1768

Integratio aequationis $dx/\sqrt{(A+Bx+Cx^2+Dx^3+Ex^4)} = dy/\sqrt{(A+By+Cy^2+Dy^3+Ey^4)}$

Leonhard Euler

INTEGRATIO AEQVATIONIS.

$$\frac{dx}{\sqrt{(A+Bx+Cx^2+Dx^3+Ex^4)}}=\frac{dy}{\sqrt{(A+By+Cy^2+Dy^3+Ey^4)}}$$

Auctore

L. EVLERO.

1.

Methodo admodum singulari atque obliqua perueneram olim ad integrationem huius aequationis, cuius integrale idque adeo completum aequatione algebraica inter x et y contineri deprehendi. Quod eo magis mirum videtur, quod vtriusque formulae seorsim integrale non solum non algebraice, sed ne per circuli quidem hyperbolaeue quadraturam exprimi potest. Tum vero id imprimis notatu dignum occurrebat, quod nulla methodus directa patebat, istud integrale algebraicum eruendi. Nulla autem occasio magis idonea videtur, fines Analyseos proferendi, quam si, quod methodo obliqua quasi per ambages elicuerimus, idem methodo directa inuestigare annitamur. Cum igitur nuper curuas definiuerim, quas corpus ad duo centra virium fixa attractum percurrit, easque ad similem aequationem perduxerim, inde vicissim hu-

 ius

ius aequationis integrationem petere licebit; quod quomodo sit praestandum, hic explicare constitui.

2. Ac primo quidem obseruo aequationem propositam semper in eiusmodi formam transfundi posse, in qua coefficientes B et D euanescant, quod quidem de alterutro ex elementis satis est notum. Vt autem ambo simul ad nihilum redigi queant, id talis formae est proprium: posito enim $x=\frac{mz+a}{nz+b}$, prior forma, cuiquidem altera est similis, abit in hanc:

$$\frac{(mb-na)\,dz}{\sqrt{(A(nz+b)^4+B(nz+b)^3(mz+a)+C(nz+b)^2(mz+a)^2+D(nz+b)(mz+a)^3+E(mz+a)^4)}}$$

in cuius denominatore terminos tam ipsa quantitate z quam eius cubo z^3 affectos destruere licebit. Prior conditio praebet hanc aequationem:

$$4Anb^3+Bmb^3+3Bnabb+2Cmabb+2Cnaab+3Dmaab+Dna^3+4Ema^3=0$$

posterior vero hanc:

$$4An^3b+Bn^3a+3Bmnnb+2Cmmna+2Cmmnb+3Dmmna+Dm^3b+4Em^3a=0$$

vnde tam ratio $a:b$ quam ratio $m:n$ elici potest.

3. Ponamus enim $a=bp$ et $m=nq$, vt habeamus has aequationes:

$$4A+Bq+3Bp+2Cpq+2Cpp+3Dppq+Dp^3+4Ep^3q=0$$

$$4A+Bp+3Bq+2Cpq+2Cqq+3Dpqq+Dq^3+4Epq^3=0$$

quarum

quarum differentia per $p-q$ diuisa praebet

$$2B+2C(p+q)+D(pp+4pq+qq)+4Epq(p+q)=0.$$

Tum vero prior per q demta posteriore per p multiplicata dat diuisione per $p-q$ facta:

$$-4A-B(p+q)+Dpq(p+q)+4Eppqq=0$$

statuamus nunc $p+q=r$ et $pq=s$, et ex aequationibus

$$2B+2Cr+Drr+2Ds+4Ers=0$$
$$-4A-Br+Drs+4Ess=0$$

elidendo $r=\frac{4(A-Ess)}{Ds-B}$ adipiscimur hanc aequationem cubicam:

$$\left.\begin{matrix}+D^3\\-4CDE\\+8BEE\end{matrix}\right\}s^3\left.\begin{matrix}-BDD\\+4BCE\\-8ADE\end{matrix}\right\}s^2\left.\begin{matrix}-BBD\\+4ACD\\-8ABE\end{matrix}\right\}s\begin{matrix}+B^3\\-4ABC=0\\+8AAD\end{matrix}$$

vnde incognita s definitur, quod igitur triplici modo fieri poterit.

4. Cum igitur sine detrimento scopi praefixi coefficientes B et D nihilo aequales assumere liceat, quaestio nostra in integrali huius aequationis inueniendo versatur

$$\frac{dx}{\sqrt{(A+Cxx+Dx^4)}}=\frac{dy}{\sqrt{(A+Cyy+Dy^4)}}$$

quam hoc modo repraesentemus:

$$\frac{dx}{dy}=\sqrt{\frac{A+Cxx+Dx^4}{A+Cyy+Dy^4}}$$

vnde relationem inter variabiles x et y generatim elici oportet, id quod sequenti modo praestare conabor.

