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Solutio facilis problematum quorundam geometricorum difficillimorum

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SOLVTIO FACILIS
PROBLEMATVM QVORVMDAM GEOMETRICORVM DIFFICILLIMORVM.

Auctore
L. EVLERO.

1.

In omni triangulo quatuor potissimum dantur puncta, quae in Geometria considerari solent.

1. Intersectio ternorum perpendiculorum, quae ex singulis angulis in latera opposita demittuntur.

2. Intersectio ternarum rectarum, quae ex singulis angulis ductae latera opposita bisecant; quod punctum simul est centrum grauitatis trianguli.

3. Intersectio ternarum rectarum, quae singulos angulos bifariam secant, in quod punctum incidit centrum circuli triangulo inscripti.

4. Intersectio ternarum rectarum ad singula latera normalium eaque bisecantium, in quo puncto reperitur centrum circuli triangulo circumscripti.

2. Ex his quatuor punctis, si dentur positione terna quaecunque, euidens est, triangulum inde determinari, nisi forte illa puncta in vno coalescant, quod cum eueniat in triangulo aequilatero, hoc casu omnia triangula aequilatera problemati aeque satisfa-

tisfacient. Hinc igitur quatuor naſcentur problemata, prout quodque eorum quatuor punctorum, pro trianguli determinatione praetermittitur, quae quemadmodum commodiſſime reſolui queant, hic oſtendere conſtitui.

3. Problemata autem haec ſolutu eſſe difficillima, mox experietur, quicunque ea fuerit aggreſſus, cum vix perſpiciatur, cuiusmodi quantitates incognitas in calculum introduci oporteat, vt ſaltem ad aequationes ſolutionem continentes perueniatur. Totum ergo negotium ad idoneam quantitatum incognitarum electionem reducitur, in quo id imprimis eſt cauendum, ne in calculos taediosiſſimos et omnino inextricabiles delabamur. Tum vero omnibus difficultatibus feliciter ſuperatis inſignes quaedam affectiones inter illa quatuor puncta ſe prodent, quarum cognitio in Geometria haud leuis momenti eſt cenſenda.

Tab. II. Fig. 1. 2. 3. 4.

4. Ne figurae nimia linearum in iis ducendarum multitudine onerentur, idem triangulum ABC quater exhibeo, in primo ſcilicet (fig. 1.) rectae AM, BN et CN in latera oppoſita ſunt normales, earumque interſectionem littera E deſigno, vbi primum ſitum eſt punctum eorum quatuor, quae commemoraui. In ſecunda figura rectae Aa, Bb et Cc latera oppoſita biſecant, quarum interſectionem indicat punctum F ſecundum, de quatuor illis punctis memoratis et centrum grauitatis trianguli.

guli. In tertia figura rectae $A\alpha$, $B\beta$, $C\gamma$ angulos A, B, C bisecant, earumque intersectio G praebet tertium punctum ante memoratum, nempe centrum circuli inscripti. Tandem in quarta figura ex singulorum laterum punctis mediis S, T, V erectae sunt perpendiculares SH, TH et VH sua intersectione H centrum circuli circumscripti exhibentes.

5. Quo horum quatuor punctorum positionem facilius definire eaque deinceps inter se comparare queam, ex singulis in latus AB pro basi assumtum quodemitto perpendicula EP, FQ, GR et HS, quorum quidem primum et quartum iam in constructione ipsa occurrunt. Tum vero voco terna trianguli latera:

$$AB = c;\ AC = b \text{ et } BC = a$$

praeterea vero etiam aream trianguli in computum duci decet, quae sit $= A$, eritque vti constat:

$$AA = \tfrac{1}{16}(a+b+c)(a+b-c)(b+c-a)(c+a-b)$$

seu $AA = \tfrac{1}{16}(2aabb + 2aacc + 2bbcc - a^4 - b^4 - c^4)$

Hinc igitur situm cuiusque horum quatuor punctorum respectu basis AB seorsim inuestigo sequenti modo. Tab. II. Fig. 1.

I. *Pro intersectione perpendiculorum* E.

6. Primo ex elementis constat fore:

$AP = \frac{cc+bb-aa}{2c}$, similique modo $BM = \frac{aa+cc-bb}{2a}$.

Deinde vero ob $\tfrac{1}{2}AM.BC = A$ erit $AM = \frac{2A}{a}$,

 vnde

vnde ſimilitudo triangulorum ABM et AEP praebet

$$AM : BM = AP : EP$$

hincque fit $EP = \frac{(cc+bb-aa)(aa+cc-bb)}{8cA}$.
Quocirca ſitus puncti E reſpectu baſis AB ita definitur vt ſit

$$AP = \frac{cc+bb-aa}{2c} \text{ et } PE = \frac{(cc+bb-aa)(aa+cc-bb)}{8cA}.$$

II. *Pro centro grauitatis* F.

Tab. II. Fig. 2.

