1767

Proprietates triangulorum, quorum anguli certam inter se tenent rationem

Leonhard Euler

PROPRIETATES TRIANGVLORVM, QVORVM ANGVLI CERTAM INTER SE TENENT RATIONEM.

Auctore

L. EVLERO.

Inter veritates geometricas eae potissimum attentione sunt dignae, quarum demonstratio ita est recondita, vt analyticae inuestigationi vix vllus locus relinqui videatur. Quae enim ita sunt comparatae, vt formula analytica facile comprehendi queant, omnino superfluum foret, memoriam earum recordatione fatigare: ad quod genus plurimae sectionum conicarum proprietates sunt referendae, quarum plerumque ingens multitudo vnica formula analytica includi potest. Elementares autem figurarum proprietates eo maiori cura memoriae sunt mandandae, quod analysis ad eas non perducat, sed iis potius ad altiora tendens superstrui debeat. Nescio, an proprietates triangulorum, quas hic euoluere constitui, elementaribus sint annumerandae, nec ne? Si enim ad earum demonstrationes geometricas spectemus, eae ita sunt intricatae, vt in elementis locum vix inuenire queant: tum vero etiam, quod hic imprimis est obseruandum, ne analysis quidem satis videtur idonea, ad earum veritatem stabiliendam;

dam; quamobrem hanc ſpeculationem attentioni geometrarum commendare non dubito.

Occaſionem autem, haec perſcrutandi, mihi praebuit prima quaſi triangulorum proprietas elementaris, qua nouimus, ſi duo anguli fuerint inter ſe aequales, etiam duo latera, ipſis ſcilicet oppoſita, inter ſe aequalia eſſe futura. Quemadmodum ergo hoc caſu ex data angulorum conditione certa relatio laterum ſequitur, ita generatim affirmare licet, quoties in triangulo certa quaedam ratio inter duos angulos datur, inde neceſſario quoque certam quandam relationem inter latera determinari. Ex quo haec naſcitur quaeſtio: *Si in triangulo, cuius anguli ſint* α, β, γ, *latera iis oppoſita litteris* a, b, c *deſignentur, haecque conditio detur, vt ſit* $\alpha:\beta = m:n$; *relationem inter latera* a, b, c *inde ortam inueſtigare*? Problema hoc ſtatim ac ratio data $m:n$ tantillum aſſumitur complicata, analytice tractatum in taedioſiſſimos calculos praecipitare tentanti mox patebit: ſin autem a caſu ſimpliciſſimo, quo $\beta = \alpha$, et $b - a = 0$, incipientes, continuo ad magis compoſitos ordine progrediamur, egregiam tandem progreſſionis legem obſeruare licebit, quae eo magis eſt notatu digna, quod per ſolam inductionem ſit inuenta, vixque demonſtrationem admittere videatur.

Problema I.

Tab. I. Fig. 2. 1. Si in triangulo ABC fuerit ang. $B = 2$ ang. A, inter eius latera $AB = c$, $AC = b$ et $BC = a$ relationem inde oriundam inueſtigare.

Solutio.

Solutio.

Angulo B per rectam BD bisecto, erit triangulum ADB isosceles, et triangulum BCD toti ACB simile,

vnde fit AC:BC=AB:BD=BC:CD,

$$\text{seu } b : a = c : \frac{ac}{b} = a : \frac{aa}{b}.$$

Ergo $BD = \frac{ac}{b}$ et $CD = \frac{aa}{b}$; hinc $AD = b - \frac{aa}{b}$.

At ob BD=AD habebimus $ac = bb - aa$, qua ergo aequatione continetur relatio quaesita inter latera trianguli, quae est vel (AC+BC)(AC-BC)=AB.BC vel $AC^2 = BC(AB+BC)$.

Coroll. 1.

2. Vltima aequatio facilem hanc suppeditat demonstrationem formulae inuentae; producto enim latere AB in C, vt sit BE=BC, erit angulus E semissis ipsius ABC, ideoque ipsi A aequalis, vnde triangula isoscelia ACE et CBE erunt similia; hinc AE:AC=CE:BC, seu AB+BC:AC=AC:BC.

Coroll. 2.

3. Viciſſim ergo, quoties inter latera trianguli ABC haec relatio deprehenditur, vt sit $AC^2 = BC.(AB+BC)$ seu $bb = aa + ac$, toties concludi oportet, angulum ABC esse duplum anguli BAC.

Scholion.

4. Haec inuersa propositio, etsi eius veritas ex praecedente necessario sequitur, tamen non ita facile geometrice demonstratur. Si scilicet fuerit $AC^2 = AB.BC + BC^2$, ostendendum est, fore angulum A semissem anguli ABC. Hunc in finem demisso ex C in AB perpendiculo CP, ex elementis constat, esse $AC^2 = AB^2 + BC^2 - 2AB.BP$; cum igitur sit $AC^2 = AB.BC + BC^2$, erit BC^2 vtrinque auferendo $AB^2 - 2AB.BP = AB.BC$, et per AB dividendo $AB - 2BP = BC$. Capiatur $PQ = BP$, vt sit $CQ = BC$, eritque $AQ = BC$ seu $AQ = CQ$, vnde angulus BQC, cui aequalis est ABC, duplus est anguli A, quae est demonstratio propositionis inversae.

Problema 2.

Tab. I. fig. 3. 5. Si in triangulo ABC angulus ABC fuerit triplus anguli A, relationem, quae hinc in latera trianguli redundat, $AB = c$, $AC = b$ et $BC = a$ definire.

Solutio.

Ex angulo B recta Bc ita ducatur, vt angulus CBc aequalis sit angulo A, ideoque angulus ABc eius duplus, sicque triangulum ABc ad casum praecedentis problematis pertineat. At triangulum BCc simile est triangulo ACB, vnde fit

$$AC : BC = AB : Bc = BC : Cc$$

$$b : a = c : \frac{ac}{b} = a : \frac{aa}{b}.$$

Ergo

Ergo $Bc = \frac{ac}{b}$, et $Cc = \frac{aa}{b}$, hincque $Ac = \frac{bb - aa}{b}$.
Iam in triangulo ABc ad analogiam ponantur latera

$$AB = \gamma, \; Ac = \beta, \text{ et } Bc = \alpha$$

et ex problemate praecedente habetur pro hoc triangulo ista proprietas:

$$\beta\beta - \alpha\alpha - \alpha\gamma = 0.$$

Ex modo inuentis autem nouimus esse

$$\gamma = c; \; \beta = \frac{bb - aa}{b}; \text{ et } \alpha = \frac{ac}{b}$$

qui valores in illa aequatione substituti praebent:

$$\frac{(bb - aa)^2}{bb} - \frac{aacc}{bb} - \frac{acc}{b} = 0, \text{ siue}$$

$$(bb - aa)^2 - acc(a + b) = 0$$

quae aequatio per $a + b$ diuisa abit in hanc:

$$(bb - aa)(b - a) - acc = 0$$

qua character indolis propositae continetur, quod angulus ABC sit triplus anguli A.

Coroll. 1.

6. Quando ergo in triangulo ABC angulus ad B triplus est anguli A, tum inter eius latera $AB = c$, $AC = b$ et $BC = a$ haec datur relatio, vt sit $(bb - aa)(b - a) - acc = 0$, seu $(b - a)^2(b + a) - acc = 0$, quae euoluta fit

$$b^3 - abb - aab + a^3 - acc = 0.$$

Coroll. 2.

Coroll. 2.

7. Ad hanc proprietatem geometrice enunciandam centro C radio $CB = a$ describatur circulus, latus AC productum secans in D et E, latus vero AB in F. Iam cum sit $AD = b + a$, et $AE = b - a$, erit $AD.AE^2 = BC.AB^2$. Ex elementis vero est $AE.AD = AF.AB$, vnde fit $AE.AF = BC.AB$, ideoque $AE:CE = AB:AF$, quam proportionem geometrice demonstrari oportet.

Coroll. 3.

8. In eadem figura cum sit angulus $CFB = ABC = 3A$, erit angulus $ACF = 2A$, et $BCD = ABC + A = 4A$, vnde arcus BD est duplus arcus EF. Ducta ergo recta BE, erit ang. $EBF = \frac{1}{2} ECF = A$, ideoque $BE = AE$. Simili modo ducta recta DF, angulus ADF quoque aequatur angulo A, ex quo fit $DF = AF$.

Coroll. 4.

9. Hinc analogia ante inuenta $AE:CE = AB:AF$ abit in istam $BE:CE = AB:DF = DF + BF:DF$ seu $BE.DF = CE.AB = BC(BF + DF)$. Quae proprietas geometrice ita ostenditur: Sumto arcu $EG = EF$, ductisque AG et BG, erit $AG = AF$ et $BG = DF$, ob arcum $FG = BD$, ideoque $BFG = BDF$ et $AG = BG$, ob $AF = DF$. Nunc vero ambo triangula isosceلia AGB et BCE sunt similia, quia ang. $CEB = 2A = BAG$; vnde sequitur:

AB:

$AB:AG=BE:CE$, seu $AB:AF=AE:BC$, vel $BC.AB=AE.AF$, quae est proprietas supra eruta.

Scholion.