5. Ponamus primo $x=n\sqrt{pq}$ et $y=n\sqrt{\frac{p}{q}}$, erit:

$dx=\frac{n(qdp+pdq)}{2\sqrt{pq}}$ et $dy=\frac{n(qdp-pdq)}{2q\sqrt{pq}}$, hincque $\frac{dx}{dy}=\frac{q(qdp+pdq)}{qdp-pdq}$. Porro autem est $\frac{A+Cxx+Dx^4}{A+Cyy+Dy^4}=\frac{qq(A+nCpq+nnDppqq)}{Aqq+nCpq+nnDpp}$, vnde fit: $\frac{qdp+pdq}{qdp-pdq}=\sqrt{\frac{A+nCpq+nnDppqq}{Aqq+nCpq+nnDpp}}$

vbi nunc numerus n ad commodum nostrum assumi potest.

6. Sit breuitatis gratia

$$\frac{A+nCpq+nnDppqq}{Aqq+nCpq+nnDpp}=\frac{P+Q}{P-Q}$$

erit $\frac{P}{Q}=\frac{A(1+qq)+2nCpq+nnDpp(1+qq)}{A(1-qq)-nnDpp(1-qq)}=\frac{(A+nnDpp)(1+qq)+2nCpq}{(A-nnDpp)(1-qq)}$.

Tum vero ob $\frac{qdp+pdq}{qdp-pdq}=\sqrt{\frac{P+Q}{P-Q}}$ obtinebimus

$$\frac{qdp}{pdq}=\frac{\sqrt{(P+Q)}+\sqrt{(P-Q)}}{\sqrt{(P+Q)}-\sqrt{(P-Q)}}=\frac{P+\sqrt{(PP-QQ)}}{Q}$$

et $\frac{pdq}{qdp}=\frac{P-\sqrt{(PP-QQ)}}{Q}$.

7. Omne iam momentum versatur in idonea substitutione; atque equidem hac vtendum obseruaui:

$q=u+\sqrt{(uu-1)}$, vnde fit $\frac{dq}{q}=\frac{du}{\sqrt{(uu-1)}}$, et porro $1+qq=2qu$; $1-qq=-2q\sqrt{(uu-1)}$, ex quo conficitur $\frac{P}{Q}=\frac{(A+nnDpp)u+nCp}{(nnDpp-A)\sqrt{(uu-1)}}$

ac nunc quidem pro n vnitatem commodissime assumi euidens est. Cum ergo sit $\frac{P}{Q}=\frac{(A+Dpp)u+Cp}{(Dpp-A)\sqrt{(uu-1)}}$ erit $\frac{\sqrt{(PP-QQ)}}{Q}=\frac{\sqrt{(4ADppuu+2Cp(A+Dpp)u+CCpp+(Dpp-A)^2)}}{(Dpp-A)\sqrt{(uu-1)}}$.

ita

ita vt noſtra aequatio integranda ſit:

$$\frac{p\,du}{dp}=\frac{(A+Dpp)u+Cp-\surd(4ADppuu+2Cpu(A+Dpp)+CCpp+(Dpp-A)^2)}{Dpp-A}.$$

8. Iſta formula irrationalis hoc modo repraeſentetur:

$$\surd\left(\left(2pu\surd AD+\frac{C(A+Dpp)}{2\surd AD}\right)^2+\frac{(4AD-CC)(Dpp-A)^2}{4AD}\right)$$

ac ponatur

$$2pu\surd AD+\frac{C(A+Dpp)}{2\surd AD}=\frac{(Dpp-A)s\surd(4AD-CC)}{2\surd AD}$$

vnde fit ipſa formula ſurda $=\frac{(Dpp-A)\surd(4AD-CC)(1+ss)}{2\surd AD}$

et

$$u=-\frac{C(A+Dpp)}{4ADp}+\frac{(Dpp-A)s\surd(4AD-CC)}{4ADp}$$

hincque

$$(A+Dpp)u+Cp=\frac{-C(Dpp-A)^2+(A+Dpp)(Dpp-A)s\surd(4AD-CC)}{4ADp}$$

ita vt iam noſtra aequatio ſit:

$$\frac{p\,du}{dp}=\frac{-C(Dpp-A)+(A+Dpp)s\surd(4AD-CC)}{4ADp}-\frac{\surd(4AD-CC)(1+ss)}{2\surd AD}.$$