7. Demiſſo ex angulo C in baſin AB perpendiculo CP habemus vt ante $AP = \frac{cc+bb-aa}{2c}$ et $CP = \frac{2A}{c}$.
Iam vero ex elementis conſtat eſſe $FQ = \frac{1}{3}CP = \frac{2A}{3c}$ et $cQ = \frac{1}{3}cP$. Cum autem ſit $Ac = \frac{1}{2}c$ erit $cP = \frac{bb-aa}{2c}$ ideoque $cQ = \frac{bb-aa}{bc}$, et conſequenter $AQ = \frac{3cc+bb-aa}{bc}$. Quam ob rem ſitus puncti F reſpectu baſis AB ita definitur, vt ſit:

$$AQ = \frac{3cc+bb-aa}{bc} \text{ et } QF = \frac{2A}{3c}.$$

III. *Pro centro circuli inſcripti* G.

Fig. 3.

8. Cum GR ſit radius circuli inſcripti, erit $\frac{1}{2}GR(a+b+c)$ area trianguli $= A$ vnde fit $GR = \frac{2A}{a+b+c}$. Tum vero poſito ſegmento $AR = x$, ſi ab AC ex A par portio reſcindatur, habebitur ibi punctum contactus, a quo proinde punctum C diſtat interuallo $= b-x$. Deinde ob $BR = c-x$ ſi a latere BC ex A aequale interuallum $c-x$ reſcindatur,

datur, ibi hoc latus a circulo tangetur, vnde punctum C ab isto puncto distabit interuallo $= a - c + x$, quod cum ex circuli natura illi $b - x$ sit aequale, erit $x = \frac{c + b - a}{2}$. Quare hoc punctum G respectu basis AB ita definitur vt sit:

$$AR = \frac{c + b - a}{2} \text{ et } RG = \frac{2A}{a + b + c}.$$

IV. *Pro centro circuli circumscripti* H. Tab. II. Fig. 4.

9. Hic quidem statim est ex constructione $AS = \frac{1}{2}c$. Tum vero ex A in BC ducto perpendiculo AM, erit $AM = \frac{2A}{a}$ et $CM = \frac{aa + bb - cc}{2a}$. Iuncta autem recta AH ex natura circuli liquet angulum AHS aequalem angulo ACB, ideoque triangulum AHS simile erit triangulo ACM vnde fit,

$$AM : CM = AS : HS$$

sicque colligitur $HS = \frac{c(aa + bb - cc)}{8A}$. Quare situs puncti H respectu basis AB ita definitur vt sit:

$$AS = \frac{1}{2}c \text{ et } SH = \frac{c(aa + bb - cc)}{8A}.$$

10. Hinc iam definire poterimus distantias inter haec quaterna puncta, si quidem in eadem figura exprimerentur, erit enim;

$$EF^2 = (AP - AQ)^2 + (PE - QF)^2$$
$$EG^2 = (AP - AR)^2 + (PE - RG)^2$$
$$EH^2 = (AP - AS)^2 + (PE - SH)^2$$

$$FG^2 = (AQ - AR)^2 + (QF - RG)^2$$
$$FH^2 = (AQ - AS)^2 + (QF - SH)^2$$
$$GH^2 = (AR - AS)^2 + RG - SH)^2.$$

Haec autem interualla ideo colligi oportet; quod ſi proponantur terna horum quatuor punctorum quaeque tanquam data, nihil aliud praeter eorum mutuas diſtantias pro cognito aſſumatur, vnde deinceps latera trianguli ſint inueſtiganda.

11. Hic autem imprimis eſt obſeruandum diſtantias illas inter quatuor noſtra puncta neceſſario ita exprimi debere, vt tria trianguli latera in expreſſiones aequaliter ingrediantur, cum nulli lateri prae reliquis reſpectu harum diſtantiarum vlla praerogatiua tribui queat. Quam ob cauſam latera trianguli ſine vllo diſcrimine contemplaturus ponam:

$$a + b + c = p;\quad ab + ac + bc = q \text{ et } abc = r$$

ita vt loco laterum iam iſtas ternas quantitates p, q et r ad ſingula aeque relatas in calculum ſim introducturus. Hinc cum ſit:

$$aa + bb + cc = pp - 2q;\quad aabb + aacc + bbcc = qq - 2pr$$
$$a^4 + b^4 + c^4 = p^4 - 4ppq + 2qq + 4pr$$

area A ita exprimitur vt ſit:

$$AA = \tfrac{1}{16}p(-p^3 + 4pq - 8r) = \frac{-p^4 + 4ppq - 8pr}{16}.$$

Hoc notato ſuperiores ſex diſtantias ad has nouas quantitates ſeorſim ſum reuocaturus.

