1766

Observationes circa integralia formularum $\int x^{p-1} dx (1-x^n)^{q/n-1}$ posito post integrationem $x=1$

Leonhard Euler

OBSERVATIONES

Circa integralia formularum

$\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$

Posito post integrationem $x=1$

Auctore

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I. FORMULAM integralem $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$, seu hoc modo expressam $\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$, hic consideraturus, assumo exponentes n, p, & q esse numeros integros positivos, quandoquidem si tales non essent, facile ad hanc formam perduci possent. Deinde hujus formulæ integrale non in genere hic perpendere constitui, sed ejus tantum valorem, quem recipit, si post integrationem statuatur $x=1$, postquam scilicet integratio ita fuerit instituta, ut integrale evanescat posito $x=0$. Primum enim nullum est dubium, quin, hoc casu $x=1$, integrale multo simplicius exprimatur; ac præterea quoties in analysi ad hujusmodi formulas pervenitur, plerumque non tam integrale indefinitum, pro quocunque valore ipsius x, quam definitum valori $x=1$, utpote præcipuo desiderari solet.

II. Constat autem casu, quo post integrationem ponitur $x=1$, integrale $\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$, hoc modo per productum infinitorum factorum exprimi, ut sit:

$$\frac{p+q}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\cdot\frac{3n(p+q+3n)}{(p+3n)(q+3n)}\cdot\ \&c.$$

cujus quidem primus factor $\frac{p+q}{pq}$ non legi sequentium adstringitur. Hoc tamen non obstante perspicuum est exponentes p & q inter se esse commutabiles, ita ut sit:

$$\int \frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^n)^{n-q}}} = \int \frac{x^{q-1}\,dx}{\sqrt[n]{(1-x^n)^{n-p}}}$$

quae quidem aequalitas etiam facile per se ostenditur. Verum productum istud infinitum nos ad alia multo majora perducet, quibus haec integralia magis illustrabuntur.

III. Ut autem brevitati in scribendo consulam, neque opus habeam scripturam hujus formulae $\int \frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ toties repetere, pro quovis exponente n ejus loco scribam $(\frac{p}{q})$, ita ut $(\frac{p}{q})$ denotet valorem formulae integralis $\int \frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^n)^{n-q}}}$, casu quo post integrationem ponitur $x = 1$. Et quoniam vidimus esse hoc casu:

$$\int \frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^n)^{n-q}}} = \int \frac{x^{q-1}\,dx}{\sqrt[n]{(1-x^n)^{n-p}}}$$

manifestum est fore $(\frac{p}{q}) = (\frac{q}{p})$, ita ut pro quovis valore exponentis n, hae expressiones $(\frac{p}{q})$ & $(\frac{q}{p})$ eandem significent quantitatem. Ita si fuerit exempli gratia $n = 4$, erit:

$$(\frac{3}{2}) = (\frac{2}{3}) = \int \frac{x^2\,dx}{\sqrt[4]{(1-x^4)^2}} = \int \frac{x\,dx}{\sqrt[4]{(1-x^4)}}$$

Per productum autem infinitum habebitur.

$$(\frac{3}{2}) = (\frac{2}{3}) = \frac{5}{2.3} \cdot \frac{4.9}{6.7} \cdot \frac{8.13}{10.11} \cdot \frac{12.17}{14.15} \cdot \text{&c.}$$

IV. Jam primum observo, si exponentes p & q fuerint majores exponente n, formulam integralem semper ad aliam reduci posse, in qua hi exponentes infra n deprimantur. Cum enim sit:

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}} = \frac{p-n}{p+q-n}\int \frac{x^{p-n-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$$

erit, recepto hic scribendi more:

$$\left(\frac{p}{q}\right) = \frac{p-n}{p+q-n}\left(\frac{p-n}{q}\right)$$

quo si fuerit $p > n$, formula ad aliam, in qua exponens p minor sit quam n revocatur, quod etiam ob commutabilitatem de altero exponente q est tenendum. Quamobrem nobis has formulas examinaturis sufficiet pro quovis exponente n exponentes p & q ipso n minores accipere, quoniam his expeditis, omnes casus quibus majores habituri essent valores, eo reduci possunt.

V. Statim autem patet casus, quibus est vel $p = n$, vel $q = n$, absolute seu algebraice esse integrabiles. Si enim fuerit $q = n$, ob $\left(\frac{p}{n}\right) = \int x^{p-1}dx = \frac{x^p}{p}$, posito $x = 1$, erit $\left(\frac{p}{n}\right) = \frac{1}{p}$; similique modo $\left(\frac{n}{q}\right) = \frac{1}{q}$. Atque hi soli sunt casus, quibus integrale nostræ formulæ absolute exhiberi potest, si quidem p & q exponentem n non excedant. Reliquis casibus omnibus integratio vel quadraturam circuli, vel adeo altiores quadraturas implicabit; quas hic accuratius perpendere animus est. Post eas igitur formulas $\left(\frac{p}{n}\right)$, seu $\left(\frac{n}{p}\right)$, quarum valor absolute est $= \frac{1}{p}$, veniunt eæ, quarum valor per solam circuli quadraturam exprimitur; tum vero sequentur eæ, quæ altiorem quandam quadraturam postulant, atque has altiores quadraturas tam ad simplicissimam formam, quam ad minimum numerum revocare conabor.

VI. Cum numeri p & q exponente n minores ponantur, eæ formulæ $\left(\frac{p}{q}\right)$ per ſolam circuli quadraturam integrabiles exiſtunt, in quibus eſt $p + q = n$. Sit enim $q = n - p$, & formula noſtra:

$$\left(\frac{p}{n-p}\right) = \left(\frac{n-p}{p}\right) = \int \frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^n)^p}} = \int \frac{x^{q-1}\,dx}{\sqrt[n]{(1-x^n)^q}}$$

hoc producto infinito exprimetur:

$$\frac{n}{p(n-p)} \cdot \frac{n \cdot 2n}{(n+p)(2n-p)} \cdot \frac{2n \cdot 3n}{(2n+p)(3n-p)} \cdot \frac{3n \cdot 4n}{(3n+p)(4n-p)} \;\&c.$$

quod hoc modo repræſentatum:

$$\frac{1}{p} \cdot \frac{nn}{nn-pp} \cdot \frac{4nn}{4nn-pp} \cdot \frac{9nn}{9nn-pp} \;\&c.$$

congruit cum eo producto, quo *ſinus* angulorum expreſſi: Quare ſi π ſumatur ad ſemicirconferentiam circuli cujus radius ſit $= 1$, ſimulque menſuram duorum angulorum rectorum exhibeat, erit:

$$\left(\frac{p}{n-p}\right) = \left(\frac{n-p}{p}\right) = \frac{\pi}{n \sin. \frac{p\pi}{n}} = \frac{\pi}{n \sin. \frac{q\pi}{n}}.$$

