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# Observationes analyticae

Leonhard Euler

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# OBSERVATIONES ANALYTICAE.

Auctore

L. EULER O.

Consideranti potestates, quae ex elevatione huius formae trinomiae  $1 + x + xx$  nascuntur, termini medii maximis coefficientibus numericisprehenduntur affecti, quorum ordo progressionis cum non parum sit reconditus, omni attentione dignus videtur; praecipue quoniam huiusmodi speculationes plerumque fructum haud spernendum in Analyfi afferre solent. Primum ergo harum potestatum simpliciores conspectui exponam:

Exponens potestatis	Potestates evolutae
0 . . . . .	1
1 . . . . .	$1 + x + x^2$
2 . . . . .	$1 + 2x + 3x^2 + 2x^3 + x^4$
3 . . . . .	$1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6$
4 . . . . .	$1 + 4x + 10x^2 + 16x^3 + 19x^4 + 16x^5 + 10x^6 + 4x^7 + x^8$
5 . . . . .	$1 + 5x + 15x^2 + 30x^3 + 45x^4 + 51x^5 + 45x^6 + 30x^7 + 15x^8 + 5x^9 + x^{10}$
	etc.

hinc si termini medii ex singulis potestatibus ordine repraesententur, haec exoritur progressio:

$1, 1x, 3x^2, 7x^3, 19x^4, 51x^5, 141x^6$  etc.

qui

qui numeri, quam lege progrediantur, haud immerito indagari videtur, ut non solum inde terminus generalis seu coefficientis dignitati indefinitae  $x^n$  conueniens innotescat, sed etiam insignes huius seriei proprietates explorentur. Hunc in finem sequentia problemata proponam, quorum resolutio deinceps ad alias speculationes non parum curiosas manuducet.

### Problema I.

I. Euoluta hac potestate indefinita  $(1+x+xx)^n$  coefficientem termini medii seu dignitatis  $x^n$  definire.

### Solutio.

Potestas proposita ita sub forma binomii representetur  $(x(1+x)+1)^n$ , quae more solito euoluta praebet:

$$x^n(1+x)^n + \frac{n}{1} x^{n-1}(1+x)^{n-1} + \frac{n(n-1)}{1 \cdot 2} x^{n-2}(1+x)^{n-2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^{n-3}(1+x)^{n-3} \text{ etc.}$$

ex cuius singulis membris, si ulterius euoluantur, terminos formae  $x^n$  elici oportet. Ac primum quidem membrum praebet  $x^n$ , cum reliquae potestates omnes ipsius  $x$  ex eius evolutione ortae futurae sint altiores. Ex secundo autem membro pro hac dignitate  $x^n$  oritur:

$$\frac{n}{1} x^{n-1} \cdot \frac{n-1}{1} x = \frac{n(n-1)}{1 \cdot 1} x^n$$

Q 3

ex

ex tertio membro simili modo consequimur :

$$\frac{n(n-1)}{1, 2} x^{n-2} \cdot \frac{(n-2)(n-3)}{1, 2} x^2 = \frac{n(n-1)(n-2)(n-3)}{1, 2, 1, 2} x^n$$

quas cunctas partes si in vnam summam colligamus obtinetur dignitatis  $x^n$  coefficientis quaesitus :

$$1 + \frac{n(n-1)}{1, 1} + \frac{n(n-1)(n-2)(n-3)}{1, 2, 1, 2} + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1, 2, 3, 1, 2, 3} \text{ etc.}$$

### Coroll. 1.

2. Haec ergo series, quae quoties  $n$  est numerus integer abrumpitur, coefficientem praebet dignitatis  $x^n$  pro serie proposita  $1 + x + 3x^2 + 7x^3 + 19x^4 + \text{etc.}$  sicque eius ope terminus quantumvis ab initio remotus statim sine praecedentibus inueniri potest.

### Coroll. 2.

3. Quod si pro  $n$  successiue numeros 1, 2, 3 etc. substituiamus, sequentes valores reperiuntur :

$n$  coefficientis ipsius  $x^n$

0 1

1 1

$$2 \cdot 1 + 2 = 3$$

$$3 \cdot 1 + 6 = 7$$

$$4 \cdot 1 + 12 + 6 = 19$$

$$5 \cdot 1 + 20 + 30 = 51$$

$$6 \cdot 1 + 30 + 90 + 20 = 141$$

$$7 \cdot 1 + 42 + 210 + 140 = 393$$

$n$  coe.