9. Inde vero colligimus:

$$du=\frac{-Cdp(Dpp-A)}{4ADpp}+\frac{sdp(A+Dpp)\surd(4AD-CC)}{4ADpp}+\frac{ds(Dpp-A)\surd(4AD-CC)}{4ADp}$$

ita vt obtineamus

$$\frac{p\,du}{dp}=\frac{-C(Dpp-A)}{4ADp}+\frac{s(A+Dpp)\surd(4AD-CC)}{4ADp}+\frac{ds(Dpp-A)\surd(4AD-CC)}{4ADdp}$$

qua formula praecedenti aequata commodiſſime vſu venit, vt plerique termini ſponte ſe tollant, indeque exſurgat haec aequatio:

$$\frac{ds(Dpp-A)\surd(4AD-CC)}{4ADdp}=\frac{-\surd(4AD-CC)(1+ss)}{2\surd AD}$$

vnde

vnde naſcitur $\frac{ds}{\surd(1+ss)}=\frac{-2dp\surd AD}{Dpp-A}=\frac{2dp\surd AD}{A-Dpp}$

cuius integrale in logarithmis eſt

$$l(s+\surd(1+ss))=l\frac{\surd A+p\surd D}{\surd A-p\surd D}+l\alpha$$

ita vt habemus

$$s+\surd(1+ss)=\frac{\alpha\surd A+\alpha p\surd D}{\surd A-p\surd D}\text{ hincque}$$

$$s=\frac{\alpha\alpha(\surd A+p\surd D)^2-(\surd A-p\surd D)^2}{2\alpha(A-Dpp)}.$$

10. Quodſi hinc regrediamur, reperiemus

$$u=\frac{-C(A+Dpp)}{4ADp}+\frac{(\surd A-p\surd D)^2-\alpha\alpha(\surd A+p\surd D)^2}{4\alpha ADp}\surd(4AD-CC)$$

vnde definiri oportet $q=u+\surd(uu-1)$. Sed quia hinc fit $u=\frac{1+qq}{2q}$, reſtituendo $p=xy$ et $q=\frac{x}{y}$, aequatio noſtra integralis completa eſt

$$\frac{xx+yy}{2xy}=\frac{-C(A+Dxxyy)}{4ADxy}+\frac{(\surd A-xy\surd D)^2-\alpha\alpha(\surd A+xy\surd D)^2}{4\alpha ADxy}\surd(4AD-CC)$$

ſeu

$$4AD(xx+yy)+2C(A+Dxxyy)=\frac{\surd(4AD-CC)}{\alpha}((\surd A-xy\surd D)^2-\alpha\alpha(\surd A+xy\surd D)^2)$$

quae euoluitur in hanc

$$\frac{4AD(xx+yy)+2C(A+Dxxyy)}{\surd(4AD-CC)}=\frac{(1-\alpha\alpha)A-2(1+\alpha\alpha)xy\surd AD+(1-\alpha\alpha)Dxxyy}{\alpha}$$

et ponendo $\alpha=\frac{\surd(4AD-CC)}{mC}$ prodit

$$4AD(xx+yy)+2C(A+Dxxyy)=\frac{((1+mm)CC-4AD)(A+Dxxyy)-2((mm-1)CC+4AD)xy\surd AD}{mC},$$

11. Ne caſus, vbi $\surd AD$ fit quantitas imaginaria, turbent, iuuabit integrationem alia via, quae ipſa deſtructione terminorum §. 9. obſeruata

vnde nafcitur $\frac{ds}{\sqrt{(1+ss)}} = \frac{-2dp\sqrt{AD}}{Dpp-A} = \frac{2dp\sqrt{AD}}{A-Dpp}$
cuius integrale in logarithmis eft

$$l(s+\sqrt{(1+ss)}) = l\frac{\sqrt{A}+p\sqrt{D}}{\sqrt{A}-p\sqrt{D}} + l\alpha$$

ita vt habemus

$$s+\sqrt{(1+ss)} = \frac{\alpha\sqrt{A}+\alpha p\sqrt{D}}{\sqrt{A}-p\sqrt{D}} \text{ hincque}$$

$$s = \frac{\alpha\alpha(\sqrt{A}+p\sqrt{D})^2-(\sqrt{A}-p\sqrt{D})^2}{2\alpha(A-Dpp)}.$$