I. *In-*

I. *Inuestigatio distantiae punctorum* E *et* F.

12. Hic primo habemus:

$$AP - AQ = \frac{cc + bb - aa}{2c} - \frac{3cc - bb + aa}{6c} = \frac{bb - aa}{3c}$$

$$PE - QF = \frac{(cc + bb - aa)(aa + cc - bb)}{8cA} - \frac{2A}{3c}$$

$$= \frac{3(cc + bb - aa)(aa + cc - bb) - 16AA}{24cA}$$

quae expressiones ad communem denominatorem reductae, fiunt:

$$AP - AQ = \frac{(bb - aa)\sqrt{(2aabb + 2aacc + 2bbcc - a^4 - b^4 - c^4)}}{12cA}$$

$$PE - QF = \frac{2c^4 - a^4 - b^4 + 2aabb - bbcc - aacc}{12cA}$$

quarum quadrata addita praebent:

$$EF^2 = \frac{1}{36AA}\left\{\begin{array}{l} + a^6 + b^6 + c^6 \\ - a^4bb - aab^4 - a^4cc - aac^4 - b^4cc^4 - bbc^4) \\ + 3aabbcc \end{array}\right.$$

vbi vtque litterae a, b, c aequaliter insunt. Est vero:

$$a^4bb + aab^4 + \text{etc.} = ppqq - 2q^3 - 2p^3r + 4pqr - 3rr$$

$$a^6 + b^6 + c^6 = p^6 - 6p^4q + 9ppqq - 2q^3 + 6p^3r$$
$$- 12pqr - 3rr$$

ex quo obtinemus:

$$EF^2 = \frac{1}{36AA}(p^6 - 6p^4q + 8ppqq + 8p^3r - 16pqr + 9rr)$$

quae expressionem ad hanc formam reducere licet:

$$EF^2 = \frac{rr}{4AA} = \frac{4}{9}(pp - 2q).$$

II. *Inuestigatio distantiae punctorum* E *et* G.

13. Hic habemus:

$$AP - AR = \frac{cc + bb - aa}{2c} - \frac{c - b + a}{2} = \frac{bb - bc - aa + ac}{2c}$$

$$PE - RG = \frac{(cc + bb - aa)(aa + cc - bb)}{8cA} - \frac{2A}{a + b + c} \text{ seu}$$

$$PE - RG = \frac{c^4 - a^4 - b^4 + 2aabb}{8cA} + \frac{p^3 - 4pq + 8r}{8A}$$

atque ad communem denominatorem reducendo:

$$AP - AR = \frac{bb - bc - aa + ac)\sqrt{(2aabb + aacc + 2bbcc - a^4 - b^4 - c^4)}}{8cA}$$

$$PE - RG = \frac{2c^4 - (a+b)c^3 - (a-b)^2cc + (a+b)(a-b)^2c - (aa-bb)^2}{8cA}$$

quorum quadratorum summa per $4cc$ diuisa ad hanc formam redit:

$$EG^2 = \frac{1}{16AA}\left\{\begin{array}{ll} +a^6 - a^5b - a^4bb + 3a^4bc - 2a^3bbc + 2a^3b^3 & \\ +b^6 - ab^5 - aab^4 + 3ab^4c - 2a^3bcc + 2a^3c^3 & \\ +c^6 - a^5c - a^4cc + 3abc^4 - 2aab^3c + 2b^3c^3 & +6aabbcc \\ -ac^5 - aac^4 \qquad -2ab^3cc & \\ -b^5c - b^4cc \qquad -2aabc^3 & \\ -bc^5 - bbc^4 \qquad -2abbc^3 & \end{array}\right.$$

Antequam hanc expressionem ad litteras p, q et r reduco, obseruo esse:

$$EG^2 \begin{array}{l} +aa - ab \\ +bb - ac \\ +cc - bc \end{array} = \frac{abc}{4AA}\left\{\begin{array}{l} +a^3 - aab + 3abc \\ +b^3 - abb \\ +c^3 - aac \\ \quad\;\; -acc \\ \quad\;\; -bbc \\ \quad\;\; -bcc \end{array}\right\}$$

$$= \frac{abc(aa + bb + cc - 2ab - 2ac - 2bc)(a + b + c)}{4AA} + \frac{9aabbcc}{4AA}.$$

Iam

Iam cum ſit $aa+bb+cc=pp-2q$ reductio eſt facilis, quippe prodit:

$$EG^2+pp-3q=\frac{pr(pp-4q)+9rr}{4AA}$$

ſicque haec diſtantia ita definitur vt ſit:

$$EG^2=\frac{r(p^3-4pq+9r)}{4AA}-pp+3q=\frac{rr}{4AA}-\frac{4r}{p}-pp+3q.$$

III. *Inueſtigatio diſtantiae punctorum* E *et* H.

14. Cum hic ſit:

$$AP-AS=\frac{cc+bb-aa}{2c}-\frac{1}{2}c$$

$$PE-SH=\frac{(cc+bb-aa)(aa+cc-bb)}{8cA}-\frac{c(aa+bb-cc)}{8A}$$

habebimus:

