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Observationes analyticae

Leonhard Euler

OBSERVATIONES ANALYTICAE.

Auctore

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Confideranti poteftates, quae ex eleuatione huius formae trinomiae $1+x+xx$ nafcuntur, termini medii maximis coefficientibus numericis deprehenduntur affecti, quorum ordo progreffionis cum non parum fit reconditus, omni attentione dignus videtur; praecipue quoniam huiusmodi fpeculationes plerumque fructum haud fpernendum in Analyfi afferre folent. Primum ergo harum poteftatum fimpliciores confpectui exponam:

Exponens poteftatis	Poteftates euolutae
0	1
1	$1+x+x^2$
2	$1+2x+3x^2+2x^3+x^4$
3	$1+3x+6x^2+7x^3+6x^4+3x^5+x^6$
4	$1+4x+10x^2+16x^3+19x^4+16x^5+10x^6+4x^7+x^8$
5	$1+5x+15x^2+30x^3+45x^4+51x^5+45x^6+30x^7+15x^8+5x^9+x^{10}$
	etc.

hinc fi termini medii ex fingulis poteftatibus ordine repraefententur, haec exoritur progreffio:

$$1,\ 1x,\ 3x^2,\ 7x^3,\ 19x^4,\ 51x^5,\ 141x^6 \text{ etc.}$$

qui

qui numeri, quanam lege progrediantur, haud immerito indagari videtur, vt non ſolum inde terminus generalis ſeu coefficiens dignitati indefinitae x^n conueniens innoteſcat, ſed etiam inſignes huius ſeriei proprietates explorentur. Hunc in finem ſequentia problemata proponam, quorum reſolutio deinceps ad alias ſpeculationes non parum curioſas manuducet.

Problema I.

1. Euoluta hac poteſtate indefinita $(1+x+xx)^n$ coefficientem termini medii ſeu dignitatis x^n definire.

Solutio.

Poteſtas propoſita ita ſub forma binomii repraeſentetur $(x(1+x)+1)^n$, quae more ſolito euoluta praebet:

$$x^n(1+x)^n + \frac{n}{1}x^{n-1}(1+x)^{n-1} + \frac{n(n-1)}{1.2}x^{n-2}(1+x)^{n-2}$$
$$+ \frac{n(n-1)(n-2)}{1.2.3}x^{n-3}(1+x)^{n-3} \text{ etc.}$$

ex cuius ſingulis membris, ſi vlterius euoluantur, terminos formae x^n elici oportet. Ac primum quidem membrum praebet x^n, cum reliquae poteſtates omnes ipſius x ex eius euolutione ortae futurae ſint altiores. Ex ſecundo autem membro pro hac dignitate x^n oritur:

$$\frac{n}{1}x^{n-1}.\frac{n-1}{1}x = \frac{n(n-1)}{1.1}x^n$$

ex tertio membro simili modo consequimur:

$$\frac{n(n-1)}{1\cdot 2}x^{n-2}\cdot\frac{(n-2)(n-3)}{1\cdot 2}x^2 = \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 1\cdot 2}x^n$$

quas cunctas partes si in vnam summam colligamus obtinetur dignitatis x^n coefficiens quaesitus:

$$1 + \frac{n(n-1)}{1\cdot 1} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 1\cdot 2} + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3} \text{ etc.}$$

Coroll. 1.

2. Haec ergo series, quae quoties n est numerus integer abrumpitur, coefficientem praebet dignitatis x^n pro serie proposita $1 + x + 3x^2 + 7x^3 + 19x^4 +$ etc. sicque eius ope terminus quantumuis ab initio remotus statim sine praecedentibus inueniri potest.

Coroll. 2.

3. Quod si pro n successiue numeros 1, 2, 3 etc. substituamus, sequentes valores reperiuntur:

n	coefficiens ipsius x^n
0	1
1	1
2	$1 + 2 = 3$
3	$1 + 6 = 7$
4	$1 + 12 + 6 = 19$
5	$1 + 20 + 30 = 51$
6	$1 + 30 + 90 + 20 = 141$
7	$1 + 42 + 210 + 140 = 393$

n | coe-

n	coefficiens ipsius x^n
8	$1+56+420+560+70=1107$
9	$1+72+756+1680+630=3139$
10	$1+90+1260+4200+3150+252=8953$
11	$1+110+1980+9240+11550+2772=25653$
12	$1+132+2970+18480+34650+16632+924=73789$

etc.