10. Quodfi hinc regrediamur, reperiemus

$$u = \frac{-C(A+Dpp)}{4ADp} + \frac{(\sqrt{A}-p\sqrt{D})^2-\alpha\alpha(\sqrt{A}+p\sqrt{D})^2}{4\alpha ADp}\sqrt{(4AD-CC)}$$

vnde definiri oportet $q = u+\sqrt{(uu-1)}$. Sed quia hinc fit $u = \frac{1+qq}{2q}$, reftituendo $p = xy$ et $q = \frac{x}{y}$, aequatio noftra integralis completa eft

$$\frac{xx+yy}{2xy} = \frac{-C(A+Dxxyy)}{4ADxy} + \frac{(\sqrt{A}-xy\sqrt{D})^2-\alpha\alpha(\sqrt{A}+xy\sqrt{D})^2}{4\alpha ADxy}\sqrt{(4AD-CC)}$$

feu

$$4AD(xx+yy)+2C(A+Dxxyy) = \frac{\sqrt{(4AD-CC)}}{\alpha}((\sqrt{A}-xy\sqrt{D})^2 - \alpha\alpha(\sqrt{A}+xy\sqrt{D})^2)$$

quae euoluitur in hanc

$$\frac{4AD(xx+yy)+2C(A+Dxxyy)}{\sqrt{(4AD-CC)}} = \frac{(1-\alpha\alpha)A-2(1+\alpha\alpha)xy\sqrt{AD}+(1-\alpha\alpha)Dxxyy}{\alpha}$$

et ponendo $\alpha = \frac{\sqrt{(4AD-CC)}}{mC}$ prodit

$$4AD(xx+yy)+2C(A+Dxxyy) = \frac{((1+mm)CC-4AD)(A+Dxxyy)-2((mm-1)CC+4AD)xy\sqrt{AD}}{mC},$$

11. Ne cafus, vbi $\sqrt{AD}$ fit quantitas imaginaria, turbent, iuuabit integrationem alia via, quae ipfa deftructione terminorum §. 9. obferuata

ta innitatur, inueftigare. Scilicet propofita aequatione:

$$\frac{dx}{dy} = \sqrt{\frac{A+Cxx+Ex^4}{A+Cyy+Ey^4}}$$

fiat $x = \sqrt{pq}$ et $q = \sqrt{\frac{p}{q}}$, vt hinc obtineatur

$$\frac{pdq}{qdp} = \frac{P-\sqrt{(PP-QQ)}}{Q}$$

exiftente $\frac{P}{Q} = \frac{(A+Epp)(1+qq)+2Cpq}{(A-Epp)(1-qq)}$.

Ponatur nunc $q = u+\sqrt{(uu-1)}$, vt fit $1+qq = 2qu$
$1-qq = 2qu-2qq = -2q\sqrt{(uu-1)}$, erit $\frac{dq}{q} = \frac{du}{\sqrt{(uu-1)}}$ et
$\frac{P}{Q} = \frac{u(A+Epp)+Cp}{(Epp-A)\sqrt{(uu-1)}}$, vnde refultat haec aequatio transformata:

$$\frac{pdu}{dp} = \frac{u(A+Epp)+Cp-\sqrt{(4AEppuu+4Cpu(A+Epp)+CCpp+(Epp-A)^2)}}{Epp-A}.$$

12. Hac aequatione in ordinem redacta et pofito breuitatis gratia membro irrationali $=\sqrt{M}$ fiet:

$$udp(A+Epp)+Cpdp-pdu(Epp-A) = dp\sqrt{M}$$

ac reiecto primum hoc membro irrationali; reperitur integrale $\frac{C+2Epu}{Epp-A} =$ Conft. cuius conftantis loco autem fumatur quantitas variabilis s, vt fit

$$2Epu+C = s(Epp-A) \text{ et } u = \frac{s(Epp-A)-C}{2Ep},$$

atque hinc membrum rationale fit:

$$\frac{-ds(Epp-A)^2}{2E} \text{ et formula irrationalis}$$

$$(Epp-A)\sqrt{\frac{Ass+Cs+E}{E}}$$

ita vt nunc fit $\frac{-ds}{2}(Epp-A) = dp\sqrt{E(Ass+Cs+E)}$
feu $\frac{ds}{\sqrt{E(Ass+Cs+E)}} + \frac{2dp}{Epp-A} = 0$ cuius integrale eft
$\frac{1}{\sqrt{AE}}l\frac{p\sqrt{E}-\sqrt{A}}{p\sqrt{E}+\sqrt{A}} + \frac{1}{\sqrt{AE}}l(As+\frac{1}{2}C+\sqrt{A(Ass+Cs+E)}) =$ Conft.