$$AP-AS=\frac{bb-aa}{2c} \text{ et } PE-SH=\frac{2c^4-(aa+bb)cc-(aa-bb)^2}{8cA}$$

et quadratis addendis diuiſione facta per $4cc$ obtinetur:

$$EH^2=\frac{1}{16AA}\left\{\begin{array}{l}+a^6-a^4bb-aac^4+3aabbcc\\ +b^6-a^4cc-b^4cc\\ +c^6-aab^4-bbc^4\end{array}\right\}$$

quae ob $16AA=-a^4-b^4-c^4+2aabb+2aacc+bbcc$ reducitur ad hanc formam:

$$EH^2=\frac{9aabbcc}{16AA}-aa-bb-cc$$

vbi ſubſtitutio facile conficitur, reſultat enim:

$$EH^2=\frac{9rr}{16AA}-pp+2q.$$

IV.

IV. *Inuestigatio distantiae punctorum* F *et* G.

15. Ex formulis supra inuentis habemus hic:

$$AQ-AR = \frac{3ac+bb-aa}{6c} - \frac{c-b+a}{2} = \frac{3(a-b)c-aa+bb}{6c}$$

$$GF-RG = \frac{2A}{3c} - \frac{2A}{a+b+c} = \frac{2A(a+b-2c)}{3c(a+b+c)}$$

quorum quadratorum summa reducitur ad hanc formam:

$$FG^2 = \frac{1}{9(a+b+c)^2}\left\{\begin{matrix} -a^4+a^3b+4aabb-5abcc \\ -b^4+ab^3+4aacc-5abbc \\ -c^4+a^3c+4bbcc-5aabc \\ +ac^3 \\ +b^3c \\ +bc^3. \end{matrix}\right.$$

Cum nunc sit:

$$a^4+b^4+c^4 = p^4-4ppq+2qq+4pr$$
$$aabb+aacc+bbcc = qq-2pr$$
$$a^3b+ab^3+a^3c+ac^3+b^3c+bc^3 = ppq-2qq-pr$$
$$abcc+abbc+aabc = pr$$

expressio inuenta hanc induit formam:

$$FG^2 = \frac{1}{9pp}(-p^4+5ppq-18pr) = \frac{-p^3+5pq-18r}{9p}$$

V. *Inuestigatio distantiae punctorum* F *et* H.

16. Pro hoc casu habemus:

$$AQ-AS = \frac{3cc+bb-aa}{6c} - \frac{1}{2}c = \frac{bb-aa}{6c}$$

$$QF-SH = \frac{2A}{3c} - \frac{c(aa+bb-cc)}{8A} = \frac{2c^4-(aa+bb)cc-(aa-bb)^2}{24cA}$$

Quod-

Quodſi has formulas cum caſu primo comparemus, deprehendimus eſſe:

$$AQ - AS = \tfrac{1}{2}(AP - AQ) \text{ et } QF - SH = \tfrac{1}{2}(PE - QF)$$

vnde manifeſtum eſt fore $FH = \tfrac{1}{2}EF$ ideoque $FH^2 = \frac{rr}{16\Delta\Delta} - \frac{1}{9}(pp - 2q)$.

VI. *Inueſtigatio diſtantiae punctorum* G *et* H.

17. Pro hoc caſu poſtremo habetur:

$$AR - AS = \frac{c + b - a}{2} - \tfrac{1}{2}c = \frac{b - a}{2}$$

$$RG - SH = \frac{2\Delta}{a+b+c} - \frac{c(aa+bb-cc)}{8\Delta} = \frac{(a+b)c^3 + (aa+bb)cc - (a+b)(aa+bb)c - (aa-bb)^2}{8(a+b+c)\Delta}$$

quarum binarum formularum quadrata ſi addantur reperitur ſequens expreſſio:

$$GH^2 = \frac{abc}{16(a+b+c)^2\Delta\Delta}\left\{\begin{array}{l} +a^5 + a^4b + ab^4 + abc^3 - 2a^3bb - 2aab^3 \\ +b^5 + a^4c + ac^4 + ab^3c - 2a^3cc - 2aac^3 \\ +c^5 + b^4c + bc^4 + a^3bc - 2b^3cc - 2bbc^3 \end{array}\right\}$$

quae per $a + b + c$ reducta abit in hanc:

$$GH^2 = \frac{abc}{16(a+b+c)\Delta\Delta}\left\{\begin{array}{l} +a^4 + aabc - 2aabb \\ +b^4 + abbc - 2aacc \\ +c^4 + abcc - 2bbcc \end{array}\right\}$$

vnde facta ſubſtitutione colligitur

$$GH^2 = \frac{r}{16p\Delta\Delta}(p^4 - 4ppq + 9pr) - \frac{r(p^3 - 4pq + 9r)}{16\Delta\Delta}$$

ſeu $GH^2 = \frac{rr}{16\Delta\Delta} - \frac{r}{p}$.