Coroll. 3.

4. Series horum numerorum ita est comparata, vt quisque terminus cum triplo praecedentis commode conferri posse videatur, ex qua comparatione sequentes differentiae nascuntur:

$$\begin{array}{l} 1,\ 1,\ 3,\ 7,\ 19,\ 51,\ 141,\ 393,\ 1107,\ 3139 \text{ etc.} \\ \quad 3,\ 3,\ 9,\ 21,\ 57,\ 153,\ 423,\ 1179,\ 3321 \\ \hline \quad 2,\ 0,\ 2,\ 2,\ 6,\ 12,\ 30,\ 72,\ 182 \text{ etc.} \end{array}$$

Scholion 1.

5. Si has differentias accuratius contemplemur, non sine ratione euenire videtur, quod eae sint numeri pronici, seu trigonales duplicati in forma $mm+m$ contenti, ac si ad istorum pronicorum numerorum radices spectemus, quae hanc seriem constituunt: Exemplum memorabile inductionis fallacis.

$$1,\ 0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13 \text{ etc.}$$

ea manifesto est recurrens, cuius quisque terminus est summa binorum praecedentium. Qui ordo cum

in

in decem primoribus terminis deprehendatur, quis dubitauerit eundem vniuerſae ſeriei tribuere? ſaepe profecto inductiones minus certae ſucceſſu non fuerunt deſtitutae. Operae ergo pretium erit hanc rationem accuratius perpendere, ſcilicet cum numerus 13 conueniat ſeriei termino x^9, in genere dignitati x^n reſpondebit numerus:

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-2}$$

cuius numerus pronicus eſt:

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-2} + \frac{1}{5}\left(\frac{1+\sqrt{5}}{2}\right)^{2n-4} + \frac{1}{5}\left(\frac{1-\sqrt{5}}{2}\right)^{2n-4} - \frac{2}{5}(-1)^{n-2}.$$

Quare ſi in ſerie propoſita bini termini contigui generatim ita exhibeantur:

$$1 + x + 3x^2 + 7x^3 + 10x^4 \ldots Px^n + Qx^{n+1}$$

erit $3P - Q =$

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1} + \frac{1}{5}\left(\frac{1+\sqrt{5}}{2}\right)^{2n-2} + \frac{1}{5}\left(\frac{1-\sqrt{5}}{2}\right)^{2n-2} - \frac{2}{5}(-1)^{n-1}$$

vnde concluditur fore:

$$P = \frac{3^n + (-1)^n}{10} + \frac{1}{5}\left(\frac{3+\sqrt{5}}{2}\right)^n + \frac{1}{5}\left(\frac{3-\sqrt{5}}{2}\right)^n + \frac{1}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

ita vt ipſa quoque ſeries propoſita foret recurrens ſcala relationis exiſtente:

$$6, -8, -8, 13, 4, -3$$

ſecun-

secundum quam erit:

$$3139 = 6.1107 - 8.393 - 8.141 + 14.51 + 4.9 - 3.7.$$

Scholion 2.

6. Verum quantumuis probabili inductione haec lex progressionis inniti videatur, dum adeo in decem primoribus terminis locum habet; tamen ea fallax deprehenditur, dum iam in termino vndecimo 8953 fallit; hoc enim a triplo praecedentis 9417 sublato residuum 464 ne numerus quidem pronicus est, multo minus radicem pronicam habet $21 = 13 + 8$, est enim $21^2 + 21 = 462$, qui numerus binario deficit ab eo 464, qui secundum legem obseruatam resultare debebat. Quam ob caussam nunc quidem in veram progressionis legem huius seriei sum inquisiturus, vt pateat quomodo quisque terminus per aliquot praecedentes reuera determinetur.

Problema 2.

7. Pro serie proposita

$$1,\ x,\ 3x^2,\ 7x^3,\ 19x^4,\ 51x^5 \text{ etc.}$$

inuestigare legem, qua quisque terminus per aliquot praecedentes determinatur.

Solutio.