*n* | coefficientis ipsius  $x^n$

$$81 + 56 + 420 + 560 + 70 = 1107$$

$$91 + 72 + 756 + 1680 + 630 = 3139$$

$$101 + 90 + 1260 + 4200 + 3150 + 252 = 8953$$

$$111 + 110 + 1980 + 9240 + 11550 + 2772 = 25653$$

$$121 + 132 + 2970 + 18480 + 34650 + 16632 + 924 = 73789$$

etc.

### Coroll. 3.

4. Series horum numerorum ita est comparata, ut quisque terminus cum triplo praecedentis commode conferri posse videatur, ex qua comparatione sequentes differentiae nascuntur:

$$\begin{array}{r}
 1, 1, 3, 7, 19, 51, 141, 393, 1107, 3139 \text{ etc.} \\
 3, 3, 9, 21, 57, 153, 423, 1179, 3321 \text{ etc.} \\
 \hline
 2, 0, 2, 2, 6, 12, 30, 72, 182 \text{ etc.}
 \end{array}$$

### Scholion I.

5. Si has differentias accuratius contemplemur, non sine ratione euenire videtur, quod eae sint numeri pronici, seu trigonales duplicati in forma  $mm + m$  contenti, ac si ad istorum pronicorum numerorum radices spectemus, quae hanc seriem constituunt:

$$1, 0, 1, 1, 2, 3, 5, 8, 13 \text{ etc.}$$

ea manifesto est recurrens, cuius quisque terminus est summa binorum praecedentium. Qui ordo cum

in

Exemplum memorabile inductionis fallacis.

in decem primoribus terminisprehendatur, quis dubitauerit eundem vniuersae seriei tribuere? saepe profecto inductiones minus certae successu non fuerunt destitutae. Operae ergo pretium erit hanc rationem accuratius perpendere, scilicet cum numerus 13 conueniat seriei termino  $x^9$ , in genere dignitati  $x^n$  respondebit numerus:

$$\frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^{n-2} - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^{n-2}$$

cuius numerus pronicus est:

$$\frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^{n-2} - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^{n-2} + \frac{1}{5} \left( \frac{1+\sqrt{5}}{2} \right)^{2n-4} + \frac{1}{5} \left( \frac{1-\sqrt{5}}{2} \right)^{2n-4} - \frac{2}{5} (-1)^{n-2}$$

Quare si in serie proposita bini termini contigui generatim ita exhibeantur:

$$1 + x + 3x^2 + 7x^3 + 10x^4 \dots Px^n + Qx^{n+1}$$

erit  $3P - Q =$

$$\frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^{n-1} - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^{n-1} + \frac{1}{5} \left( \frac{1+\sqrt{5}}{2} \right)^{2n-2} + \frac{1}{5} \left( \frac{1-\sqrt{5}}{2} \right)^{2n-2} - \frac{2}{5} (-1)^{n-1}$$

vnde concluditur fore:

$$P = \frac{3^n + (-1)^n}{10} + \frac{1}{5} \left( \frac{3+\sqrt{5}}{2} \right)^n + \frac{1}{5} \left( \frac{3-\sqrt{5}}{2} \right)^n + \frac{1}{5} \left( \frac{1+\sqrt{5}}{2} \right)^n + \frac{1}{5} \left( \frac{1-\sqrt{5}}{2} \right)^n$$

ita vt ipsa quoque series proposita foret recurrens scala relationis existente:

$$6, -8, 7, 8, 13, 4, -3$$

secun-

secundum quam erit :

$$3139 = 6.1107 - 8.393 - 8.141 + 14.51 + 4.9 - 3.7.$$

Scholion 2.

6. Verum quantumvis probabili inductione haec lex progressionis inniti videatur, dum adeo in decem primoribus terminis locum habet; tamen ea fallax deprehenditur, dum iam in termino vndecimo 8953 fallit; hoc enim a triplo praecedentis 9417 sublato residuum 464 ne numerus quidem pronicus est, multo minus radicem pronicam habet  $21 = 13 + 8$ , est enim  $21^2 + 21 = 462$ , qui numerus binario deficit ab eo 464, qui secundum legem obseruatam resultare debebat. Quam ob causam nunc quidem in veram progressionis legem huius seriei sum inquisiturus, vt pateat quomodo quisque terminus per aliquot praecedentes reuera determinetur.

Problema 2.

7. Pro serie proposita

$$1, x, 3x^2, 7x^3, 19x^4, 51x^5 \text{ etc.}$$

intelligere legem, qua quisque terminus per aliquot praecedentes determinatur.