18. En ergo ſub vno conſpectu quadrata ſex horum interuallorum:

$$\text{I.}\quad EF^2 = \frac{rr}{4AA} - \frac{4}{9}(pp - 2q)$$

$$\text{II.}\quad EG^2 = \frac{rr}{4AA} - pp + 3q - \frac{4r}{p}$$

$$\text{III.}\quad EH^2 = \frac{9rr}{16AA} - pp + 2q$$

$$\text{IV.}\quad FG^2 = -\frac{1}{9}pp + \frac{5}{9}q - \frac{2r}{p}$$

$$\text{V.}\quad FH^2 = \frac{rr}{16AA} - \frac{1}{9}(pp - 2q)$$

$$\text{VI.}\quad GH^2 = \frac{rr}{16AA} - \frac{r}{p}$$

Tab. II. Fig. 5. vbi euidens eſt, eſſe $EH = \frac{3}{2}EF$ et $FH = \frac{1}{2}EF$, ſicque punctum H per puncta E, F ſponte determinatur, ſcilicet ſi tria puncta E, F, G forment triangulum EFG tum quartum punctum H ita in recta EF producta erit ſitum vt ſit $FH = \frac{1}{2}EF$ ideoque $EH = \frac{3}{2}EF$. Hinc vero deducitur $4GH^2 + 2EG^2 = 3EF^2 + 6FG^2$, quod cum valoribus inventis apprime congruit.

19. Quo nunc has formulas ad maiorem ſimplicitatem reuocemus, ponamus $4pq - p^3 - 8r = 4s$ vt ſit $4AA = ps$ et $4q = pp + \frac{8r}{p} + \frac{4s}{p}$; tum vero faciamus:

$$\frac{rr}{ps} = R,\ \frac{r}{p} = Q \text{ et } pp = P$$

ita vt P, Q, R ſint quantitates duas dimenſiones involuentes. Quoniam igitur hinc eſt $\frac{-s}{p} = \frac{QQ}{R}$ erit $p = \sqrt{P}$; $q = \frac{1}{4}P + 2Q + \frac{QQ}{R}$, et $r = Q\sqrt{P}$, atque

que $4AA = \frac{PQQ}{R}$ et interualla nostra ita exprimentur:

I. $EF^2 = R - \frac{2}{9}P + \frac{16}{9}Q + \frac{8QQ}{9R}$

II. $EG^2 = R - \frac{1}{4}P + 2Q + \frac{3QQ}{R}$

III. $EH^2 = \frac{9}{4}R - \frac{1}{8}P + 4Q + \frac{2QQ}{R}$

IV. $FG^2 = \quad + \frac{1}{36}P - \frac{8}{9}Q + \frac{5QQ}{9R}$

V. $FH^2 = \frac{1}{4}R - \frac{1}{18}P + \frac{4}{9}Q + \frac{2QQ}{9R}$

VI. $GH^2 = \frac{1}{4}R - Q.$

20. Cum igitur horum quatuor punctorum terna nisi capiantur haec tria E, F et H iam contineant determinationem quarti, vnicum resultat problema, quod ita se habet.

Problema.

Datis positione his quatuor punctis in quolibet triangulo assignabilibus 1°. *Intersectione perpendicularium ex singulis angulis in latera opposita ductarum* E, 2°. *Centro grauitatis* F, 3°. *Centro circuli inscripti* G *et* 4° *centro circuli circumscripti* H; *construere triangulum.* Tab. II. Fig. 5.

Quod problema ex hactenus erutis horum punctorum affectionibus satis concinne resoluere licebit.

Solutio.

21. Cum positio horum quatuor punctorum per eorum distantias detur, vocemus: $GH = f$,

$$GH = f,\ FH = g \text{ et } FG = h$$

nouimusque fore $EF = 2g$ et $EH = 3g$, itemque

$$EG = \sqrt{(6gg + 3hh - 2ff)}.$$

Nunc igitur ſtatim habemus has tres aequationes

$$\text{I. } ff = \tfrac{1}{4}R - Q$$

$$\text{II. } gg = \tfrac{1}{4}R - \tfrac{1}{18}P + \tfrac{4}{9}Q + \tfrac{2QQ}{9R}$$

$$\text{III. } hh = \tfrac{1}{36}P - \tfrac{8}{9}Q + \tfrac{6QQ}{9R}$$

ex quarum reſolutione colligimus:

$$R = \frac{4f^4}{3gg + 6hh - 2ff};\ Q = \frac{3ff(ff - gg - 2hh)}{3gg + 6hh - 2ff}$$

$$\text{et } P = \frac{27f^4}{3gg + 6hh - 2ff} - 12ff - 15gg + 6hh$$

vnde fit $\frac{QQ}{R} = \frac{9(ff - gg - 2hh)^2}{4(3gg + 6hh - 2ff)}$.

