz^2+Dz^4)}}$ exprimatursit, extendi queant. Sumto scilicet in tali curva arcu quocunque, ab alio quouis puncto arcus abscindi poterit, qui ab illo arcu differat quantitate geometricè assignabili. Tum vero etiam abscindi poterunt eiusmodi arcus, qui ab arcus proposito duplo, triplo, vel quouis multiplo differant quantitate geometricè assignabili. Quin etiam illud punctum, unde arcus abscindi oportet, ita capi poterit, ut haec differentia plane in nihilum abeat.

3. Quaecunque ergo circa arcus parabolicos iam olim sunt praestita, eadem quoque in omnibus curvis, quarum rectificatio ad istam formulam integram est reducibilis, pari successu expediri poterunt. Cum autem Comes Fagnanus ad has mirabiles comparationes per substitutiones admodum molestas, et quarum ratio inventionis ne quidem perspiciatur, pervenerit; ego methodum planam aperui, quae quasi sponte ad easdem comparationes manducat. Atque ista methodus etiam multo vberius hoc negotium conficit, quod generalissime omnes comparationes in se complectitur; aequivalent enim integrationi completae, quae simul constantem arbitrariam involvit, dum illae substitutiones tantum

tum integrationes particulares referre sunt censenda, quam ob causam mihi quidem huius methodi beneficium multo longius progressi licuit, vti ex aliquot specimenibus, quae iam descriptae, luculenter apparet.

4. Quemadmodum autem in his formulis, quas pertractavi, ista expressio surda $\sqrt{(A+Cz+Cz'+Ez')}$ implicatur, quae quidem iam casus solutissimi difficillimos complectitur, ita eadem ad expressiorem surdam magis complicatam hanc: $\sqrt{(A+2Bz+Cz'+2Dz'+Ez')}$ extendi posse observavi; quae multo amplior campus aperitur similes comparationes in pluribus aliis lineis curvis instituenti. Neque vero haec inuestigatio tantum in lineis curvis tam eximium praestat usum, sed etiam in Analysis et calculo integrali gravissima incrementa largiri videtur, ad quae plenius excolenda vi viam serenam, euolutiones ad hanc formulam generaliorempertinentes diligentius exponam. Hunc in finem proposita sit sequens aequatio relationem inter binas variables x et y exprimens.

$$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + \alpha xy(x+y) + \zeta xy^2y.$$

Aequatio Canonica expendenda

5. Haec aequatio praeter binas variables x et y continet sex quantitates constantes, quae autem cum eandem earum ratio spectetur, ad quinque reducuntur, ita vt quinque determinationes ab arbitrio

trio nostro pendentes recipere sit censenda. Deinde etsi haec aequatio ratione variabilium ad quatuor dimensiones exurgit, tamen utraque seorsim nusquam ultra duas ascendit, ita vt utriusque valor per resolutionem aequationis quadraticae exhiberi queat: id quod praefens institutum necessario possulat. Denique ambae variables x et y in hac aequationem aequaliter ingrediuntur, et etiam si permutentur nullam mutationem inducunt, vt utraque per alteram formula omnino simili exprimitur. Arque ob has rationes membra $x'+y'$, $x''+y''$; et $xy(x+y)$, vti et altiores dimensiones omitti oportuit.

6. Quodsi iam ex hac aequatione tam valorem ipsius x quam ipsius y extrahamus, reperiemus:

$$x = \frac{-\epsilon - \delta y - \epsilon y^2 \pm \sqrt{(\epsilon + \delta y + \epsilon y^2)^2 - (\alpha + 2\beta y + \gamma y^2)(\gamma + 2\epsilon y + \zeta y^2)}}{\gamma + 2\epsilon y + \zeta y^2}$$

Ponamus breuitatis gratia:

$$\begin{aligned} &+ \sqrt{((\epsilon + \delta y + \epsilon y^2)^2 - (\alpha + 2\beta y + \gamma y^2)(\gamma + 2\epsilon y + \zeta y^2))} = Y \\ &+ \sqrt{((\epsilon + \delta x + \epsilon x^2)^2 - (\alpha + 2\beta x + \gamma x^2)(\gamma + 2\epsilon x + \zeta x^2))} = X \end{aligned}$$

vt habeamus:

$$x = \frac{-\epsilon - \delta y - \epsilon y^2 + Y}{\gamma + 2\epsilon y + \zeta y^2} \quad \text{et} \quad y = \frac{-\epsilon - \delta x - \epsilon x^2 + X}{\gamma + 2\epsilon x + \zeta x^2}$$

ideoque:

$$\begin{aligned} Y &= \epsilon + \delta y + \epsilon y^2 + x(\gamma + 2\epsilon y + \zeta y^2) \\ X &= \epsilon + \delta x + \epsilon x^2 + y(\gamma + 2\epsilon x + \zeta x^2) \end{aligned}$$

7. Hanc aequationem canonicam differentie-
mus, ac prodibit aequatio differentialis per bina-
rum diuisa:

$$0 = \delta dx + \epsilon x dx + \delta y dx + 2\epsilon y dx + \epsilon y^2 dx + \zeta xy dx \\ + \beta dy + \gamma y dy + \delta xy dy + 2\epsilon xy dy + \epsilon xy^2 dy + \zeta xy dy$$

quae cum reducat ad hanc formam

$$0 = \delta dx (\delta + \delta y + \epsilon y) + \epsilon dx (\gamma + 2\epsilon y + \zeta y y) \\ + dy (\beta + \delta x + \epsilon x x) + y dy (\gamma + 2\epsilon x + \zeta x x)$$

quoniam coefficientes ipsorum dx et dy sunt eae
ipsae quantitates, quas modo pro formulis radicali-
bus X et Y exhibuimus, ista aequatio differentialis
erit

$$0 = Y dx + X dy \text{ seu } \frac{dx}{X} + \frac{dy}{Y} = 0$$

in qua cum variables x et y sint separatae, si qui-
dem pro X et Y valores illos surdos substituamus,
per integrationem inde hanc aequationem finitam
obtinuimus.

$$\int \frac{dx}{X} + \int \frac{dy}{Y} = \text{Const.}$$

8. Cum igitur haec aequatio integralis cer-
tam quandam relationem inter variables x et y ex-
primat, ea a relatione in aequatione contenta di-
uisa esse non potest, siquae ipsa aequatio canonica
continebit istam aequationem integram. Est ergo
in aequatione differentiali $\frac{dx}{X} + \frac{dy}{Y} = 0$, neutra pars
est integrabilis, atque adeo neque per circuli qua-
draturam neque logarithmos expediri potest, tamen
inte-

integratio algebraicam relationem inter ambas varia-
biles x et y praebet, propterea quod haec aequatio
integrata cum ipsa aequatione canonica conuenit.
Quin etiam dico, aequationem canonicam non solum
casum particularem integram praebere, cuiusmodi
casus saepe aequationibus maxime complicatis satis-
faciunt, sed eam adeo integrale completum secun-
dum omnem extensionem exhibere.

9. Ad hoc ostendendum, in quo sine dubio
summa vis huius integrationis agnoscenda debet, notasse
sufficit in aequatione canonica una constante plus
contineri quam in aequatione differentiali. Vidi-
mus enim aequationem canonicam quinque inuol-
vere constantes arbitrarias; unde examinamus, quot
huiusmodi constantes aequatio differentialis comple-
tatur. Manifestum autem est eam huiusmodi ha-
bere formam

$$\frac{Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5}{x} + \frac{Ay + By^2 + Cy^3 + Dy^4 + Ey^5}{y} = 0$$

in qua quidem etiam quinque constantes $A, B, C,$
 D, E inesse videntur: verum euidentis est, vnam-
quamque per diuisionem tolli posse, ita ut re ve-
ra quatuor tantum inesse sint censendae. Quare
cum aequatio integralis quin ue contineat, una ar-
bitrario nostro relinquatur, quod est manifestum in-
dicium integralis completi.

10. Vicinque autem isti quinque coefficientes
 A, B, C, D, E se habeant, semper coefficientes
aequationis canonicae iis conformiter ita definiti
possunt,

possunt, ut vnus maneat indeterminatus. Diuidamus enim aequationem differentialem per quantitatem indefinitam p , quae iam sublata est censenda, ut re vera fuerit:

$$X = \gamma (A p + 2 B p x + C p x x + 2 D p x^2 + E p x^3).$$

Iam euoluamus quoque secundum potestates ipsius x valorem primitiuum ipsius X , qui erit

$$X = \gamma \left(\begin{array}{l} \epsilon \delta + 2 \delta \delta + 2 \delta \epsilon \\ - a \gamma - 2 a \epsilon \\ - 2 \delta \gamma \end{array} \right) \left\{ \begin{array}{l} + \delta \delta \\ + \alpha \zeta \\ - 4 \delta \epsilon \end{array} \right\} \left\{ \begin{array}{l} + 2 \delta \epsilon \\ x^2 - 2 \delta \zeta \\ - 2 \gamma \epsilon \end{array} \right\} \left\{ \begin{array}{l} x^3 + \epsilon \epsilon \\ x^2 \\ - \gamma \zeta \end{array} \right\} x^3$$

atque istae literae $a, \epsilon, \gamma, \delta, \epsilon, \zeta$ ita definiantur, ut haec forma cum priori congruens redatur, sic enim patebit vnam determinationem adhuc arbitrio nostro relinquui.

11. Satisfieri igitur oportet sequentibus quinque aequationibus:

- I. $\epsilon \delta - a \gamma = A p$
- II. $\delta \delta - a \epsilon - \delta \gamma = B p$
- III. $\delta \delta - a \zeta - 2 \delta \epsilon - \gamma \gamma = C p$
- IV. $\delta \epsilon - \delta \zeta - \gamma \epsilon = D p$
- V. $\epsilon \epsilon - \gamma \zeta = E p$.