Considerentur generatim aliquot huius seriei termini se mutuo sequentes:

$$1,\ x,\ 3x^2,\ 7x^3 \ \ldots\ldots\ Px^n,\ Qx^{n+1},\ Rx^{n+2}$$

et quoniam in problemate praecedente vidimus esse:

$$P = 1 + \frac{n(n-1)}{1 \cdot 1} + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 1 \cdot 2 \cdot 2} + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3} \text{ etc.}$$

erit simili modo:

$$Q = 1 + \frac{(n+1)n}{1 \cdot 1} + \frac{(n+1)n(n-1)(n-2)}{1 \cdot 1 \cdot 2 \cdot 2} + \frac{(n+1)n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3} \text{ etc.}$$

$$R = 1 + \frac{(n+2)(n+1)}{1 \cdot 1} + \frac{(n+2)(n+1)n(n-1)}{1 \cdot 1 \cdot 2 \cdot 2} + \frac{(n+2)(n+1)n(n-1)(n-2)(n-3)}{1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3} \text{ etc.}$$

vnde quamlibet a sequente subtrahendo colligimus:

$$Q - P = \frac{2n}{1} + \frac{2n(n-1)(n-2)}{1 \cdot 1 \cdot 2} + \frac{2n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 1 \cdot 2 \cdot 2 \cdot 3} \text{ etc.}$$

$$R - Q = \frac{2(n+1)}{1} + \frac{2(n+1)n(n-1)}{1 \cdot 1 \cdot 2} + \frac{2(n+1)n(n-1)(n-2)(n-3)}{1 \cdot 1 \cdot 2 \cdot 2 \cdot 3} \text{ etc.}$$

hinc capiamus hanc formam:

$$\frac{n+2}{n+1}(R - Q) = \frac{2(n+2)}{1} + \frac{2(n+2)n(n-1)}{1 \cdot 1 \cdot 2} + \frac{2(n+2)n(n-1)(n-2)(n-3)}{1 \cdot 1 \cdot 2 \cdot 2 \cdot 3} \text{ etc.}$$

a qua illa $Q - P$ subtrahatur vt fiat:

$$\frac{n+2}{n+1}(R - Q) - (Q - P) = 4 + \frac{4n(n-1)}{1 \cdot 1} + \frac{4n(n-1)(n-2)(n-3)}{1 \cdot 1 \cdot 2 \cdot 2} + \text{ etc.}$$

quae series cum sit $= 4P$, habebimus:

$$R = Q + \frac{(n+1)(Q-P)}{n+2} + \frac{4(n+1)P}{n+2} \text{ seu}$$

$$R = \frac{(2n+3)Q + 3(n+1)P}{n+2}.$$

Coroll. 1.

8. En ergo legem qua quisque seriei terminus per binos praecedentes determinatur, quae ita se habet:

$$R = Q + \frac{n+1}{n+2}(Q + 3P)$$

vnde etiam ex binis sequentibus Q et R praecedens P ita definitur:

$$P = \frac{(n+2)R - (2n+3)Q}{3(n+1)}.$$

Coroll.

Coroll. 2.

9. Quo appareat, quomodo haec lex in ſerie propoſita locum habeat, per caſus aliquot eam illuſtremus:

ſi $n=0$, $3 = 1 + \frac{1}{2}(1 + 3.1)$
ſi $n=1$, $7 = 3 + \frac{2}{3}(3 + 3.1)$
ſi $n=2$, $19 = 7 + \frac{3}{4}(7 + 3.3)$
ſi $n=3$, $51 = 19 + \frac{4}{5}(19 + 3.7)$
ſi $n=4$, $141 = 51 + \frac{5}{6}(51 + 3.19)$
etc.

Coroll. 3.

10. Quoniam igitur in relationem, quae inter ternos terminos contiguos intercedit, ipſe exponens n ingreditur, hinc facile colligitur hanc ſeriem non ad genus recurrentium eſſe referendam.

Coroll. 4.

11. Inter quatuor autem continuos terminos P, Q, R et S relatio ab exponente n libera exhiberi poteſt, cum enim ex ternis prioribus ſit

$$n = \frac{2R - 3Q - 3P}{3P + 2Q - R}$$

erit ſimili modo $n + 1 = \frac{2S - 3R - 3Q}{3Q + 2R - S}$

vnde concludimus fore

$$S = R + Q + \frac{3P(Q + R) + 2QR}{6P + 3Q - R}$$

quae eſt relatio conſtans, qua per ternos quosque terminos contiguos ſequens definitur.

Scholion 1.