VII. Ceteris caſibus, quibus neque $p = n$, neque $q = n$, neque $p + q = n$, integrale etiam neque abſolute, neque per quadraturam circuli exhiberi poteſt, ſed aliam quandam altiorem quadraturam complectitur. Neque vero ſinguli caſus diverſi peculiarem hujuſmodi quadraturam exigunt, ſed plures dantur reductiones, quibus diverſas formulas inter ſe comparare licet. Hæ autem reductiones derivantur ex producto infinito ſupra exhibito cum enim ſit:

$$\left(\frac{p}{q}\right) = \frac{p+q}{pq} \cdot \frac{n(p+q+n)}{(p+n)(q+n)} \cdot \frac{2n(p+q+2n)}{(p+2n)(q+2n)} \;\&c.$$

erit ſimili modo:

$$\left(\frac{p+q}{r}\right) = \frac{p+q+r}{(p+q)r} \cdot \frac{n(p+q+r+n)}{(p+n+n)(r+n)} \cdot \frac{2n(p+q+r+2n)}{(p+q+2n)(r+2n)} \;\&c,$$

quibus in ſe invicem ductis obtinetur:

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right) = \frac{p+q+r}{p\,q\,r}\cdot\frac{nn(p+q+r+n)}{(p+n)(q+n)(r+n)}\cdot\frac{4nn(p+q+r+2n)}{(p+2n)(q+2n)(r+2n)}\ \&c.$$

ubi ternæ quantitates p, q, r ſunt inter ſe permutabiles.

VIII. Hinc ergo permutandis his quantitatibus p, q, r conſequimur ſequentes reductiones.

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right) = \left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right) = \left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right).$$

unde ex datis aliquot formulis plures aliæ determinari poſſunt. Veluti ſi ſit $q+r=n$, ſeu $r=n-q$, ob $\left(\frac{q+r}{p}\right) = \frac{1}{p}$ & $\left(\frac{q}{r}\right) = \frac{\pi}{n\,\text{ſin.}\,\frac{q\pi}{n}}$ erit: $\left(\frac{p}{q}\right)\left(\frac{p+q}{n-q}\right)$

$$= \frac{\pi}{n\,p\,\text{ſin.}\,\frac{q\pi}{n}},\ \text{nec non}\ \left(\frac{p}{n-q}\right)\left(\frac{n+p-q}{q}\right) = \frac{\pi}{np\,\text{ſin.}\,\frac{q\pi}{n}}$$

Deinde ſi ſit $p+q+r=n$, ſeu $r=n-p-q$, erit:

$$\frac{\pi}{n\,\text{ſin.}\,\frac{r\pi}{n}}\left(\frac{p}{q}\right) = \frac{\pi}{n\,\text{ſin.}\,\frac{q\pi}{n}}\left(\frac{p}{r}\right) = \frac{\pi}{n\,\text{ſin.}\,\frac{p\pi}{n}}\left(\frac{q}{r}\right)$$

unde inſignes reductiones aliarum ad alias oriuntur, quibus multitudo quadraturarum ad noſtrum ſcopum neceſſariarum vehementer diminuitur.

IX. Præterea vero pro p, q, r numeris determinatis aſſumendis, ſequentes adipiſcimur productorum ex binis formulis æqualitates:

$$\left(\frac{1}{1}\right)\left(\frac{2}{2}\right) = \left(\frac{2}{1}\right)\left(\frac{3}{1}\right)$$

$$\left(\frac{1}{1}\right)\left(\frac{3}{2}\right) = \left(\frac{3}{1}\right)\left(\frac{4}{1}\right)$$

$$\left(\frac{2}{1}\right)\left(\frac{3}{3}\right) = \left(\frac{3}{1}\right)\left(\frac{4}{2}\right) = \left(\frac{3}{2}\right)\left(\frac{5}{1}\right)$$

$$\left(\frac{2}{2}\right)\left(\frac{4}{3}\right) = \left(\frac{3}{2}\right)\left(\frac{5}{2}\right)$$

$\left(\frac{3}{1}\right)$

$$\left(\frac{3}{1}\right)\left(\frac{4}{3}\right) = \left(\frac{3}{3}\right)\left(\frac{6}{1}\right)$$
$$\left(\frac{3}{2}\right)\left(\frac{5}{3}\right) = \left(\frac{3}{3}\right)\left(\frac{6}{2}\right)$$
$$\left(\frac{2}{2}\right)\left(\frac{4}{4}\right) = \left(\frac{4}{2}\right)\left(\frac{6}{2}\right)$$
$$\left(\frac{3}{1}\right)\left(\frac{4}{4}\right) = \left(\frac{4}{1}\right)\left(\frac{5}{3}\right) = \left(\frac{4}{3}\right)\left(\frac{7}{1}\right)$$
$$\left(\frac{2}{1}\right)\left(\frac{5}{3}\right) = \left(\frac{5}{1}\right)\left(\frac{6}{2}\right) = \left(\frac{5}{2}\right)\left(\frac{7}{1}\right)$$
$$\left(\frac{1}{1}\right)\left(\frac{6}{2}\right) = \left(\frac{6}{1}\right)\left(\frac{7}{1}\right)$$
&c.

ubi quidem plures occurrunt, quæ jam in reliquis continentur.

X. His quasi principiis præmissis formulam generalem $\int \frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^n)^{n-q}}}$, in qua numeros p & q exponentem n non superare assumo, in classes ex exponente n petitas distinguam, ita ut valores $n = 1$, $n = 2$, $n = 3$, $n = 4$ &c. classes primam, secundam, tertiam &c. sint præbituri.

Ac prima quidem classis, qua $n = 1$, unicam formulam complectitur $\left(\frac{1}{1}\right)$, cujus valor est $= 1$. Secunda classis, qua $n = 2$, has formulas $\left(\frac{1}{1}\right)$, $\left(\frac{2}{1}\right)$ & $\left(\frac{2}{2}\right)$ continet, quarum evolutio per se est manifesta. Tertia classis, qua $n = 3$ has habet:

$$\left(\frac{1}{1}\right), \left(\frac{2}{1}\right), \left(\frac{3}{1}\right), \left(\frac{2}{2}\right), \left(\frac{3}{2}\right), \left(\frac{3}{3}\right).$$

Quarta vero classis, qua $n = 4$, istas:

$$\left(\frac{1}{1}\right), \left(\frac{2}{1}\right), \left(\frac{3}{1}\right), \left(\frac{4}{1}\right), \left(\frac{2}{2}\right), \left(\frac{3}{2}\right), \left(\frac{4}{2}\right), \left(\frac{3}{3}\right), \left(\frac{4}{3}\right), \left(\frac{4}{4}\right);$$

sicque in sequentibus classibus formularum numerus secundum numeros trigonales crescit. Has igitur classes ordine percurramus.