Ponamus ad abbreviandum $\delta - \gamma = \lambda$ seu $\delta = \gamma + \lambda$ et incipiamus a II et IV,

II. $\epsilon \lambda - a \epsilon = B p$ et IV. $\epsilon \lambda - \delta \zeta = D p$

unde

unde definiemus ϵ et e ita ut sit:

$$\epsilon = \frac{D a + B \lambda}{\lambda \alpha - a \zeta} p \quad \text{et} \quad e = \frac{B \zeta + D \lambda}{\lambda \lambda - a \zeta} p.$$

At I et V coniunctae dant:

$$\epsilon \delta \zeta - a \epsilon \epsilon = A p \zeta - E p a = \frac{B B \zeta - D D \epsilon}{\lambda \lambda - a \zeta} p p$$

unde eruitur $p = \frac{(\lambda \lambda - a \zeta)(A \zeta - E a)}{B B \zeta - D D \epsilon}$,

qui valor in alterutra substituitus praebet

$$\gamma = \frac{(A \zeta - E a)(A D D - B B E \lambda \lambda + B D A \zeta - E a) \lambda + A B B \zeta - D D E a \epsilon}{(B B \zeta - D D \epsilon)^2}.$$

12. Superest igitur III. aequatio, quae ob

$\delta = \gamma + \lambda$ transit in

$$2 \gamma \lambda + \lambda \lambda - a \zeta - 2 \delta \epsilon = C p.$$

Cum nunc substituto valore ipsius p fit

$$\epsilon = \frac{(A \zeta - E a)(D a + B \lambda)}{B B \zeta - D D \epsilon} \quad \text{et} \quad e = \frac{(A \zeta - E a)(B \zeta + D \lambda)}{B B \zeta - D D \epsilon}$$

si isti valores pro γ, ϵ, e et p substituantur, tota aequatio per $\lambda \lambda - a \zeta$ diuidi poterit, quo facto reperietur

$$\lambda = \frac{C(A \zeta - E a)(B B \zeta - D D \epsilon) - B D(A \zeta - E a)^2 - (B B \zeta - D D \epsilon)^2}{2(A \zeta - E a)(A D D - B B E)}.$$

Quoniam igitur nunc omnibus conditionibus est satisfactum, arbitrio nostro adhuc relinquuntur duo coefficientes a et ζ , seu potius eorum ratio mutua, quam ergo pro lubitu definire licet. Ex quo manifestum est, in aequatione integrali seu ipsa canonica inesse constantem arbitrariam ab aequatione differentiali non pendentem.

Alia resolutio eandem formulam.

13. Quia istam valorum applicatio fieri nequit casibus, quibus $AED - BEE = 0$, aliam resolutionem tunc incommode non exoptari staram. Posito autem $\delta = \gamma + \lambda$, statuo porro: $\lambda - \alpha\zeta = \mu$ seu $\lambda = \mu + \alpha\zeta$, atque vi ante ex aequationibus II et IV habebimus:

$$\xi = \frac{p}{\mu}(D\alpha + B\lambda); \quad \varepsilon = \frac{p}{\mu}(B\zeta + D\lambda).$$

Tum vero quia I et V conjunctae dant

$$A\zeta - E\alpha = (BB\zeta - DD\alpha)\frac{p}{\mu}.$$

hinc definitio rationem inter α et ζ , seu quoniam alterutram pro lubitu accipere licet, vitamque hoc modo vi sit:

$$\alpha = \mu A - BBp \quad \text{et} \quad \zeta = \mu E - DDp$$

hincque $\lambda = \mu + (\mu A - BBp)(\mu E - DDp)$. At alterutra I et V, valoribus hactenus inuentis substituitis, praebet:

$$\gamma = \frac{p}{\mu} (2BD\lambda + (ADD + BBE)\mu) - \frac{BBDDp^2}{\mu} - \frac{p}{\mu}.$$

14. Quodsi iam hi valores in aequatione III substituantur, ea ad formam quidem admodum prolixam reducitur: verum negotium commodius absoluetur, si valores pro α et ζ inuenti in formula vltima praecedentis resolutionis substituantur, tum enim prodibit:

$$\lambda = \frac{\mu}{\mu} + BDp - \frac{1}{2}C\mu$$

cuius

cuius quadratum cum superiori ipsius λ valore coequatum praebet:

$$\mu(\mu - Cp)^2 + 4(BD - AE)p\mu + 4(ADD - BCD + BBE)p^2 = 4pp$$

ad quam resoluendam ponamus $\mu = pM$ eritque

$$p = \frac{M(M - C)^2 + M(BD - AE) + (ADD - BCD + BBE)}{M}$$

et $\mu = \frac{M(M - C)^2 + M(BD - AE) + (ADD - BCD + BBE)}{M}$ atque iam M est constans illa arbitraria integrale reddens completum.

15. Hoc modo omnes coefficients α , ξ , γ , δ etc. eodem denominatore affecti prodibunt, qui ergo si per eundem multiplicentur sequenti modo sese habebunt:

$$\alpha = 4(AM - BB); \quad \xi = 2B(M - C) + 4AD; \quad \gamma = 4AE - (M - C)^2$$

$$\zeta = 4(EM - DD); \quad \varepsilon = 2D(M - C) + 4BE; \quad \delta = MM - CC + 4(AE + BD)$$

ac si illum denominatorem breuitatis gratia statuerimus:

$$M(M - C)^2 + 4M(BD - AE) + 4(ADD - BCD + BBE) = \Delta$$

aequatio nostra canonica

$$0 = \alpha + 2\xi(x + y) + \gamma(xx + yy) + 2\delta xy + 2\varepsilon y(x + y) + \zeta xyy$$

resoluita dabit:

$$x + \delta y + \varepsilon xy + x(\gamma + 2\varepsilon y + \zeta y^2) = \frac{1}{2} \sqrt{\Delta(A + 2Bx + Cxx + 2Dx^2 + Ex^3)}$$

$$x + \delta y + \varepsilon yy + x(\gamma + 2\varepsilon y + \zeta y^2) = \frac{1}{2} \sqrt{\Delta(A + 2By + Cyy + 2Dy^2 + Ey^3)}$$

simul-

simulque est integrale completum huius aequationis differentialis :

$$0 = \frac{dx}{\pm\sqrt{(A+1B+Cx+Dx^2+Ex^3)} + \frac{dy}{\pm\sqrt{(A+1B+Cy+Dy^2+Ey^3)}}$$

quia constantem arbitrariam M inuoluit, quae in aequationem differentialem non ingreditur.

Inuestigatio casuum, quibus formula

$$\frac{Pdx}{x} + \frac{Qdy}{y} \text{ fit integrabilis.}$$

16. Designat hic P functionem ipsius x et Q similem functionem ipsius y, et quia haec formula integrabilis esse debet, fit V eius integrale, ut habeamus :

$$\frac{Pdx}{x} + \frac{Qdy}{y} = dV \text{ et } \int \frac{Pdx}{x} + \int \frac{Qdy}{y} = V.$$

Cum autem fit $\frac{dx}{x} + \frac{dy}{y} = 0$, ideoque $\frac{dy}{y} = -\frac{dx}{x}$, erit

$$dV = \frac{(P-Q)dx}{x} = \frac{(P-Q)dx}{\epsilon + \delta x + \epsilon xx + \gamma(\gamma + \epsilon \epsilon x + \zeta \zeta x)}$$

Casus ergo inuestigari oportet, quibus haec formula integrationem admittit.

17. Quoniam vero nulla est ratio, cur hic differentiale dx potius insit, quam dy, tertiam variabilem introductam, quae ad utramque aequaliter referatur, siquidem quantitas V utramque aequaliter inuoluere debet. Statuamus ergo $x+y=s$, et in aequatione differentiali (§. 7.) pro dy scribamus ds-dx; sicque prodibit :

0 =

$$0 = + dx(\epsilon + \delta y + \epsilon y y) + x dx(\gamma + 2 \epsilon y + \zeta y y) - dx(\delta + \delta x + \epsilon x x) - y dx(\gamma + 2 \epsilon x + \zeta x x) + ds(\delta + \delta x + \epsilon x x) + y ds(\gamma + 2 \epsilon x + \zeta x x)$$

vnde dx per ds ita definitur, ut fit :

$$dx = \frac{ds(\epsilon + \delta x + \epsilon x x) + y ds(\gamma + 2 \epsilon x + \zeta x x)}{\delta(x-y) + (\epsilon x - \gamma y) - \gamma x \frac{ds}{dx} + \zeta x(x-y)} \text{ siue}$$

$$dx = \frac{ds}{x-y} \cdot \frac{\epsilon + \delta x + \epsilon x x + \gamma(\gamma + 2 \epsilon x + \zeta x x)}{\delta - \gamma + \epsilon(x+y) + \zeta x y}$$

quo valore substituto fiet :

$$dV = \frac{(P-Q)ds}{(x-y)(\delta - \gamma + \epsilon(x+y) + \zeta x y)}$$

18. Cum P et Q sint similes functiones ipsarum x et y, manifestum est P-Q per x-y fore diu sibile, et fractionem $\frac{P-Q}{x-y}$ utramque variabilem x et y aequaliter esse complexuram. Quia vero posuimus $x+y=s$, ponamus insuper $xy=t$, ut fit :

$$dV = \frac{P-Q}{x-y} \cdot \frac{ds}{\delta - \gamma + \epsilon s + \zeta t}$$

At ob $xx+yy=ss-2t$ aequatio canonica inducet hanc formam :

$$0 = \alpha + 2 \delta s + \gamma s s + 2(\delta - \gamma)t + 2 \epsilon s t + \zeta t t$$

ex qua elicitur :

$$t = \frac{-\delta + \gamma - \epsilon s + \sqrt{(\delta - \gamma)^2 - \alpha \zeta + s(\delta - \gamma)\epsilon s - 2 \delta \zeta s + \epsilon s s s - \gamma \zeta s}}{\zeta}$$

ita ut fit :

$$\delta - \gamma + \epsilon s + \zeta t = \sqrt{((\delta - \gamma)^2 - \alpha \zeta + 2((\delta - \gamma)\epsilon - \delta \zeta)s + (\epsilon \epsilon - \gamma \zeta)s s)}$$

Statuamus hanc formulam irrationalem :

$$\sqrt{((\delta - \gamma)^2 - \alpha \zeta + 2((\delta - \gamma)\epsilon - \delta \zeta)s + (\epsilon \epsilon - \gamma \zeta)s s)} = S$$

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ut fit

$$f = \frac{-\delta - \gamma}{2} = \frac{u + s}{2} \text{ et } dV = \frac{P - Q}{x - y}, \frac{dI}{s}.$$

19. Ut hinc iam casus integrabilis cruanus, ponamus:

$$P = a + bx + cx^2 + dx^3 + ex^4$$

$$Q = a + by + cy^2 + dy^3 + ey^4$$

erique

$$\frac{P - Q}{x - y} = b + c(x - y) + d(xx + xy + yy) + e(x^3 + xxy + xyy + y^3)$$

sive introductis nouis variabilibus s et t

$$\frac{P - Q}{x - y} = b + ts + d(ss - t^2) + e(ss - 2t),$$

At pro t valore substituto habebimus ob $\lambda = \delta - \gamma$

$$\frac{P - Q}{x - y} = \frac{b + cs + dss + e s^2}{\frac{1}{2}(\lambda s + \frac{1}{2}\lambda e s^2)} = \frac{(d + 2es)s}{\frac{1}{2}(\lambda s + \frac{1}{2}\lambda e s^2)}$$

unde consequimur:

$$dV = \frac{d(\lambda s + \frac{1}{2}\lambda e s^2) + (2e + \lambda d + \frac{1}{2}\lambda e) ds + \frac{1}{2}\lambda e ds^2}{\frac{1}{2}(\lambda s + \frac{1}{2}\lambda e s^2)^2}$$

quam formulam integrabilem esse oportet.