12. Inuenta lege, qua noſtrae progreſſionis quisque terminus a binis praecedentibus pendet, iam multo facilius hanc progreſſionem quousque lubuerit continuare licet. Ita cum dignitates x^{11} et x^{12} affectae ſint numeris 25653 et 73789, ſequentis x^{13} ob $n = 11$ coefficiens erit:

$$73789 + \tfrac{12}{13}(73789 + 3.25653) = 212941$$

et dignitatis x^{14}

$$212941 + \tfrac{13}{14}(212941 + 3.73789) = 616227$$

vnde noſtra progreſſio ad dignitatem viceſimam vsque continuata ita ſe habebit:

1	
$1x$	$25653x^{11}$
$3x^2$	$73789x^{12}$
$7x^3$	$212941x^{13}$
$19x^4$	$616227x^{14}$
$51x^5$	$1787607x^{15}$
$141x^6$	$5196627x^{16}$
$393x^7$	$15134931x^{17}$
$1107x^8$	$44152809x^{18}$
$3139x^9$	$128996853x^{19}$
$8953x^{10}$	$377379369x^{20}$

circa

circa quos numeros obſeruo nullum eorum eſſe per 5 diuiſibilem, dignitatum vero $x^{5\alpha+2}$ coefficientes eſſe per 3. diuiſibiles, dignitatum $x^{7\alpha+2}$ per 7. neque vero hinc quicquam circa indolem horum numerorum concludere licet. Verum ex lege progreſſionis hic inuenta eius ſummam, ſiquidem in infinitum continuetur, definire poterimus, cui fini ſequens problema deſtinatur.

Scholion 2.

13. Si noſtrae progreſſionis quilibet terminus a triplo antecedentis ſubtrahatur, differentiae talem progreſſionem conſtituunt:

1.2; 2. 1; 3. 2; 4.3; 5.6; 6.12; 7.26; 8.58; 9.134; 10.317; 11.766; 12.1883; 13.4698; 14.11871; 15.30330; 16.78249; 17.203622; 18.533955

pro qua generatim ſtatuamus:

$$mp;\ (m+1)q;\ (m+2)r$$

vbi primum notatu dignum occurrit, quod horum terminorum factores priores in ſerie numerorum naturali progrediantur, poſteriores vero ita ſint comparati, vt quilibet ex binis praecedentibus hoc modo conficiatur:

$$r = \frac{3mp + 2(m+1)q}{m+4}.$$

Problema 3.

14. Si series nostra

$$1 + x + 3x^2 + 7x^3 + 19x^4 + \text{etc.}$$

in infinitum continuetur, eius summam inuestigare.

Solutio.

Cum relatio cuiusque termini ad binos antecedentes sit definita, statuamus:

$$s = 1 + x + 3x^2 + \ldots + Px^n + Qx^{n+1} + Rx^{n+2} + \text{etc.}$$

vbi notetur esse $(n+2)R - (2n+3)Q - 3(n+1)P = 0$ cui conditioni vt satisfaciamus, sumamus differentiale:

$$\frac{ds}{dx} = 1 + 6x + \ldots + nPx^{n-1} + (n+1)Qx^n \\ + (n+2)Rx^{n+1} \text{ etc.}$$

quod multiplicatum per $1 - 2x - 3xx$ praebet:

$$\frac{ds}{dx}(1 - 2x - 3xx) =$$

$$\begin{array}{llllll} 1 + 6x + 21xx & \ldots & + nPx^{n-1} & + (n+1)Qx^n & + (n+2)Rx^{n+1} \\ - 2 - 12 & & & - 2nP & - (2n+2)Q \\ - 3 & & & & - 3nP \end{array}$$

quae series reducitur ad hanc:

$$1 + 4x + 6xx \ldots (Q + 3P)x^{n+1} + \text{etc.}$$

At ipsa series proposita per $1 + 3x$ multiplicata dat

$$s(1 + 3x) = 1 + 4x + 6xx \ldots (Q + 3P)x^{n+1}$$

vnde manifestum est fore:

$$\frac{ds}{dx}(1 - 2x - 3xx) = s(1 + 3x) \text{ ideoque}$$

$\frac{ds}{s}$

$\frac{ds}{s} = \frac{dx(1+3x)}{1-2x-3xx}$, cuius integratio praebet

$s = \frac{1}{\sqrt{(1-2x-3xx)}} = \frac{1}{\sqrt{(1+x)(1-3x)}}$

quae est ipsa summa seriei propositae in infinitum continuatae.

Coroll. 1.