Solutio.

Considerentur generatim aliquot huius seriei termini se mutuo sequentes :

$$1, x, 3x^2, 7x^3 \dots \dots Px^n, Qx^{n+1}, Rx^{n+2}$$

Tom. XI. Nou. Comm.

R

et

et quoniam in problemate praecedente vidimus esse:

$$P = 1 + \frac{n(n-1)}{1 \cdot 1} + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 1 \cdot 2 \cdot 2} + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3} \text{ etc.}$$

erit simili modo:

$$Q = 1 + \frac{(n+1)n}{1 \cdot 1} + \frac{(n+1)(n-1)(n-2)}{1 \cdot 1 \cdot 2 \cdot 2} + \frac{(n+1)n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3} \text{ etc.}$$

$$R = 1 + \frac{(n+1)(n+1)}{1 \cdot 1} + \frac{(n+2)(n+1)(n-1)}{1 \cdot 1 \cdot 2 \cdot 2} + \frac{(n+2)(n+1)(n-1)(n-2)(n-3)}{1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3} \text{ etc.}$$

vnde quamlibet a sequente subtrahendo colligimus:

$$Q - P = \frac{2n}{1} + \frac{2n(n-1)(n-2)}{1 \cdot 1 \cdot 2 \cdot 2} + \frac{2n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3} \text{ etc.}$$

$$R - Q = \frac{2(n+1)}{1} + \frac{2(n+1)(n-1)}{1 \cdot 1 \cdot 2 \cdot 2} + \frac{2(n+1)(n-1)(n-2)(n-3)}{1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3} \text{ etc.}$$

hinc capiamus hanc formam:

$$\frac{n+2}{n+1}(R - Q) = \frac{2(n+2)}{1} + \frac{2(n+1)(n-1)}{1 \cdot 1 \cdot 2 \cdot 2} + \frac{2(n+2)(n-1)(n-2)(n-3)}{1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3} \text{ etc.}$$

a qua illa  $Q - P$  subtrahatur vt fiat:

$$\frac{n+2}{n+1}(R - Q) - (Q - P) = 4 + \frac{4n(n-1)}{1 \cdot 1} + \frac{4n(n-1)(n-2)(n-3)}{1 \cdot 1 \cdot 2 \cdot 2} + \text{ etc.}$$

quae series cum sit  $= 4P$ , habebimus:

$$R = Q + \frac{(n+1)(Q-P)}{n+2} + \frac{4(n+1)P}{n+2} \text{ seu}$$

$$R = \frac{(2n+3)Q + 3(n+1)P}{n+2}$$

### Coroll. I.

8. En ergo legem qua quisque seriei terminus per binos praecedentes determinatur, quae ita se habet:

$$R = Q + \frac{n+1}{n+2}(Q + 3P)$$

vnde etiam ex binis sequentibus  $Q$  et  $R$  praecedens  $P$  ita definitur:

$$P = \frac{(n+2)R - (2n+3)Q}{3(n+1)}$$

Coroll.



## Coroll. 2.

9. Quo appareat, quomodo haec lex in serie proposita locum habeat, per casus aliquot eam illustremus:

$$\begin{aligned} \text{si } n=0, & \quad 3 = 1 + \frac{1}{2}(1 + 3 \cdot 1) \\ \text{si } n=1, & \quad 7 = 3 + \frac{2}{3}(3 + 3 \cdot 1) \\ \text{si } n=2, & \quad 19 = 7 + \frac{3}{4}(7 + 3 \cdot 3) \\ \text{si } n=3, & \quad 51 = 19 + \frac{4}{5}(19 + 3 \cdot 7) \\ \text{si } n=4, & \quad 141 = 51 + \frac{5}{6}(51 + 3 \cdot 19) \\ & \quad \text{etc.} \end{aligned}$$

## Coroll. 3.

10. Quoniam igitur in relationem, quae inter ternos terminos contiguos intercedit, ipse exponent  $n$  ingreditur, hinc facile colligitur hanc seriem non ad genus recurrentium esse referendam.

## Coroll. 4.

11. Inter quatuor autem continuos terminos  $P, Q, R$  et  $S$  ratio ab exponente  $n$  libera exhiberi potest, cum enim ex ternis prioribus fit

$$n = \frac{2R - 3Q - 3P}{5P + 2Q - R}$$

$$\text{erit simili modo } n + 1 = \frac{2S - 3R - 3Q}{3Q + 2R - S}$$

vnde concludimus fore

$$S = R + Q + \frac{3P(Q + R) + 2QR}{5P + 2Q - R}$$

R 2

quae

quae est ratio constans, qua per ternos quosque terminos contiguos sequens definitur.