$$Classis\ 2^{da}\ formæ \int \frac{x^{p-1}\,dx}{\sqrt[2]{(1-x^2)^{2-q}}} = \left(\frac{p}{q}\right)$$

Perspicuum hic quidem est istas formulas vel absolute, vel per quadraturam circuli exprimi: nam hæ $\left(\frac{2}{1}\right)$ & $\left(\frac{2}{2}\right)$ absolute dantur, & reliqua $\left(\frac{1}{1}\right)$ ob $1+1=2$ est $\frac{\pi}{2\,\text{sin.}\,\frac{\pi}{2}} = \frac{\pi}{2}$; si ergo brevitatis causa ponamus $\frac{\pi}{2} = \alpha$, uti scilicet in sequentibus classibus faciemus, omnes formulæ hujus classis ita definiuntur:

$$\left(\frac{2}{1}\right) = 1,\quad \left(\frac{2}{2}\right) = \frac{1}{2}$$

$$\left(\frac{1}{1}\right) = \alpha.$$

$$Classis\ 3^a\ formæ \int \frac{x^{p-1}\,dx}{\sqrt[3]{(1-x^3)^{3-q}}} = \left(\frac{p}{q}\right)$$

Cum hic sit $n = 3$, formula quadraturam circuli involvens est $\left(\frac{2}{1}\right) = \frac{\pi}{3\,\text{sin.}\,\frac{\pi}{3}}$, ponamus ergo $\left(\frac{2}{1}\right) = \alpha$ reliquæ autem formulæ, quæ non absolute dantur, altiorem quadraturam involvunt, & quidem unicam $\left(\frac{1}{1}\right)$, quam litera A indicemus, qua concessa valores omnium formularum hujus classis assignari poterimus:

$$\left(\frac{3}{1}\right) = 1, \quad \left(\frac{3}{2}\right) = \frac{1}{2}, \quad \left(\frac{3}{3}\right) = \frac{1}{3}$$

$$\left(\frac{2}{1}\right) = \alpha, \quad \left(\frac{2}{2}\right) = \frac{\alpha}{A}$$

$$\left(\frac{1}{1}\right) = A.$$

Classis 4^{ta} *formæ* $\int \frac{x^{p-1}dx}{\sqrt[4]{(1-x^4)^{4-q}}} = \left(\frac{p}{q}\right)$

Cum hic sit $n = 4$, duas habemus formulas a quadratura circuli pendentes, quarum valores, quia sunt cogniti, ita indicemus

$$\left(\frac{3}{1}\right) = \frac{\pi}{4 \text{ sin. } \frac{\pi}{4}} = \alpha \quad \& \quad \left(\frac{2}{2}\right) = \frac{\pi}{4 \text{ sin. } \frac{2\pi}{4}} = \beta.$$

Præterea vero unica opus est formula altiorem quadraturam involvente, qua concessa reliquas omnes cognoscemus. Ponamus enim $\left(\frac{2}{1}\right) = A$, & omnes formulæ hujus classis ita determinabuntur:

$$\left(\frac{4}{1}\right) = 1, \left(\frac{4}{2}\right) = \frac{1}{2}, \left(\frac{4}{3}\right) = \frac{1}{3}, \left(\frac{4}{4}\right) = \frac{1}{4}$$

$$\left(\frac{3}{1}\right) = \alpha, \left(\frac{3}{2}\right) = \frac{\beta}{A}, \left(\frac{3}{3}\right) = \frac{\alpha}{2A}$$

$$\left(\frac{2}{1}\right) = A, \left(\frac{2}{2}\right) = \beta$$

$$\left(\frac{1}{1}\right) = \frac{\alpha A}{\beta}.$$

Classis $5^{tæ}$ *formæ* $\int \frac{x^{p-1}\,dx}{\sqrt[5]{(1-x^5)^{5-q}}} = \left(\frac{p}{q}\right)$.

Cum hic sit $n = 5$, notemus statim formulas a quadratura circuli pendentes:

$$\left(\frac{4}{1}\right) = \frac{\pi}{5 \text{ sin. } \frac{\pi}{5}} = \alpha,\ \left(\frac{3}{2}\right) = \frac{\pi}{5 \text{ sin. } \frac{2\pi}{5}} = \beta$$

Duabus autem insuper novis quadraturis opus est huic classi peculiaribus, quas ita designemus:

$$\left(\frac{3}{1}\right) = A,\ \&\ \left(\frac{2}{2}\right) = B$$

ex quibus reliquæ omnes ita definientur:

$$\left(\frac{5}{1}\right) = 1,\ \left(\frac{5}{2}\right) = \frac{1}{2},\ \left(\frac{5}{3}\right) = \frac{1}{3},\ \left(\frac{5}{4}\right) = \frac{1}{4},\ \left(\frac{5}{5}\right) = \frac{1}{5}$$

$$\left(\frac{4}{1}\right) = \alpha,\ \left(\frac{4}{2}\right) = \frac{\beta}{A},\ \left(\frac{4}{3}\right) = \frac{\beta}{2B},\ \left(\frac{4}{4}\right) = \frac{\alpha}{3A},$$

$$\left(\frac{3}{1}\right) = A,\ \left(\frac{3}{2}\right) = \beta,\ \left(\frac{3}{3}\right) = \frac{\beta\beta}{\alpha B},$$

$$\left(\frac{2}{1}\right) = \frac{\alpha B}{\beta},\ \left(\frac{2}{2}\right) = B,$$

$$\left(\frac{1}{1}\right) = \frac{\alpha A}{\beta}.$$

Classis $6^{tæ}$ *formæ* $\int \frac{x^{p-1}\,dx}{\sqrt[6]{(1-x^6)^{6-q}}} = \left(\frac{p}{q}\right)$.

Hic est $n = 6$, & formulæ quadraturam circuli involventes sunt:

$$\left(\frac{5}{1}\right) = \frac{\pi}{6 \text{ sin. } \frac{\pi}{6}} = \alpha,\ \left(\frac{4}{2}\right) = \frac{\pi}{6 \text{ sin. } \frac{2\pi}{6}} = \beta,\ \left(\frac{3}{3}\right) = \frac{\pi}{6 \text{ sin. } \frac{3\pi}{6}} = \gamma.$$

Reliquarum vero omnium valores insuper a binis hisce quadraturis pendent:

$$\left(\frac{4}{1}\right) = A \ \& \ \left(\frac{3}{2}\right) = B,$$

atque ita se habere deprehenduntur:

$\left(\frac{6}{1}\right) = 1, \left(\frac{6}{2}\right) = \frac{1}{2}, \left(\frac{6}{3}\right) = \frac{1}{3}, \left(\frac{6}{4}\right) = \frac{1}{4}, \left(\frac{6}{5}\right) = \frac{1}{5}, \left(\frac{6}{6}\right) = \frac{1}{6}$