13. Haec aequatio ergo redit ad hanc formam:

$$As+\tfrac{1}{2}C+\sqrt{A}(Ass+Cs+E)=\alpha\frac{p\sqrt{E}+\sqrt{A}}{p\sqrt{E}-\sqrt{A}}=T$$

vnde elicitur

$$AE=TT-T(2As+C)+\tfrac{1}{4}CC \text{ seu}$$

$$2As+C=\frac{TT+\frac{1}{4}CC-AE}{T}=\frac{\alpha\alpha(p\sqrt{E}+\sqrt{A})^2+(\frac{1}{4}CC-AE)(p\sqrt{E}-\sqrt{A})^2}{\alpha(Epp-A)}$$

Cum nunc sit $p=xy$ et $q=\frac{x}{y}$ erit $u=\frac{xx+yy}{2xy}$ et $s=\frac{E(xx+yy)+C}{Exxyy-A}$, ex quo efficitur

$$\frac{2AE(xx+yy)+CExxyy+AC}{Exxyy-A}=T+\frac{CC-4AE}{4T}$$

existente $T=\alpha.\frac{xy\sqrt{E}+\sqrt{A}}{xy\sqrt{E}-\sqrt{A}}=\alpha.\frac{Exxyy+A+2xy\sqrt{AE}}{Exxyy-A}$

et $\frac{1}{T}=\frac{1}{\alpha}.\frac{Exxyy+A-2xy\sqrt{AE}}{Exxyy-A}$ ideoque

$$2AE(xx+yy)+CExxyy+AC=\alpha(Exxyy+A)+2\alpha xy\sqrt{AE}$$
$$+\frac{CC-4AE}{4\alpha}(Exxyy+A)-\frac{2(CC-4AE)}{4\alpha}xy\sqrt{AE}.$$

14. Ne vnquam haec expressio inuoluat imaginaria, constantis α formam ita immutemus vt sit

$$\alpha+\frac{CC-4AE}{4\alpha}=F \text{ seu } 4\alpha\alpha=4\alpha F-CC+4AE$$

hincque $2\alpha=F+\sqrt{(FF+4AE-CC)}$ et

$$\frac{1}{2\alpha}=\frac{F-\sqrt{(FF+4AE-CC)}}{CC-4AE}$$

vnde fit $2\alpha-\frac{(CC-4AE)}{2\alpha}=2\sqrt{(FF+4AE-CC)}$ et

$$2AE(xx+yy)=(F-C)(Exxyy+A)+2xy\sqrt{AE(FF+4AE-CC)}$$

sit nunc $F-C=2G$ erit

$$AE(xx+yy)=G(A+Exxyy)+2xy\sqrt{AE(AE+CG+GG)}$$

quae

quae est aequatio integralis completa huius differentialis:

$$\frac{dx}{\sqrt{(A+Cxx+Ex^4)}}=\frac{dy}{\sqrt{(A+Cyy+Ey^4)}}.$$

vbi constans G ita accipi debet, vt formula irrationalis $\sqrt{AE(AE+CG+GG)}$ non fiat imaginaria.

15. Forma haec integralis adhuc commodior reddi potest ponendo $G=Eff$, sicque fiet aequatio integralis:

$$A(xx+yy)=ff(A+Exxyy)+2xy\sqrt{A(A+Cff+Ef^4)}$$

vbi f est constans arbitraria. Hinc autem elicitur

$$y=\frac{x\sqrt{A(A+Cff+Ef^4)}\pm f\sqrt{A(A+Cxx+Ex^4)}}{A-Effxx}$$

similique modo

$$x=\frac{y\sqrt{A(A+Cff+Ef^4)}\pm f\sqrt{A(A+Cyy+Ey^4)}}{A-Effyy}$$

Quae formulae cum iis, quas olim dederam, perfecte consentiunt.

16. Integrale hic quidem aequationis differentialis propositae methodo directa sum consecutus, verumtamen diffiteri non possum, hoc per multas ambages esse praestitum, ita vt vix sit expectandum, cuiquam has operationes in mentem venire potuisse. Ex quo haec ipsa methodus, qua hic sum vsus, plurimum in recessu habere videtur, neque vllum est dubium, quin eam diligentius scrutando aditus ad multa alia praeclara aperiatur, ac fortasse alia noua methodus idem praestandi detegatur,

tur, vnde non contemnenda ſubſidia ad Analyſin perficiendam hauriri queant.

17. Operationes hic adhibitae aliquantum variari poſſunt, quod probe perpendiſſe vſu non carebit. Propoſitam ſcilicet aequationem differentialem ita refero:

$$\frac{ydx}{xdy} = \sqrt{\frac{Ayy + Cxxyy + Ex^4yy}{Axx + Cxxyy + Exxy^4}} = \sqrt{\frac{P+Q}{P-Q}}$$

vt ſit $\frac{P}{Q} = \frac{(A + Exxyy)(xx+yy) + 2Cxxyy}{(A - Exxyy)(yy - xx)}$ eritque

$$\frac{ydx + xdy}{ydx - xdy} = \frac{\sqrt{(P+Q)} + \sqrt{(P-Q)}}{\sqrt{(P+Q)} - \sqrt{(P-Q)}} = \frac{P + \sqrt{(PP-QQ)}}{Q}$$

tum etiam $\frac{ydx - xdy}{ydx + xdy} = \frac{P - \sqrt{(PP-QQ)}}{Q}$.