### Scholion I.

12. Inuenta lege, qua nostrae progressionis quisque terminus a binis praecedentibus pendet, iam multo facilius hanc progressionem quousque lubuerit continuare licet. Ita cum dignitates  $x^{11}$  et  $x^{12}$  affectae sint numeris 25653 et 73789, sequentis  $x^{13}$  ob  $n=11$  coefficientis erit:

$$73789 + \frac{12}{11}(73789 + 3 \cdot 25653) = 212941$$

et dignitatis  $x^{14}$

$$212941 + \frac{13}{12}(212941 + 3 \cdot 73789) = 616227$$

vnde nostrae progressio ad dignitatem vicefimam vsque continuata ita se habebit:

1	
$1x$	$25653x^{11}$
$3x^2$	$73789x^{12}$
$7x^3$	$212941x^{13}$
$19x^4$	$616227x^{14}$
$51x^5$	$1787607x^{15}$
$141x^6$	$5196627x^{16}$
$393x^7$	$15134931x^{17}$
$1107x^8$	$44152809x^{18}$
$3139x^9$	$128996853x^{19}$
$8953x^{10}$	$377379369x^{20}$

circa

circa quos numeros obseruo nullum eorum esse per 5 diuisibilem, dignitatum vero  $x^{3\alpha+2}$  coefficientes esse per 3. diuisibiles, dignitatum  $x^{7\alpha+2}$  per 7. neque vero hinc quicquam circa indolem horum numerorum concludere licet. Verum ex lege progressionis hic inuenta eius summam, siquidem in infinitum continuetur, definire poterimus, cui fini sequens problema destinatur.

### Scholion 2.

13. Si nostrae progressionis quilibet terminus a triplo antecedentis subtrahatur, differentiae talem progressionem constituunt:

1. 2; 2. 1; 3. 2; 4. 3; 5. 6; 6. 12; 7. 26; 8. 58; 9. 134;  
 10. 317; 11. 766; 12. 1883; 13. 4698; 14. 11871;  
 15. 30330; 16. 78249; 17. 203622; 18. 533955

pro qua generatim statuamus:

$$mp; (m+1)q; (m+2)r$$

vbi primum notatu dignum occurrit, quod horum terminorum factores priores in serie numerorum naturali progrediantur, posteriores vero ita sint comparati, vt quilibet ex binis praecedentibus hoc modo conficiatur:

$$r = \frac{3mp + 2(m+1)q}{m+1}$$

## Problema 3.

14. Si series nostra

$$1 + x + 3x^2 + 7x^3 + 19x^4 + \text{etc.}$$

in infinitum continuetur, eius summam inuefigare;

## Solutio.

Cum relatio cuiusque termini ad binos antecedentes sit definita, statuamus:

$$s = 1 + x + 3x^2 + \dots + Px^n + Qx^{n+1} + Rx^{n+2} + \text{etc.}$$

$$\text{ubi notetur esse } (n+2)R - (2n+3)Q - 3(n+1)P = 0$$

cui conditioni vt satisfaciamus, fumamus differentiale:

$$\frac{d^2 s}{dx^2} = 1 + 6x + \dots + nPx^{n-1} + (n+1)Qx^n + (n+2)Rx^{n+1} \text{ etc.}$$

quod multiplicatum per  $1 - 2x - 3xx$  praebet:

$$\begin{aligned} \frac{d^2 s}{dx^2} (1 - 2x - 3xx) = \\ 1 + 6x + 21xx \dots + nPx^{n-1} + (n+1)Qx^n + (n+2)Rx^{n+1} \\ - 2 - 12 \qquad \qquad \qquad - 2nP \qquad - (2n+2)Q \\ - 3 \qquad \qquad \qquad \qquad \qquad \qquad - 3nP \end{aligned}$$

quae series reducitur ad hanc:

$$1 + 4x + 6xx \dots (Q + 3P)x^{n+1} + \text{etc.}$$

At ipsa series proposita per  $1 + 3x$  multiplicata dat

$$s(1 + 3x) = 1 + 4x + 6xx \dots (Q + 3P)x^{n+1}$$

vnde manifestum est fore:

$$\frac{d^2 s}{dx^2} (1 - 2x - 3xx) = s(1 + 3x) \text{ ideoque}$$

$\frac{ds}{s} = \frac{dx(1+3x)}{1-2x-3xx}$ , cuius integratio praebet

$$s = \frac{1}{\sqrt{(1-2x-3xx)}} = \frac{1}{\sqrt{(1+x)(1-3x)}}$$

quae est ipsa summa seriei propositae in infinitum continuatae.