20. Quo hoc facilius praestemus, recordemur

ex §. 13 et 14. esse $(\delta - \gamma)^2 - a\zeta = \lambda\lambda - a\zeta = \mu$;

$(\delta - \gamma)^2 - \xi\zeta = D\mu$; et $e\epsilon - \gamma\zeta = E\mu$ unde fit $S =$

$$S = \frac{e\sqrt{M} + 2D\mu + E\mu s}{\gamma\Delta}.$$

Ponamus porro breuitatis gratia:

$$b + \frac{\lambda d}{2} = b; c + \frac{e\lambda + 2d}{2} = g; d + \frac{e\lambda}{2} = f$$

ut

ut fit

$$dV = \frac{(b + \frac{e\lambda + 2d}{2} + f)s + (g + e\lambda)s^2}{\gamma\sqrt{M} + 2D\mu + E\mu s} = \frac{(g + f)s + (g + e\lambda)s^2}{\gamma\sqrt{M} + 2D\mu + E\mu s}$$

Naturae partis prioris integrale:

$$(g + f)s + (g + e\lambda)s^2 \sqrt{M} + 2D\mu + E\mu s$$

erique differentialium comparatione infitura:

$$b = 2\mathcal{G}M + 1\mathcal{G}D; g = 4\mathcal{G}M + 6\mathcal{G}D + 2\mathcal{G}E$$

$$f = 10\mathcal{H}D + 4\mathcal{G}E; e = 6\mathcal{H}E$$

unde pro integrabilitate requiritur ut fit:

$$0 = eD(3EM - 5DD) + fE(3DD - EM) - 2gDEH + 2bHE'$$

21. Hac autem conditione impleta, erit:

$$\mathcal{G} = \frac{b}{2D} - \frac{fM}{4DE} + \frac{3eM}{12EE}; \mathcal{G} = \frac{f}{2E} - \frac{4eD}{12EE}; \mathcal{H} = \frac{e}{2E}$$

et integrale quaesitum reperitur:

$$V = (\mathcal{G} + \mathcal{G}s + \mathcal{H}ss)\sqrt{M} + 2D\mu + E\mu s - \frac{(d + e\lambda)s^2}{2}, \text{ vel}$$

$$V = \frac{1}{2}(\mathcal{G} + \mathcal{G}s + \mathcal{H}ss)\Delta S - \frac{(d + e\lambda)s^2}{2}$$

Cum nunc fit $S = \lambda + e(x + y)^2 + \zeta xy$, si pro s seruiamus $x + y$, valor integralis V ita per x et y exprimitur, ut fit

$$V = \frac{1}{2}(\mathcal{G} + \mathcal{G}(x + y) + \mathcal{H}(x + y)^2)(\lambda + e(x + y) + \zeta xy) - \frac{d(e + \lambda)(x + y)^2}{2}$$

Quare ut pro V prodeat quantitas algebraica, coeeficientes b, g, f et e non pro lubrica assumere licet, sed certam quandam relationem inter eos statui oportet, quae vltima aequalitate §. praec. exprimitur. Ceterum huc assumi, non esse $E = 0$, si enim

esset

esset $E=0$, valor ipsius V semper algebraice exhiberi posset, vri ex elementis integrationis est manifestum.

22. Verum si coefficientes b, c, d, e etc. vitaeque assumamus, tum expressio $\int \frac{dx}{x} + \int \frac{Qdy}{y}$ non quidem semper algebraice exhiberi poterit; atamen eius valor altiore quadraturam non inuoluet, quam in formula $\int \frac{dx}{\sqrt{(M+2Ds+Es^2)}}$ contentam, quae propter ea semper vel per logarithmos vel per arcus circulares exhiberi poterit. Cum igitur sit

$$X = \sqrt{b(A+2Bx+Cxx+2Dx^2+Ex^3)}$$

et $\sqrt{b} = \frac{1}{\sqrt{\Delta}}$ erit $X = \frac{1}{\sqrt{\Delta}} \sqrt{(A+2Bx+Cxx+2Dx^2+Ex^3)}$ unde inuenio valore ipsius V habebitur sequens integratio:

$$\int \frac{dx (a+bx+cx^2+dx^3+ex^4)}{\sqrt{(A+2Bx+Cx^2+2Dx^3+Ex^4)}} + \int \frac{dy (a+by+cy^2+dy^3+ey^4)}{\sqrt{(A+2By+Cy^2+2Dy^3+Ey^4)}} = \frac{1}{\sqrt{\Delta}}$$

At substituitis superioribus valoribus crit:

$$\frac{1}{\sqrt{\Delta}} = \int \frac{b+2dx+(c+2d+2d^2+2e)dx+2es^2}{s\sqrt{(M+2Ds+Es^2)}} ds = \frac{(d+e)s\sqrt{\Delta}}{s(EM-DD)}$$

Existente $s = x+y$. Atque hinc sequentia problema resolui poterunt.

Problema I.

23. Inuenire integrale completum huius aequationis differentialis:

$$\frac{dx}{\sqrt{(A+2Bx+2Cx^2+2Dx^3+Ex^4)}} = \frac{dy}{\sqrt{(A+2By+2Cy^2+2Dy^3+Ey^4)}}$$

Solutio.

Solutio.

Statim apparet huic aequationi differentiali satisfacere casum $y=x$, qui autem non nisi integrale particulare largitur. Verum ad integrale completum inueniendum, quod praeter constantes A, B, C, D, E nouam constantem arbitriam M inuoluat, ponamus secundum §. 15 breuitatis gratia: $a=4(AM-BB)$; $b=2B(M-C)+4AD$; $\gamma=4AE-(M-C)^2$ $\xi=4(EM-DD)$; $\epsilon=2D(M-C)+4BE$; $\delta=MM-CC+4(AE-BD)$ atque aequatio integralis completa erit

$$0 = a + 2\delta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\epsilon xy(x+y) + \xi xxyy$$

quae ergo est algebraica. Hinc autem siue y per x , siue vicissim x per y sequenti modo definitur, posito item breuitatis ergo:

$$\Delta = M(M-C)^2 + 4M(BD-AE) + 4(ADD+BBE) - 4BCD$$

vt sit

$$\text{vel } y = \frac{-\delta - \delta x - \epsilon x x + \gamma \sqrt{\Delta(A+2Bx+2Cx^2+2Dx^3+Ex^4)}}{\gamma + 2\epsilon x + \xi x x}$$

$$\text{vel } x = \frac{-\epsilon - \delta y - \epsilon y y + 2\gamma \sqrt{\Delta(A+2By+2Cy^2+2Dy^3+Ey^4)}}{\gamma + 2\epsilon y + \xi y y}$$

scilicet ratione signorum ambiguum in utraque expressione vel signa superiora vel inferiora capi debent, ita vt si in altera formulae surdae tribuatur signum +, in altera formulae surdae signum - tribui debeat. Quae ratio ex §. 15. intelligitur, vbi in aequatione differentiali formulis surdis signa ambigua sunt adiuncta.

COROLL. 1.

24. Quamquam igitur aequationis differentialis propositae, in qua ambae variables x et y se invicem sunt separatae, neutrum membrum integrationem absolutam admittit, atque adeo neque per, logarithmos neque arcus circulares in genere exprimi potest, tamen vera relatio inter variables x et y aequatione algebraica exhiberi potest.

COROLL. 2.

25. Quemadmodum scilicet si duo arcus quantitate constante differunt, est neuter algebraice exprimitur, tamen eorum sinus inter se algebraicam tenent rationem, quae satisfacit aequationi differentiali $\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{\sqrt{1-x^2}}$, ita quoque aequationis differentialis propositae multoque latius parentis integrale completum algebraice exhiberi potest.

Scholion.

26. Vis huius solutionis facilius percipitur, si eam ad casus magis restrictos applicemus, inter quos ii praecipue sunt notatu digni, ubi signum radicale vel unico vel duobus tantum terminis praefigitur ac si unicus tantum terminus reperitur, ratio per se est manifesta.

I. Sit enim $B=0$; $C=0$; $D=0$, et $E=0$ ut integranda sit aequatio:

$$\frac{dy}{\sqrt{A}} = \frac{dx}{\sqrt{A}} \text{ siue } dy = dx \text{ erit}$$

$$a=4AM; \quad \delta=0; \quad \gamma=-MM; \quad \delta=MM; \quad \epsilon=0; \quad \zeta=0;$$

ideoque

ideoque

ideoque aequatio integralis:

$$0 = 4AM - MM(xx + yy) + 2MMxy$$

$$\text{seu } x - y = 2\sqrt{\frac{A}{M}} \text{ vel } y = x \pm \text{Const.}$$

II. Sit $A=0$; $C=0$; $D=0$; et $E=0$, ut integranda sit aequatio:

$$\frac{dy}{\sqrt{By}} = \frac{dx}{\sqrt{Bx}} \text{ seu } \frac{dy}{y} = \frac{dx}{x} \text{ erit}$$

$$a = -4BB; \quad \delta = 2BM; \quad \gamma = -MM; \quad \delta = MM; \quad \epsilon = 0 \text{ et } \zeta = 0$$

ideoque aequatio integralis, ob $\Delta = M^2$

$$0 = -4BB + 4BM(x + y) - MM(xx + yy) + 2MMxy$$

$$\text{seu } y = \frac{-2BM - MMx \pm \sqrt{4BM^2x^2 + 2\sqrt{\frac{B}{M}}x}}{-MM} = x + \frac{\sqrt{B}}{M} \pm 2\sqrt{\frac{B}{M}}x$$

hincque $\sqrt{y} = \sqrt{x} + \text{Const.}$ vti est perspicuum.