Faciamus nunc hanc ſubſtitutionem:

$$x = p\left(\sqrt{\tfrac{q+1}{2}} - \sqrt{\tfrac{q-1}{2}}\right) \text{ et } y = p\left(\sqrt{\tfrac{q+1}{2}} + \sqrt{\tfrac{q-1}{2}}\right)$$

erit $xy = pp$; $xx + yy = 2ppq$; $yy - xx = 2pp\sqrt{(qq-1)}$ deinde $\frac{dx}{x} = \frac{dp}{p} - \frac{dq}{2\sqrt{(qq-1)}}$ et $\frac{dy}{y} = \frac{dp}{p} + \frac{dq}{2\sqrt{(qq-1)}}$; vnde fit

$$\frac{ydx}{xdy} = \frac{\frac{dp}{p} - \frac{dq}{2\sqrt{(qq-1)}}}{\frac{dp}{p} + \frac{dq}{2\sqrt{(qq-1)}}} \text{ et } \frac{ydx - xdy}{ydx + xdy} = \frac{-pdq}{2dp\sqrt{(qq-1)}}$$

atque $\frac{P}{Q} = \frac{2(A + Ep^4)ppq + 2Cp^4}{2(A - Ep^4)pp\sqrt{(qq-1)}} = \frac{(A + Ep^4)q + Cpp}{(A - Ep^4)\sqrt{(qq-1)}}$

vnde fit $\frac{\sqrt{(PP-QQ)}}{Q} = \frac{\sqrt{(4AEp^4qq + 2Cppq(A + Ep^4) + CCp^4 + (A - Ep^4)^2)}}{(A - Ep^4)\sqrt{(qq-1)}}$.

18. Sit $pp = r$ eritque ob $\frac{dp}{p} = \frac{dr}{2r}$

$$0 = \frac{rdq}{dr} + \frac{(A + Err)q + Cr - \sqrt{(4AErrqq + 2Crq(A + Err) + CCrr + (A - Err)^2)}}{A - Err}$$

ſiue

$$rdq(A - Err) + qdr(A + Err) + Crdr = dr\sqrt{(4AErrqq + 2Crq(A + Err) + CCrr + (A - Err)^2)}.$$

Quan-

Quantitas vinculo radicali implicata ita exhibeatur

$$\frac{1}{4AE}(16AAEErrqq + 8ACErq(A + Err) + 4ACCErr + 4AE(A - Err)^2)$$

$$= \frac{1}{4AE}((4AErq + C(A + Err))^2 + (4AE - CC)(A - Err)^2).$$

Ponamus ergo $4AErq + C(A + Err) = s(A - Err)\sqrt{(4AE - CC)}$

eritque formula ſurda $= \frac{(A - Err)\sqrt{(4AE - CC)(1 + ss)}}{2\sqrt{AE}}$, et ob

$s\sqrt{(4AE - CC)} = \frac{4AErq - C(A + Err)}{A - Err}$

erit differentiando:

$$ds\sqrt{(4AE - CC)} = \frac{4AAE(rdq + qdr) - 4AEEr^3dq + 4AEErrqdr + 4ACErdr}{(A - Err)^2}$$

ideoque

$$rdq(A - Err) + qdr(A + Err) + Crdr = \frac{ds(A - Err)^2\sqrt{(4AE - CC)}}{4AE}$$

quod cum ſit ipſum prius membrum noſtrae aequationis, cui aequalis eſt $\frac{dr(A - Err)\sqrt{(4AE - CC)(1 + ss)}}{2\sqrt{AE}}$ habebimus

$\frac{ds(A - Err)}{2\sqrt{AE}} = dr\sqrt{(1 + ss)}$ et $\frac{2dr\sqrt{AE}}{A - Err} = \frac{ds}{\sqrt{(1 + ss)}}$ cuius integrale eſt $l s + \sqrt{(1 + ss)} = \alpha . \frac{\sqrt{A} + r\sqrt{E}}{\sqrt{A} - r\sqrt{E}}$ vnde fit

$$1 = \alpha\alpha\left(\frac{\sqrt{A} + r\sqrt{E}}{\sqrt{A} - r\sqrt{E}}\right)^2 - 2\alpha s . \frac{\sqrt{A} + r\sqrt{E}}{\sqrt{A} - r\sqrt{E}}.$$

Eſt vero $s = \frac{4AEqr + C(A + Err)}{(A - Err)\sqrt{(4AE - CC)}}$

atque $r = pp = xy$ et $q = \frac{xx + yy}{2xy}$, hincque

$$s = \frac{2AE(xx + yy) + C(A + Exxyy)}{(A - Exxyy)\sqrt{(4AE - CC)}}.$$

19.