10. Inuenta ergo proprietas concinnius hoc modo geometrice demonstrabitur:

Centro C radioque CB descripto circulo latus AC in D et E, latus vero AB in F secante, ductisque BE et CF, ob angulum $CFB=CBF=3A$, erit angulus $CEB=2A=CBE$, et quia angulus $ABE=A$, erit $BE=AE$. Tum sumto arcu $EG=EF$, ductisque AG et BG, erit vtique $AG=AF$, et tam $BAG=2A$, quam $ABG=2A$, ideoque $BG=AG=AF$. Simile ergo erit triangulum AGB triangulo BEC, vnde fit $AB:AG=BE:BC$, et quia $AG=AF$, et $BE=AE$, erit $AB:AF=AE:BC$. Ex elementis vero est $AF:AE=AD:AB$, vnde fit componendo

$AB:AE=AE.AD:BC.AB$, seu $AE^2.AD=BC.AB^2$

quae aequatio dat $(AC-BC)^2(AC+BC)=BC.AB^2$, quae est proprietas supra inuenta, et nunc geometrice demonstrata.

Problema 3.

11. Si in triangulo ABC angulus ABC fuerit quadruplus anguli A, inter eius latera $AB=c$, $AC=b$ et $BC=a$, relationem illa conditione determinatam inuestigare. Tab. I. Fig. 4.

 Solutio.

Solutio.

Ex angulo quadruplo B ducatur recta Bc abscindens angulum $CBc = A$, vt in triangulo ABc angulus ad B triplus ſit anguli A, hocque triangulum ad caſum problematis praecedentis pertineat. Triangulum autem BCc ſimile erit triangulo ACB, vnde colligitur vt ante:

$$Bc = \frac{ac}{b}.\ Cc = \frac{aa}{b},\ \text{hincque } Bc = \frac{bb-aa}{b}.$$

Ponantur iam pro triangulo ABc latera $AB = \gamma$, $Ac = \beta$ et $Bc = \alpha$, et inter haec latera per problema praecedens haec relatio intercedet, vt ſit:

$$\beta^3 - \alpha\beta\beta - \alpha\alpha\beta - \alpha(\gamma\gamma - \alpha\alpha) = 0.$$

Hic igitur loco α, β, γ valores illi $\frac{ac}{b}$, $\frac{bb-aa}{b}$ et c ſubſtituantur, ſeu ad fractiones tollendas, quia ibi dimenſionum numerus vbique eſt idem, hi valores per b multiplicati, quaſi eſſet $\alpha = ac$, $\beta = bb - aa$ et $\gamma = bc$, ſcribantur; ſicque exorietur haec aequatio:

$$(bb-aa)^3\ ac(bb-aa)^2 - aacc(bb-aa) - ac^3(bb-aa = 0$$

quae cum manifeſto diuiſorem habeat $bb - aa$, erit aequatio relationem quaeſitam exprimens:

$$(bb-aa)^2 - ac(bb-aa) - aacc - ac^3 = 0.$$

Coroll.

12. Aequatio haec euoluta, et ſecundum poteſtates ipſius b diſpoſita, abit in hanc formam:

$$b^4 - a(2a+c)bb - a(ac-aa)(a+c) = 0$$

qua deinceps erit vtendum.

Pro-

Problema 4.

13. Si in triangulo ABC angulus ABC fuerit quintuplus anguli A, inter eius latera AB$=c$, AC$=b$ et BC$=a$ relationem ista conditione determinatam inuestigare.

Solutio.

Ducta iterum recta Bc, angulum CBc ipsi A aequalem abscindente, vt triangulum BCc toti ACB simile fiat, triangulum vero ABc ad casum praecedentem sit referendum, pro quo si ponamus latera AB$=\gamma$, A$c=\beta$ et B$c=\alpha$, erit vti modo inuenimus:

$$\beta^4-\alpha(2\alpha+\gamma)\beta\beta-\alpha(\gamma\gamma-\alpha\alpha)(\alpha+\gamma)=0$$

At vero hic, vti ante ostendimus, has substitutiones fieri oportet: $\alpha=ac$, $\beta=bb-aa$, et $\gamma=bc$, vnde oritur haec aequatio:

$$(bb-aa)^4-acc(2a+b)(bb-aa)^2-ac^4(bb-aa)(a+b)=0$$

quae diuisa per $(bb-aa)(b+a)$ induit hanc formam:

$$(bb-aa)^2(b-a)-acc(2a+b)(b-a)-ac^4=0$$

et facta euolutione prodit

$$b^5-ab^4-2aab^3-a(cc-2aa)bb-aa(cc-aa)b-a(cc-aa)^2=0.$$

Problema 5.

14. Si in triangulo ABC angulus ABC fuerit sextuplus anguli A, inter eius latera AB$=c$, AC$=b$ et BC$=a$ relationem ista conditione determinatam inuestigare.

Solutio.

Ex superioribus satis iam est perspicuum, hanc relationem inueniri, si in ea, quam modo sumus adepti, loco litterarum a, b, c scribamus has formulas: ac, $bb-aa$, et bc; sicque prodit:

$$(bb-aa)^5 - ac(bb-aa)^4 - 2aacc(bb-aa)^3 - ac^3(bb-2aa)(bb-aa)^2 - aac^4(bb-aa)^2 - ac^5(bb-aa)^2 = 0$$

quae aequatio per $(bb-aa)^2$ diuisa induit hanc formam:

$$(bb-aa)^3 - ac(bb-aa)^2 - 2aacc(bb-aa) - ac^3(bb-2aa) - aac^4 - ac^5 = 0$$

euolutione autem facta obtinet

$$b^6 - a(3a+c)b^4 + a(3a^3+2aac-2acc-c^3)bb - a(cc-aa)^2(c+a) = 0 \text{ seu}$$

$$b^6 - a(c+3a)b^4 - a(c+a)(cc+ac-3aa)bb - a(cc-aa)^2(c+a) = 0.$$

Coroll. 1.

15. Si hic simili modo fiat substitutio $a = ac$, $b = bb-aa$ et $c = bc$, oritur aequatio pro triangulo ABC, in quo angulus ad B est septuplus anguli A, quae ergo erit:

$$(bb-aa)^6 - acc(b+3a)(bb-aa)^4 - ac^4(b+a)(bb+ab-3aa)(bb-aa)^2 - ac^6(bb-aa)^2(b+a) = 0$$

quae iam per $(bb-aa)^2(b+a)$ diuisionem admittit, et dat

$$(bb-aa)^3(b-a) - acc(b+3a)(bb-aa)(b-a) - ac^4(bb+ab-3aa) - ac^6 = 0$$

seu

seu:

$$b^7-ab^6-3aab^5-a(cc-3aa)b^4-aa(2cc-3aa)b^3-a(cc-aa)(cc-3aa)bb-aa(cc-aa)^2b-a(cc-aa)^3=0.$$

Coroll. 2.

16. Ope eiusdem substitutionis hinc obtinetur aequatio pro triangulo ABC, in quo angulus ad B est octuplus anguli A; scilicet:

$$(bb-aa)^7-ac(bb-aa)^6-3aacc(bb-aa)^5-ac^3(bb-3aa)(bb-aa)^4-aac^4(2bb-3aa)(bb-aa)^3-ac^5(bb-aa)(bb-3aa)(bb-aa)^2-aac^6(bb-aa)^2(bb-aa)-ac^7(bb-aa)^3=0$$

quae aequatio per $(bb-aa)^3$ diuisa praebet:

$$(bb-aa)^4-ac(bb-aa)^3-3aacc(bb-aa)^2-ac^3(bb-3aa)(bb-aa)-aac^4(2bb-3aa)-ac^5(bb-3aa)-aac^6-ac^7=0$$

ex cuius euolutione nascitur haec forma:

$$b^8-a(c+4a)b^6-a(c^3+3acc-3aac-6a^3)b^4-a(c+a)(cc-aa)(cc+ac-4aa)b^2-a(c+a)(cc-aa)^3=0.$$

Scholion.

17. Nunc igitur rem in genere considerando, si angulus ABC ad angulum A teneat rationem $=n:1$, vt sit $ABC=n.BAC$, positis lateribus $AB=c$, $AC=b$, et $BC=a$, contemplemus aequationes pro casibus simplicioribus hactenus inuentas, quas propterea ita ordine exhibeamus, litterisque maiusculis designemus:

si	erit
$n=1$	$b-a=0$ . . A
$n=2$	$bb-a(a+c)=0$ . . B
$n=3$	$b^3-abb-aab-a(cc-aa)=0$ . . C
$n=4$	$b^4-a(c+2a)bb-a(c+a)(cc-aa)=0$ . . D
$n=5$	$b^5-ab^4-2aab^3-a(cc-2aa)bb-aa(cc-aa)b-a(cc-aa)^2=0$ . . E
$n=6$	$b^6-a(c+3a)b^4-a(c+a)(cc+ac-3aa)bb-a(c+a)(cc-aa)^2=0$ . . F
$n=7$	$b^7-ab^6-3aab^5-a(cc-3aa)b^4-aa(2cc-3aa)b^3-a(cc-aa)(cc-3aa)bb$ $-aa(cc-aa)^2b-a(cc-aa)^3=0$ . . G
$n=8$	$b^8-a(c+4a)b^6-a(c^3+3acc-3aac-ba^2)b^4-a(c+a)(cc-aa)(cc+ac$ $-4aa)b^2-a(c+a)(cc-aa)^3=0$ . . H.