III. Sit $A=0$; $B=0$; $D=0$; $E=0$, ut integranda sit haec aequatio:

$$\frac{dy}{\sqrt{Cy}} = \frac{dx}{\sqrt{Cx}} \text{ seu } \frac{dy}{y} = \frac{dx}{x}; \text{ erit}$$

$$a = 0; \quad \delta = 0; \quad \gamma = -(M-C)^2; \quad \delta = MM - CC; \quad \epsilon = 0, \text{ et } \zeta = 0$$

ideoque aequatio integralis

$$0 = -(M-C)^2(xx + yy) + 2(MM - CC)xy \text{ seu } y = nx,$$

IV. Sit $A=0$; $B=0$; $C=0$; et $E=0$ ut integranda sit haec aequatio:

$$\frac{dy}{\sqrt{Dy}} = \frac{dx}{\sqrt{Dx}} \text{ seu } \frac{dy}{y} = \frac{dx}{x}; \text{ erit}$$

$$a = 0; \quad \delta = 0; \quad \gamma = -MM; \quad \delta = MM; \quad \epsilon = 2DM; \quad \zeta = -4DD$$

H 2

ideo-

ideoque aequatio integralis:

$$0 = -MM'(xx+yy) + 2MM'xy + 4DM'xy(x+yy) - 4DD'xyxy$$

quae ob $\Delta = M^2$ dat

$$y = \frac{-MM'x - 2DM'xy + \sqrt{V^2 DM'^2 x^2}}{-MM' + 4DM'x - 4DD'x^2}$$

$$\text{feu } \sqrt{y} = \frac{M + \sqrt{V^2 DM'^2}}{M - 4DD'x} \sqrt{x} = \frac{\sqrt{Mx}}{\sqrt{M - 4DD'x}}$$

vel $\frac{1}{\sqrt{y}} = \frac{1}{\sqrt{x}} \pm \sqrt{\frac{4DD'}{M}}$, vii rei natura postulat.

V. Sit $A = 0$; $B = 0$; $C = 0$; et $D = 0$ vt intergranda fit haec aequatio:

$$\frac{dy}{y^2} = \frac{dx}{\sqrt{E}x^2} \text{ feu } \frac{dy}{y} = \frac{dx}{x}; \text{ erit}$$

$\alpha = 0$; $\beta = 0$; $\gamma = -MM$; $\delta = MM$; $\epsilon = 0$; et $\zeta = 4EM$ ideoque aequatio integralis:

$$0 = -MM'(xx+yy) + 2MM'xy + 4EM'xyxy$$

hincque $y - x = 2xy\sqrt{\frac{E}{M}}$ feu $y' = \frac{1}{2} \pm 2\sqrt{\frac{E}{M}}$.

Quando autem signum radicale complectitur duos terminos, varios casus, qui huc pertinent, sequentibus exemplis euoluemus.

Exemplum 1.

27. Si sit $C = 0$; $D = 0$ et $E = 0$, vt intergranda fit aequatio:

$$\frac{dy}{\sqrt{(A+B'y)}} = \frac{dx}{\sqrt{(A+B'x)}}$$

inuenire aequationem integram completam.

Erit

Erit ergo $\alpha = 4(A-M-BB)$; $\beta = 2BM$; $\gamma = -MM$; $\delta = MM$; $\epsilon = 0$; $\zeta = 0$; vnde aequatio integralis:

$$0 = 4(A'M-BB) + 4BM'(x+yy) - MM'(xx+yy) + 2MM'xy$$

et ob $\Delta = M^2$

$$y = \frac{-BM - MM'x + \sqrt{V^2 M^2 (A+B'x)}}{-MM' + 4BM'x - 4DD'x^2} = \frac{-B + Mx}{M} \pm 2\sqrt{\frac{A+B'x}{M}}$$

Vnde ponendo $A = f$; $2B = g$; et $M = c$ legitur

Theorema 1.

28. Huius aequationis differentialis $\frac{dy}{\sqrt{(f+gy)}}$ = $\frac{dx}{\sqrt{(f+gx)}}$; Integrale completum est:

$$0 = 4cf - gg + 2cg(x+yy) - cc(xx+yy) + 2ccxy$$

vnde fit:

$$y = x + \frac{g}{c} \mp 2\sqrt{f + \frac{g}{c}x} \text{ et } x = y + \frac{g}{c} \pm 2\sqrt{f + \frac{g}{c}y}.$$

Exemplum 2.

29. Si sit $B = 0$, $D = 0$ et $E = 0$, vt intergranda fit aequatio:

$$\frac{dy}{\sqrt{(A+Cy)}} = \frac{dx}{\sqrt{(A+C'x)}}$$

inuenire aequationem integram completam.

Erit ergo: $\alpha = 4AM$, $\beta = 0$, $\gamma = -(M-C)'$, $\delta = MM-CC$, $\epsilon = 0$ et $\zeta = 0$, vnde aequatio integralis quaesita erit:

$$0 = 4AM - (M-C)'(xx+yy) + 2(MM-CC)xy$$

H 3

et

et ob $\Delta = M(M-C)^2$ erit

$$y = \frac{-(-MM-CC)x^2 + (-M-C)VM(A+CCx)}{M-C} = \frac{(M+CC)x^2 + VM(A+CCx)}{M-C}$$

Quare ponendo $A = f$; $C = g$; et $M = c$, sequitur

Theorema 2.

30. Huius aequationis differentialis $\sqrt{(f+gxy)}$
 $= \frac{dx}{\sqrt{(f+gxy)}}$ integrale completum est:

$$0 = 4cf - (c-g)^2 (xx+yy) + 2(cg-eg)xy$$

unde fit:

$$y = \frac{(c+g)x \pm \sqrt{c(f+gxx)}}{c-g} \text{ et } x = \frac{(c+g)y \mp \sqrt{c(f+gxy)}}{c-g}$$

Exemplum 3.

31. Si fit $B = 0$; $C = 0$; et $E = 0$, vt inter-
 granda fit haec aequatio:

$$\frac{dy}{\sqrt{(A+2Dy^2)}} = \frac{dx}{\sqrt{(A+2Dx^2)}}$$

invenire aequationem integram completam.

Erit ergo $\alpha = 4AM$; $\beta = 4AD$; $\gamma = -M^2$; $\delta = M^2$;
 $\epsilon = 2DM$ et $\zeta = -4DD$; unde aequatio integralis
 quaesita est

$$0 = 4AM + 8AD(x+y) - M^2(xx+yy) + 2M^2xy$$

$$+ 4DMxy(x+y) - 4DDxyy$$

et cum fit $\Delta = M^2 + 4ADD$ erit

$$y = \frac{-4AD - MMx \pm \sqrt{(M^2 + 4ADD)(A + 2Dx^2)}}{-MM + 4DMx - 4DDx^2}$$

fiue:

fiue:

$$y = \frac{4AD + MMx \pm \sqrt{(M^2 + 4ADD)(A + 2Dx^2)}}{M - 2Dx^2}$$

Quare si ponatur $A = f$; $2D = g$ et $M = c$ sequitur

Theorema 3.

32. Huius aequationis differentialis $\sqrt{(f+gxy)}$
 $= \frac{dx}{\sqrt{(f+gxy)}}$ integrale completum est:

$$0 = 4cf + 4fg(x+y) - c^2(xx+yy) + 2ccy^2 + 2cgxy(x+y) - ggxyy$$

unde fit:

$$y = \frac{2fg + ccx + cgxy \pm \sqrt{(c^2 + fgg)(f+gxy)}}{(c-gx)^2} \text{ et}$$

$$x = \frac{2fg + ccy + ccgy \mp \sqrt{(c^2 + fgg)(f+gxy)}}{(c-gy)^2}$$

Exemplum 4.

33. Si fit $B = 0$; $C = 0$; $D = 0$ vt aequatio
 intergranda fit:

$$\frac{dy}{\sqrt{(A+Ex^2)}} = \frac{dx}{\sqrt{(A+Ex^2)}}$$

invenire aequationem integram completam.

Erit ergo $\alpha = 4AM$; $\beta = 0$; $\gamma = 4AE - MM$;
 $\delta = MM + 4AE$; $\epsilon = 0$ et $\zeta = 4EM$, unde ae-
 quatio integralis quaesita est:

$$0 = 4AM + (4AE - MM)(xx+yy) + 2(4AE + MM)xy$$

$$+ 4EMxyy$$

et cum fit $\Delta = M^2 - 4AEM$ erit

$$y = \frac{-(-MM + 4AE)x \pm \sqrt{(M^2 - 4AEM)(A + Ex^2)}}{4AE - MM + 4EMx}$$

Quare

Quare si ponatur $A = f$; $E = g$; et $M = 2c$ sequitur

Theorema 4.

34. Huius aequationis differentialis $\frac{dy}{\sqrt{(f+g^2)}}$ integrale completum est:

$$0 = 2cf - (cc - fg)(xx + yy) + 2(cc + fg)xy + 2igxyy$$

vnde fit

$$y = \frac{+ (cc + fg)x \pm \sqrt{2c(cc - fg)(f + g^2)}}{6c - \sqrt{g - 10gxx}}$$

$$x = \frac{+ (cc + fg)y \pm \sqrt{2c(cc - fg)(f + g^2)}}{cc - \sqrt{g - 10gxx}}$$

Exemplum 5.

35. Si fit $A = 0$, $C = 0$ et $D = 0$ ut integranda fit haec aequatio:

$$\frac{dy}{\sqrt{(2By + B^2)}} = \frac{dx}{\sqrt{(2Bx + B^2)}}$$

invenire aequationem integram completam.