### Coroll. 1.

15. Liqueat ergo seriei huius summam esse imaginariam nisi sumatur  $x < \frac{1}{3}$ , casu autem  $x = \frac{1}{3}$  fieri infinitam. At ipsi  $x$  valores negativos tribuendo, puta  $x = -y$ , summa fit finita sumendo  $y < 1$ , at casu  $y > 1$  imaginaria euadit. Ita statuendo  $x = -\frac{1}{2}$  fit

$$\frac{2}{\sqrt{5}} = 1 - \frac{1}{2} + \frac{3}{4} - \frac{7}{8} + \frac{19}{16} - \frac{51}{32} + \frac{141}{64} - \text{etc.}$$

### Coroll. 2.

16. Nunc ergo nouimus seriem nostram quae resultare fit formula irrationalis  $(1-2x-3xx)^{-\frac{1}{2}}$  more solito in seriem euolatur: quae formula cum ita repraesentari possit  $s = ((1-x)^2 - 4xx)^{-\frac{1}{2}}$  prodit:

$$s = \frac{x}{1-x} + \frac{2x^2}{x} \cdot \frac{2x}{(1-x)^2} + \frac{2 \cdot 6}{1 \cdot 2} \cdot \frac{x^3}{(1-x)^3} + \frac{2 \cdot 6 \cdot 10}{1 \cdot 2 \cdot 3} \cdot \frac{x^4}{(1-x)^4} + \text{etc.}$$

ex cuius ulteriori euolutione oritur:

$$s = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} \\ + 2.1 + 2.3 + 2.6 + 2.10 + 2.15 + 2.21 + 2.28 + 2.36 + 2.45 \\ + 6.1 + 6.5 + 6.15 + 6.35 + 6.70 + 6.126 + 6.210 \\ + 20.1 + 20.7 + 20.28 + 20.84 + 20.210 \\ + 70.1 + 70.9 + 70.45 \\ + 252.1$$

Coroll.

## Coroll. 3.

17. Hinc colligimus in genere dignitatis  $x^n$  coefficientem numericum ita expressum iri :

$$+ \frac{2}{1 \cdot 2} \frac{n(n-1)}{1 \cdot 2} + \frac{2 \cdot 6}{1 \cdot 2 \cdot 3} \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.}$$
 quae forma non discrepat ab ea, quam problemate primo inuenimus.

## Scholion.

18. Quodsi formam huius summae accuratius perpendamus, haud difficulter inde methodum multo latius patentem elicimus, cuius ope adeo haec potestas generalior  $(a+bx+cx^2)^n$  ita pertractari poterit, vt non solum termini medii singularum potestatum sed etiam termini a mediis vtrinque aequidistantes assignari queant. Hanc ergo methodum in sequente problemate sum expositurus.

## Problema 4.

19. Si trinomii  $a+bx+cx^2$  singulae potestates euoluantur, indeque tam termini medii, quam a mediis aequidistantes seorsim in series disponantur, singularum harum serierum naturam et summam inuestigare.

## Solutio.

Consideretur formula ista  $\frac{1}{1-y(a+bx+cx^2)}$  quae euoluta praebet :

$$1 + y(a+bx+cx^2) + yy^2(a+bx+cx^2)^2 + y^3(a+bx+cx^2)^3 \text{ etc.}$$

vbi

vbi cum trinomiali propositi singulae potestates occurrant, a explicatis oriatur:

$$\begin{aligned}
 & \text{I} \\
 & y(a + bx + cxx) \\
 & y^2(a^2 + 2abx + 2acx^2 + 2bcx^3 + ccx^4) \\
 & \quad + bb \\
 & y^3(a^3 + 3a^2bx + 3a^2cx^2 + 6abcx^3 + 3bbc^2x^4 + 3bccx^5 + c^3x^6) \\
 & \quad + 3ab^2 + b^3 + 3acc \\
 & \text{etc.}
 \end{aligned}$$

hinc si primo termini medii, tum vero termini a mediis vtrinque aequidistantes capiantur, nascentur sequentes series:

$$\begin{aligned}
 & 1 + bxy + (2ac + bb)xyy + (6abc + b^3)x^2y^2 + \text{etc.} \\
 & y(a + cxx)(1 + 2bxy + (3ac + 3bb)xyy + \text{etc.}) \\
 & y^2(a^2 + c^2x^4)(1 + 3bxy + \text{etc.}) \\
 & y^3(a^3 + c^3x^6)(1 + 4bxy + \text{etc.}) \\
 & y^4(a^4 + c^4x^8)(1 + 5bxy + \text{etc.}) \\
 & \text{etc.}
 \end{aligned}$$