22. His valoribus inuentis inueſtigentur tres ſequentes expreſſiones:

$$p = \sqrt{P},\ q = \tfrac{1}{4}P + 2Q + \tfrac{QQ}{R}, \text{ et } r = Q\sqrt{P}$$

indeque formetur haec aequatio cubica:

$$z^3 - pzz + qz - r = 0$$

cuius tres radices dabunt tria latera trianguli quaeſiti, quo pacto eius conſtructio facillima habetur.

Exemplum.

23. Sumtis lateribus trianguli $a = 5$, $b = 6$ et $c = 7$, vt ſit area $A = 6\sqrt{6}$, inde colliguntur diſtantiae quaternorum punctorum:

EF²

$$EF^2=\tfrac{155}{72};\ EG^2=\tfrac{11}{8};\ EH^2=\tfrac{155}{32};\ FG^2=\tfrac{1}{9};\ FH^2=\tfrac{155}{288};\ GH^2=\tfrac{35}{32}$$

vnde ſitus horum punctorum talis prodit vti in fig. 6. repraeſentatur. Cum igitur habeamus: Tab. II. Fig. 6.

$$ff=\tfrac{35}{32};\ gg=\tfrac{155}{288}\ \text{et}\ hh=\tfrac{1}{9}$$

videamus num ſolutio inuenta ad triangulum aſſumtum perducat.

24. Hinc autem fit $3gg+6hh-2ff=\tfrac{3}{32}$, tum vero $ff-gg-2hh=\tfrac{1}{3}$; $4ff+5gg-2hh=\tfrac{219}{32}$; colligitur

$$R=\tfrac{1225}{24};\ Q=\tfrac{35}{3};\ P=324\ \text{et}\ \frac{QQ}{R}=\tfrac{24}{9}=\tfrac{8}{3}$$

vnde nanciſcimur:

$$p=\sqrt{P}=18;\ q=107\ \text{et}\ r=\tfrac{35}{3}.18=5.6.7=210$$

et aequatio cubica hinc oritur:

$$z^3-18zz+107z-210=0$$

cuius tres radices manifeſto ſunt 5, 6, 7 quae ſunt ipſa tria latera trianguli ſatisfacientis.

Caſus quo quatuor puncta in directum ſunt ſita.

25. Hoc ergo caſu cum ſit: Fig. 7.

$$FH=g;\ FG=h;\ GH=f;\ EF=2g;\ EH=3g\ \text{et}\ EG=2g-h$$

erit

erit $g = f - b$, vnde facta hac ſubſtitutione colligimus:

$$R = \frac{4f^4}{(3b-f)^2}; \quad Q = \frac{3ffb(2f-3b)}{(f-3b)^2}; \quad P = \frac{3b(4f-3b)^2}{(f-3b)^2}$$

ideoque $\frac{QQ}{R} = \frac{9bb(2f-3b)^2}{4(f-3b)^2}$.

Ex his vero porro elicimus:

$$p = \frac{(4f-3b)\sqrt{3b(4f-3b)}}{f-3b}$$

$$q = \frac{3fb(4f-3b)(5f-6b)}{(f-3b)^2}$$

$$r = \frac{3ffb(2f-3b)(4f-3b)\sqrt{3b(4f-3b)}}{(f-3b)^3}.$$

26. Cum iam radices huius aequationis cubicae:

$$z^3 - pzz + qz - r = 0$$

praebeant tria latera a, b, c trianguli quaeſiti, ponamus ad eam concinniorem reddendam

$$z = \frac{y\sqrt{3b(4f-3b)}}{f-3b}$$

et prodibit haec aequatio:

$$y^3 - (4f-3b)yy + f(5f-6b)y - ff(2f-3b) = 0$$

cuius radices manifeſto ſunt

$$f, \quad f, \quad \text{et } 2f - 3b.$$

Quocirca trianguli quaeſiti, quod ſit iſosceles, latera erunt:

$$a = b = \frac{f\sqrt{3b(4f-3b)}}{f-3b}; \text{ et } c = \frac{(2f-3b)\sqrt{3b(4f-3b)}}{f-3b}.$$

27. Hic autem caſus per ſe ſolutu et facilis, cum recta illa, in qua ſunt puncta data triangulum

lum in duas partes ſimiles neceſſario ſecet, ideoque triangulum ſit iſoſceles. Poſito autem ſtatim a principio $b=a$, fit $A=\frac{1}{4}c\sqrt{(4aa-cc)}$ et $AP=AQ=AR=AS=\frac{1}{2}c$, tum vero

$$PE=\frac{c^3}{4A};\ QF=\frac{2A}{3c};\ RG=\frac{2A}{2a+c};\ SH=\frac{c(2aa-cc)}{8A}$$

vnde ob puncta P, Q, R, S coincidentia in baſis puncto medio, quod ſit O, interualla inter haec puncta erunt:

$$OF-OE=\frac{2(aa-cc)}{3\sqrt{(4aa-cc)}};\ OG-OE=\frac{c(a-c)}{\sqrt{(4aa-cc)}};$$
$$OH-OE=\frac{aa-cc}{\sqrt{(4aa-cc)}}$$

$$OF-OG=\frac{(a-c)(2a-c)}{3\sqrt{(4aa-cc)}};\ OH-OF=\frac{aa-cc}{3\sqrt{(4aa-cc)}};$$
$$OH-OG=\frac{a(a-c)}{\sqrt{(4aa-cc)}}.$$