Hic igitur ſtatim conſtat, has formulas nonniſi alternatim ſumtas commode inter ſe comparari poſſe; quandoquidem in iis, quae numeris paribus reſpondent, littera b tantum pares habet dimenſiones, in imparibus autem eiusdem litterae b, praeter pares, etiam impares dimenſiones occurrunt, dum contra hoc caſu littera c tantum pares dimenſiones obtinet. Hinc iſtas formulas, prouti n eſt numerus vel par vel impar, ſeorſim percurramus, in legem progreſſionis inquiſituri, vbi quidem primo oſtendam, vtroque caſu has formulas ſeriem recurrentem conſtituere, cuius quisque terminus per binos praecedentes determinatur; deinde vero etiam formam generalem exhibere conabor.

Problema 6.

18. Si in triangulo ABC fuerit angulus $B=2iA$, denotante $2i$ numerum integrum parem quem-

quemcunque, naturam relationis, quae inter trianguli latera $AB = c$, $AC = b$ et $BC = a$ intercedit, inuestigare.

Solutio.

Hic igitur considerari oportet progressionem earum alternarum formularum, quas ante litteris B, D, F, H etc. designauimus, quae ita se habent:

si	inuenta est formula
$i = 1$	$B = bb - a(c + a) = 0$
$i = 2$	$D = b^4 - a(c + 2a)bb - a(c + a)(cc - aa) = 0$
$i = 3$	$F = b^6 - a(c + 3a)b^4 - a(c + a)(cc + ac - 3aa)bb$
	$- a(c + a)(cc - aa)^2 = 0$
$i = 4$	$H = b^8 - a(c + 4a)b^6 - a(c^3 + 3acc - 3aac - 6a^3)b^4$
	$- a(c + a)(cc - aa)(cc + ac - 4aa)bb - a(c + a)(cc - aa)^3 = 0$

quarum formularum lex quo facilius obseruetur, eas etiam secundum potestates litterae c disponamus:

si	erit
$i = 1$	$B = (bb - aa) - ac = 0$
$i = 2$	$D = (bb - aa)^2 - ac(bb - aa) - aacc - ac^3 = 0$
$i = 3$	$F = (bb - aa)^3 - ac(bb - aa)^2 - 2aacc(bb - aa)$
	$- ac^3(bb - 2aa) - aac^4 - ac^5 = 0$
$i = 4$	$H = (bb - aa)^4 - ac(bb - aa)^3 - 3aacc(bb - aa)^2$
	$- ac^3(bb - 3aa)(bb - aa) - aac^4(2bb - 3aa)$
	$- ac^5(bb - 3aa) - aac^6 - ac^7 = 0.$

Hic primum obseruo, si a quauis formula praecedens per $bb - aa$ multiplicata subtrahatur, residua multo simpliciora esse proditura; erit enim:

D-B

$D - B(bb - aa) = -aacc - ac^3$

$F - D(bb - aa) = -aacc(bb - aa) + a^3c^3 - aac^4 - ac^5$

$H - F(bb - aa) = -aacc(bb - aa)^2 + a^3c^3(bb - aa)$
$\quad - aac^4(bb - 2aa) + 2a^3c^5 - aac^6 - ac^7$

ſubtrahatur inſuper a qualibet praecedens per cc multiplicata, reperieturque:

$D - B(bb - aa + cc) = -bbcc$

$F - D(bb - aa + cc) = -bbcc(bb - aa) + abbc^3$

$H - F(bb - aa + cc) = -bbcc(bb - aa)^2 + abbc^3(bb - aa)$
$\quad + aabbc^4 + abbc^5$

quae formae per $-bbcc$ diuiſae praebent

$$\frac{B(bb - aa + cc) - D}{bbcc} = 1$$

$$\frac{D(bb - aa + cc) - F}{bbcc} = bb - aa - ac = B$$

$$\frac{F(bb - aa + cc) - H}{bbcc} = (bb - aa)^2 - ac(bb - aa) - aacc - ac^3 = D$$

vbi profecto caſu non euenire videtur, vt primo quidem vnitas, tum vero ipſae litterae B et D prodeant; pro inductione quidem hinc ſtabilienda hi duo caſus certe minime ſufficerent, verum calculo ad ſequentem formulam K continuato, nonſolum idem contingit, ſed etiam pro formulis ordine imparibus A, C, E, G deinceps eadem lex progreſſionis deprehendetur. Quam ob rem non dubito, huic inductioni innixus pronunciare, formulas has B, D, F, H etc. ſeriem conſtituere recurrentem, cuius ſcala relationis $bb - aa - cc$, $-bbcc$, hincque terminum antecedentem ipſi $i = 0$ reſpondentem eſſe vnitatem. Ita his formulis ita diſpoſitis:

$i \ldots 0$,	1,	2,	3,	4,	5,	6,	7	
I,	B,	D,	F,	H,	K,	M,	O	etc.

erit

erit primo quidem $B = bb - aa - ac$, tum vero secundum legem feriei recurrentis:

$$D = (bb - aa + cc)B - bbcc.1 \text{ pro } n = 4$$
$$F = (bb - aa + cc)D - bbcc B \text{ pro } n = 6$$
$$H = (bb - aa + cc)F - bbcc D \text{ pro } n = 8$$
$$K = (bb - aa + cc)H - bbcc F \text{ pro } n = 10$$
$$M = (bb - aa + cc)K - bbcc H \text{ pro } n = 12$$
etc.

vnde has formulas, quousque lubuerit, continuare licet.

Coroll. 1.

19. Si ergo haec feries formetur: $1 + Bz + Dz^2 + Fz^3 +$ etc. ea ex euolutione huiusmodi fractionis:

$$\frac{1 + \Delta z}{1 - (bb + aa + cc)z + bbcczz}$$

nafcitur, vbi quidem eft $\Delta = -ac - cc$, haecque fractio adeo illius feriei in infinitum prolatae fummam exhibet.

Coroll. 2.

20. Hinc porro in genere formulam indefinite numero i conuenientem exhibere licet, quippe quae ita exprimetur:

$$\mathfrak{A}\left(\frac{bb - aa + cc + \sqrt{(a^4 + b^4 + c^4 - 2aabb - 2aacc - 2bbcc)}}{2}\right)^i$$
$$\mathfrak{B}\left(\frac{bb - aa + cc - \sqrt{(a^4 + b^4 + c^4 - 2aabb - 2aacc - 2bbcc)}}{2}\right)^i$$

vbi

vbi quidem, applicatione ad duas primores facta, fit $\mathfrak{A}+\mathfrak{B}=1$

et $\frac{bb-aa+cc}{2}+\frac{1}{2}(\mathfrak{A}-\mathfrak{B})\surd \ldots\ldots = bb-aa-ac$

seu $\mathfrak{A}-\mathfrak{B}=\frac{bb-(a+c)^2}{\surd \ldots} = -\surd\frac{(a+c+b)(a+c-b)}{(a+b-c)(b+c-a)}$.

Scholion.

Tab. I. Fig. 5. 21. Formula haec generalis eo maiori cura euolui meretur, quod adhuc soli inductioni innititur, ideoque vberiori confirmatione indiget. Sint igitur trianguli ABC latera $AB=c$, $AC=b$, $BC=a$, et anguli $A=\alpha$, $B=\beta$, $C=\gamma$, vbi quidem assumimus esse $\beta=2i\alpha$. Nunc vero est

$\cos\alpha=\frac{bb-aa+cc}{2bc}$ et $\sin.\alpha=\frac{\surd(2aabb+2aacc+2bbcc-a^4-b^4-c^4)}{2bc}$

ex quo formula nostra inuenta induet hanc formam:

$$\mathfrak{A}(bc\cos.\alpha+bc\surd-1\sin.\alpha)^i+\mathfrak{B}(bc\cos.\alpha-bc\surd-1\sin.\alpha)^i$$

quae per principia nota transfunditur in hanc:

$$\mathfrak{A}b^ic^i(\cos.i\alpha+\surd-1\sin.i\alpha)+\mathfrak{B}b^ic^i(\cos.i\alpha-\surd-1\sin.i\alpha)$$

Cum vero sit $\mathfrak{A}+\mathfrak{B}=1$, et $\mathfrak{A}-\mathfrak{B}=\frac{bb-(a+c)^2}{2bc\surd-1.\sin.\alpha}$, ex formula $\cos.\beta=\cos.2i\alpha=\frac{aa+bc-bb}{2ac}$, sequitur fore

$$1+\cos.2i\alpha=\frac{(a+c)^2-bb}{2ac}\text{, ideoque } \mathfrak{B}-\mathfrak{A}=\frac{a(1+\cos.2i\alpha)}{b\surd-1.\sin.\alpha}$$

at $\sin.\alpha:\sin.2i\alpha=a:b$, vnde $\mathfrak{B}-\mathfrak{A}=\frac{1+\cos.2i\alpha}{\surd-1.\sin.2i\alpha}$, seu $\mathfrak{B}-\mathfrak{A}=\frac{\cos.i\alpha}{\surd-1.\sin.i\alpha}$. Quo circa habebitur:

$$\mathfrak{A}=\frac{-\cos.i\alpha+\surd-1.\sin.i\alpha}{2\surd-1.\sin i\alpha}\text{, et } \mathfrak{B}=\frac{\cos.i\alpha+\surd-1.\sin.i\alpha}{2\surd-1\sin.i\alpha}$$

sicque

sicque formula inuenta fit

$$\frac{b^i c^i}{2\sqrt{-1}.\,\text{fin.}\,i\alpha}\big((\text{cof.}\,i\alpha+\sqrt{-1}.\,\text{fin.}\,i\alpha)(-\text{cof.}\,i\alpha+\sqrt{-1}.\,\text{fin.}\,i\alpha)$$
$$+(\text{cof.}\,i\alpha-\sqrt{-1}.\,\text{fin.}\,i\alpha)(\text{cof.}\,i\alpha+\sqrt{-1}.\,\text{fin.}\,i\alpha)\big)$$

quae cum fponte in nihilum abeat, euidens eft, cafu, quo angulus $\beta=2i\alpha$, formulam inuentam nihilo effe aequalem, ideoque inductionem veritati confentaneam.