Omissis ergo his multiplicatoribus, quia in seriebus ipsis adsunt potestates producti  $xy$ , ponamus  $xy = z$  et indicemus istas series hoc modo:

$$\begin{aligned}
 1 + bz + (2ac + bb)zz + (6abc + b^3)z^3 &= P \\
 1 + 2bz + (3ac + 3bb)zz + \text{etc.} &= Q \\
 1 + 3bz + \text{etc.} &= R \\
 1 + 4bz + \text{etc.} &= S
 \end{aligned}$$

etc.

Tom. XI. Nou. Comm.

S

ita

ita vt iam ob  $y = \frac{z}{x}$ , habeamus:

$$\frac{1}{1 - bz - z\left(\frac{a}{x} + cx\right)} = P + z\left(\frac{a}{x} + cx\right)Q + zz\left(\frac{a^2}{xx} + ccxx\right)R \\ + z^3\left(\frac{a^3}{x^3} + c^3x^3\right)S + \text{etc.}$$

multiplicetur vtrinque per  $1 - bz - z\left(\frac{a}{x} + cx\right)$ , et quoniam quantitates P, Q, R etc. a sola  $z$  pendent, omnia membra secundum potestates ipsius  $x$  tam positua quam negatiua disponantur: quo facto obtinebimus:

$$1 = P(1 - bz) + Qz(1 - bz)cx + Rzz(1 - bz)c^2x^2 + Sz^3(1 - bz)c^3x^3 \\ - 2Qacz - Pz \cdot cx - Qzz \cdot ccx^2 - Rz^3c^3x^3 \\ - Rz^3 \cdot accx - Sz^4 \cdot ac^3x^2 - Tz^5ac^4x^3 \\ + Qz(1 - bz)\frac{a}{x} + Rzz(1 - bz)\frac{a^2}{x^2} + Sz^3(1 - bz)\frac{a^3}{x^3} \\ - Pz \cdot \frac{a}{x} - Qzz \cdot \frac{a^2}{xx} - Rz^3 \cdot \frac{a^3}{x^3} \\ - Rz^3ac \cdot \frac{a}{x} - Sz^4ac \cdot \frac{a^2}{xx} - Tz^5ac \cdot \frac{a^3}{x^3}$$

vbi euident est potestates negatiuas ipsius  $x$  iisdem conditionibus ad nihilum redigi ac posituas. Hinc erga sequentes determinaciones adipiscimur:

$$Q = \frac{P(1 - bz) - 1}{zacz} \\ R = \frac{Q(1 - bz) - P}{acz} \\ S = \frac{R(1 - bz) - Q}{acz} \\ T = \frac{S(1 - bz) - R}{acz}$$

etc.

Vide.



Videmus ergo quantitates P, Q, R, S etc. secundum seriem recurrentem progredi, cuius scala relationis est:

$$\frac{1-bz}{acz^2}; \quad -\frac{1}{acz^2}$$

hinc si illis quantitatibus indices tribuantur:

$$P^0, Q^1, R^2, S^3, \dots, Z^n$$

ita ut Z sit ea, quae indici n conuenit, ex natura recurrentiae erit:

$$Z = A \left( \frac{1-bz - \sqrt{(1-bz)^2 - 4acz^2}}{2acz^2} \right)^n + B \left( \frac{1-bz + \sqrt{(1-bz)^2 - 4acz^2}}{2acz^2} \right)^n$$

vbi cum constet quantitatem Z eiusmodi serie exprimi, ut sit:

$$Z = 1 + (n+1)bz + \dots zz + \dots z^5 + \dots z^4 \text{ etc.}$$

vnde manifestum est necessario esse debere B=0, quia alioquin termini ex posteriori membro orientur potestatibus negatiuis ipsius z affecti. Facto ergo B=0, erit:

$$Z = A \left( \frac{1-bz - \sqrt{(1-bz)^2 - 4acz^2}}{2acz^2} \right)^n$$

Iam fiat n=0, ac necesse est prodire A=P, posito autem n=1, effici debet:

