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De motu rectilineo trium corporum se mutuo attrahentium

Leonhard Euler

DE MOTV RECTILINEO
TRIVM CORPORVM SE MVTVO ATTRAHENTIVM.

Auctore

L. EVLERO.

A B C

O

1.

Sint A, B, C maſſae trium corporum eorumque diſtantiae a puncto fixo O ad datum tempus t ponantur

$$OA = x, \quad OB = y \quad \text{et} \quad OC = z$$

vbiquidem ſumitur $y > x$ et $z > y$. Hinc motus principia praebent has tres aequationes:

$$\text{I. } \frac{ddx}{dt^2} = \frac{B}{(y-x)^2} + \frac{C}{(z-x)^2};$$
$$\text{II. } \frac{ddy}{dt^2} = \frac{-A}{(y-x)^2} + \frac{C}{(z-y)^2}$$
$$\text{III. } \frac{ddz}{dt^2} = \frac{-A}{(z-x)^2} - \frac{B}{(z-y)^2}$$

vnde facile deducuntur binae aequationes integrabiles:

prior $\quad A\,dx + B\,dy + C\,dz = E\,dt$ et $Ax + By + Cz = Et + F$

poſterior $\frac{A\,dx^2 + B\,dy^2 + C\,dz^2}{dt^2} = G + \frac{2AB}{y-x} + \frac{2AC}{z-x} + \frac{2BC}{z-y}.$

Hinc autem ob defectum tertiae aequationis integralis parum ad motus cognitionem concludere licet.

2. Sta-

2. Statuamus $x = y - p$ et $z = y + q$, vt p et q ſint quantitates poſitiuae: et prima integralis praebet:

$$(A+B+C)y - Ap + Cq = Et + F \text{ ideoque}$$

$$y = \frac{Ap - Cq + Et + F}{A+B+C}; \quad dy = \frac{A\,dp - C\,dq + E\,dt}{A+B+C}$$

$$x = \frac{-(B+C)p - Cq + Et + F}{A+B+C}; \quad dx = \frac{-(B+C)\,dp - C\,dq + E\,dt}{A+B+C}$$

$$z = \frac{Ap + (A+B)q + Et + F}{A+B+C}; \quad dz = \frac{A\,dp + (A+B)\,dq + E\,dt}{A+B+C}.$$

Hinc integralis ſecunda hanc induit formam:

$$\frac{A(B+C)dp^2 + C(A+B)dq^2 + 2AC\,dp\,dq + EE\,dt^2}{(A+B+C)\,dt^2} = G + \frac{2AB}{p} + \frac{2AC}{p+q} + \frac{2BC}{q}$$

3. Faciamus vero easdem ſubſtitutiones in primis aequationibus differentio-differentialibus, quae iam ad duas reuocabantur:

$$\frac{-(B+C)\,ddp - C\,ddq}{(A+B+C)\,dt^2} = \frac{B}{pp} + \frac{C}{(p+q)^2}$$

$$\frac{A\,ddp + (A+B)\,ddq}{(A+B+C)\,dt^2} = \frac{-A}{(p+q)^2} - \frac{B}{qq}$$

vnde colligitur: $\frac{ddp + ddq}{dt^2} = \frac{-A-C}{(p+q)^2} - \frac{B}{pp} - \frac{B}{qq}$.

Deinde vtrumque elementum ddp et ddq ſeorſim ita exprimitur:

1°. $\frac{ddp}{dt^2} = \frac{-A-B}{pp} - \frac{C}{(p+q)^2} + \frac{C}{qq}$

2°. $\frac{ddq}{dt^2} = \frac{A}{pp} - \frac{A}{(p+q)^2} - \frac{B-C}{qq}$

vnde oritur vna aequatio integralis ad hanc formam perducta:

$$\frac{B(A\,dp^2 + C\,dq^2) + AC(dp+dq)^2}{(A+B+C)\,dt^2} = G + \frac{2AB}{p} + \frac{2AC}{p+q} + \frac{2BC}{q}$$

quandoquidem in G poſtremus ille terminus EE comprehenditur.