Problema 7.

22. Si in triangulo ABC fuerit angulus $\beta=(2i+1)\alpha$, exiftente $2i+1$ numero impare quocunque integro, naturam relationis, quae hinc inter latera trianguli a, b, c intercedit, inueftigare.

Solutio.

Ex ferie ergo formularum fupra (17) exhibita, eas alternas confiderari oportet, quae litteris A, C, E, G etc. funt defignatae, et ordine expofitae, ita fe habent:

fi	formula inuenta eft
$i=0$	$A=b-a=0$
$i=1$	$C=b^3-abb-aab-a(cc-aa)=0$
$i=2$	$E=b^5-ab^4-2aab^3-a(cc-2aa)bb-aa(cc-aa)b-a(cc-aa)^2=0$
$i=3$	$G=b^7-ab^6-3aab^5-a(cc-3aa)b^4-aa(2cc-3aa)b^3-a(cc-aa)(cc-3aa)bb-aa(cc-aa)^2b-a(cc-aa)^3=0$

quae eaedem secundum potestates ipsius c dispositae ita repraesententur:

si	erit
$i = 0$	$A = (b-a) = 0$
$i = 1$	$C = (b-a)(bb-aa) - acc = 0$
$i = 2$	$E = (b-a)(bb-aa)^2 - acc(bb+ab-2aa) - ac^4 = 0$
$i = 3$	$G = (b-a)(bb-aa)^3 - acc(bb-aa)(bb+2ab-3aa)$ $-ac^4(bb+ab-3aa) - ac^6 = 0,$

Atque ex his colligimus primo:

$$(bb-aa)A - C = acc$$
$$(bb-aa)C - E = aacc(b-a) + ac^4$$
$$(bb-aa)E - G = aacc(bb-aa)(b-a) + aac^4(b-2a) + ac^6$$

tum vero porro:

$$(bb-aa+cc)A - C = bcc$$
$$(bb-aa+cc)C - E = bbcc(b-a) = bbccA$$
$$(bb-aa+cc)E - G = bbcc(b-a)(bb-aa) - abbc^4 = bbccC.$$

Vnde iam multo maiori fiducia concludimus, has formulas A, C, E, G etc. seriem recurrentem constituere, cuius scala relationis sit $bb-aa+cc$, et terminum primo A praecedentem censendum esse $= \frac{1}{b}$. Quare ex cognitis duobus primis $A = b-a$ et $C = (b-a)(bb-aa) - acc$ sequentes hac lege formantur.

$$E = (bb-aa+cc)C - bbccA \quad \text{pro } n = 5$$
$$G = (bb-aa+cc)E - bbccC \quad \text{pro } n = 7$$
$$I = (bb-aa+cc)G - bbccE \quad \text{pro } n = 9$$
$$L = (bb-aa+cc)I - bbccG \quad \text{pro } n = 11$$
$$N = (bb-aa+cc)L - bbccI \quad \text{pro } n = 13$$

etc.

Coroll.

Coroll. I.

23. His igitur formulis ita secundum numeros i dispositis, erit

i 0, 1, 2, 3, 4, 5, 6
formula A, C, E, G, I, L, N etc.

et terminus indefinite numero i conueniens erit, vt ante, huius formae:

$$\mathfrak{A}\left(\frac{bb - aa + cc + \sqrt{(a^4 + b^4 + b^4 - 2aabb - 2aacc - 2bbcc)}}{2}\right)^i$$
$$+ \mathfrak{B}\left(\frac{bb - aa + cc - \sqrt{(a^4 + b^4 + c^4 - 2aabb - 2aacc - 2bbcc)}}{2}\right)^i.$$

Coroll. I.

24. Coefficientes $\mathfrak{A}$ et $\mathfrak{B}$ ex binis terminis initialibus A et C ita definiuntur, vt primo fit $\mathfrak{A} + \mathfrak{B} = A = b - a$, tum vero $\frac{(b-a)(bb-aa+cc)}{2} + \frac{1}{2}$

$(\mathfrak{A} - \mathfrak{B})\sqrt{(\ldots)} = (b-a)(bb-aa) - acc$

ergo $(\mathfrak{A} - \mathfrak{B})\sqrt{(\ldots)} = (b-a)(bb-aa - (b+a)cc$

$= (b+a)(bb - 2ab + aa - cc$

hincque $\mathfrak{A} - \mathfrak{B} = \frac{(b+a)((b-a)^2 - cc)}{\sqrt{(a^4 + b^4 + c^4 - 2aabb - 2aacc - 2bbcc)}}$.

Scholion.

25. Euoluamus hanc formulam generalem pari modo, quo ante fecimus (21), eritque prorsus vt ante forma nostra general s:

$\mathfrak{A}(bc\cos.\alpha + bc\sqrt{-1}.\sin.\alpha)^i + \mathfrak{B}(bc\cos.\alpha - bc\sqrt{-1}.\sin.\alpha)$

quae pariter in hanc abit:

$\mathfrak{A}b^i c^i(\cos.i\alpha + \sqrt{-1}.\sin.i\alpha) + \mathfrak{B}b^i c^i(\cos.i\alpha - \sqrt{-1}.\sin.i\alpha)$ vbi

vbi eſt $\mathfrak{A} + \mathfrak{B} = b - a$, et $\mathfrak{A} - \mathfrak{B} = \frac{(b+a)((b-a)^2 - cc)}{2bc\sqrt{-1}.\sin.\alpha}$.
Nunc vero, ob ang. $\gamma = 180° - 2(i+1)\alpha$, erit
$\cos.2(i+1)\alpha = \frac{cc-aa-bb}{2ab}$ et $1 + \cos.2(i+1)\alpha = \frac{cc-(b-a)^2}{2ab}$
vnde $\mathfrak{B} - \mathfrak{A} = \frac{a(b+a)(1+\cos.2(i+1)\alpha)}{c\sqrt{-1}.\sin.\alpha}$.
At eſt $a : c = \sin.\alpha : \sin.2(i+1)\alpha$, ideoque $\mathfrak{B} - \mathfrak{A}$
$= \frac{(b+a)\cos.(i+1)\alpha}{\sqrt{-1}.\sin.(i+1)\alpha}$, vnde colligimus

$$\mathfrak{A} = -\frac{(b+a)\cos.(i+1)\alpha + (b-a)\sqrt{-1}.\sin.(i+1)\alpha}{2\sqrt{-1}.\sin.(i+1)\alpha} \text{ et}$$
$$\mathfrak{B} = \frac{(b+a)\cos.(i+1)\alpha + (b-a)\sqrt{-1}.\sin.(i+1)\alpha}{2\sqrt{-1}.\sin.(i+1)\alpha}.$$

Quoniam latera a et b ſunt ſinibus angulorum oppoſitorum proportionalia, ſtatuamus $a = 2f\sin.\alpha$, et $b = 2f\sin.(2i+1)\alpha$, eritque

$$a\cos.(i+1)\alpha = f(\sin.(i+2)\alpha - \sin.i\alpha);\ a\sin.(i+1)\alpha = f(\cos.i\alpha - \cos.(i+2)\alpha)$$
$$b\cos.(i+1)\alpha = f(\sin.(3i+2)\alpha + \sin.i\alpha);\ b\sin.(i+i)\alpha = f(\cos.i\alpha - \cos.(3i+2)\alpha)$$

vnde colligimus:

$$(a+b)\cos.(i+1)\alpha = f(\sin.(i+2)\alpha + \sin.(3i+2)\alpha) \text{ et}$$
$$(b-a)\sin.(i+1)\alpha = f(\cos.(i+2)\alpha - \cos.(3i+2)\alpha).$$

Quare cum ſit generatim

$$\sin.\mu + \sin.\nu = 2\sin.\tfrac{\mu+\nu}{2}\cos.\tfrac{\nu-\mu}{2} \text{ et}$$
$$\cos.\mu - \cos.\nu = 2\sin.\tfrac{\mu+\nu}{2}\sin.\tfrac{\nu-\mu}{2}$$

habebimus;

$$(a+b)\cos.(i+1)\alpha = 2f\sin.(2i+2)\alpha\cos.i\alpha \text{ et}$$
$$(b-a)\sin.(i+1)\alpha = 2f\sin.(2i+2)\alpha\sin.i\alpha$$

ac propterea adipiſcimur:

$$\mathfrak{A} = \frac{f\,\text{ſin.}(2i+2)\alpha}{\sqrt{-1}.\,\text{ſin.}(i+1)\alpha}\,(-\text{coſ.}\,i\alpha + \sqrt{-1}.\,\text{ſin.}\,i\alpha) \text{ et}$$

$$\mathfrak{B} = \frac{f\,\text{ſin.}(2i+2)\alpha}{\sqrt{-1}.\,\text{ſin.}(i+1)\alpha}\,(\text{coſ.}\,i\alpha + \sqrt{-1}.\,\text{ſin.}\,i\alpha)$$

vnde perſpicuum eſt, fore

$$\mathfrak{A}(\text{coſ.}\,i\alpha + \sqrt{-1}.\,\text{ſin.}\,i\alpha) + \mathfrak{B}(\text{coſ.}\,i\alpha - \sqrt{-1}.\,\text{ſin.}\,i\alpha) = 0,$$

quo ipſo veritas inductionis noſtrae euincitur. His autem obſeruatis, nunc demum ſolutionem noſtri Problematis directe aggredi licet.