15. Liquet ergo seriei huius summam esse imaginariam nisi sumatur $x < \frac{1}{3}$, casu autem $x = \frac{1}{3}$ fieri infinitam. At ipsi x valores negatiuos tribuendo, puta $x = -y$, summa fit finita sumendo $y < 1$, at casu $y > 1$ imaginaria euadit. Ita statuendo $x = -\frac{1}{2}$ fit

$\frac{2}{\sqrt{5}} = 1 - \frac{1}{2} + \frac{3}{4} - \frac{7}{8} + \frac{19}{16} - \frac{51}{32} + \frac{141}{64} -$ etc.

Coroll. 2.

16. Nunc ergo nouimus seriem nostram quoque resultare si formula irrationalis $(1-2x-3xx)^{-\frac{1}{2}}$ more solito in seriem euoluatur: quae formula cum ita repraesentari possit $s = ((1-x)^2 - 4xx)^{-\frac{1}{2}}$, prodit:

$s = \frac{1}{1-x} + \frac{2x}{1} \cdot \frac{xx}{(1-x)^3} + \frac{2 \cdot 6}{1 \cdot 2} \cdot \frac{x^4}{(1-x)^5} + \frac{2 \cdot 6 \cdot 10}{1 \cdot 2 \cdot 3} \cdot \frac{x^6}{(1-x)^7} +$ etc.

ex cuius vlteriori euolutione oritur:

$$\begin{array}{rcccccccccc} s = 1 & + x & + x^2 & + x^3 & + x^4 & + x^5 & + x^6 & + x^7 & + x^8 & + x^9 & + x^{10} \\ & & +2.1 & +2.3 & +2.6 & +2.10 & +2.15 & +2.21 & +2.28 & +2.36 & +2.45 \\ & & & & +6.1 & +6.5 & +6.15 & +6.35 & +6.70 & +6.126 & +6.210 \\ & & & & & & +20.1 & +20.7 & +20.28 & +20.84 & +20.210 \\ & & & & & & & & +70.1 & +70.9 & +70.45 \\ & & & & & & & & & & +252.1 \end{array}$$

Coroll.

Coroll. 3.

17. Hinc colligimus in genere dignitatis x^n coefficientem numericum ita expressum iri:

$$+\frac{2}{1}\cdot\frac{n(n-1)}{1.\ 2}+\frac{2.\ 6}{1.\ 2}\cdot\frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}+\text{etc.}$$

quae forma non discrepat ab ea, quam problemate primo inuenimus.

Scholion.

18. Quodsi formam huius summae accuratius perpendamus, haud difficulter inde methodum multo latius patentem elicimus, cuius ope adeo haec potestas generalior $(a+bx+cxx)^n$ ita pertractari poterit, vt non solum termini medii singularum potestatum sed etiam termini a mediis vtrinque aequidistantes assignari queant. Hanc ergo methodum in sequente problemate sum expositurus.

Problema 4.

19. Si trinomii $a+bx+cxx$ singulae potestates euoluantur, indeque tam termini medii, quam a mediis aequidistantes seorsim in series disponantur, singularum harum serierum naturam et summam inuestigare.

Solutio.

Consideretur formula ista $\frac{1}{1-y(a+bx+cxx)}$ quae euoluta praebet:

$$1+y(a+bx+cxx)+yy(a+bx+cxx)^2+y^3(a+bx+cxx)^3\ \text{etc.}$$

vbi

vbi cum trinomii propofiti fingulae poteftates occurrant, a explicatis orietur:

$$1$$
$$y(a + bx + cxx)$$
$$y^2(a^2 + 2abx + 2acx^2 + 2bcx^3 + ccx^4)$$
$$+ bb$$
$$y^3(a^3 + 3a^2bx + 3a^2cx^2 + 6abcx^3 + 3bbcx^4 + 3bccx^5 + c^3x^6)$$
$$+ 3ab^2 + b^3 + 3acc$$
etc.

hinc fi primo termini medii, tum vero termini a mediis vtrinque aequidiftantes capiantur, nafcentur fequentes feries:

$$1 + bxy + (2ac + bb)xxyy + (6abc + b^3)x^3y^3 + \text{etc.}$$
$$y(a + cxx)(1 + 2bxy + (3ac + 3bb)xxyy + \text{etc.})$$
$$y^2(a^2 + c^2x^4)(1 + 3bxy + \text{etc.})$$
$$y^3(a^3 + c^3x^6)(1 + 4bxy + \text{etc.})$$
$$y^4(a^4 + c^4x^8)(1 + 5bxy + \text{etc.})$$
etc.