 4. Cum

4. Cum igitur solutio sit perducta ad duas aequationes differentio - differentiales inter p, q et t; insigne lucrum obtineri est censendum, si has aequationes ad duas alias primi tantum gradus differentiales reuocare liceret. Hoc autem singulari artificio sequentem in modum praestari posse comperi. Statuo $q = pu$, et binae aequationes differentio-differentiales ita repraesententur:

$$d.\frac{dp}{dt} = \frac{dt}{pp}\left(-A-B-\frac{C}{(u+1)^2}+\frac{C}{uu}\right)$$
$$d.\frac{udp+pdu}{dt} = \frac{dt}{pp}\left(A-\frac{A}{(u+1)^2}-\frac{B-C}{uu}\right).$$

Iam artificium in hac substitutione consistit, vt ponam $\frac{dp}{dt} = \frac{r}{\sqrt{p}}$ et $\frac{dq}{dt} = \frac{udp+pdu}{dt} = \frac{s}{\sqrt{p}}$; mox enim patebit his substitutionibus binas variabiles p et t ex calculo elidi posse ita vt tantum hae tres r, s et u relinquantur, per prima differentialia determinandae. Statim vero aequatio illa integralis supra inuenta adeo ad formam finitam redit hanc $\frac{B(Arr+Css)+AC(r+s)^2}{A+B+C} = Gp+2AB+\frac{2AC}{u+1}+\frac{2BC}{u}$ quae insignem vsum afferre poterit.

5. Cum sit $\frac{dp}{dt} = \frac{r}{\sqrt{p}}$ erit $dt = \frac{dp\sqrt{p}}{r}$, vnde nostrae aequationes differentio-differentiales praebent:

$$\frac{dr}{\sqrt{p}}-\frac{rdp}{2p\sqrt{p}} = \frac{dp}{pr\sqrt{p}}\left(-A-B-\frac{C}{(u+1)^2}+\frac{C}{uu}\right)$$
$$\frac{ds}{\sqrt{p}}-\frac{sdp}{2p\sqrt{p}} = \frac{dp}{pr\sqrt{p}}\left(A-\frac{A}{(u+1)^2}-\frac{B-C}{uu}\right)$$

vnde fit:

$$dr = \frac{rdp}{2p}+\frac{dp}{pr}\left(-A-B-\frac{C}{(u+1)^2}+\frac{C}{uu}\right)$$
$$ds = \frac{sdp}{2p}+\frac{dp}{pr}\left(A-\frac{C}{(u+1)^2}-\frac{B-C}{uu}\right).$$

Praeterea

Praeterea vero habebitur:

$$u\,dp + p\,du = \frac{s\,dt}{\sqrt{p}} = \frac{s\,dp}{r}$$

ficque fit $\frac{dp}{p} = \frac{r\,du}{s - ru}$, quo valore ibi fubftituto fit

$$dr(s-ru) = \tfrac{1}{2}rr\,du + du\left(-A - B - \frac{C}{(u+1)^2} + \frac{C}{uu}\right)$$

$$ds(s-ru) = \tfrac{1}{2}rs\,du + du\left(A - \frac{A}{(u+1)^2} - \frac{B-C}{uu}\right)$$

ex quarum combinatione nafcitur:

$$\tfrac{1}{2}r(r\,ds - s\,dr) + ds\left(-A - B - \frac{C}{(u+1)^2} + \frac{C}{uu}\right)$$
$$- dr\left(A - \frac{A}{(u+1)^2} - \frac{B-C}{uu}\right) = 0.$$

6. En ergo duas aequationes fimpliciter differentiales inter ternas variabiles r, s et u, vnde fi liceret r et s per u determinare, haberetur folutio problematis completa. Inde enim primo innotefceret quantitas p ex formula $\frac{dp}{p} = \frac{r\,du}{s-ru}$ hincque porro $q = pu$. Deinde vero tempus t daretur ex aequatione $dt = \frac{dp\sqrt{p}}{r} = \frac{p\,du\sqrt{p}}{s-ru}$; Ex quibus tandem pro dato tempore t colligerentur diftantiae x, y, z ex formulis §. 2 datis.

7. Cum binae aequationes differentiales inuentae fint:

$$dr(s-ru) = \tfrac{1}{2}rr\,du + du\left(-A - B - \frac{C}{(u+1)^2} + \frac{C}{uu}\right)$$

$$ds(s-ru) = \tfrac{1}{2}rs\,du + du\left(A - \frac{A}{(u+1)^2} - \frac{B-C}{uu}\right).$$

ftatim patet ambabus fatisfieri fumendo quantitatem u conftantem et $s - ru = 0$, vnde folutio obtinetur

 parti-

Solutio particularis. Sit ergo $u = \alpha$ et $s = \alpha r$, et aequatio ex combinatione nata praebet:

$$-(A+B)\alpha - \frac{C\alpha}{(\alpha+1)^2} + \frac{C}{\alpha} = A - \frac{A}{(\alpha+1)^2} - \frac{B-C}{\alpha\alpha} \text{ vel}$$

$$0 = A\left(\alpha + 1 - \frac{1}{(\alpha+1)^2}\right) + B\left(\alpha - \frac{1}{\alpha\alpha}\right) + C\left(\frac{\alpha}{(\alpha+1)^2} - \frac{1}{\alpha} - \frac{1}{\alpha\alpha}\right)$$

$$\text{seu } 0 = \frac{A((\alpha+1)^3 - 1)}{(\alpha+1)^2} + \frac{B(\alpha^3-1)}{\alpha\alpha} + \frac{C(\alpha^3 - (\alpha+1)^3)}{\alpha\alpha(\alpha+1)^2}$$

ideoque

$$C(1+3\alpha+3\alpha\alpha) = A\alpha^3(\alpha\alpha+3\alpha+3) + B(\alpha+1)^2(\alpha^3-1).$$