Problema 8.

26. Si in triangulo ABC angulus B ad angulum A rationem teneat quamcunque multiplam, vt n ad 1, relationem, quae inde inter latera trianguli $AB = c$, $AC = b$, $BC = a$ intercedit, analytice inueſtigare.

Solutio.

Poſito angulo $A = \alpha$, vt ſit angulus $B = n\alpha$, erit, vti ex angulorum doctrina conſtat:

$$\text{coſ.}\,n\alpha + \sqrt{-1}.\,\text{ſin.}\,n\alpha = (\text{coſ.}\,\alpha + \sqrt{-1}.\,\text{ſin.}\,\alpha)^n \text{ et}$$

$$\text{coſ.}\,n\alpha - \sqrt{-1}.\,\text{ſin.}\,n\alpha = (\text{coſ.}\,\alpha - \sqrt{-1}.\,\text{ſin.}\,\alpha)^n, \text{ ideoque}$$

$$\frac{\text{coſ.}\,n\alpha + \sqrt{-1}.\,\text{ſin.}\,n\alpha}{\text{coſ.}\,n\alpha - \sqrt{-1}.\,\text{ſin.}\,n\alpha} = \left(\frac{\text{coſ.}\,\alpha + \sqrt{-1}\,\text{ſin.}\,\alpha}{\text{coſ.}\,\alpha - \sqrt{-1}\,\text{ſin.}\,\alpha}\right)^n.$$

Iam prout n eſt numerus par vel impar, duo caſus ſunt euoluendi.

Sit primo $n = 2i$, et vtrinque radix quadrata extrahatur, fietque:

$$\frac{\text{coſ.}\,i\alpha + \sqrt{-1}.\,\text{ſin.}\,i\alpha}{\text{coſ.}\,i\alpha - \sqrt{-1}.\,\text{ſin.}\,i\alpha} = \left(\frac{\text{coſ.}\,\alpha + \sqrt{-1}.\,\text{ſin}\,\alpha}{\text{coſ.}\,\alpha - \sqrt{-1}.\,\text{ſin.}\,\alpha}\right)^i$$

nunc

nunc vero eſt $\cos. 2i\alpha = \frac{aa + cc - bb}{2ac}$, ideoque
$\cos. i\alpha = \frac{1}{2}\sqrt{\frac{(a+c)^2 - bb}{ac}}$ et $\sin. i\alpha = \frac{1}{2}\sqrt{\frac{bb - (c-a)^2}{ac}}$
deinde $\cos. \alpha = \frac{bb - aa + cc}{2bc}$ et $\sin. \alpha = \frac{\sqrt{(2aabb + 2aacc + 2bbcc - a^4 - b^4 - c^4)}}{2bc}$.
Sit breuitatis gratia

$$\Delta = \sqrt{(a^4 + b^4 + c^4 - 2aabb - 2aacc - 2bbcc)}$$

et noſtra aequatio fit

$$\frac{(a+c)^2 - bb + \Delta}{(a+c)^2 - bb - \Delta} = \left(\frac{bb - aa + cc + \Delta}{bb - aa + cc - \Delta}\right)^i \text{ ſeu}$$

$$((a+c)^2 - bb - \Delta)(bb - aa + cc + \Delta)^i - ((a+c)^2 - bb + \Delta)(bb - aa + cc - \Delta)^i = 0$$

quae per 2Δ diuiſa conuenit cum forma ſupra inuenta. Sit deinde $n = 2i + 1$, et multiplicando aequationem per $\frac{\cos. \alpha - \sqrt{-1}.\sin. \alpha}{\cos. \alpha + \sqrt{-1}.\sin. \alpha}$ orietur,

$$\frac{\cos. 2i\alpha - \sqrt{-1}.\sin. 2i\alpha}{\cos. 2i\alpha - \sqrt{-1}.\sin. 2i\alpha} = \left(\frac{\cos. \alpha + \sqrt{-1}.\sin. \alpha}{\cos. \alpha - \sqrt{-1}.\sin. \alpha}\right)^{2i}$$

et quadratam radicem extrahendo:

$$\frac{\cos. i\alpha + \sqrt{-1}.\sin. i\alpha}{\cos. i\alpha - \sqrt{-1}.\sin. i\alpha} = \left(\frac{\cos. \alpha + \sqrt{-1}.\sin. \alpha}{\cos. \alpha - \sqrt{-1}.\sin. \alpha}\right)^i.$$

Cum nunc ſit $\gamma = 180^\circ - 2(i+1)\alpha$, erit

$$\cos. 2(i+1)\alpha = \frac{cc - bb - aa}{2ab} \text{ hincque}$$

$\cos. (i+1)\alpha = \frac{1}{2}\sqrt{\frac{cc - (b-a)^2}{ab}}$ et $\sin. (i+1)\alpha = \frac{1}{2}\sqrt{\frac{(b+a)^2 - cc}{ab}}$.
Eſt vero $\cos. \alpha = \frac{bb - aa + cc}{2bc}$ et $\sin. \alpha\sqrt{-1} = \frac{\Delta}{2bc}$ ſeu $\sin. \alpha = \frac{\Delta}{2bc\sqrt{-1}}$,
vbi notetur eſſe $\frac{\Delta}{\sqrt{-1}} = \sqrt{(cc - (b-a)^2((b+a)^2 - cc)}$;
quamobrem elicietur

$$\cos. i\alpha = \frac{1}{4bc\sqrt{ab}}((bb - aa + cc)\sqrt{(cc - (b-a)^2)} + ((b+a)^2 - cc)\sqrt{(cc - (b-a)^2)}$$

ſeu

ſeu $\cos. i\alpha = \frac{b+a}{2c\sqrt{ab}} \sqrt{(cc-(b-a)^2)}$, tum vero

$$\sin. i\alpha = \frac{1}{4bc\sqrt{ab}}((bb-aa+cc)\sqrt{((b+a)^2-cc)} - (cc-(b-a)^2)\sqrt{((b+a)^2-cc)})$$

ſeu $\sin. i\alpha = \frac{b-a}{2c\sqrt{ab}} \sqrt{((b+a)^2-cc)}$. Quibus ſubſtitutis erit:

$$\frac{(b+a)(cc-(b-a)^2)+(b-a)\Delta}{(b+a)(cc-(b-a)^2)-(b-a)\Delta} = \left(\frac{bb-ae+cc+\Delta}{bb-aa+cc-\Delta}\right)^i$$

et aequatio hinc ſupra inuenta colligitur:

$$\left(b+a-\frac{(b-a)\Delta}{cc-(b-a)^2}\right)(bb-aa+cc+\Delta)^i-\left(b+a+\frac{(b-a)\Delta}{cc-(b-a)^2}\right)(bb-aa+cc-\Delta)^i=0$$

dummodo haec ducatur in $\frac{cc-(b-a)^2}{2\Delta}$, atque ex hac forma ſimul natura ſeriei recurrentis intelligitur.

Coroll. 1.

27. Pro caſu ergo quo in triangulo ABC angulus $B=2iA$ aequatio laterum relationem exprimens eſt:

$$\left(1+\frac{bb-(a+c)^2}{\Delta}\right)(bb-aa+cc+\Delta)^i+\left(1-\frac{bb+(a+c)^2}{\Delta}\right)(bb-aa+cc-\Delta)^i=0$$

pro caſu autem, quo angulus $B=(2i+1)A$, habetur:

$$\left(b-a-\frac{(b+a)(cc-(b-a)^2)}{\Delta}\right)(bb-aa+cc+\Delta)^i+\left(b-a+\frac{(b+a)(oc-(b-a)^2)}{\Delta}\right)(bb-aa+cc-\Delta)^i=0.$$

Coroll. 2.

28. Quodſi ergo has conſtituamus formas:

$$\tfrac{1}{1}\left(\frac{bb-aa+cc+\Delta}{2}\right)^i+\tfrac{1}{2}\left(\frac{bb-aa+cc-\Delta}{2}\right)^i=\mathrm{V}$$

$$\frac{1}{2\Delta}\left(\frac{bb-aa+cc+\Delta}{2}\right)^i-\frac{1}{2\Delta}\left(\frac{bb-aa+cc-\Delta}{2}\right)^i=\mathrm{W}$$

quarum vtraque eſt rationalis non obſtante formula irrationali:

$$\Delta=\sqrt{}(a^4+b^4+c^4-2aabb-2aacc-2bbcc)$$
$$=\sqrt{}((bb-aa+cc)^2-4bbcc)$$

pro caſu $\mathrm{B}=2i\mathrm{A}$ erit

$$\mathrm{V}+(bb-(a+c)^2)\mathrm{W}=0$$

pro caſu vero $\mathrm{B}=(2i+1)\mathrm{A}$ erit

$$(b-a)\mathrm{V}+(b+a)((b-a)^2-cc)\mathrm{W}=0.$$

Coroll. 3.