Erit ergo: $a = -4BB$; $g = 2BM$; $\gamma = -MM$; $\delta = MM$, $\epsilon = 4BE$ et $\zeta = 4EM$, hincque aequatio integralis quae fita:

$$0 = -4BB + 4BM(x+y) - MM(xx+yy) + 2MMxy + 8BExy(x+y) + 4EMxxyy$$

et cum fit $\Delta = M^2 + 4BBE$ erit

$$y = \frac{2BM + MMx + 4BExx \pm \sqrt{(M^2 + 4BBE)(2Bx + g^2)}}{MN - 2BE - 4EMx}$$

Quare si ponatur $2B = f$; $E = g$; $M = c$; $x = xx$ et $y = yy$ sequitur

Theore-

Theorema 5.

36. Huius aequationis differentialis $\frac{dy}{\sqrt{(f+g^2x^2)}}$ integrale completum est:

$$0 = -ff + 2cf(xx + yy) - cc(x^2 + y^2) + 2ccxyy + 4fgxyyy + (xx + yy) + 4cggx^2y^2$$

vnde fit

$$yy = \frac{cf + ccxx + 2fgx^2 \pm \sqrt{(c^2 + ffg)(f + g^2)}}{cc - \sqrt{g - 10gxx}}$$

$$et \quad xx = \frac{cf + ccyy + 2fgy^2 \pm \sqrt{(c^2 + ffg)(f + g^2)}}{cc - \sqrt{g - 10gxy}}$$

Scholion 1.

37. Probabile hinc videtur etiam huius aequationis differentialis:

$$\frac{dy}{\sqrt{(f+g^2y^n)}} = \frac{dx}{\sqrt{(f+g^2x^n)}}$$

atque adeo huius latissime patentis:

$$\frac{dy}{\sqrt{(a+by^2+cy^3+dy^4+ey^5+fy^6)}} = \frac{dx}{\sqrt{(a+bx^2+cx^3+dx^4+ex^5+fx^6)}}$$

ad quocunque dimensiones variables x et y in vinculis radicalibus affurgant, aequationem dari integram completam algebraicam. Hoc enim assertum non solum verum est ostensum, quando potestates ipsarum x et y quartum ordinem non superant, sed etiam casu $n = 6$, vti vidimus, priorum formularum integratio completa algebraice succedit. Interim tamen nullus adhuc modus patet pro casu $n = 5$ integrale completum aequationis $\frac{dy}{\sqrt{(f+g^2y^5)}}$ exhibendi, multo minus id ad casus, Tom. XII. Nqv. Corp. I

quibus n fenarium superat extendere licet, etiam si pro casibus $n=1$, $n=2$, $n=3$, $n=4$, et $n=6$ sit in promptu. His autem de successu in reliquis casibus vix dubitare licet, tamen restrictio necessaria videtur, ut exponents n sit numerus integer, nisi forte et eos casus fractionum adicere lubuerit, quibus utraque formula per se sit integrabilis, uti evenit si n sit fractio unitatem pro numeratore habens. Præterea vero certum est, veritatem non nisi pro signo radicali quadrato subsistere posse; ne-

que enim hæc æquatio
$$\frac{dy}{\sqrt{f+gx^2}} = \frac{dx}{\sqrt{f+gx^2}}$$

neque hæc
$$\frac{dy}{\sqrt{f+gy^2}} = \frac{dx}{\sqrt{f+gx^2}}$$
 aliæque harum similes integralia completa algebraica admittunt; quia hæc formulæ, ad rationalitatem perductæ, tam logarithmos quam quadraturam circuli mixtum involuunt, atque ex talium quantitatum heterogenearum comparatione æquatio algebraica resultare nequit. Hæc eadem vero ratio dubitationem superiorem quoque decedit; ac iam audacter pronuntiare possumus hanc æquationem differentialem:

$$\frac{dy}{\sqrt{(a+by^2+c)^2 + d^2 + e^2y^2}} = \frac{dx}{\sqrt{(a+bx^2+c)^2 + d^2 + e^2x^2}}$$

generaliter per æquationem algebraicam integrari non posse; inde enim sequeretur integratio algebraica huius æquationis

$$\frac{dy}{A+By+Cy^2+Dy^3} = \frac{dx}{A+Bx+Cx^2+Dx^3}$$

quod

quod vtiqve esset absurdum; multo minus igitur integratio in æquationibus magis compositis succederet. Verum nequidem integrabilis ad potestatem quintam vsque extendi potest; nam posito $g=0$ si etiam factuatur $a=0$, et pro y et x scribatur y^2 et x^2 , prodit hæc æquatio differentialis:

$$\frac{dy}{\sqrt{(b+cy^2+fy^3)^2 + d^2 + e^2y^2}} = \frac{dx}{\sqrt{(b+cx^2+fx^3)^2 + d^2 + e^2x^2}}$$

in qua si radicis extractio succedat, continetur hæc:

$$\frac{dy}{A+By^2+Cy^3} = \frac{dx}{A+Bx^2+Cx^3}$$

quam ingenere integrationem algebraicam non admittere est manifestum.

Scholion 2.

38 Nunc igitur pro certo affirmare licet, ex hoc genere æquationem differentialem latissime patentem, quæ quidem generaliter algebraice integrari queat, esse eam ipsam, quam hæcenus tractavimus:

$$\frac{dy}{\sqrt{(A+By+Cy^2+Dy^3+Ey^4)^2 + d^2 + e^2y^2}} = \frac{dx}{\sqrt{(A+Bx+Cx^2+Dx^3+Ex^4)^2 + d^2 + e^2x^2}}$$

et cuius æquationem integralem completam assignavimus. Quam ob causam hæc æquatio multo magis est notari digna, quod in hoc genere est generalissima, quæ integrationem algebraicam admittat. Quoniam igitur eius integrationem iam exposui, operæ pretium erit eius usum in comparatione hinc

nearum

nearum curvarum, quarum elementa per huiusmodi formulas exprimuntur, vberius offendere, si quidem in iis omnia continentur, quae in hoc genere praestari possunt. Atque hac ipsa consideratio nos quoque integrationem huiusmodi aequationum

$$\frac{Mdx}{\sqrt{(A+By+Cz^2+Dy^2+Ez^2) - \sqrt{(A+Bz+Cz^2+Dz^2+Ez^2)}}$$

mandaret, si quidem m et n fuerint numeri integri.

Problema 2.

Tab. I. 39. Si linea curva habeatur, cuius arcus si-
 Fig. 4. ve abscissae, siue applicatae, siue cordae, siue alii cui-
 cunque rectae variabili z ad curvam relatae respon-
 dens sit $= \int \frac{Mdx}{\sqrt{(A+Bz+Cz^2+Dz^2+Ez^2)}}$; deturque in
 hac curva arcus quicumque AB , ab alio quouis
 puncto P arcum abscindere PQ , qui aequalis sit illi
 arcui AB .

Solutio.

Ex coefficientibus datis A, B, C, D, E quae-
 rantur hi alii:

$$\begin{aligned} a &= 4(A M - B B); & \epsilon &= 2B(M - C) + 4AD; & \gamma &= 4AE \\ & & & & & - (M - C)^2 \\ \zeta &= 4(EM - DD); & \epsilon &= 2D(M - C) + 4BE; & \delta &= MM - CC \\ & & & & & + 4(AE + BD) \end{aligned}$$

vbi M denotat novam constantem arbitratiam, at-
 que vidimus hanc aequationem algebraicam:

$$\begin{aligned} 0 &= a + 2\epsilon(x+y) + \gamma(xx+yy) + 2\delta xy + 2\epsilon xy(x+y) \\ & \quad + \zeta xxyy \end{aligned}$$

congruere cum hac transcendente:

$$\int \frac{Mdx}{\sqrt{(A+Bz+Cz^2+Dz^2+Ez^2)}} - \int \frac{Mdx}{\sqrt{(A+Bz+Cz^2+Dz^2+Ez^2)}} = \text{Const.}$$

vbi quantitas constans ita designari debet, vt illi M
 sit constantia. Si iam ponamus in curva propo-
 sita variabilem z puncto Z respondere, curvaeque
 initium in puncto Δ statui, atque ad abbreviandum
 hunc arcum ΔZ ita indicemus $\Pi : z$ vt fit

$$\int \frac{Mdx}{\sqrt{(A+Bz+Cz^2+Dz^2+Ez^2)}} = \Pi : z$$

erit ex aequatione superiori

$$\Pi : z - \Pi : x = \text{Const.}$$

Respondent nunc punctis A et B rectae a et b ,
 punctis vero P et Q rectae p et q , vt sint arcus

$$\Delta A = \Pi : a; \quad \Delta B = \Pi : b; \quad \Delta P = \Pi : p \quad \text{et} \quad \Delta Q = \Pi : q$$

ideoque

$$\text{arcus } AB = \Pi : b - \Pi : a \quad \text{et} \quad \text{arcus } PQ = \Pi : q - \Pi : p$$

ac loco x et y scribamus p et q vt fit

$$\begin{aligned} 0 &= a + 2\epsilon(p+q) + \gamma(pp+qq) + 2\delta pq + 2\epsilon pq(p+q) \\ & \quad + \zeta pppq \end{aligned}$$

erit $\Pi : q - \Pi : p = \text{Const.}$ Quod si ergo constantem
 M ita assumamus, vt facto $p = a$ prodeat $q = b$,
 habebimus:

$$\Pi : q - \Pi : p = \Pi : b - \Pi : a$$

ideoque arcum $PQ =$ arcui AB vt requiritur.
 Constans igitur M , vel si ponamus $M - C = L$ vt
 sit

fit $M=C+L$, constans L ex sequenti aequatione debet defini: $0=4AC-4BB+4AL+2(2BL+4AD)(a+b)+(4AE-LL)(aa+bb)$

$$+2(LL+2CL+4AE+4BD)ab+2(2DL+4BF)ab(a+b)+4(CE-DD+EL)abbb$$

vnde fit:

$$LL = \frac{4L(A+B/a+b) + Cab + Dab(a+b) + Eaabbb}{(b-a)^2}$$

et radice extracta:

$$L = \frac{2(A+Ma+b) + Cab + Dab(a+b) + Eaabbb}{(b-a)^2}$$

sicque exit

$$M = \frac{2A+2B(a+b) + C(ae+bb) + Dade(a+b) + 2Eaabb}{(b-a)^2}$$

$$\pm \frac{1}{(b-a)^2} \sqrt{(A+2Ba+Ca+2Da^2+Ea^3)(A+2Bb+Cbb+2Db^2+Eb^3)}$$

Quo valore inuento si iam designantur valores coefficientium $a, \xi, \gamma, \delta, \epsilon, \zeta$, quoniam ex dato curvae puncto P datur variabilis p , ex ea valor idoneus variabilis q , cui curvae punctum Q responderet, determinabitur per hanc aequationem

$$0 = a + \xi(p+q) + \gamma(pp+qq) + 2\delta pq + 2\epsilon p^2(p+q) + \zeta p^2 q^2$$

ex qua si breuitatis gratia ponamus:

$$\Delta = N/(M-C)^2 + 4M(BD-AE) + 4(ADD+BBE) - 4BCD$$

habebitur:

$$q = \frac{-\epsilon - \delta p - \epsilon p^2 + 2\gamma \sqrt{\Delta} + 2\sqrt{AVA} + 2BB + Cpp + 2Dp^2 + Epp^2}{\gamma + 2\epsilon p + \zeta p^2}$$

sicque

sicque dato arcu AB et puncto P assignabitur punctum Q ut arcus PQ aequalis fiat arcui AB. Reperientur autem ob signum ambiguum bina puncta Q, quorum alterum citra alterum ultra punctum P erit situm.