Quare quantitatem α ex hac aequatione quinti gradus definiri oportet:

$$(A+B)\alpha^5 + (3A+2B)\alpha^4 + (3A+B)\alpha^3 - (B+3C)\alpha^2 - (2B+3C)\alpha - B - C = 0.$$

Deinde vero relatio inter r et p ex hac aequatione est definienda:

$$dr = \frac{r\,dp}{2p} + \frac{dp}{pr}\left(-A - B - \frac{C}{(\alpha+1)^2} + \frac{C}{\alpha\alpha}\right)$$

seu posito $A + B + \frac{C}{(\alpha+1)^2} - \frac{C}{\alpha\alpha} = \frac{1}{2}D$ ex hac

$$2\,dr = \frac{dp}{p}\left(r - \frac{D}{r}\right) \text{ seu } \frac{dp}{p} = \frac{2r\,dr}{rr-D} \text{ ita vt sit}$$

$$p = \mathfrak{C}(rr-D), \text{ tum vero } q = \alpha\beta(rr-D) \text{ et } dt = \frac{dp\sqrt{p}}{r}$$

seu $dt = 2\mathfrak{C}\,dr\sqrt{\mathfrak{C}(rr-D)}$ hinc

$$t = \mathfrak{C}r\sqrt{\mathfrak{C}(rr-D)} - \mathfrak{C}^2 D \int \frac{dr}{\sqrt{\mathfrak{C}(rr-D)}}.$$

8. Casus hic particularis, quo solutio succedit euolutionem diligentiorem meretur. Primum ergo obseruo ex aequatione illa quinti gradus pro α semper valorem realem positiuum, eumque vnicum elici, cum vnica signorum variatio occurrat, neque

igitur

igitur hic vlla ambiguitas locum habet, ſed valor ipſius α tanquam determinatus ſpectari poteſt, pendens a maſſis trium corporum A, B, C. Inuento autem numero α colligitur quantitas $D = 2(A+B) - \frac{2C(2\alpha+1)}{\alpha\alpha(\alpha+1)^2}$, vbi animaduerto quantitatem D nunquam euaneſcere poſſe. Si enim eſſet $D = 0$ foret $B = \frac{C(2\alpha+1)}{\alpha\alpha(\alpha+1)^2} - A$ quo valore ſubſtituto prodiret:

$$C(1+3\alpha+3\alpha\alpha) = A\alpha^3(\alpha\alpha+3\alpha+3) + \frac{C(2\alpha+1)(\alpha^3-1)}{\alpha\alpha} - A(\alpha+1)^2(\alpha^3-1)$$

ſeu $\frac{C}{\alpha\alpha}(\alpha^4+2\alpha^3+\alpha\alpha+2\alpha+1) = A(\alpha^4+2\alpha^3+\alpha\alpha+2\alpha+1)$

ideoque $C = A\alpha\alpha$ et $B = \frac{A(2\alpha+1)}{(\alpha+1)^2} - A = \frac{-A\alpha\alpha}{(\alpha+1)^2}$ foretque adeo maſſa B negatiua, quod eſt abſurdum. Multo minus quantitas D vnquam fieri poteſt negatiua. Poſito enim:

$$B = \frac{C(2\alpha+1)}{\alpha\alpha(\alpha+1)^2} - A - \Delta, \text{ proueniret}$$
$$\frac{C}{\alpha\alpha} = A - \frac{\Delta(\alpha+1)^2(\alpha^3-1)}{\alpha^4+2\alpha^3+\alpha^2+2\alpha+1} \text{ hincque}$$
$$B = \frac{C(2\alpha+1)}{\alpha\alpha(\alpha+1)^2} - \frac{C}{\alpha\alpha} - \frac{\Delta(\alpha^5+3\alpha^4+3\alpha^3)}{\alpha^4+2\alpha^3+\alpha\alpha+2\alpha+1}$$

ſeu B multo magis eſſet quantitas negatiua, cum valor ipſius α neceſſario ſit poſitiuus.