Omiffis ergo his multiplicatoribus, quia in feriebus ipfis adfunt poteftates producti xy, ponamus $xy = z$ et indicemus iftas feries hoc modo:

$$1 + bz + (2ac + bb)zz + (6abc + b^3)z^3 = P$$
$$1 + 2bz + (3ac + 3bb)zz + \text{etc.} = Q$$
$$1 + 3bz + \text{etc.} = R$$
$$1 + 4bz + \text{etc.} = S$$
etc.

 ita

ita vt iam ob $y = \frac{z}{x}$, habeamus:

$$\frac{1}{1 - bz - z(\frac{a}{x} + cx)} = P + z(\frac{a}{x} + cx)Q + zz(\frac{aa}{xx} + ccxx)R + z^3(\frac{a^3}{x^3} + c^3x^3)S + \text{etc.}$$

multiplicetur vtrinque per $1 - bz - z(\frac{a}{x} + cx)$, et quoniam quantitates P, Q, R etc. a ſola z pendent, omnia membra ſecundum poteſtates ipſius x tam poſitiua quam negatiua diſponantur: quo facto obtinebimus:

$$\begin{aligned}1 = P(1-bz) &+ Qz(1-bz)cx + Rzz(1-bz)c^2x^2 + Sz^3(1-bz)c^3x^3 \\ -2Qaczz &- Pz.cx \quad - Qzz.ccx^2 \quad - Rz^3c^3x^3 \\ &- Rz^3.accx - Sz^4.ac^3x^2 \quad - Tz^5ac^4x^3 \\ &+ Qz(1-bz)\tfrac{a}{x} + Rzz(1-bz)\tfrac{a^2}{x^2} + Sz^3(1-bz)\tfrac{a^3}{x^3} \\ &- Pz.\tfrac{a}{x} \quad - Qzz.\tfrac{aa}{xx} \quad - Rz^3.\tfrac{a^3}{x^3} \\ &- Rz^3ac.\tfrac{a}{x} \quad - Sz^4ac.\tfrac{aa}{xx} \quad - Tz^5ac.\tfrac{a^3}{x^3}\end{aligned}$$

vbi euidens eſt poteſtates negatiuas ipſius x iisdem conditionibus ad nihilum redigi ac poſitiuas. Hinc erga ſequentes determinationes adipiſcimur:

$$Q = \frac{P(1 - bz) - 1}{2aczz}$$
$$R = \frac{Q(1 - bz) - P}{aczz}$$
$$S = \frac{R(1 - bz) - Q}{aczz}$$
$$T = \frac{S(1 - bz) - R}{aczz}$$

etc.

Vide-

Videmus ergo quantitates P, Q, R, S etc. secundum seriem recurrentem progredi, cuius scala relationis est:

$$\frac{1-bz}{aczz}\,; \quad -\frac{1}{aczz}$$

hinc si illis quantitatibus indices tribuantur:

$$\overset{0}{P},\ \overset{1}{Q},\ \overset{2}{R},\ \overset{3}{S}\ \ldots\ldots\ \overset{n}{Z}$$

ita vt Z sit ea, quae indici n conuenit, ex natura recurrentiae erit:

$$Z = A\left(\frac{1-bz-\sqrt{((1-bz)^2-4aczz)}}{2aczz}\right)^n + B\left(\frac{1-bz+\sqrt{((1-bz)^2-4aczz)}}{2aczz}\right)^n$$

vbi cum constet quantitatem Z eiusmodi serie exprimi, vt sit:

$$Z = 1+(n+1)bz + \ldots zz + \ldots z^3 + \ldots z^4 \text{ etc.}$$

vnde manifestum est necessario esse debere $B=0$, quia alioquin termini ex posteriori membro orientur potestatibus negatiuis ipsius z affecti. Facto ergo $B=0$, erit:

$$Z = A\left(\frac{1-bz-\sqrt{(1-2bz+(bb-4ac)zz)}}{2aczz}\right)^n$$

Iam fiat $n=0$, ac necesse est prodire $A=P$, posito autem $n=1$, effici debet:

$$A.\,\frac{1-bz-\sqrt{(1-2bz+(bb-4ac)zz)}}{2aczz} = Q.$$

Cum igitur sit $A=P$ et $2aczzQ+1=P(1-bz)$ sequitur fore:

$$P(1-bz)-P\sqrt{(1-2bz+(bb-4ac)zz)} = P(1-bz)-1$$

 ideoque

ideoque $P = \frac{1}{\sqrt{(1-2bz+(bb-4ac)zz)}}$.