$\left(\frac{5}{1}\right) = \alpha, \left(\frac{5}{2}\right) = \frac{\beta}{A}, \left(\frac{5}{3}\right) = \frac{\gamma}{2B}, \left(\frac{5}{4}\right) = \frac{\beta}{3B}, \left(\frac{5}{5}\right) = \frac{\alpha}{4A},$

$\left(\frac{4}{1}\right) = A, \left(\frac{4}{2}\right) = \beta, \left(\frac{4}{3}\right) = \frac{\beta\gamma}{\alpha B}, \left(\frac{4}{4}\right) = \frac{\beta\gamma A}{2\alpha BB},$

$\left(\frac{3}{1}\right) = \frac{\alpha B}{\beta}, \left(\frac{3}{2}\right) = B, \left(\frac{3}{3}\right) = \gamma.$

$\left(\frac{2}{1}\right) = \frac{\alpha B}{\gamma}, \left(\frac{2}{2}\right) = \frac{\alpha BB}{\gamma A},$

$\left(\frac{1}{1}\right) = \frac{\alpha A}{\beta}.$

$$\textit{Classis } 7^{mae} \textit{ formae } \int \frac{x^{p-1}\,dx}{\sqrt[7]{(1-x^7)^{7-q}}} = \left(\frac{p}{q}\right).$$

Quia $n = 7$, formulæ a quadratura circuli pendentes ita designentur:

$$\left(\frac{6}{1}\right) = \frac{\pi}{7 \sin. \frac{\pi}{7}} = \alpha, \left(\frac{5}{2}\right) = \frac{\pi}{7 \sin. \frac{2\pi}{7}} = \beta, \left(\frac{4}{3}\right) = \frac{\pi}{7 \sin. \frac{3\pi}{7}} = \gamma$$

præterea vero hæ quadraturæ introducantur:

$$\left(\frac{5}{1}\right) = A, \left(\frac{4}{2}\right) = B, \left(\frac{3}{3}\right) = C$$

quibus datis omnes formulæ ita determinabuntur:

$\left(\frac{7}{1}\right) = 1, \left(\frac{7}{2}\right) = \frac{1}{2}, \left(\frac{7}{3}\right) = \frac{1}{3}, \left(\frac{7}{4}\right) = \frac{1}{4}, \left(\frac{7}{5}\right) = \frac{1}{5}, \left(\frac{7}{6}\right) = \frac{1}{6}, \left(\frac{7}{7}\right) = \frac{1}{7},$

$\left(\frac{6}{1}\right) = \alpha, \left(\frac{6}{2}\right) = \frac{\beta}{A}, \left(\frac{6}{3}\right) = \frac{\gamma}{2B}, \left(\frac{6}{4}\right) = \frac{\gamma}{3C}, \left(\frac{6}{5}\right) = \frac{\beta}{4B}, \left(\frac{6}{6}\right) = \frac{\alpha}{5A},$

$\left(\frac{5}{1}\right) = A, \left(\frac{5}{2}\right) = \beta, \left(\frac{5}{3}\right) = \frac{\beta\gamma}{\alpha B}, \left(\frac{5}{4}\right) = \frac{\gamma\gamma A}{2\alpha BC}, \left(\frac{5}{5}\right) = \frac{\beta\gamma A}{3\alpha BC},$

$$(\frac{4}{1}) = \frac{\alpha B}{\beta},\ (\frac{4}{2}) = B,\ (\frac{4}{3}) = \gamma,\ (\frac{4}{4}) = \frac{\gamma\gamma}{\alpha C};$$

$$(\frac{3}{1}) = \frac{\alpha C}{\gamma},\ (\frac{3}{2}) = \frac{\alpha BC}{\gamma A},\ (\frac{3}{3}) = C;$$

$$(\frac{2}{1}) = \frac{\alpha B}{\gamma},\ (\frac{2}{2}) = \frac{\alpha\beta BC}{\gamma\gamma A};$$

$$(\frac{1}{1}) = \frac{\alpha A}{\beta}.$$

Classis 8^{vae} *formæ* $\int \frac{x^{p-1}\,dx}{\sqrt[8]{(1-x^8)^{8-q}}} = (\frac{p}{q})$

Quia hic eſt $n = 8$, formula quadraturam circuli implicantes erunt:

$$(\frac{7}{1}) = \frac{\pi}{8 \text{ ſin. } \frac{\pi}{8}} = \alpha,\ (\frac{6}{2}) = \frac{\pi}{8 \text{ ſin. } \frac{2\pi}{8}} = \beta,$$

$$(\frac{5}{3}) = \frac{\pi}{8 \text{ ſin. } \frac{3\pi}{8}} = \gamma,\ (\frac{4}{4}) = \frac{\pi}{8 \text{ ſin. } \frac{4\pi}{8}} = \delta$$

Nunc vero tres frequentes formulæ tanquam cognitæ ſpectentur:

$$(\frac{6}{1}) = A,\ (\frac{5}{2}) = B,\ \&\ (\frac{4}{3}) = C$$

atque ex his omnes formulæ hujus claſſis ita determinabuntur.

$$(\frac{8}{1}) = 1,\ (\frac{8}{2}) = \frac{1}{2},\ (\frac{8}{3}) = \frac{1}{3},\ (\frac{8}{4}) = \frac{1}{4},\ (\frac{8}{5}) = \frac{1}{5},$$
$$(\frac{8}{6}) = \frac{1}{6},\ (\frac{8}{7}) = \frac{1}{7},\ (\frac{8}{8}) = \frac{1}{8};$$

$$(\frac{7}{1}) = \alpha,\ (\frac{7}{2}) = \frac{\beta}{A},\ (\frac{7}{3}) = \frac{\gamma}{2B},\ (\frac{7}{4}) = \frac{\delta}{3C},\ (\frac{7}{5}) = \frac{\gamma}{4C},$$
$$(\frac{7}{6}) = \frac{\beta}{5B},\ (\frac{7}{7}) = \frac{\alpha}{6A};$$

$(\frac{6}{1}) = A$, $(\frac{6}{2}) = \beta$, $(\frac{6}{3}) = \frac{\beta\gamma}{\alpha B}$, $(\frac{6}{4}) = \frac{\gamma\delta A}{2\alpha BC}$, $(\frac{6}{5}) = \frac{\gamma\delta A}{3\gamma CC}$,

$(\frac{6}{6}) = \frac{\beta\gamma A}{4\alpha BC}$;

$(\frac{5}{1}) = \frac{\alpha B}{\beta}$, $(\frac{5}{2}) = B$, $(\frac{5}{3}) = \gamma$, $(\frac{5}{4}) = \frac{\gamma\delta}{\alpha C}$, $(\frac{5}{5}) = \frac{\gamma\gamma\delta A}{2\alpha\beta CC}$;

$(\frac{4}{1}) = \frac{\alpha C}{\beta}$, $(\frac{4}{2}) = \frac{\alpha BC}{\gamma A}$, $(\frac{4}{3}) = C$, $(\frac{4}{4}) = \delta$;

$(\frac{3}{1}) = \frac{\alpha C}{\delta}$, $(\frac{3}{2}) = \frac{\alpha\beta CC}{\gamma\delta A}$, $(\frac{3}{3}) = \frac{\alpha CC}{\delta A}$;

$(\frac{2}{1}) = \frac{\alpha B}{\gamma}$, $(\frac{2}{2}) = \frac{\alpha\beta BC}{\gamma\delta A}$;

$(\frac{1}{1}) = \frac{\alpha A}{\beta}$;

Hinc iſtas reductiones ad ſequentes claſſes, quouſque libuerit, continuare licet. Quemadmodum ergo hinc in genere ſingularum formularum integralia ſe ſint habitura exponamus.