28. Hic duos caſus contemplari conuenit prout fuerit vel $a>c$ vel $a<c$, nam ſi $a=c$, ſeu triangulum aequilaterum, omnia quatuor puncta in vnum coaleſcunt;

I. Si $a>c$ puncta erunt diſpoſita vti fig. 8. refert, vbi eſt $HF=\frac{1}{3}EH$ ſeu $EF=\frac{2}{3}EH$ et $EG<\frac{1}{2}EH$ hocque caſu punctum baſis medium O in recta HE producta vltra E cadit: vt ſit $OE=\frac{cc}{2\sqrt{(4aa-cc)}}$. Tab. II. Fig. 8.

II. Si $a<c$, puncta erunt diſpoſita vti fig. 9. refert, vbi eſt iterum $HF=\frac{1}{3}EH$ ſeu $EF=\frac{2}{3}EH$ at $EG>\frac{1}{2}EH$ Hoc autem caſu punctum baſis medium O in recta EH producta vltra H cadit; vt ſit Fig. 9.

HO

$HO = \frac{2aa - cc}{2\sqrt{(4aa - cc)}}$, vnde ſi $2aa < cc$ punctum O adeo intra H et E cadit.

29. Datis ergo in recta punctis tribus E, G et H, ita vt G intra extrema E et H ſit ſitum, videndum eſt vtrum ſit $EG < \frac{1}{2}EH$ an $EG > \frac{1}{2}EH$.

Tab. II. Priori caſu quo $EG < \frac{1}{2}EH$ ſolutio ita ſe habet.

Fig. 8. Sit $EH = 2d$ et $EG = d - e$, hincque reperitur

$$a = b = \frac{(d+e)}{2e}\sqrt{(d+3e)(3d+e)}$$

$$c = \frac{d-e}{2e}\sqrt{(d+3e)(3d+e)}, \text{ et } OE = \frac{(d-e)^2}{4e}.$$

Fig. 9. Poſteriori caſu $EG > \frac{1}{2}EH$ ſolutio erit haec:

ſit $EH = 2d$ et $EG = d + e$, hincque colligitur

$$a = b = \frac{d-e}{2e}\sqrt{(d-3e)(3d-e)}; \; c = \frac{d+e}{2e}\sqrt{(d-3e)(3d-e)}$$

et $OE = \frac{(d+e)^2}{2e}$, vnde patet hunc caſum locum habere non poſſe, ſi d intra limites $3e$ et $\frac{1}{3}e$ contineatur. Cum enim eſſe debet $2a > c$ neceſſe eſt ſit $d > 3e$.

Fig. 5. 30. Ex hoc caſu colligere licet, etiam in genere ſolutionem concinniorem eſſe prodituram, ſi omiſſo puncto F tria puncta E, G et H conſiderentur. Ponamus ergo:

$$EG = e, \; GH = f \text{ et } EH = k$$

eritque $FH = g = \frac{1}{3}k$, $EF = \frac{2}{3}k$ et $FG = b = \sqrt{(\frac{1}{3}ee + \frac{2}{3}ff - \frac{2}{9}kk)}$

hincque adipiſcimur $R = \frac{4f^4}{2ff + 2ee - kk}$, $Q = \frac{ff(kk - ff - 2ee)}{2ee + 2ff - kk}$

et

et $P = \frac{27f^4}{2ee-4ff-kk} + 2ee - 8ff - 3kk = \frac{4e^4+11f^4+3k^4-12eeff+2ffkk-8eekk}{2ee+2ff-kk}$
tum vero $\frac{QQ}{R} = \frac{(kk-ff-2ee)^2}{4(2ee+2ff-kk)}$, vnde fit
$p = \sqrt{P}$, $q = \frac{2e^4+f^4+k^4-6eeff-3eekk+2ffkk}{2ee+2ff-kk}$ et $r = Q\sqrt{P}$
et aequationis $z^3 - pzz + qz - r = 0$ radices dant latera trianguli quaesiti: quae aequatio posito $z = y\sqrt{P}$ abit in hanc:

$$y^3 - yy + \frac{(2e^4+f^4+k^4-6eeff+3eekk+2ffkk)y - ff(kk-2ee-ff)}{4e^4+11f^4+3k^4-12eeff-8eekk+2ffkk} = 0.$$