19. Idem expedire possumus sine substitutione noua; statim enim ac peruenimus ad hanc aequationem:

$$rdq(A-Err)+qdr(A+Err)+Crdr = dr\sqrt{\frac{(4AErq+C(A+Err))^2+(4AE-CC)(A-Err)^2}{4AE}}$$

notetur esse membrum prius $=\frac{(A-Err)^2}{4AE}\, d.\frac{4AErq+C(A+Err)}{A-Err}$ posterius vero ita exprimi posse

$$\frac{dr(A-Err)}{2\sqrt{AE}}\sqrt{(4AE-CC+(\frac{4AErq+C(A+Err)}{A-Err})^2)}$$

vnde posito breuitatis gratia $\frac{4AErq+C(A+Err)}{A-Err}=v$ erit $\frac{(A-Err)^2}{4AE}dv=\frac{dr(A-Err)}{2\sqrt{AE}}\sqrt{(4AE-CC+vv)}$ ideoque $\frac{dv}{\sqrt{(4AE-CC+vv)}}=\frac{2dr\sqrt{AE}}{A-Err}$.

20. Aliud specimen huius reductionis daturus, considerabo hanc aequationem:

$$\frac{dx}{\sqrt{(Bx+Cxx+Dx^3)}}=\frac{dy}{\sqrt{(By+Cyy+Dy^3)}}$$

quam ita repraesento

$$\frac{ydx}{xdy}=\sqrt{\frac{Bxyy+Cxxyy+Dx^3yy}{Bxxy+Cxxyy+Dxxy^3}}=\sqrt{\frac{P+Q}{P-Q}}$$

vt sit $\frac{P}{Q}=\frac{Bxy(y+x)+2Cxxyy+Dxxyy(x+y)}{Bxy(y-x)-Dxxyy(x-y)}$

seu $\frac{P}{Q}=\frac{(B+Dxy)(x+y)+2Cxy}{(B-Dxy)(y-x)}$

eritque $\frac{ydx-xdy}{ydx+xdy}=\frac{P-\sqrt{(PP-QQ)}}{Q}$.

21. Statuatur nunc $x=p(u+\sqrt{(uu-1)})$ et $y=p(u-\sqrt{(uu-1)})$ erit $\frac{dx}{x}=\frac{dp}{p}+\frac{du}{\sqrt{(uu-1)}}$ et $\frac{dy}{y}=\frac{dp}{p}-\frac{du}{\sqrt{(uu-1)}}$, hincque $\frac{ydx-xdy}{ydx+xdy}=\frac{pdu}{dp\sqrt{(uu-1)}}$. Deinde

inde ob $xy=pp$, et $x+y=2pu$; $y-x=-2p\sqrt{(uu-1)}$ erit $\frac{P}{Q}=-\frac{(B+Dpp)u+Cp}{(B-Dpp)\sqrt{(uu-1)}}$ ideoque

$$\frac{pdu}{dp}=\frac{(B+Dpp)u+Cp-\sqrt{(4BDppuu+2Cpu(B+Dpp)+CCpp+(B-Dpp)^2)}}{Dpp-B}$$

vnde fit

$$udp(B+Dpp)-pdu(Dpp-B)+Cpdp=dp\sqrt{(\ldots)}.$$

Prius membrum est $(B-Dpp)^2\, d\frac{pu+\frac{C}{4BD}(B+Dpp)}{B-Dpp}$ seu

$$\frac{(B-Dpp)^2}{4BD}\, d.\frac{4BDpu+C(B+Dpp)}{B-Dpp}$$

at quantitas signo radicali inuoluta ita scribi potest

$$\frac{1}{4BD}(16BBDDppuu+8BCDpu(B+Dpp)+4BCCDpp+4BD(B-Dpp)^2)$$

$$=\frac{1}{4BD}((4BDpu+C(B+Dpp))^2+(4BD-CC)(B-Dpp)^2)$$

vnde membrum irrationale erit

$$\frac{B-Dpp}{2\sqrt{BD}}\sqrt{(4BD-CC+(\frac{4BDpu+C(B+Dpp)}{B-Dpp})^2)}.$$

Quare posito breuitatis gratia $\frac{4BDpu+C(B+Dpp)}{B-Dpp}=s$ erit

$$\frac{(B-Dpp)^2}{4BD}ds=\frac{(B-Dpp)dp}{2\sqrt{BD}}\sqrt{(4BD-CC+ss)}$$ vnde fit

$$\frac{ds}{\sqrt{(4BD-CC+ss)}}=\frac{2dp\sqrt{BD}}{B-Dpp}$$ et integrando

$$s+\sqrt{(4BD-CC+ss)}=\alpha.\frac{\sqrt{B}+p\sqrt{D}}{\sqrt{B}-p\sqrt{D}}$$ ideoque

$$4BD-CC=\alpha\alpha(\frac{\sqrt{B}+p\sqrt{D}}{\sqrt{B}-p\sqrt{D}})^2-2\alpha s.\frac{\sqrt{B}+p\sqrt{D}}{\sqrt{B}-p\sqrt{D}}.$$