COROLL. I.

40. Inuento valore q sumit modo a puncto Q vltimus abscindi poterit arcus QR arcui AB aequalis. Posita enim variabili puncto R respondente r , capiatur:

$$r = \frac{-\epsilon - \delta q - \epsilon q^2 + 2\gamma \sqrt{\Delta} + 2Bq + Cqq + Dq^2 + E q^3}{\gamma + 2\epsilon q + \zeta q^2}$$

sicque a puncto P simul abscindetur arcus PR duplus arcus dati AB.

COROLL. 2.

41. Quoniam r hinc duplicem obtinet valorem, notandum est alterum iterum in p abire, quia ante animaduertimus esse:

$$p = \frac{-\epsilon - \delta q - \epsilon q^2 + 2\gamma \sqrt{\Delta} + 2Bq + Cqq + Dq^2 + E q^3}{\gamma + 2\epsilon q + \zeta q^2}$$

quare ut arcus PR euadat duplus, idem signum quod in valore ipsius q fuerit electum, in valore ipsius r capi oportet.

COROLL. 3.

42. Pari modo ultra R reperietur punctum S, vt denuo arcus RS aequalis, sicque angulus PS triplicis

triplex euadat arcus AB, inuenta enim Variabili r valor Variabilis s puncto S respondens hac formula exprimitur :

$$s = \frac{-E - \delta r - \epsilon r r + \gamma \sqrt{A + 2B r + C r r + 2D r^2 + E r^3}}{\gamma + \epsilon r + \delta r r}$$

hocque modo quousque libuerit ulterius progredi licet.

Coroll. 4.

43. Hac ergo repetita operatione a dato puncto P arcus abscindi poterit, qui se habeat ad arcum AB, vt numerus quicumque integer m ad unitatem. Quare si ab alio puncto abscindatur arcus, qui sit ad eundem AB vt alius numerus integer n ad unitatem, duo habebuntur arcus rationem quacunque numeri ad numerum tenentes.

Coroll. 5.

44. Omnium igitur curvarum, quarum arcus Variabili cuiuspiam z respondens huiusmodi formula $\int \frac{a dx}{\sqrt{A + Bz + Cz^2 + Dz^3 + Ez^4}}$ exprimitur, haec est proprietas, vt earum arcus simili modo inter se comparari possint, quo arcus circuli inter se comparare licet. Atque ob rationes supra allegatas haec similitudo cum circulo vix ad alias curvas, nisi quarum significatio ad hanc formulam reduci possit, extendi videtur.

Exem-

Exemplum.

45. Proposita sit linea curva, cuius arcus ad quampiam rectam Variabilem v relatus hac formula integrali $\int \frac{dv}{\sqrt{(a - v^2)}}$ exprimitur, cuiusmodi curvae algebraicae infinitae exhiberi possunt, in qua a puncto P arcus abscindi oporteat PQ, PR, PS, ad datum arcum AB rationem tenentes vel aequalitatis, vel duplam, vel triplam.

Quia haec expressio in nostra forma generali non continetur, eo reducatur ponendo $v^2 = z$ seu $v = \sqrt{z}$; sic enim arcus huic nouae Variabili z respondens erit $= \int \frac{dz}{\sqrt{z(z - a^2)}}$. Fiat ergo $\alpha = 1$ et $A = 0$; $B = 1$; $C = 0$; $D = 0$ et $E = -1$, vnde obtinetur :

$$\alpha = 1; \quad \epsilon = M; \quad \gamma = -MM; \quad \delta = MM; \quad \zeta = -4M$$

ideoque constituta aequatione :

$$0 = -1 + 2M(p+q) - MM(pp+qq) + 2MMpq - 4pq(p+q) - 4Mppqq$$

vnde fit :

$$q = \frac{M + MMp - 2pp + \sqrt{M^2 - 1}(p - p^2)}{MM + 4p + 4Mpp}$$

erit :

$$\int \frac{dq}{\sqrt{q(q-a^2)}} - \int \frac{dp}{\sqrt{p(p-a^2)}} = \text{Const.}$$

$$\text{seu } \Pi : q - \Pi : p = \Pi : b - \Pi : a$$

si quidem a, b, p, q sint valores Variabilis z qui arcibus $\Delta A, \Delta B, \Delta P$ et ΔQ conueniunt. At

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iam

iam constantis M ex datis a et b ita definiti debet
 ut fit

$$0 = -1 + 2M(a+b) - MM(b-a)^2 - 4b(a+b) - 4Maab$$

$$M = \frac{a+b - 2aabb + \sqrt{(a-a^2)(b-b^2)}}{(b-a)^2}$$

$$\text{et } \sqrt{(M-1)} = \frac{\sqrt{a(-a)(1+b+b^2)} + \sqrt{b(1-b)(1+a+a^2)}}{(b-a)}$$

$$\sqrt{(M^2-1)} = \frac{(a+b - \sqrt{a^2+b^2})\sqrt{(a-a^2)(b-b^2)} + (b-a - \sqrt{a^2+b^2})\sqrt{(a-a^2)(b-b^2)}}{(b-a)^2}$$

Invento hoc modo valore constantis M ex data
 quantitate p inuenitur q, atque hinc porro valor
 variabilis r puncto R respondens scilicet:

$$r = \frac{M + MMq - 2qq \pm \sqrt{(M^2-1)(q-q^2)}}{M^2 + q + Mq}$$

siquae a puncto P arcus quicumque multiplex arcus
 dati AB abicindi poterit.

Scholion.

46. Circa huiusmodi curvas singularis affectio
 notari meretur, si enim breuitatis gratia ponamus
 $\sqrt{(a-a^2)} = a$ et $\sqrt{(b-b^2)} = b$ ut fit:

$$M = \frac{a+b - 2aabb + ab}{(b-a)^2} \text{ et } \sqrt{(M^2-1)} = \frac{(a+3b-4ab^2)\sqrt{a} + (b+3a-4a^2)\sqrt{b}}{(b-a)^2}$$

utraq; quantitas radicalis a et b tam affirmative
 quam negative capi potest; unde pro M geminus
 valor habetur; ex quo pro $q = \frac{M + M^2p - 2pp \pm \sqrt{(M^2-1)(p-p^2)}}{M^2 + p + Mpp}$
 ob novam signi ambiguitatem quaterni valores re-
 sultant. Binos quidem natura rei offendit, quia
 punctum Q tam ante quam post punctum P capi
 potest

potest, sed quia quatuor reperiuntur, id indicio
 est curvam duplici ramo esse praeditam, et in
 utroque arcus aequales exhiberi. Consideremus ca-
 sum quo punctum P in ipso puncto A capitur ita
 ut fit $p = a$; et $q = \frac{M + MMa - 2aa \pm \sqrt{(M^2-1)}}{M^2 + a + Ma}$

quae forma substituto pro M valore statim duos va-
 lores praebet aequales $q = b$; at duo reliqui diversi
 continentur in
 $q = \frac{a^2 + aab - aab^2 + 2a^2b - 2a^2b^2 + aab^2 + a^2(1-b-1a^2b)}{a^2 + aab + ab^2 + a^2 - 1a^2b - 1a^2b^2 + 1a^2b^2 + 1a^2b^2 - 1a^2b^2 + 1a^2b^2}$
 qui duo valores semper sunt diversi, nisi sit vel
 $b = a$ vel $a = \pm \sqrt{a}$; illo casu prodit $q = a = b$,
 hoc vero reperitur $q = \pm \frac{b}{a}$. Punctum ergo cur-
 vae quod respondet quantitati $\pm \frac{1}{\sqrt{a}}$ singulari pro-
 prietate erit praeditum.

Problema 3.

47. Invenire integrale completum huius ac-
 quationis differentialis:

$$\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4} - \sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4}$$

Solutio.

Idud integrale quaesitum ex praecedenti pro-
 blemate colligi potest. Capiatur enim punctum P in ipso
 puncto B, ut sit $p = b$; et consideretur tantum pun-
 ctum A ut fixum, B vero seu P ut variabile, ex
 quo continuo assignari debeat punctum Q, ut sit
 arcus

arcus A Q duplus arcus AP. Posita ergo variabili p loco b sumatur :

$$M = \frac{A + B(a+p) + C(ca + pp) + Dq(a+p) + Eacpp}{(p-a)^2} \\ + \frac{1}{(p-a)^2} \sqrt{(A + 2Ba + Ca + 2Da^2 + Ea^3)(A + 2Bp + Cp + 2Dp^2 + Ep^3)}$$

ita vt iam M sit functio variabilis p et constantis a .