29. Quodſi pro ſingulis valoribus numeri integri i ambae formae V et W euoluantur, binae exorientur ſeries recurrentes per eandem ſcalam relationis $bb-aa+cc$, $-bbcc$ continuandae, ex quibus deinceps ambae illae triangulorum proprietates facile exhibentur.

Scholion.

30. Quo has ſeries ſuccinctius exprimamus ſit breuitatis gratia $bb-aa+cc=ff$, et pro ſerie priori $\mathrm{V}=\left(\frac{ff+\Delta}{2}\right)^i+\left(\frac{ff-\Delta}{2}\right)^i$

Ob $\Delta = \sqrt{(f^4 - 4bbcc)}$ et ſcalam relationis $ff, -bbcc$ inueniemus:

ſi	valores ipſius V
$i=0$	2
$i=1$	ff
$i=2$	$f^4 - 2bbcc$
$i=3$	$f^6 - 3bbccff$
$i=4$	$f^8 - 4bbccf^4 + 2b^4c^4$
$i=5$	$f^{10} - 5bbccf^6 + 5b^4c^4ff$
$i=6$	$f^{12} - 6bbccf^8 + 9b^4c^4f^4 - 2b^6c^6$
$i=7$	$f^{14} - 7bbccf^{10} + 14b^4c^4f^6 - 7b^6c^6f^2$

vnde generatim colligitur fore . . . . . . $V =$

$$f^{2i}\ ibbccf^{2i-4} + \frac{i(i-3)}{1.\,2} b^4c^4f^{2i-8} - \frac{i(i-4)(i-5)}{1.\,2.\,3} b^6c^6f^{2i-12} + \text{etc.}$$

Deinde pro altera forma $W = \frac{1}{\Delta}\left(\frac{ff+\Delta}{2}\right)^i - \frac{1}{\Delta}\left(\frac{ff-\Delta}{2}\right)^i$ ſequens naſcetur ſeries:

ſi	valores ipſius W
$i=0$	0
$i=1$	1
$i=2$	ff
$i=3$	$f^4 - bbcc$
$i=4$	$f^6 - 2bbccff$
$i=5$	$f^8 - 3bbccf^4 + b^4c^4$
$i=6$	$f^{10} - 4bbccf^6 + 3b^4c^4ff$
$i=7$	$f^{12} - 5bbccf^8 + 6b^4c^4f^4 - b^6c^6$

vnde in genere haec forma erit . . . . $W =$

$$f^{2i-2} - (i-2)bbccf^{2i-6} + \frac{(i-3)(i-4)}{1.\,2} b^4c^4f^{2i-10} - \frac{(i-4)(i-5)(i-6)}{1.\,2.\,3} b^6c^6f^{2i-14} + \text{etc.}$$

vbi probe notandum eſt, has ambas expreſſiones generales tantum vsque ad terminos euaneſcentes proferri debere, etiamſi deinceps denuo termini finiti redeant. Caeterum hinc patet, fore . $\frac{1}{2}(V+ffW)=$

$$f^{2i}-(i-1)bbccf^{2i-4}+\frac{(i-2)(i-3)}{1.\ 2}b^4c^4f^{2i-8}$$
$$-\frac{(i-3)(i-4)(i-5)}{1.\ 2.\ 3}b^6c^6f^{2i-12}+\text{etc.}$$

atque . . . . $\frac{1}{2}(V-ffW)=$

$$-bbccf^{2i-4}+(i-3)b^4c^4f^{2i-8}-\frac{(i-4)(i-5)}{1.\ 2}b^6c^6f^{2i-12}$$
$$+\frac{(i-5)(i-6)(i-7)}{1.\ 2.\ 3}b^8c^8f^{2i-16}-\text{etc.}$$

Hinc iam pro caſu triangulorum, vbi angulus $B=2iA$; ob $bb=ff+aa-cc$, aequatio laterum relationem exprimens erit:

$$\tfrac{1}{2}(V+ffW)-c(a+c)W=0$$

pro altero autem caſu, vbi angulus $B=(2i+1)A$, poſito hic $cc=ff-bb+aa$, aequatio laterum relationem exprimens erit:

$$\tfrac{1}{2}b(V-ffW)-\tfrac{1}{2}a(V+ffW)+b(bb-aa)W=0.$$

Verum etiam alio modo hae expreſſiones generales absque introductione quantitatis $ff=bb-aa+cc$ repraeſentari poſſunt, vti in ſequenti problemate videbimus.

Problema 9.

31. Si in triangulo ABC angulus B ad angulum A rationem teneat quamcunque multiplam, vt n ad 1, aequationem, qua relatio inter latera

AB

AB$=c$, AC$=b$ et BC$=a$ exprimitur, in genere exhibere.

Solutio.

Si aequationes pro ſingulis caſibus ſupra inventas attentius conſideremus, haud difficulter legem certam in terminorum progreſſu obſeruabimus, ex indole progreſſionis demonſtrata facile confirmandam. Duos autem caſus hic diſtingui oportet, prout numerus ille n fuerit par, vel impar. Pro vtroque autem caſu aequatio quaeſita ſequenti modo exhiberi poterit:

Pro caſu, quo $n=2i$.

aequatio laterum relationem exprimens ita ſe habet:

$$\frac{b^{2i}}{a} = \begin{array}{l} +c\,b^{2i-2}+c(cc-(i-1)aa)b^{2i-4}+c(c^4-2(i-2)aacc+\frac{(i-2)(i-1)}{1.\ 2}a^4)b^{2i-6} \\ +iab^{2i-2}+a((i-1)cc-\frac{(i-1)i}{1.\ 2}aa)b^{2i-4}+a((i-2)c^4-\frac{2(i-2)(i-1)}{1.\ 2.}a^2c^2+\frac{(i-2)(i-1)i}{1.\ 2.\ 3}a^4)b^{2i-6} \end{array}$$

$$+c(c^6-3(i-3)a^2c^4+\frac{3(i-3)(i-2)}{1.\ 2}a^4c^2-\frac{(i-3)(i-2)(i-1)}{1.\ 2.\ 3}a^6)b^{2i-8}$$

$$+a((i-3)c^6-\frac{3(i-3)(i-2)}{1.\ 2}a^2c^4+\frac{3(i-3)(i-2)(i-1)}{1.\ 2.\ 3}a^4c^2-\frac{(i-3)(i-2)(i-1)i}{1.\ 2.\ 3.\ 4}a^6)b^{2i-8}$$

$$+c(c^8-4(i-4)a^2c^6+\frac{6(i-4)(i-3)}{1.\ 2}a^4c^4-\frac{4(i-4)(i-3)(i-2)}{1.\ 2.\ 3}a^6c^2+\frac{(i-4)(i-3)(i-2)(i-1)}{1.\ 2.\ 3.\ 4}a^8)b^{2i-10}$$

$$+a((i-4)c^8-\frac{4(i-4)(i-3)}{1.\ 2.}a^2c^6+\frac{6(i-4)(i-3)(i-2)}{1.\ 2.\ 3.}a^4c^4-\frac{4(i-4)(i-3)(i-2)(i-1)}{1.\ 2.\ 3.\ 4.}a^6c^2$$
$$+\frac{(i-4)(i-3)(i-2)(i-1)i}{1.\ 2.\ 3.\ 4.\ 5}a^8)b^{2i-10}$$

cuius lex continuationis ſatis eſt manifeſta.

Pro caſu, quo $n=2i+1$.

aequatio laterum relationem exprimens ita ſe habet:

$$\begin{aligned}\frac{b^{2i+1}}{a} = {} & +b^{2i}+(cc-iaa)b^{2i-2}+\left(c^4-2(i-1)aacc+\tfrac{(i-1)i}{1.\,2}a^4\right)b^{2i-4} \\ & +iab^{2i-1}+a\left((i-1)cc-\tfrac{(i-1)}{1.\,2}aa\right)b^{2i-3}+a\left((i-2)c^4-\tfrac{2(i-2)(i-1)}{1.\,2}a^2c^2+\tfrac{(i-2)(i-1)i}{1.\,2.\,3}a^4\right)b^{2i-5} \\ & +\left(c^6-3(i-2)a^2c^4+3\tfrac{(i-2)(i-1)}{1.\,2}a^4c^2-\tfrac{(i-2)(i-1)i}{1.\,2.\,3}a^6\right)b^{2i-6} \\ & +a\left((i-3)c^6-3\tfrac{(i-3)(i-2)}{1.\,2}a^2c^4+3\tfrac{(i-3)(i-2)(i-1)}{1.\,2.\,3}a^4c^2-\tfrac{(i-3)(i-2)(i-1)}{1.\,2.\,3.\,4}a^6\right)b^{2i-7}\end{aligned}$$

etc.

quae aequatio commodius hac forma, ſecundum poteſtates ipſius c diſpoſita, repraeſentari poteſt:

$$\begin{aligned}\frac{b^{2i+1}}{a} = {} & c^{2i}+c^{2i-2}\left\{\begin{matrix} bb-iaa \\ +ab \end{matrix}\right\}+c^{2i-4}\left\{\begin{matrix} b^4-2(i-1)aabb+\tfrac{(i-1)i}{1.\,2}a^4 \\ +2ab^3-(i-1)a^3b \end{matrix}\right\} \\ & +c^{2i-6}\left\{\begin{matrix} b^6-3(i-2)a^2b^4+\tfrac{3(i-2)(i-1)}{1.\,2}a^4b^2-\tfrac{(i-2)(i-1)i}{1.\,2.\,3}a^6 \\ +3ab^5-3(i-2\ a^3b^3+\tfrac{(i-2)(i-1)}{1.\,2}a^5b \end{matrix}\right\} \\ & +c^{2i-8}\left\{\begin{matrix} b^8-4(i-3)a^2b^6+\tfrac{6(i-3))i-2)}{1.\,2}a^4b^4-\tfrac{4(i-3)(i-2)(i-1)}{1.\,2.\,3}a^6b^2+\tfrac{(i-3)(i-2)(i-1)i}{1.\,2.\,3.\,4}a^8 \\ +4ab^7-6(i-3(a^3b^5+\tfrac{4(i-3)(i-2)}{1.\,2}a^5b^3-\tfrac{(i-3)(i-2)(i-1)}{1.\,2.\,3}a^8 \end{matrix}\right\}\end{aligned}$$

etc.