Evolutio formæ generalis $\int \frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^n)^{n-q}}} = (\frac{p}{q})$

Primo ergo abſolute integrabiles ſunt hæ formulæ:

$(\frac{n}{1}) = 1$, $(\frac{n}{2}) = \frac{1}{2}$, $(\frac{n}{3}) = \frac{1}{3}$, $(\frac{n}{4}) = \frac{1}{4}$, &c.

deinde formulæ a quadratura circuli pendentes ſunt:

$(\frac{n-1}{1}) = \alpha$, $(\frac{n-2}{2}) = \beta$, $(\frac{n-3}{3}) = \gamma$, $(\frac{n-4}{4}) = \delta$ &c.

quarum quantitatum progreſſio tandem in ſe revertitur cum ſit etiam:

$(\frac{4}{n-4}) = \delta$, $(\frac{3}{n-3}) = \gamma$, $(\frac{2}{n-2}) = \beta$, $(\frac{1}{n-1}) = \alpha$.

Præterea vero altiores quadraturæ in ſubſidium vocari debent, quæ ita repræſententur:

$$\left(\frac{n-2}{1}\right) = A,\ \left(\frac{n-3}{2}\right) = B,\ \left(\frac{n-4}{3}\right) = C,\ \left(\frac{n-5}{4}\right) = D \text{ \&c.}$$

quarum numerum quovis caſu ſponte determinatur, quia hæ formulæ tandem in ſe revertuntur.

His autem formulis admiſſis omnes omnino ad eandem claſſem pertinentes definiri poterunt. Habebimus autem a formula $\left(\frac{n-1}{1}\right) = \alpha$, uti ſupra iſtas formulas ordinavimus, deorſum deſcendendo:

$$\left(\frac{n-1}{1}\right) = \alpha,\ \left(\frac{n-2}{1}\right) = A,\ \left(\frac{n-3}{1}\right) = \frac{\alpha B}{\beta},\ \left(\frac{n-4}{1}\right) = \frac{\alpha C}{\gamma},$$

$$\left(\frac{n-5}{1}\right) = \frac{\alpha D}{\delta},\ \left(\frac{n-6}{1}\right) = \frac{\alpha E}{\varepsilon} \text{ \&c.}$$

qui valores retro ſumti ita ſe habent:

$$\left(\frac{1}{1}\right) = \frac{\alpha A}{\beta},\ \left(\frac{2}{1}\right) = \frac{\alpha B}{\gamma},\ \left(\frac{3}{1}\right) = \frac{\alpha C}{\delta},\ \text{\&c.}$$

Tum vero ab eadem formula $\left(\frac{n-1}{1}\right) = \alpha$ horizontaliter progrediendo definiuntur iſtæ formulæ:

$$\left(\frac{n-1}{1}\right) = \alpha,\ \left(\frac{n-1}{2}\right) = \frac{\beta}{A},\ \left(\frac{n-1}{3}\right) = \frac{\gamma}{2B},\ \left(\frac{n-1}{4}\right) = \frac{\delta}{3C} \text{ \&c.}$$

quarum ultima erit $\left(\frac{n-1}{n-1}\right) = \frac{\alpha}{(n-2)A}$, penultima $\left(\frac{n-1}{n-2}\right) = \frac{\beta}{(n-3)B}$, antepenultima $\left(\frac{n-1}{n-3}\right) = \frac{\gamma}{(n-4)C}$ &c.

Simili modo a formula $\left(\frac{n-2}{2}\right) = \beta$ tam deſcendendo, quam progrediendo horizontaliter valores aliarum impetrabimus, ac deſcendendo quidem:

$$\left(\frac{n-2}{2}\right) = \beta,\ \left(\frac{n-3}{2}\right) = B,\ \left(\frac{n-4}{2}\right) = \frac{\alpha BC}{\gamma A},\ \left(\frac{n-5}{2}\right) = \frac{\alpha\beta CD}{\gamma\delta A},$$

$$\left(\frac{n-6}{2}\right) = \frac{\alpha\beta DE}{\delta\varepsilon A},\ \left(\frac{n-7}{2}\right) = \frac{\alpha\beta EF}{\varepsilon\zeta A} \text{ \&c.}$$

ubi

ubi erit ultima $(\frac{2}{2}) = \frac{\alpha\beta BC}{\gamma\delta A}$, penultima $(\frac{3}{2}) = \frac{\alpha\beta CD}{\delta\epsilon A}$ &c.; at horizontaliter progrediendo:

$$(\frac{n-2}{2}) = \beta,\ (\frac{n-2}{3}) = \frac{\beta\gamma}{\alpha B},\ (\frac{n-2}{4}) = \frac{\gamma\delta A}{2\alpha BC},\ (\frac{n-2}{5}) = \frac{\delta\epsilon A}{3\alpha CD},\ (\frac{n-2}{6}) = \frac{\epsilon\zeta A}{4\alpha DE},\ (\frac{n-2}{7}) = \frac{\zeta\eta A}{5\alpha EF} \text{ \&c.}$$

quarum erit ultima $(\frac{n-2}{n-2}) = \frac{\beta\gamma A}{(n-4)\alpha BC}$, penultima $(\frac{n-2}{n-3}) = \frac{\gamma\delta A}{(n-5)\alpha CD}$ &c.