22. Fundamentum ergo harum reductionum in hoc consistit, vt primo ponatur $x=pq$ et $y=\frac{p}{q}$, tum vero pro q eiusmodi formula accipiatur, qua partes $x\pm y$, $xx\pm yy$, etc. quae in formula $\frac{P}{Q}$ insunt,

funt, quam fimpliciffimae reddantur. Veluti in cafu § 17. fumfimus $q = \sqrt{\frac{u+1}{2}} + \sqrt{\frac{u-1}{2}}$, feu $qq = u + \sqrt{(uu-1)}$, in vltimo vero $q = u + \sqrt{(uu-1)}$: ibi nempe opus non erat, vt $x+y$ rationaliter exprimatur, vnde fufficiebat ipfi qq formam $u + \sqrt{(uu-1)}$ tribui, hic vero neceffe erat, vt $x+y$ rationalem confequatur valorem.

23. Denique cafum fimpliciorem praetermittere non poffum, quo proponitur haec aequatio $\frac{dx}{\sqrt{(A+Cxx)}} = \frac{dy}{\sqrt{(A+Cyy)}}$, quam ita refero $\frac{ydx}{xdy} = \sqrt{\frac{Ayy+Cxxyy}{Axx+Cxxyy}} = \sqrt{\frac{P+Q}{P-Q}}$; pofito ergo $x = p(\sqrt{\frac{q+1}{2}} - \sqrt{\frac{q-1}{2}})$ et $y = p(\sqrt{\frac{q+1}{2}} + \sqrt{\frac{q-1}{2}})$ fiet $\frac{-pdq}{2dp\sqrt{(qq-1)}} = \frac{P-\sqrt{(PP-QQ)}}{Q}$ exiftente

$$\frac{P}{Q} = \frac{Aq+Cpp}{A\sqrt{(qq-1)}} \text{ et } \frac{\sqrt{(PP-QQ)}}{Q} = \frac{\sqrt{(2ACppq+CCp^4+AA)}}{A\sqrt{(qq-1)}}$$

vnde fumto $pp = r = xy$ erit

$$0 = \frac{rdq}{dr} + \frac{Aq+C-\sqrt{(2ACrq+CCrr+AA)}}{A} \text{ hincque}$$

$$\frac{A(rdq+qdr)+Crdr}{\sqrt{(2ACrq+CCrr+AA)}} = dr \text{ cuius integrale eft}$$

$$Cr+F = \sqrt{(2ACrq+CCrr+AA)} \text{ feu}$$

$$FF+2CFr = 2ACrq+AA$$

eft vero $r = xy$ et $q = \frac{xx+yy}{2xy}$ vnde aequatio integralis eft $FF+2CFxy = AA+AC(xx+yy)$.

Sicque haec comparatio inter x et y, quae alias per logarithmos vel arcus circulares oftendi folet, hic algebraice eft eruta.

DE

DE ARCVBVS CVRVARVM AEQVE AMPLIS EORVMQVE COMPARATIONE.

Auctore

L. EVLERO.

1.

Amplitudinem arcus cuiuscunque lineae curuae cum Celeb. *Ioh. Bernoullio* b. m. voco angulum, quem rectae ad eius terminos normales inter fe conftituunt. Ita fi fuerit AM arcus lineae cuiuscunque curuae, atque ad eius terminos A et M rectae normales ducantur AO et MO in O concurrentes, angulus AOM erit amplitudo arcus AM. Haec amplitudinis idea perquam ingeniofe ad curuas dimetiendas eft introducta, propterea quod non vti ceterae relationes, quibus natura curuarum per coordinatas exprimi folet, ab hypothefibus arbitrariis pendet; dum enim relatio inter coordinatas, prouti axis eiusque initium diuerfimode accipitur, plurimum variare poteft manente eadem linea curva, notio amplitudinis nulli huiusmodi varietati eft obnoxia, nifi forte quod alio atque alio puncto curuae A pro initio affumto angulus AOM quantitate conftante augeri diminuiue queat, vnde ta- Tab. I. Fig. 1.

 men