Scholion.

32. His conſiderationibus doctrina triangulorum non mediocriter amplificari videtur, dum ſtatim atque in quopiam triangulo ratio inter binos eius angulos innoteſcit, ſimul relatio certa inter eius latera exhiberi poteſt. Cum autem haec nimis ſint generalia, quandoquidem ex hac relatione vnicum

cum latus per bina reliqua determinatur, conueniet, has proprietates generales inuentas ad certam triangulorum ſpeciem accommodari, vbi quidem triangula iſoſcelia prae caeteris ſunt notatu digna, quia in iis ſaepenumero ratio inter angulum verticalem et angulos ad baſin praeſcribi ſolet, quoties ſcilicet polygona regularia ſunt conſtruenda. Duo autem caſus hic euoluendi occurrunt, prout vel angulus ad baſin eſt multiplus anguli verticalis, vel angulus verticalis multiplus anguli ad baſin; quos ambos in ſequentibus problematibus ſum expediturus.

Problema 10.

33. Si in triangulo iſoſcele BAC angulus ad baſin fuerit multiplus anguli verticalis A in ratione $n:1$, inueſtigare relationem inter baſin $BC=a$ et latera $AB=AC=b$. Tab. I. Fig. 6.

Solutio.

Primum obſeruandum eſt; ob hanc rationem ipſos angulos dari; poſita enim menſura duorum angulorum rectorum $=\pi$, et angulo verticali $A=\alpha$, ob $\alpha+2n\alpha=\pi$, fit $\alpha=\frac{\pi}{2n+1}$. Iam formulis ante inuentis huc transferendis, erit $c=b$, et binis caſibus ſeorſim tractatis, prout n eſt numerus vel par vel impar, quorum vtroque formulae ſeriem recurrentem conſtituunt, cuius ſcala relationis eſt $2bb-aa$, $-b^4$, primo ponendo $n=2i$, habebimus:

ſi

ſi	has aequationes
$i=0$	$1=0$
$i=1$	$B=bb-ab-aa=0$
$i=2$	$D=b^4-2ab^3-3aabb+a^3b+a^4=0$
$i=3$	$F=b^6-3ab^5-6aab^4+4a^3b^3+5a^4bb-a^5b-a^6=0$
$i=4$	$H=b^8-4ab^7-10aab^6+10a^3b^5+15a^4b^4-6a^5b^3-7a^6b^2+a^7b+a^8=0$

vnde concludimus, in genere fore:

$$0=b^{2i}-iab^{2i-1}-\frac{i(i+1)}{1.2}a^2b^{2i-2}+\frac{i(ii-1)}{1.2.3}a^3b^{2i-3}+\frac{i(ii-1)(i+2)}{1.2.3.4}a^4b^{2i-4}$$
$$-\frac{i(ii-1)(ii-4)}{1.2.3.4.5}a^5b^{2i-5}-\frac{i(ii-1)(ii-4)(i+3)}{1.2.3.4.5.6}a^6b^{2i-6}\text{ etc.}$$

Pro altero caſu, quo $n=2i+1$, habebimus:

ſi	has aequationes
$i=0$	$A=b-a=0$
$i=1$	$C=b^3-2abb-aab+a^3=0$
$i=2$	$E=b^5-3ab^4-3aab^3+4a^3b^2+a^4b-a^5=0$
$i=3$	$G=b^7-4ab^6-6aab^5+10a^3b^4+5a^4b^3-6a^5b^2-a^6b+a^7=0$

vnde concludimus in genere fore:

$$0=b^{2i+1}-(i+1)ab^{2i}-\frac{i(i+1)}{1.2}a^2b^{2i-1}+\frac{i(i+1)(i+2)}{1.2.3}a^3b^{2i-2}$$
$$-\frac{(i-1)i(i+1)(i+2)}{1.2.3.4}a^4b^{2i-3}$$
$$-\frac{(i-1)i(i+1)(i+2)(i+3)}{1.2.3.4.5}a^5b^{2i-4}+\frac{(i-2)(i-1)(i+1)(i+2)(i+3)}{1.2.3.4.5.6}a^6b^{2i-5}\text{ etc.}$$

quae forma, ſi ponamus $n=2i-1$, commodius ita exhibetur:

$$0=b^{2i-1}-iab^{2i-2}-\frac{i(i-1)}{1.2}a^2b^{2i-3}+\frac{i(ii-1)}{1.2.3}a^3b^{2i-4}+\frac{i(ii-1)(i-2)}{1.2.3.4}a^4b^{2i-5}$$
$$-\frac{i(ii-1)(ii-4)}{1.2.3.4.5}a^5b^{2i-6}-\frac{i(ii-1)(ii-4)(i-3)}{1.2.3.4.5.6}a^6b^{2i-7}\text{ etc}$$

Coroll.

Coroll. 1.

34. Cum in formulis generalibus supra exhibitis poni debeat $c=b$, erit pro casu $n=2i$ aequatio generalis:

$$\mathfrak{A}\left(\frac{2bb-aa+aa\surd(aa-4bb)}{2}\right)^i+\mathfrak{B}\left(\frac{2bb-aa-a\surd(aa-4bb)}{2}\right)^i=0$$

existente $\mathfrak{A}+\mathfrak{B}=1$ et $\mathfrak{A}-\mathfrak{B}=\frac{-a-2b}{\surd(aa-4bb)}$, hincque aequatio nostra:

$$\left(\frac{a+2b}{\surd(aa-4bb)}-1\right)\left(\frac{2bb-aa+a\surd(aa-4bb)}{2}\right)^i=\left(\frac{a+2b}{\surd(aa-4bb)}+1\right)\left(\frac{2bb-aa-a\surd(aa-4bb)}{2}\right)^i$$

Coroll. 2.

35. Pro casu autem $n=2i+1$, ob $c=b$, ex §. 23. adipiscimur hanc aequationem:

$$\mathfrak{A}\left(\frac{2bb-aa+a\surd(aa-4bb)}{2}\right)^i+\mathfrak{B}\left(\frac{2bb-aa-a\surd(aa-4bb)}{2}\right)^i=0$$

vbi est $\mathfrak{A}+\mathfrak{B}=b-a$ et $\mathfrak{A}-\mathfrak{B}=\frac{(b+a)(a-2b)}{\surd(aa-4bb)}$: ideoque

$$\left(b-a+\frac{(b+a)(a-2b)}{\surd(aa-4bb)}\right)\left(\frac{2bb-aa+a\surd(aa-4bb)}{2}\right)^i=\left(\frac{(b+a)(a-2b)}{\surd(aa-4bb)}-b+a\right)\left(\frac{2bb-aa-a\surd(aa-4bb)}{2}\right)^i.$$

Coroll. 3.

36. Si pro i scribamus $i-1$, vt sit $n=2i-1$, haec oritur aequatio:

$$\left(\frac{a-2b}{\surd(aa-4bb)}+1\right)\left(\frac{2bb-aa+a\surd(aa-4bb)}{2}\right)^i=\left(\frac{a-2b}{\surd(aa-4bb)}-1\right)\left(\frac{2bb-aa-a\surd(aa-4bb)}{2}\right)^i$$

 hinc

hinc autem prodeunt ſuperiores aequationes per $2b$ multiplicatae.

Scholion.

37. Formae generales hic exhibitae a ſumma poteſtate ipſius b incipiunt; eaedem vero etiam ita inuerſae repraeſentari poſſunt, vt a ſumma poteſtate ipſius a incipiant. [illegible] colligimus hanc aequationem:

$$0 = a^{2i} - (2i-1)a^{2i-2}b^2 + \frac{(2i-2)(2i-3)}{1 \cdot 2}a^{2i-4}b^4 - \frac{(2i-3)(2i-4)(2i-5)}{1 \cdot 2 \cdot 3}a^{2i-6}b^6 \text{ etc.}$$
$$+ a^{2i-1}b - (2i-2)a^{2i-3}b^3 + \frac{(2i-3)(2i-4)}{1 \cdot 2}a^{2i-5}b^5 - \frac{(2i-4)(2i-5)(2i-6)}{1 \cdot 2 \cdot 3}a^{2i-7}b^7 \text{ etc.}$$

Pro caſu autem poſteriori, quo $n = 2i + 1$, iſtam:

$$0 = \begin{array}{l} + a^{2i-1} - 2ia^{2i-1}b^2 + \frac{(2i-1)(2i-2)}{1 \cdot 2}a^{2i-3}b^4 - \frac{(2i-2)(2i-3)(2i-4)}{1 \cdot 2 \cdot 3}a^{2i-5}b^6 \\ - a^{2i}b + (2i-1)a^{2i-2}b^3 - \frac{(2i-2)(2i-3)}{1 \cdot 2}a^{2i-4}b^5 + \frac{(2i-3)(2i-4)(2i-5)}{1 \cdot 2 \cdot 3}a^{2i-6}b^7 \text{ etc.} \end{array}$$

Verum notandum eſt, has expreſſiones tantum eo vsque continuari debere, quoad ad terminum euaneſcentem perueniatur, et ſequentes, etiamſi non euaneſcant, tamen reiici oportere, cui cautioni formae ſuperiores non ſunt obnoxiae, ex quo eae quoque ad caſus, vbi i non eſt numerus integer, extendi poſſunt, vbi quidem aequatio ſerie infinita conſtabit.