Deinde posito breuitatis gratia $M - C = L$ seu

$$L = \frac{A + B(a+p) + Cca + Dq(a+p) + Eacpp}{(p-a)^2} + \frac{1}{(p-a)^2} \sqrt{(A + 2Ba + Ca + 2Da^2 + Ea^3)(A + 2Bp + Cp + 2Dp^2 + Ep^3)}$$

definiatur q per hanc aequationem :

$$0 = 4AC - 4BB + 4AL + 2(2BL + 4AD)(p+q) + (4AE - LL)(pp + qq) \\ + 2(LL + 2CL + 4AE + 4BD)pq + 2(2DL + 4BE)pq(p+q) + 4(CE - DD + EL)ppqq$$

erique ob $b = p$:

$$\text{II} : q - \text{II} : p = \text{II} : p - \text{II} : a \text{ seu } \text{II} : q = 2 \text{II} : p - \text{II} : a$$

quae aequatio differentiatia dat :

$$\frac{1}{\sqrt{(A + 2Bp + Cp + 2Dp^2 + Ep^3)}} = \frac{1}{\sqrt{(A + 2Bp + Cp + 2Dp^2 + Ep^3)}} \frac{d}{dp}$$

cuius propterea integralis est illa aequatio algebraica inter p et q exhibita, quam simul patet esse integram completam, quoniam continet quantitatem constantem a , quae in aequatione differentiali non inest.

Coroll. I.

48. Si recinente L valorem exhibitum, inventaque variabili q per p , ex q simili modo quaeratur

ratur r , vt sit $\text{II} : r - \text{II} : q = \text{II} : p - \text{II} : a$ erit $\text{II} : r = 3 \text{II} : p - 2 \text{II} : a$, unde prodit aequatio differentialis :

$$\frac{dr}{\sqrt{(A + 2Br + Cr + 2Dr^2 + Er^3)}} = \frac{1}{\sqrt{(A + 2Bp + Cp + 2Dp^2 + Ep^3)}} dp$$

cuius ergo aequatio integralis completa est :

$$0 = 4(AC - BB + AL) + 2(2BL + 4AD)(q+r) + (4AE - LL)(qq + rr) \\ + 2(LL + 2CL + 4AE + 4BD)qr + 2(2DL + 4BE)q(r+r) + 4(CE - DD + EL)qqr$$

Coroll. 2

49. Quo haec magis contrahamus, postquam ex coefficientibus datis A, B, C, D, E et variabili p vna cum constanti arbitraria a ita fuerit definita quantitas L vt sit :

$$\text{II}(\text{II} - a)^2 = \frac{A + B(a+p) + Ccp + Dq(a+p) + Eacpp}{\sqrt{(A + 2Bp + Cp + 2Dp^2 + Ep^3)}} \\ + \frac{1}{\sqrt{(A + 2Bp + Cp + 2Dp^2 + Ep^3)}} \frac{d}{dp}$$

hinc determinentur sequentes coefficientes variables :

$$a = 4(AC - BB + AL); \epsilon = 2BL + 4AD; \gamma = 4AE - LL \\ \zeta = 4(CE - DD + EL); \epsilon = 2DL + 4BE; \delta = LL + 2CL + 4AE + 4BD.$$

Coroll. 3.

50. His iam quantitatibus inuentis erit huius aequationis differentialis :

$$\frac{d}{dp} \frac{1}{\sqrt{(A + 2Bp + Cp + 2Dp^2 + Ep^3)}} = \frac{1}{\sqrt{(A + 2Bp + Cp + 2Dp^2 + Ep^3)}} \frac{d}{dp}$$

aequatio integralis completa :

$$0 = a + 2\epsilon(p+q) + \gamma(pp + qq) + 2\delta pq + 2\epsilon pq(p+p) + \zeta pppqq$$

K 3

Coroll.

Coroll. 4.

51. Porro huius aequationis differentialis :

$$\frac{dx}{\sqrt{A+Bx+Cx^2+Dx^3+Ex^4}} = \frac{rdp}{\sqrt{A+Bp+Cp^2+Dp^3+Ep^4}}$$

aequatio integralis completa erit

$$0 = a + 2\delta(q+r) + \gamma(qq+rr) + 2\delta qr + 2\epsilon q^2(r+s) + \zeta qqr^2$$

postquam scilicet variabilis q ope praecedentis aequationis ex p fuerit determinata.

Coroll. 5.

52. Simili modo progrediendo huius aequationis differentialis :

$$\frac{dx}{\sqrt{A+Bx+Cx^2+Dx^3+Ex^4}} = \frac{rdp}{\sqrt{A+Bp+Cp^2+Dp^3+Ep^4}}$$

aequatio integralis completa erit

$$0 = a + 2\delta(r+s) + \gamma(rr+ss) + 2\delta rs + 2\epsilon rs(r+s) + \zeta rrs^2$$

postquam ex praecedentibus aequationibus r per q , et q per p fuerint definitae.

Coroll. 6.

53. Hoc modo quousque libuerit ulterius progredi licet, siquae ingenerae aequatio integralis inveniri poterit completa huius differentialis :

$$\frac{dx}{\sqrt{A+Bx+Cx^2+Dx^3+Ex^4}} = \frac{rdp}{\sqrt{A+Bp+Cp^2+Dp^3+Ep^4}}$$

quicumque numerus integer pro m assumatur

Proble-

Problema 4.

54. Si m et n fuerint numeri integri quicunque, invenire aequationem integram completam huius differentialis :

$$\frac{dx}{\sqrt{A+Bx+Cx^2+Dx^3+Ex^4}} = \frac{rdp}{\sqrt{A+Bp+Cp^2+Dp^3+Ep^4}}$$

Solutio.

Quaeratur primum ope praeced. Probl. aequatio integralis completa istius differentialis :

$$\frac{dx}{\sqrt{A+Bx+Cx^2+Dx^3+Ex^4}} = \frac{rdp}{\sqrt{A+Bp+Cp^2+Dp^3+Ep^4}}$$

quae erit algebraica ac praeter variables p et x constantem arbitrariam a involvens. Deinde simili modo quaeratur aequatio integralis completa huius differentialis :

$$\frac{dx}{\sqrt{A+Bx+Cx^2+Dx^3+Ex^4}} = \frac{rdp}{\sqrt{A+Bp+Cp^2+Dp^3+Ep^4}}$$

quae etiam erit algebraica inter binas variables y et p , insuperque constantem arbitrariam b complectetur. Ex his duabus aequationibus eliminetur variabilis p , ut obtineatur aequatio algebraica inter x et y , quae erit integralis completa huius differentialis :

$$\frac{dx}{\sqrt{A+Bx+Cx^2+Dx^3+Ex^4}} = \frac{rdp}{\sqrt{A+Bp+Cp^2+Dp^3+Ep^4}}$$

Quia autem duas constantes arbitrarias a et b continebit, alterutri pro lubitu valorem determinatum tribuere licet, vel inter eas datam rationem statueri:

re, pro integrali enim completa sufficit ut una constans arbitraria introducatur.

Scholion.

55. Si m et n sint numeri modice magni, nemo certe aequationem algebraicam inter x et y euolutam exhibebit: cum enim tot eliminationibus sit opus, evidens est, ad aequationem plurimorum terminorum, in qua variables x et y ad summam dimensionum exsurgant, perueniri oportere. Atque adeo in casu problematis 3. Vbi est $m=2$ et $n=1$, nemo facile eliminationis opus perficiet. Neque vero hoc etiam opus est, cum ad nostrum insitutum sufficiat offendisse, aequationem integram esse algebraicam, eiusque constructionem geometricè absolui posse; tantum enim abest, ut alienae variables q , r , s etc. quae in subsidium sunt vocatae, calculum turbent, ideoque eliminari debeant, ut potius ad constructionem commode instituendam absolutione sint necessariae. Atque haec sunt fere quae de curvis, quarum rectificatio hac formula:

$$\int \frac{Mdz}{\sqrt{(A+Bz+Cz^2+Dz^3+Ez^4)}}$$

exprimitur, tradi operae pretium videbatur, quae eo redeunt, ut earum arcus inter se perinde atque arcus circulares comparari queant; siquidem proposito arcu quocunque AB, a puncto dato P arcus abscindi possunt, qui ad illum rationem teneant rationalem quamcunque. Consideremus igitur etiam

curvas,

quarum rectificatio tali formula exprimitur:

$$\int \frac{de(M+Bz+Cz^2+Dz^3+Ez^4)}{\sqrt{(A+Bz+Cz^2+Dz^3+Ez^4)}}$$

de quibus curvis quoque affectiones egregie circa comparationem arduum notari mereantur; quæ in suam euolutio formularum huc pertinentium supra §. 16 et seqq. est instituta. Similis scilicet comparatio inter arcus huiusmodi constructum suscipi potest, quæ iam prædem inter arcus parabolæ fieri potest est offensa; atque inde sequentium problematum solutionem derivare licebit.

Problema 5.

56. Proposita curva, cuius arcus indefinitè Tab. I. variabili cuiuspiam z respondens hac formula exprimitur:

$$\int \frac{dz(M+Bz+Cz^2+Dz^3+Ez^4)}{\sqrt{(A+Bz+Cz^2+Dz^3+Ez^4)}}$$

si in ea detur arcus quicumque AB, a dato puncto P arcum abscindere PQ qui ab illo arcu AB distinet linea siue geometricè assignabili, siue a circuli hyperbolæque quadratura pendente.

Solutio.

Sit in curva proposita AZ arcus variabili z respondens, qui breuitatis gratia ita exprimitur $\Pi:z$, ut sit:

$$\Pi:z = \int \frac{dz(M+Bz+Cz^2+Dz^3+Ez^4)}{\sqrt{(A+Bz+Cz^2+Dz^3+Ez^4)}}.$$

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Punctis

Punctis autem A, B, P, Q respondeant variabilis α valores a, b, p, q : ut sit

$$\Delta A = \Pi : a; \Delta B = \Pi : b; \Delta P = \Pi : p \text{ et } \Delta Q = \Pi : q$$

hincque erit

$$\text{arcus datus } AB = \Pi : b - \Pi : a$$

et arcus quaesitus PQ = $\Pi : q - \Pi : p$.