Porro a formula $(\frac{n-3}{n-3}) = \gamma$ descendendo pervenimus ad has formulas:

$$(\frac{n-3}{3}) = \gamma,\ (\frac{n-4}{3}) = C,\ (\frac{n-5}{3}) = \frac{\alpha CD}{\delta A},\ (\frac{n-6}{3}) = \frac{\alpha\beta CDE}{\delta\epsilon AB},$$
$$(\frac{n-7}{3}) = \frac{\alpha\beta\gamma DEF}{\delta\epsilon\zeta AB},\ (\frac{n-8}{3}) = \frac{\alpha\beta\gamma EFG}{\epsilon\zeta\eta AB} \text{ \&c.}$$

& horizontaliter progrediendo:

$$(\frac{n-3}{3}) = \gamma,\ (\frac{n-3}{4}) = \frac{\gamma\delta}{\alpha C},\ (\frac{n-3}{5}) = \frac{\gamma\delta\epsilon A}{2\alpha\beta CD},\ (\frac{n-3}{6}) = \frac{\delta\epsilon\zeta AB}{3\alpha\beta CDE},$$
$$(\frac{n-3}{7}) = \frac{\epsilon\zeta\eta AB}{4\alpha\beta DEF},\ (\frac{n-3}{8}) = \frac{\zeta\eta\theta AB}{5\alpha\beta EFG} \text{ \&c.}$$

Pari modo a formula $(\frac{n-4}{4}) = \delta$ descendendo nanciscimur:

$$(\frac{n-4}{4}) = \delta,\ (\frac{n-5}{4}) = D,\ (\frac{n-6}{4}) = \frac{\alpha DE}{\epsilon A},\ (\frac{n-7}{4}) = \frac{\alpha\beta DEF}{\epsilon\zeta AB},$$
$$(\frac{n-8}{4}) = \frac{\alpha\beta\gamma DEFG}{\epsilon\zeta\eta ABC},\ (\frac{n-9}{4}) = \frac{\alpha\beta\gamma\delta EFGH}{\epsilon\zeta\eta\theta ABC} \text{ \&c.}$$

& horizontaliter progrediendo:

$$(\frac{n-4}{4}) = \delta,\ (\frac{n-4}{5}) = \frac{\delta\epsilon}{\alpha D},\ (\frac{n-4}{6}) = \frac{\delta\epsilon\zeta A}{2\alpha\beta DE},\ (\frac{n-4}{7}) = \frac{\delta\epsilon\zeta\eta AB}{3\alpha\beta\gamma DEF},$$

$$\left(\frac{n-4}{8}\right) = \frac{\varepsilon\zeta\eta\theta}{4\alpha\beta\gamma}\frac{ABC}{DEFG}, \left(\frac{n-4}{9}\right) = \frac{\zeta\eta\theta\iota}{5\alpha\beta\gamma}\frac{ABC}{EFGH} \text{ \&c.}$$

Atque hac ratione tandem omnium formularum valores reperiuntur.

Accommodemus has generales reductiones ad

$$\textit{Classem } 9^{nae} \textit{ formulæ } \int \frac{x^{p-1}dx}{\sqrt[9]{(1-x^9)^{9-q}}} = \left(\frac{p}{q}\right)$$

Ubi ob $n = 9$ formulæ quadraturam circuli involventes erunt:

$$\left(\frac{8}{1}\right) = \alpha, \left(\frac{7}{2}\right) = \beta, \left(\frac{6}{3}\right) = \gamma, \left(\frac{5}{4}\right) = \delta;$$

hinc $\varepsilon = \delta$, $\zeta = \gamma$, $\eta = \beta$, $\theta = \alpha$.

Deinde novæ quadraturæ huc requisitæ ponantur:

$$\left(\frac{7}{1}\right) = A, \left(\frac{6}{2}\right) = B, \left(\frac{5}{3}\right) = C, \left(\frac{4}{4}\right) = D;$$

sicque erit $E = C$, $F = B$, & $G = A$; atque his quatuor valoribus concessis omnium formularum nonæ classis valores assignari poterunt, quos simili ordine, ut hactenus fecimus, repræsentemus.

$$\left(\frac{9}{1}\right) = 1, \left(\frac{9}{2}\right) = \frac{1}{2}, \left(\frac{9}{3}\right) = \frac{1}{3}, \left(\frac{9}{4}\right) = \frac{1}{4}, \left(\frac{9}{5}\right) = \frac{1}{5},$$

$$\left(\frac{9}{6}\right) = \frac{1}{6}, \left(\frac{9}{7}\right) = \frac{1}{7}, \left(\frac{9}{8}\right) = \frac{1}{8}, \left(\frac{9}{9}\right) = \frac{1}{9};$$

$$\left(\frac{8}{1}\right) = \alpha, \left(\frac{8}{2}\right) = \frac{\beta}{A}, \left(\frac{8}{3}\right) = \frac{\gamma}{2B}, \left(\frac{8}{4}\right) = \frac{\delta}{3C}, \left(\frac{8}{5}\right) = \frac{\delta}{4D},$$

$$\left(\frac{8}{6}\right) = \frac{\gamma}{5C}, \left(\frac{8}{7}\right) = \frac{\beta}{6B}, \left(\frac{8}{8}\right) = \frac{\alpha}{7A};$$

$$\left(\frac{7}{1}\right) = A, \left(\frac{7}{2}\right) = \beta, \left(\frac{7}{3}\right) = \frac{\beta\gamma}{\alpha B}, \left(\frac{7}{4}\right) = \frac{\gamma\delta A}{2\alpha BC}, \left(\frac{7}{5}\right) = \frac{\delta\delta A}{3\alpha CD},$$

$$\left(\frac{7}{6}\right) = \frac{\gamma\delta A}{4\alpha CD}, \left(\frac{7}{7}\right) = \frac{\beta\gamma\alpha}{5\alpha BC};$$

$$\left(\frac{6}{1}\right) = \frac{\alpha B}{\beta},\ \left(\frac{6}{2}\right) = B,\ \left(\frac{6}{3}\right) = \gamma,\ \left(\frac{6}{4}\right) = \frac{\gamma\delta}{\alpha C},\ \left(\frac{6}{5}\right) = \frac{\gamma\delta\delta A}{2\alpha\beta CD},$$
$$\left(\frac{6}{6}\right) = \frac{\gamma\delta\delta AB}{3\alpha\beta CCD};$$

$$\left(\frac{5}{1}\right) = \frac{\alpha C}{\gamma},\ \left(\frac{5}{2}\right) = \frac{\alpha BC}{\gamma A},\ \left(\frac{5}{3}\right) = C,\ \left(\frac{5}{4}\right) = \delta,\ \left(\frac{5}{5}\right) = \frac{\delta\delta}{\alpha D};$$

$$\left(\frac{4}{1}\right) = \frac{\alpha D}{\delta},\ \left(\frac{4}{2}\right) = \frac{\alpha\beta CD}{\gamma\delta A},\ \left(\frac{4}{3}\right) = \frac{\alpha CD}{\delta A},\ \left(\frac{4}{4}\right) = D;$$

$$\left(\frac{3}{1}\right) = \frac{\alpha C}{\delta},\ \left(\frac{3}{2}\right) = \frac{\alpha\beta CD}{\delta\delta A},\ \left(\frac{3}{3}\right) = \frac{\alpha\beta CCD}{\delta\delta AB};$$