Quocirca feriei noftrae P, Q, R, S . . . Z terminus generalis eft:

$$Z = \frac{1}{\sqrt{(1-2bz+(bb-4ac)zz)}}\left(\frac{1-bz-\sqrt{(1-2bz+(bb-4ac)zz)}}{2aczz}\right)^n.$$

Pofito ergo $y=1$ vt fit $x=z$, fi omnes poteftates trinomii $a+bz+czz$ euoluantur, feries terminorum intermediorum $1+bz+(2ac+bb)zz$ etc. erit $= P$

terminorum autem a mediis, n locis in antecedentia remotorum fumma eft $= a^n Z$, totidem vero locis in confequentia remotorum fumma $= c^n z^n Z$. At omnium harum ferierum iunctim fumtarum fumma eft $= \frac{1}{1-a-bz-czz}$.

Coroll. 1.

20 Quantitates ergo P, Q, R, S etc. progreffionem geometricam conftituunt, cuius primus terminus eft $P = \frac{1}{\sqrt{(1-2bz+(bb-4ac)zz)}}$, et denominator progreffionis:

$$\frac{1-bz-\sqrt{(1-2bz+(bb-4ac)zz)}}{2aczz}.$$

Coroll. 2.

21. Si fumamus $a=1$, $b=1$ et $c=1$, prodir cafus ante tractatus quo poteftates trinomii $1+z$

$1+z+zz$ confiderauimus, quarum termini medii feriem conftituunt, cuius fumma eft $= \frac{1}{\sqrt{(1-2z-3zz)}}$, vti fupra inuenimus.

Problema 5.

22. Formulam in praecedente problemate inventam:

$$\frac{1}{\sqrt{(1-2bz+(bb-4ac)zz)}}\left(\frac{1-bz-\sqrt{(1-2bz+(bb-4ac)zz)}}{2aczz}\right)^n.$$

in feriem conuertere, cuius termini fecundum dignitates ipfius z procedant.

Solutio.

Sit breuitatis gratia $bb-4ac=e$, ac ponatur

$$s=\frac{1}{\sqrt{(1-2bz+ezz)}}\left(\frac{1-bz-\sqrt{(1-2bz+ezz)}}{2aczz}\right)^n$$

quam relationem inter z et s per differentiationem ab irrationalitate liberari oportet. Hunc in finem ftatuatur:

$\frac{1-bz-\sqrt{(1-2bz+ezz)}}{2aczz}=v$ vt fit $acvvzz-(1-bz)v+1=0$

vnde differentiando fit:

$dv(2acvzz-1+bz)+vdz(2acvz+b)=0$ feu

$dv\sqrt{(1-2bz+ezz)}=\frac{vdz}{z}(1-\sqrt{(1-2bz+ezz)})$

ideoque $\frac{dv}{v}=\frac{dz}{z\sqrt{(1-2bz+ezz)}}-\frac{dz}{z}$.

Hinc illa aequatione logarithmice differentiata prodit

$$\frac{ds}{s}=\frac{dz(b-ez)}{1-2bz+ezz}-\frac{ndz}{z}+\frac{ndz}{z\sqrt{(1-2bz+ezz)}}.$$

 Pona-

Ponamus tantisper $\frac{dt}{t} = \frac{ds}{s} + \frac{n\,dz}{z} - \frac{dz(b-ez)}{1-2bz+ezz}$, vt ſit $\frac{dt}{t} = \frac{n\,dz}{z\sqrt{(1-2bz+ezz)}}$, vnde quadrata ſumendo colligimus: $zzdt^2(1-2bz+ezz) = nn\mathrm{H}dz^2$, quae aequatio denuo differentiata poſito elemento dz conſtante dat:

$$zzddt(1-2bz+ezz) + zdt\,dz(1-3bz+2ezz) = nnt\,dz^2$$

ſeu $\frac{ddt}{t} + \frac{dz(1-3bz+2ezz)}{z(1-2bz+ezz)} \cdot \frac{dt}{t} - \frac{nn\,dz^2}{zz(1-2bz+ezz)} = 0.$