Item primam ex coefficientibus A, B, C, D, E et constanti arbitraria M deinceps designanda formentur quantitates sequentes :

$$\alpha = 4(AM - BB); \epsilon = 2B(M - C) + 4AD; \gamma = 4AE - (M - C); \\ \xi = 4(EM - DD); \epsilon = 2D(M - C) + 4BE; \delta = MM - CC \\ + 4(AE + BD)$$

tam vero porro statuantur :

$$\Delta = M(M - C)^2 + 4M(BD - AE) + 4(ADD + BBE) - 4BCD$$

atque inter p et q haec constituantur relatio ut sit

$$0 = \alpha + 2\epsilon(p + q) + \gamma(pp + qq) + 2\delta pq + 2\epsilon p q(p + q) \\ + \xi p p q q$$

ex qua data variabili p altera q puncto Q respondens ita definitur ut sit

$$q = \frac{-\epsilon - \delta p - \epsilon p p + \gamma \sqrt{\Delta(A + 2Bp + Cp^2 + 2Dp^2 + Ep^3)}}{\gamma + 2\epsilon p + \xi p p}$$

vnde innotebit curvae punctum Q ita, ut differentia inter arcus AB et PQ sit vel geometricae assignabilis, vel saltem a quadratura circuli seu hyperbolae pendeat, cuius rei ratio in indole coefficientium $\alpha, \epsilon, \gamma, \delta, \xi$ numeratoris est sita. Quomodo

modo igitur differentia ista exprimitur, videmus: quia valorem ipsius q iam invenimus, ponamus $p + q = s$,

et ex §. 19 colligimus fore posito $\delta - \gamma = \lambda$

$$\Pi : q - \Pi : p \text{ Const.} = \frac{\lambda D + \epsilon D^2 \sqrt{\Delta}}{\xi \sqrt{M + 2Bp + Cp^2}} ds$$

quod integrale manifestum est vel esse algebraicum, vel a quadratura circuli hyperbolae pendere. Si istud integrale brevioris gratia = S; cuius valor posito $s = a + b$ fiat = I, et pro constante designanda statuantur $p = a$ et $q = b$, serique debet

$$\text{Const.} = \Pi : b - \Pi : a + \frac{(D + \epsilon(a + b)) \sqrt{\Delta}}{\xi} - I.$$

ex quo habebitur :

$$\text{arcus PQ} - \text{arcu AB} = \frac{D(a + b) + \epsilon(a + b)^2}{\xi} \sqrt{\Delta} - \frac{Dp + \epsilon p^2 - \epsilon p p q^2}{\xi} \sqrt{\Delta} \\ - I + \int \frac{(D + \lambda D + \epsilon \epsilon + \epsilon D + \lambda \epsilon) \sqrt{\Delta} + (D + \epsilon D + \epsilon \epsilon)^2}{\xi \sqrt{(M + 2Bp + Cp^2)}} ds.$$

At constans arbitraria M etiam ita designari debet, ut posito $p = a$ fiat $q = b$; quocirca erit :

$$M = \frac{(D + \epsilon)^2}{\xi} (2A + 2B(a + b) + C(aa + bb) + 2Daa(a + b) + 2Eaabb) \\ + \frac{(D + \epsilon)^2}{\xi} \sqrt{(A + 2Ba + Caa + 2Da^2 + Ea^2)(A + 2Bb + Cbb + 2Db^2 + Eb^2)}.$$

Hinc ergo cognita constans haec M, et ex puncto P definito puncto Q, differentia arcuum AB et PQ vel geometricae vel per quadraturam circuli hyperbolae assignari poterit.

L 2

Coroll.

Coroll. 1.

57. Ex datis ergo punctis A et B, seu variabilis 2 valoribus a et b primum constans arbitraria M ita definiatur, vt sit

$$M = \frac{1}{(a-b)^2} (2A + 2B(a+b) + C(aa+bb) + 2Dab(a+b) + 2Eaabb + \frac{1}{(a-b)^2} \sqrt{(A+2Ba+Caa+2Da^2+Ea^2)(A+2Bb+Cbb + 2D)b^2 + Ebb^2})$$

Tum hinc definitis modo praecepto coefficientibus a, b, γ, δ, ε, ζ ex dato puncto P punctum Q per hanc aequationem determinetur :

$$0 = a + 2\epsilon(p+q) + \gamma(pp+qq) + 2\delta pq + 2\epsilon pq(p+q) + \zeta pppq$$

atque arcuum PQ et AB differentia erit vel algebraica vel a circuli hyperbolae quadratura pendens.

Coroll. 2.

58. Ad istam autem arcuum differentiam assignandam capi debet positio $p+q = s$ hoc integrale, vbi $\lambda = \delta - \gamma = 2M(M-C) + 4BD$

$$s = \sqrt{\frac{(D+\lambda D) + (\epsilon C + D) + \lambda \epsilon C + (\epsilon D + \lambda \epsilon D) + \epsilon C^2}{\epsilon \sqrt{(M + D\epsilon + B\epsilon)}}} ds$$

cuius valor positio $s = a + b$ sit = 1, quo facto erit

$$\text{arc. PQ} - \text{arc. AB} = \frac{1}{\zeta} \Delta (\mathfrak{D}(a+b) + \mathfrak{E}(a+b)^2 - \mathfrak{D}s - \mathfrak{E}s^2) - 1 + S$$

existente $\Delta = M(M-C)^2 + 4M(BD - AE) + 4(ADD + BBE) - 4BCD$.

Coroll.

Coroll. 3.

59. Si eveniret vt esset $\zeta = 0$, determinatio puncti Q maneret vt ante, sed pro arcuum PQ et AB differentia assignanda recurri deberet ad priores operationes. Scilicet ex $p+q = s$, quaeratur vt sit :

$$0 = a + 2\epsilon s + \gamma s^2 + 2\lambda s + 2\epsilon s^2 + \zeta s^3$$

eritque

$$\text{arc. PQ} - \text{arc. AB} = 2 \sqrt{\frac{d(\mathfrak{D} + \epsilon C + \mathfrak{D}(\epsilon - 1) + \mathfrak{E}(\epsilon(\epsilon - 1))) \sqrt{\Delta}}{\sqrt{\lambda - \epsilon \zeta + 2(\lambda \epsilon - \zeta^2) s^2 + (\epsilon^2 - \zeta^2) s^3}}}$$

integrali hoc ita accepto vt euascat positio $s = a + b$ Vbi notandum est, esse :

$$\sqrt{(\lambda - \epsilon \zeta + 2(\lambda \epsilon - \zeta^2) s^2 + (\epsilon^2 - \zeta^2) s^3)} = 2 \sqrt{\Delta (M + 2Ds + Es^2)} = \lambda + \epsilon s + \zeta s^2$$

Coroll. 4.

60. Hinc etiam colligere licet, quanam sit futura differentia arcuum AB et PQ, si formulae elementum curvae exhibentis numerator ad plures terminos extendatur, vt sit arcus curvae :

$$\int \frac{dx (\mathfrak{D} + \mathfrak{E}x + \mathfrak{D}x^2 + \mathfrak{E}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \mathfrak{D}x^6 + \mathfrak{E}x^7 + \mathfrak{D}x^8 + \mathfrak{E}x^9)}{\sqrt{(A + Bx + Cx^2 + Dx^3 + Ex^4)}}$$

reliquis enim invariantibus vt ante, esse

$$\text{arc. PQ} - \text{arc. AB} = \frac{d(\mathfrak{D} + \mathfrak{E}x + \mathfrak{D}x^2 + \mathfrak{E}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \mathfrak{D}x^6 + \mathfrak{E}x^7 + \mathfrak{D}x^8 + \mathfrak{E}x^9)}{\sqrt{(M + Dx + Ex^2)}}$$

sequentia scilicet numeratoris membra erunt :

$$\mathfrak{D}(s^8 - 4s^7 + 3s^6) + \mathfrak{E}(s^7 - 5s^6 + 6s^5 - s^4) \text{ etc.}$$

L. 3. Coroll.

Coroll. 5.

61. Si a puncto Q simili modo abscindatur R, vt sit

$$0 = a + a\delta(q+r) + \gamma(qg+r^2) + 2\delta qr + 2eqr(q+r) + \zeta\zeta qqr$$

ponaturque $q+r=U$ et $qr=0$, ita vt sit

$$0 = a + 2\delta U + \gamma U U + 2\lambda v + 2\epsilon U v + \zeta v v$$

seu $\lambda + \epsilon U + \zeta v = 2\sqrt{\Delta}(M + 2Du + Euv)$ erit

$$\text{arc. PR} = 2 \text{arc. AB} = \int \frac{ds(\delta + \epsilon s + 2\delta s^2 - 1 + \epsilon(s^2 - 2s)) + \pi\epsilon s}{\sqrt{M + 2Ds + Es^2}} + \int \frac{du(\delta + \epsilon u + 2\delta u^2 - v) + \epsilon(\delta u^2 - uv) + \pi\epsilon u}{\sqrt{M + 2Du + Euv}}$$

his integralibus ita sumtis vt evanescant posito $s=a+b$ et $u=a+b$.

Coroll. 6.

62. Simili modo a puncto P abscindi potest arcus PS, qui triplum dati arcus AB superet quantitate sine geometricae assignabili sine a circuli hyperbolae quadratura pendente: hisque casibus punctum P ita assumi poterit, vt isse excessus plane evanescat, quod quidem semper praestare licebit, si excessus sit algebraicus; sin autem sit transcendens, insuper alter terminus arcus dati A vel B huic scopo conformiter determinabitur.

DE VSV

ALGORITHMI INFINITESIMALIS

IN ARTE CONIECTANDI SPECIMEN.

Auctore

DANIELE BERNOLLII.

S. I.

Cum nuper de argumento cogitarem, cuius examen in proximam differam occasionem, in quaestionem incidi, quae ad artem coniectandi pertinet, quia vero ipsa haec quaestio coniecturalis altero argumento videri potest profus aliena, non incongruum putavi eam seorsum praemittere, siquae disquisitionibus proximis veluti viam sternere, et quidem tanto libentius, vt hoc facerem, animum induxi, quod ipsa methodus nostra mox explicanda aliquid conferre posse videatur ad noua in arte coniectandi principia formanda ac stabilienda, nondum quod sciam adhibita tanquam magis geometrarum attentione digna. Quoties nempe sit, vt forte coniecturae variae rerum status permittetur, veluti cum schedulae, diversis numeris inscriptis distinctae, successu ex vna extrahuntur, vna post alteram atque leges quaeruntur pro variis inde natis mutationibus determinandis, calculi infinitesimalis vtiliter adhiberi possunt ad negotium istud perficien-