$$\left(\frac{2}{1}\right) = \frac{\alpha B}{\gamma},\ \left(\frac{2}{2}\right) = \frac{\alpha\beta BC}{\gamma\delta A};$$

$$\left(\frac{1}{1}\right) = \frac{\alpha A}{\beta}.$$

Ordo harum formularum etiam in genere diagonaliter a finiſtra ad dextram procedendo notari meretur, ubi quidem duo genera progreſſionum occurrunt, prout vel a prima ſerie verticali, vel a ſuprema horizontali incipimus; hoc modo primum a ſerie verticali incipiendo:

$$\left(\frac{n-1}{1}\right) = \alpha,\ \left(\frac{n-2}{2}\right) = \frac{\beta}{\alpha}\times\left(\frac{n-1}{1}\right),\ \left(\frac{n-3}{3}\right) = \frac{\gamma}{\beta}\times\left(\frac{n-2}{2}\right),\ \left(\frac{n-4}{4}\right) = \frac{\delta}{\gamma}\times\left(\frac{n-3}{3}\right)\ \ \&c.$$

$$\left(\frac{n-2}{1}\right) = A,\ \left(\frac{n-3}{2}\right) = \frac{B}{A}\times\left(\frac{n-2}{1}\right),\ \left(\frac{n-4}{3}\right) = \frac{C}{B}\times\left(\frac{n-3}{2}\right),\ \left(\frac{n-5}{4}\right) = \frac{D}{C}\times\left(\frac{n-4}{3}\right)$$

$$\left(\frac{n-3}{1}\right) = \frac{\alpha B}{\beta},\ \left(\frac{n-4}{2}\right) = \frac{\beta C}{\gamma A}\times\left(\frac{n-3}{1}\right),\ \left(\frac{n-5}{3}\right) = \frac{\gamma D}{\delta B}\times\left(\frac{n-4}{2}\right),\ \left(\frac{n-6}{4}\right) = \frac{\delta E}{\varepsilon C}\times\left(\frac{n-5}{3}\right)$$

$$\left(\frac{n-4}{1}\right) = \frac{\alpha C}{\gamma},\ \left(\frac{n-5}{2}\right) = \frac{\beta D}{\delta A}\times\left(\frac{n-4}{1}\right),\ \left(\frac{n-6}{3}\right) = \frac{\gamma E}{\varepsilon B}\times\left(\frac{n-5}{2}\right),\ \left(\frac{n-7}{4}\right) = \frac{\delta F}{\zeta C}\times\left(\frac{n-6}{3}\right)$$

$$\left(\frac{n-5}{1}\right) = \frac{\alpha D}{\delta},\ \left(\frac{n-6}{2}\right) = \frac{\beta E}{\varepsilon A}\times\left(\frac{n-5}{1}\right),\ \left(\frac{n-7}{3}\right) = \frac{\gamma F}{\zeta B}\times\left(\frac{n-6}{2}\right),\ \left(\frac{n-8}{4}\right) = \frac{\delta G}{\eta C}\times\left(\frac{n-7}{3}\right)$$

$$\left(\frac{n-6}{1}\right) = \frac{\alpha E}{\varepsilon},\ \left(\frac{n-7}{2}\right) = \frac{\beta F}{\zeta A}\times\left(\frac{n-6}{1}\right),\ \left(\frac{n-8}{3}\right) = \frac{\gamma G}{\eta B}\times\left(\frac{n-7}{2}\right),\ \left(\frac{n-9}{4}\right) = \frac{\delta H}{\theta C}\times\left(\frac{n-8}{3}\right)$$

&c.

Deinde a ſuprema horizontali incipiendo:

$$\left(\frac{n}{1}\right) = 1, \quad \left(\frac{n-1}{2}\right) = \frac{\beta}{A} \times \left(\frac{n}{1}\right), \quad \left(\frac{n-2}{3}\right) = \frac{\gamma A}{\alpha B} \times \left(\frac{n-1}{2}\right), \quad \left(\frac{n-3}{4}\right) = \frac{\delta B}{\beta C} \times \left(\frac{n-2}{3}\right)$$

$$\left(\frac{n}{2}\right) = \frac{1}{2}, \quad \left(\frac{n-1}{3}\right) = \frac{\gamma}{B} \times \left(\frac{n}{2}\right), \quad \left(\frac{n-2}{4}\right) = \frac{\delta A}{\alpha C} \times \left(\frac{n-1}{3}\right), \quad \left(\frac{n-3}{5}\right) = \frac{\varepsilon B}{\beta D} \times \left(\frac{n-2}{4}\right)$$

$$\left(\frac{n}{3}\right) = \frac{1}{3}, \quad \left(\frac{n-1}{4}\right) = \frac{\delta}{C} \times \left(\frac{n}{3}\right), \quad \left(\frac{n-2}{5}\right) = \frac{\varepsilon A}{\alpha D} \times \left(\frac{n-1}{4}\right), \quad \left(\frac{n-3}{6}\right) = \frac{\zeta B}{\beta E} \times \left(\frac{n-2}{5}\right)$$

$$\left(\frac{n}{4}\right) = \frac{1}{4}, \quad \left(\frac{n-1}{5}\right) = \frac{\varepsilon}{D} \times \left(\frac{n}{4}\right), \quad \left(\frac{n-2}{6}\right) = \frac{\zeta A}{\alpha E} \times \left(\frac{n-1}{5}\right), \quad \left(\frac{n-3}{7}\right) = \frac{\eta B}{\beta F} \times \left(\frac{n-2}{6}\right)$$

$$\left(\frac{n}{5}\right) = \frac{1}{5}, \quad \left(\frac{n-1}{6}\right) = \frac{\zeta}{E} \times \left(\frac{n}{5}\right), \quad \left(\frac{n-2}{7}\right) = \frac{\eta A}{\alpha F} \times \left(\frac{n-1}{6}\right), \quad \left(\frac{n-3}{8}\right) = \frac{\theta B}{\beta G} \times \left(\frac{n-2}{7}\right)$$

&c.

Ubi lex, qua hæ formulæ a ſe invicem pendent, ſatis eſt perſpicua; ſi modo notemus, in utraque litterarum ſerie α, β, γ, δ &c. & A, B, C, D &c. terminos primum antecedentes inter ſe eſſe æquales.

Concluſio.