$$A \cdot \frac{1-bz - \sqrt{(1-bz)^2 - 4acz^2}}{2acz^2} = Q.$$

Cum igitur sit A=P et 2acz^2Q+1=P(1-bz) sequitur fore:

$$P(1-bz) - P\sqrt{(1-bz)^2 - 4acz^2} = P(1-bz) - 1$$

S 2

ideoque

ideoque  $P = \frac{1}{\sqrt{(1-2bz+(bb-4ac)zz)}}$

Quocirca seriei nostrae P, Q, R, S . . . Z terminus generalis est:

$$Z = \frac{1}{\sqrt{(1-2bz+(bb-4ac)zz)}} \left( \frac{1-bz-\sqrt{(1-2bz+(bb-4ac)zz)}}{2acz} \right)^n$$

Posito ergo  $y=1$  ut sit  $x=z$ , si omnes potestates trinomiali  $a+bz+cz^2$  euoluantur, series terminorum intermediorum  $1+bz+(2ac+bb)zz$  etc. erit  $=P$

terminorum autem a mediis,  $n$  locis in antecedentia remotorum summa est  $=a^n Z$ , totidem vero locis in consequentia remotorum summa  $=c^n z^n Z$ . At omnium harum serierum iunctim sumtarum summa est  $=\frac{1}{1-a-bz-cz^2}$ .

### Coroll. 1.

20 Quantitates ergo P, Q, R, S etc. progressionem geometricam constituunt, cuius primus terminus est  $P = \frac{1}{\sqrt{(1-2bz+(bb-4ac)zz)}}$ , et denominator progressionis:

$$\frac{1-bz-\sqrt{(1-2bz+(bb-4ac)zz)}}{2acz}$$

### Coroll. 2.

21. Si sumamus  $a=1$ ,  $b=1$  et  $c=1$ , prodir casus ante tractatus quo potestates trinomiali

$$1+z$$

$1 + z + z^2$  considerauimus, quarum termini medii seriem constituunt, cuius summa est  $= \frac{1}{\sqrt{(1-2z+z^2)}}$ , vti supra inuenimus.

**Problema 5.**

22. Formulam in praecedente problemate inventam :

$$\frac{1}{\sqrt{(1-2bz+(bb-4ac)zz)}} \left( \frac{1-bz-\sqrt{(1-2bz+(bb-4ac)zz)}}{2acz} \right)^n$$

in seriem conuertere, cuius termini secundum dignitates ipsius  $z$  procedant.

**Solutio.**

Sit breuitatis gratia  $bb-4ac=e$ , ac ponatur

$$s = \frac{1}{\sqrt{(1-2bz+ezz)}} \left( \frac{1-bz-\sqrt{(1-2bz+ezz)}}{2acz} \right)^n$$

quam relationem inter  $z$  et  $s$  per differentiationem ab irrationalitate liberari oportet. Hunc in finem statuatur :

$$\frac{1-bz-\sqrt{(1-2bz+ezz)}}{2acz} = v \text{ vt fit } acvzz - (1-bz)v + 1 = 0$$

vnde differentiando fit :

$$dv(2acvzz - 1 + bz) + vdz(2acvz + b) = 0 \text{ feu}$$

$$dv\sqrt{(1-2bz+ezz)} = \frac{v dz}{z} (1 - \sqrt{(1-2bz+ezz)})$$

$$\text{ideoque } \frac{dv}{v} = \frac{dz}{z\sqrt{(1-2bz+ezz)}} - \frac{dz}{z}$$

Hinc illa aequatione logarithmice differentiata prodit

$$\frac{ds}{s} = \frac{dz(b-ez)}{1-2bz+ezz} - \frac{ndz}{z} + \frac{ndz}{z\sqrt{(1-2bz+ezz)}}$$

Ponamus tantisper  $\frac{dt}{t} = \frac{ds}{s} + \frac{ndz}{z} - \frac{dz(b-ez)}{1-2bz+ezz}$ , vt  
fit  $\frac{dt}{t} = \frac{ndz}{z\sqrt{(1-2bz+ezz)}}$ , vnde quadrata fumendo  
colligimus:  $zzdt^2(1-2bz+ezz) = nnHdz^2$ , quae  
aequatio denuo differentiata posito elemento  $dz$  con-  
stante dat:

$$zzddt(1-2bz+ezz) + zdt dz(1-3bz+2ezz) = nntdz^2$$

$$\text{feu } \frac{ddt}{t} + \frac{dz(1-3bz+2ezz)}{z(1-2bz+ezz)} \cdot \frac{dt}{t} - \frac{nn dz^2}{zz(1-2bz+ezz)} = 0.$$