Iam cum ſit $\frac{ddt}{t} = d.\frac{dt}{t} + \frac{dt^2}{tt}$ erit

$$\frac{ddt}{t} = \frac{dds}{s} - \frac{ds^2}{ss} - \frac{ndz^2}{zz} + \frac{dz^2(e-2bb+2bez-eezz)}{(1-2bz+ezz)^2} + \frac{nndz^2}{zz} - \frac{2ndz^2(b-ez)}{z(1-2bz+ezz)}$$
$$+ \frac{ds^2}{ss} + \frac{2ndzds}{sz} - \frac{2dzds(b-ez)}{s(1-2bz+ezz)} + \frac{dz^2(bb-2bez+eezz)}{(1-2bz+ezz)^2}.$$

Facta ergo ſubſtitutione ſuperior aequatio in hanc abit formam:

$$\frac{dds}{s} + \frac{2ndz}{z}\cdot\frac{ds}{s} - \frac{2dz(b-ez)}{1-2bz+ezz}\cdot\frac{ds}{s} + \frac{n(n-1)dz^2}{zz} - \frac{2ndz^2(b-ez)}{z(1-2bz+ezz)} + \frac{dz^2(e-bb)}{(1-2bz+ezz)^2}$$
$$+ \frac{dz(1-3bz+2ezz)}{z(1-2bz+ezz)}\cdot\frac{ds}{s} + \frac{ndz^2(1-3bz+2ezz)}{zz(1-2bz+ezz)} - \frac{dz^2(b-(e+3bb)z+5bezz-2eez^3)}{z(1-2bz+ezz)^2}$$
$$- \frac{nndz^2}{zz(1-2bz+ezz)} = 0$$

vbi ſi termini per $(1-2bz+ezz)^2$ diuiſi in vnam ſummam colligantur, fractio per $1-2bz+ezz$ deprimi poterit, vnde facta reductione adipiſcimur:

$$\frac{dds}{s} + \frac{2ndz}{z}\cdot\frac{ds}{s} + \frac{dz(1-5bz+4ezz)}{z(1-2bz+ezz)}\cdot\frac{ds}{s} - \frac{nndz^2(2b-ez)}{z(1-2bz+ezz)} - \frac{3ndz^2(b-ez)}{z(1-2bz+ezz)}$$
$$- \frac{dz^2(b-2ez)}{z(1-2bz+ezz)} = 0$$

quae ordinata euadit:

$$zdds(1-2bz+ezz) + dzds(2n+1-(4n+5)bz+2(n+2)ezz)$$
$$-sdz^2((n+1)(2n+1)b-(n+1)(n+2)zz) = 0.$$

Cum

Cum nunc conſtet poſito $z=0$ fieri $s=1$, fingamus hanc feriem :

$$s=1+Az+Bzz+Cz^3+Dz^4+Ez^5+\text{etc.}$$

qua ferie ſubſtituta ſequens forma ad nihilum eſt redigenda :

$$\begin{array}{r}
2Bz+6Czz \qquad +12Dz^3 \qquad +20Ez^4 \\
-4Bb \qquad -12Cb \qquad -24Db \\
+2Be \qquad +\ 6Ce \\
(2n+1)A+2(2n+1)B+3(2n+1)C+4(2n+1)D+5(2n+1)E \\
-(4n+5)A-2(4n+5)Bb-3(4n+5)Cb-4(4n+5)Db \\
+2(n+2)Ae+4(n+2)Be+6(n+2)Ce \\
-(n+1)(2n+1)b-(n+1)(2n+1)Ab-(n+1)(2n+1)Bb-(n+1)(2n+1)Cb-(n+1)(2n+1)De \\
+(n+1)(n+2)e+(n+1)(n+2)Ae+(n+1)(n+2)Be+(n+1)(n+2)Ce
\end{array}$$

vnde colligimus has determinationes :

$$A=(n+1)b$$

$$B=\frac{(n+2)((2n+3)Ab-(n+1)e)}{2(2n+2)}$$

$$C=\frac{(n+3)((2n+5)Bb-(n+2)Ae)}{3(2n+3)}$$

$$D=\frac{(n+4)((2n+7)Cb-(n+3)Be)}{4(2n+4)}$$

$$E=\frac{(n+5)((2n+9)Db-(n+4)Ce)}{5(2n+5)}$$

etc

vbi notetur eſſe $e=bb-4ac$. Sicque feriei quaeſitae finguli termini per binos praecedentes determinantur.

DE