Proble-

Problema II.

38. Si in triangulo isoscele ABC angulus verticalis B sit multiplus anguli ad basin A in ratione $n:1$, vt sit $B=nA$, inuestigare relationem inter basin $AC=b$ et latera $BA=BC=a$. Tab. I. Fig. 7.

Solutio.

Positis angulis ad basin $A=C=\alpha$, vt sit verticalis $B=n\alpha$, erit $(n+2)\alpha=\pi$, ideoque $\alpha=\frac{\pi}{n+2}$ et $B=\frac{n\pi}{n+2}$. In formulis ergo supra inuentis poni debet $c=a$, ita vt iam scala relationis sit $bb, -aabb$. Quare casus iterum binos distinguendo, prout n fuerit numerus par vel impar, habebimus:

Pro casu $n=2i$.

si	has aequationes
$i=0$	$1=0$
$i=1$	$B=bb-2aa=0$
$i=2$	$D=b^4-3aabb=0$
$i=3$	$F=b^6-4aab^4+2a^4bb=0$
$i=4$	$H=b^8-5aab^6+5a^4b^4=0$
$i=5$	$K=b^{10}-6aab^8+9a^4b^6-2a^6b^4=0$
$i=6$	$M=b^{12}-7aab^{10}+14a^4b^8-7a^6b^6=0$
$i=7$	$O=b^{14}-8aab^{12}+20a^4b^{10}-16a^6b^8+2a^8b^6=0$
$i=8$	$Q=b^{16}-9aab^{14}+27a^4b^{12}-30a^6b^{10}+9a^8b^8=0$

etc.

quae ad has formas ſimpliciores reducuntur:

$i=1$	$bb-2aa=0$
$i=2$	$bb-3aa=0$
$i=3$	$b^4-4aabb+2a^4=0$
$i=4$	$b^4-5aabb+5a^4=0$
$i=5$	$b^6-6aab^4+9a^4b^2-2a^6=0$
$i=6$	$b^6-7aab^4+14a^4b^2-7a^6=0$
$i=7$	$b^8-8aab^6+20a^4b^4-16a^6bb+2a^8=0$
$i=8$	$b^8-9aab^6+27a^4b^4-30a^6bb+9a^8=0$

etc.

Hic ergo iterum duos caſus diſcerni conuenit, prout numerus i ſit par vel impar.

Si ſit $i=2\lambda-1$ et $n=4\lambda-2$ erit aequatio:

$$0=b^{2\lambda}-2\lambda aab^{2\lambda-2}+\frac{2\lambda(2\lambda-3)}{1\cdot 2}a^4b^{2\lambda-4}-\frac{2\lambda(2\lambda-4)(2\lambda-5)}{1\cdot 2\cdot 3}a^6b^{2\lambda-6}$$
$$+\frac{2\lambda(2\lambda-5)(2\lambda-6)(2\lambda-7)}{1\cdot 2\cdot 3\cdot 4}a^8b^{2\lambda-8}-\text{etc.}$$

et ordine inuerſo ita ſe habebit:

$$0=a^{2\lambda}-\frac{\lambda\lambda}{1\cdot 2}a^{2\lambda-2}b^2+\frac{\lambda\lambda(\lambda\lambda-1)}{1\cdot 2\cdot 3\cdot 4}a^{2\lambda-4}b^4-\frac{\lambda\lambda(\lambda\lambda-1)(\lambda\lambda-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}a^{2\lambda-6}b^6+\text{etc.}$$

Sin autem ſit $i=2\lambda$, et $n=4\lambda$, erit aequatio:

$$0=b^{2\lambda}-(2\lambda+1)aab^{2\lambda-2}+\frac{(2\lambda+1)(2\lambda-2)}{1\cdot 2}a^4b^{2\lambda-4}-\frac{(2\lambda+1)(2\lambda-3)(2\lambda-4)}{1\cdot 2\cdot 3}a^6b^{2\lambda-6}$$
$$+\frac{(2\lambda+1)(2\lambda-4)(2\lambda-5)(2\lambda-6)}{1\cdot 2\cdot 3\cdot 4}a^8b^{2\lambda+8}-\text{etc.}$$

quae ordine inuerſo ita ſe habebit:

$$0=(2\lambda+1)a^{2\lambda}-\frac{(2\lambda+1)\lambda(\lambda+1)}{1\cdot 2\cdot 3}a^{2\lambda-2}b^2+\frac{(2\lambda+1)\lambda(\lambda\lambda-1)(\lambda+2)}{1\cdot 2\cdot 3\cdot 4\cdot 5}a^{2\lambda-4}b^4$$
$$-\frac{(2\lambda+1)\lambda(\lambda\lambda-1)(\lambda\lambda-4)(\lambda+3)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}a^{2\lambda-6}b^6\ \text{etc.}$$

ſeu

ſeu per $2\lambda+1$ diuidendo hoc modo:

$$0=a^{2\lambda}-\frac{\lambda(\lambda+1)}{2\cdot 3}a^{2\lambda-2}b^2+\frac{\lambda(\lambda\lambda-1)(\lambda+2)}{2\cdot 3\cdot 4\cdot 5}a^{2\lambda-4}b^4-\frac{\lambda(\lambda\lambda-1)(\lambda\lambda-4)(\lambda+3)}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}a^{2\lambda-6}b^6$$
$$+\frac{\lambda(\lambda\lambda-1)(\lambda\lambda-4)(\lambda\lambda-9)(\lambda+4)}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}a^{2\lambda-8}b^8-\text{etc.}$$

Nunc igitur ad alterum caſum progrediamur.

Pro caſu $n=2i+1$.

ſi	erit aequatio
$i=0$	$b-a=0$
$i=1$	$b^3-abb-aab=0$
$i=2$	$b^5-ab^4-2aab^3+a^3bb=0$
$i=3$	$b^7-ab^6-3aab^5+2a^3b^4+a^4b^3=0$
$i=4$	$b^9-ab^8-4aab^7+3a^3b^6+3a^4b^5-a^5b^4=0$
$i=5$	$b^{11}-ab^{10}-5aab^9+4a^3b^8+6a^4b^7-3a^5b^6-a^6b^5=0$

quae reducuntur ad has formas ſimpliciores:

$i=0$	$b-a=0$
$i=1$	$bb-ab-aa=0$
$i=2$	$b^3-abb-2aab+a^3=0$
$i=3$	$b^4-ab^3-3aabb+2a^3b+a^4=0$
$i=4$	$b^5-ab^4-4aab^3+3a^3bb+3a^4b-a^5=0$
$i=5$	$b^6-ab^5-5aab^4+4a^3b^3+6a^4bb-3a^5b-a^6=0$

vnde in genere concluditur fore:

$$0=\begin{cases}+b^{i+1}-ia^2b^{i-1}+\frac{(i-1)(i-2)}{1\cdot 2}a^4b^{i-3}-\frac{(i-2)(i-3)(i-4)}{1\cdot 2\cdot 3}a^6b^{i-5}\\ -ab^i+(i-1)a^3b^{i-2}-\frac{(i-2)(i-3)}{1\cdot 2}a^5b^{i-4}+\frac{(i-3)(i-4)(i-5)}{1\cdot 2\cdot 3}a^7b^{i-6}\text{etc.}\end{cases}$$

Inuerſe autem duos caſus contemplari conuenit:

I. Si $i = 2(\lambda - 1)$, et $n = 4\lambda - 3$, erit

$$0 = a^{2\lambda-1} - \tfrac{\lambda}{1} a^{2\lambda-2} b^2 - \frac{\lambda(\lambda-1)}{1 \cdot 2} a^{2\lambda-3} b^2 + \frac{\lambda(\lambda\lambda-1)}{1 \cdot 2 \cdot 3} a^{2\lambda-4} b^3 + \frac{\lambda(\lambda\lambda-1)(\lambda-2)}{1 \cdot 2 \cdot 3 \cdot 4} a^{2\lambda-5} b^4 \text{ etc.}$$

II. Si $i = 2\lambda - 1$, et $n = 4i - 1$, erit

$$0 = a^{2\lambda} + \tfrac{\lambda}{1} a^{2\lambda-1} b - \frac{\lambda(\lambda+1)}{1 \cdot 2} a^{2\lambda-2} b^2 - \frac{\lambda(\lambda\lambda-1)}{1 \cdot 2 \cdot 3} a^{2\lambda-4} b^4 + \frac{\lambda(\lambda\lambda-1)(\lambda+2)}{1 \cdot 2 \cdot 3 \cdot 4} a^{2\lambda-6} b^6 \text{ etc.}$$

ſicque pro omnibus caſibus, quibus n eſt numerus integer, aequationes inter latera a et b eruimus.

SOLV-