Iam cum fit  $\frac{ddt}{t} = d \frac{dt}{t} + \frac{dt^2}{t^2}$  erit

$$\frac{ddt}{t} = \frac{dds}{s} - \frac{ds^2}{s^2} - \frac{ndz^2}{zz} + \frac{dz^2(e-2bb+2bez-eez)}{(1-2bz+ezz)^2} + \frac{undz^2}{zz} - \frac{2ndz^2(b-ez)}{z(1-2bz+ezz)}$$

$$+ \frac{ds^2}{s^2} + \frac{2ndzds}{sz} - \frac{2dzds(b-ez)}{s(1-2bz+ezz)} + \frac{dz^2(bb-2bez+eez)}{(1-2bz+ezz)^2}$$

Facta ergo substitutione superior aequatio in hanc  
abit formam:

$$\frac{dds}{s} + \frac{2ndz}{z} \cdot \frac{ds}{s} - \frac{2dz(b-ez)}{1-2bz+ezz} \cdot \frac{ds}{s} + \frac{n(n-1)dz^2}{zz} - \frac{2ndz^2(b-ez)}{z(1-2bz+ezz)} + \frac{dz^2(e-bb)}{(1-2bz+ezz)^2}$$

$$+ \frac{dz(1-3bz+2ezz)}{z(1-2bz+ezz)} \cdot \frac{ds}{s} + \frac{ndz^2(1-3bz+2ezz)}{zz(1-2bz+ezz)} - \frac{dz^2(b-(e+3bb)z+5bez-2eez)}{z(1-2bz+ezz)^2}$$

$$- \frac{undz^2}{zz(1-2bz+ezz)} = 0$$

vbi si termini per  $(1-2bz+ezz)^2$  diuifi in vnam  
summam colligantur, fractio per  $1-2bz+ezz$   
deprimi poterit, vnde facta reductione adipiscimur:

$$\frac{dds}{s} + \frac{2ndz}{z} \cdot \frac{ds}{s} + \frac{dz(1-3bz+2ezz)}{z(1-2bz+ezz)} \cdot \frac{ds}{s} - \frac{undz^2(2b-ez)}{z(1-2bz+ezz)} - \frac{5ndz^2(b-ez)}{z(1-2bz+ezz)}$$

$$- \frac{dz^2(b-2ez)}{z(1-2bz+ezz)} = 0$$

quae ordinata euadit:

$$zdds(1-2bz+ezz) + dzds(2n+1-(4n+5)bz+2(n+2)ezz)$$

$$- sdz^2((n+1)(2n+1)b-(n+1)(n+2)zz) = 0.$$

Cum

Cum nunc constet posito  $z=0$  fieri  $s=1$ , fingamus hanc seriem :

$$s=1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \text{etc.}$$

qua ferie substituta sequens forma ad nihilum est redigenda :

$$\begin{array}{r} 2Bz + 6Cz^2 \\ - 4Bb \\ + 12Dz^3 \\ - 12Cb \\ + 2Be \\ + 20Ez^4 \\ - 24Db \\ + 6Ce \end{array}$$

$$\begin{aligned} & (2n+1)A + 2(2n+1)B + 3(2n+1)C + 4(2n+1)D + 5(2n+1)E \\ & - (4n+5)A - 2(4n+5)Bb - 3(4n+5)Cb - 4(4n+5)Db \\ & + 2(n+2)Ae + 4(n+2)Be + 6(n+2)Ce \\ & - (n+1)(2n+1)b - (n+1)(2n+1)Ab - (n+1)(2n+1)Bb - (n+1)(2n+1)Cb - (n+1)(2n+1)De \\ & + (n+1)(n+2)e + (n+1)(n+2)Ae + (n+1)(n+2)Be + (n+1)(n+2)Ce \end{aligned}$$

vnde colligimus has determinationes :

$$\begin{aligned} A &= (n+1)b \\ B &= \frac{(n+2)((2n+3)Ab - (n+1)e)}{2(2n+3)} \\ C &= \frac{(n+3)((2n+5)Bb - (n+2)Ae)}{3(2n+5)} \\ D &= \frac{(n+4)((2n+7)Cb - (n+3)Be)}{4(2n+7)} \\ E &= \frac{(n+5)((2n+9)Db - (n+4)Ce)}{5(2n+9)} \end{aligned}$$

etc

vbi notetur esse  $e=bb-4ac$ . Sicque ferie quaefitae finguli termini per binos praecedentes determinantur.