9. Cum ergo D neceſſario ſit quantitas poſitiua, ponatur $D = aa$, ſi etiam numerus α ſpectetur vt datus, maſſae trium corporum ita ſe habebunt, vt ſit:

 B =

$$B = \frac{\alpha^3(\alpha\alpha + 3\alpha + 3)aa}{2(\alpha^4 + 2\alpha^3 + \alpha\alpha + 2\alpha + 1)} - \frac{C}{[\alpha + 1]^2} \text{ et}$$
$$A = \frac{C}{\alpha\alpha} - \frac{(\alpha + 1)^2(\alpha^3 - 1)aa}{2(\alpha^4 + 2\alpha^3 + \alpha\alpha + 2\alpha + 1)}$$

ex quo necesse est vt quantitas $\frac{2C(\alpha^4 + 2\alpha^3 + \alpha\alpha + 2\alpha + 1)}{\alpha\alpha(\alpha+1)^2 aa}$ intra hos limites $(\alpha + 1)^3 - 1$ et $\alpha^3 - 1$ contineatur. Introducta ergo quantitate aa cum numero α in calculum, duo casus sunt perpendendi, prout $\mathfrak{E}$ fuerit quantitas positiua vel negatiua, quos seorsim euoluamus.

Casus I. 10. Sit primo $\mathfrak{E} = + nn$, erit $p = nn(rr - aa)$ et $q = \alpha nn(rr - aa)$, vnde cum constantes, E et F nihilo aequales statuere liceat, loca trium corporum A, B, C, quorum iam centrum grauitatis in O existit, ita per r definiuntur, vt sit:

$$x = OA = \frac{-nn(rr - aa)}{A + B + C}(B + C + C\alpha)$$
$$y = OB = \frac{nn(rr - aa)}{A + B + C}(A - C\alpha)$$
$$z = OC = \frac{nn(rr - aa)}{A + B + C}(A + (A + B)\alpha).$$

At relatio inter r et tempus t ita se habet:

$$t = n^3 r \sqrt{(rr - aa)} - n^3 aa \int \frac{dr}{\sqrt{rr - aa)}} \text{ seu}$$
$$t = n^3 r \sqrt{(rr - aa)} - n^3 aa\, l \frac{r + \sqrt{(rr - aa)}}{\Delta}.$$

sumta constante $\Delta = a$, tempore $t = 0$, erat $r = a$, tumque omnia corpora in centro grauitatis erant coniuncta, vnde quasi celeritatibus infinitis erant explosa, vt eae fuerint inter se vt quantitates $-B - C - C\alpha$, $A - C\alpha$, $A + (A + B)\alpha$ tum vero labente tempore t quantitas r continuo magis increscit; quouis

quouis autem tempore celeritas cuiusque corporis ex formula $\frac{dt}{dr} = 2n^3 \sqrt{(rr - aa)}$ innotescit. Notandum autem distantias corporum perpetuo inter se eandem proportionem conseruare.

11. Sit nunc $\mathfrak{E} = -nn$ erit $p = nn(aa - rr)$ et $q = \alpha nn(aa - rr)$ et per r loca corporum vt ante ita determinantur, vt sit: Casus II.

$$x = OA = \frac{-nn(aa - rr)}{A + B + C}(B + (1 + \alpha)C)$$
$$y = OB = \frac{nn(aa - rr)}{A + B + C}(A - C\alpha)$$
$$z = OC = \frac{nn(aa - rr)}{A + B + C}(A(\alpha + 1) + B\alpha).$$

Pro tempore autem t obtinetur, $dt = 2n^3 dr \sqrt{(aa - rr)}$

seu $t = n^3 r \sqrt{(aa - rr)} + n^3 aa \int \frac{dr}{\sqrt{(aa - rr)}}$

hinc $t = n^3 r \sqrt{(aa - rr)} + n^3 aa \,\text{Ang. sin.} \frac{r}{a}$.

Quodsi ergo ponatur Ang. sin. $\frac{r}{a} = \Phi$, vt sit $r = a \sin. \Phi$ erit $t = n^3 aa(\Phi + \sin. \Phi \cos. \Phi)$, et distantiae inter se proportionales ad quoduis tempus sunt vt $\cos. \Phi^2$. Vnde si initio quo $t = 0$ fuerit $\Phi = 0$, sicque $r = 0$, et $\frac{dt}{dr} = 2n^3 a$, erant tum distantiae:

$$x = \frac{-nnaa}{A + B + C}(B + (1 + \alpha)C)$$
$$y = \frac{nnaa}{A + B + C}(A - C\alpha)$$
$$z = \frac{nnaa}{A + B + C}(A(\alpha + 1) + B\alpha)$$

ibique corpora in quiete. Sumto autem $\Phi = 90^\circ$, seu elapso tempore $t = n^3 aa. 90^\circ$, corpora in centro grauitatis conueniunt celeritate infinita.

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