31. Hic autem obseruo quantitates has datas e, f, k non solum ita assumi oportere, vt triangulum constituant, sed quoniam latera trianguli quaesiti a, b, c tanquam positiua spectari possunt, etiam tam P quam Q et R valores positiuos recipere debent. Non solum ergo esse debet $kk < 2ee + 2ff$ sed etiam $kk > 2ee + ff$, tum vero vt P fiat positiuum, necesse est

fit $3kk > 4ee + ff + 2\sqrt{(e^4 + 11eeff - 8f^4)}$

qua conditione cum illis collata sequitur esse debere

$$ff > \tfrac{8}{19}ee \text{ et } ff < \tfrac{11+\sqrt{153}}{16}ee \text{ seu } ff < \tfrac{19}{13}ee$$

alioquin problema nullam admitteret solutionem.

32. *Exempl.* Sit $ee = ff$ erit $R = \frac{4f^4}{4ff-kk}$;
$Q = \frac{ff(kk-3ff)}{4ff-kk}$
et $P = \frac{27f^4}{4ff-kk} - 6ff - 3kk = \frac{3(kk-ff)^2}{4ff-kk}$; atque $\frac{QQ}{R} = \frac{(kk-3ff)^2}{4(4ff-kk)}$
ideoque $p = \frac{(kk-ff)\sqrt{3}}{\sqrt{(4ff-kk)}}$; $q = \frac{k^4-ffkk-3f^4}{4ff-kk}$; et $r = \frac{ff(kk-3ff)(kk-ff)\sqrt{3}}{4(ff-kk)\sqrt{(4ff-kk)}}$

ſit iam $ee = ff = 2$ et $kk = 7$; fietque

$$p = 5\sqrt{3};\ q = 23,\ \text{et}\ r = 10\sqrt{3}$$

et latera trianguli quaeſiti erunt radices huius aequationis cubicae $z^3 - 5zz\sqrt{3} + 23z - 10\sqrt{3} = 0$, quae poſito;

$$z = \frac{y}{\sqrt{3}}\ \text{abit in}\ y^3 - 15yy + 69y - 90 = 0$$

cuius vna radix eſt $y = 6$, vnde binae reliquae ſunt

$$y = \frac{9 + \sqrt{21}}{2}\quad \text{et}\ y = \frac{9 - \sqrt{21}}{2}$$

ſicque trianguli quaeſiti latera ſunt:

$$a = \frac{3\sqrt{3} + \sqrt{7}}{2};\ b = \frac{3\sqrt{3} - \sqrt{7}}{2};\ c = 2\sqrt{3}.$$

33. Verum etiam generalius manente $ee = ff$ quomodocunque accipiatur k, ſi ponatur $z = \frac{y}{\sqrt{3(4ff-kk)}}$ habetur haec aequatio reſoluenda:

$$y^3 - 3(kk - ff)yy + 3(k^4 - ffkk - 3f^4)y$$
$$- 9ff(kk - 3ff)(kk - ff) = 0$$

cui primo ſatisfacit $y = 3ff$, et duae reliquae radices ex hac aequatione

$$yy - 3(kk - 2ff)y + 3(kk - 3ff)(kk - ff) = 0$$

quae ſunt $y = \frac{3(kk - 2ff) \pm k\sqrt{3(4ff - kk)}}{2}$

ſicque tria latera trianguli quaeſiti fiunt:

$$a = \frac{(kk - 2ff)\sqrt{3}}{2\sqrt{(4ff - kk)}} + \frac{1}{2}k$$

$$b = \frac{(kk - 2ff)\sqrt{3}}{2\sqrt{(4ff - kk)}} - \frac{1}{2}k$$

$$c = \frac{ff\sqrt{3}}{\sqrt{(4ff - kk)}}.$$

34. Re-

34. Reliquis casibus negotium non tam facile expeditur, quia aequatio cubica factores non admittit. Quod vt exemplo ostendatur sit $ee = 3$, $ff = 2$ et $kk = 9$; vnde trianguli latera sunt radices huius aequationis cubicae $z^3 - zz\sqrt{71} + 22z - 2\sqrt{71} = 0$; ad quam resoluendam quaeratur angulus α, cuius cosinus sit $= +\sqrt{\frac{71}{75}}$, qui erit acutus, quo inuento latera trianguli erunt:

$a = \frac{1}{3}\sqrt{71} + \frac{2}{3}\sqrt{5}.\cos.(60° - \frac{1}{3}\alpha)$ et $c = \frac{1}{3}\sqrt{71} - \frac{2}{3}\sqrt{5}.\cos.\frac{1}{3}\alpha$

$b = \frac{1}{3}\sqrt{71} + \frac{2}{3}\sqrt{5}.\cos.(60° + \frac{1}{3}\alpha)$ vbi est proxime $\alpha = 11°.\ 32'.\ 13''$

sicque per anguli trisectionem problema semper satis expedite resoluetur.