Cum igitur formulas ſecundæ claſſis, ſola conceſſa circuli quadratura, exhibere valeamus, formulæ tertiæ claſſis inſuper requirunt quadraturam contentam vel hac formula

$$\int \frac{dx}{\sqrt[3]{(1-x^3)^2}} = A, \quad \text{vel hac} \int \frac{x\,dx}{\sqrt[3]{(1-x^3)}} = \frac{\alpha}{A}$$

quandoquidem, data una, ſimul altera datur. Quod ſi iſtas formulas per productum infinitum exprimamus, earum valor reperitur:

$$\int \frac{dx}{\sqrt[3]{(1-x^3)^2}} = \frac{2}{1} \cdot \frac{3.5}{4.4} \cdot \frac{6.8}{7.7} \cdot \frac{9.11}{10.10} \cdot \frac{12.14}{13.13} \text{ \&c.}$$

unde ejus quantitas vero proxime ſatis expedite colligi poteſt; ſimili modo eſt:

$$\frac{x\,dx}{\sqrt[3]{(1-x^3)}} = 1 \cdot \frac{3.7}{5.5} \cdot \frac{6.10}{8.8} \cdot \frac{9.13}{11.11} \cdot \frac{12.16}{14.14} \text{ \&c.}$$

Deinde omnes formulas quartæ classis integrare poterimus si modo, præter circuli quadraturam, una ex his quatuor formulis fuerit cognita: $(\frac{2}{1})$, $(\frac{1}{1})$, $(\frac{3}{2})$, $(\frac{3}{3})$, quæ præbent has formas:

$$\int \frac{x\,dx}{\sqrt[4]{(1-x^4)^3}} = \frac{1}{2}\int \frac{dx}{\sqrt[4]{(1-xx)^3}} = \int \frac{dx}{\sqrt{(1-x^4)}} = A;$$

$$\int \frac{dx}{\sqrt[4]{(1-x^4)^3}} = \frac{\alpha A}{\beta}; \int \frac{xx\,dx}{\sqrt[4]{(1-x^4)}} = \frac{\alpha}{2A};$$

$$\int \frac{xx\,dx}{\sqrt{(1-x^4)}} = \int \frac{x\,dx}{\sqrt[4]{(1-x^4)}} = \frac{1}{2}\int \frac{dx}{\sqrt[4]{(1-xx)}} = \frac{\beta}{A};$$

at per productum infinitum erit

$$A = \frac{3}{1.2} \cdot \frac{4.7}{5.6} \cdot \frac{8.11}{9.10} \cdot \frac{12.15}{13.14} \cdot \frac{16.19}{17.18} \text{ \&c.}$$

Quinta classis postulat duas quadraturas altiores: $(\frac{3}{1}) = A$, & $(\frac{2}{2}) = B$, quarum loco aliæ binæ ab his pendentes assumi possunt, quæ quidem faciliores videantur, etsi ob 5 numerum primum aliæ aliis vix simpliciores reputari queant.

Pro sexta classe etiam duæ quadraturæ requiruntur: $(\frac{4}{1}) = A$ & $(\frac{3}{2}) = B$. Verum hic loco alterius ea, quæ in tertia classe opus erat, assumi potest, ut unica tantum nova sit adhibenda. Cum enim sit

$$(\frac{2}{2}) = \int \frac{x\,dx}{\sqrt[6]{(1-x^6)^4}} = \frac{1}{2}\int \frac{dx}{\sqrt[3]{(1-x^3)^2}} = \frac{\alpha BB}{\gamma A}$$

erit $\frac{2\alpha BB}{\gamma A} = \int \frac{dx}{\sqrt[3]{(1-x^3)^2}}$, quæ est formula ad classem tertiam requisita. Hac ergo data, si insuper innotescat formula:

$(\frac{3}{2}) = \int \frac{x\,dx}{\sqrt{(1-x^6)}} = \frac{1}{2}\int \frac{dx}{\sqrt{(1-x^3)}} = B$, vel etiam haec $(\frac{4}{3}) = \int \frac{xx\,dx}{\sqrt[3]{(1-x^6)}} = \frac{1}{3}\int \frac{dx}{\sqrt[3]{(1-xx)}} = \frac{\beta\gamma}{\alpha B}$, quae sunt simplicissimae in hoc genere, reliquae omnes per has definiri poterunt. His autem combinatis patet fore:

$$\int \frac{dx}{\sqrt{(1-x^3)}} \cdot \int \frac{dx}{\sqrt[3]{(1-xx)}} = \frac{6\beta\gamma}{\alpha} = \frac{\pi}{\sqrt{3}}.$$

Simili modo ex formulis quartae classis colligitur:

$$\int \frac{dx}{\sqrt{(1-x^4)}} \cdot \int \frac{dx}{\sqrt[4]{(1-x^2)}} = \frac{\pi}{2}$$

cujusmodi Theorematum ingens multitudo hinc deduci potest: inter quae hoc imprimis est notabile:

$$\int \frac{dx}{\sqrt[m]{(1-x^n)}} \cdot \int \frac{dx}{\sqrt[n]{(1-x^m)}} = \frac{\pi \text{ fin. } \frac{(m-n)\pi}{mn}}{(m-n) \text{ fin. } \frac{\pi}{m} \cdot \text{fin. } \frac{\pi}{n}}$$

quod, si m & n sint numeri fracti, in hanc formam transmutatur:

$$\int \frac{x^{q-1}\,dx}{\sqrt[r]{(1-x^p)^s}} \cdot \int \frac{x^{s-1}\,dx}{\sqrt[p]{(1-x^r)^q}} = \frac{\pi \text{ fin. } (\frac{s}{r} - \frac{q}{p})\pi}{(ps-qr) \text{ fin. } \frac{q}{p}\pi \cdot \text{fin. } \frac{s}{r}\pi}$$

In genere vero est;

$$(\frac{n-p}{q}) \cdot (\frac{n-q}{p}) = \frac{(\frac{n-p}{p})\ (\frac{n-q}{q})}{(q-p) \cdot (\frac{n-q+p}{q-p})}$$

quod hanc formam praebet:

$$\int \frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^n)^q}} \cdot \int \frac{x^{q-1}\,dx}{\sqrt[n]{(1-x^n)^p}} = \frac{\pi \text{ fin. } \frac{(q-p)\pi}{n}}{n(q-p) \text{ fin. } \frac{p\pi}{n} \cdot \text{ fin. } \frac{q\pi}{n}}$$

unde non folum præcedentia Theoremata, fed alia plura facile derivantur. Pofito enim $n = \frac{pq}{m}$ habebimus:

$$\int \frac{x^{p-1}\,dx}{\sqrt[p]{(1-x^q)^m}} \cdot \int \frac{x^{q-1}\,dx}{\sqrt[q]{(1-x^p)^m}} = \frac{\pi \text{ fin. } \left(\frac{m}{p} - \frac{m}{q}\right)\pi}{m(q-p) \text{ fin. } \frac{m\pi}{q} \cdot \text{ fin. } \frac{m\pi}{p}}$$

quam ita latius extendere licet:

$$\int \frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^m)^q}} \cdot \int \frac{x^{q-1}\,dx}{\sqrt[m]{(1-x^n)^p}} = \frac{\pi \text{ fin. } \left(\frac{q}{n} - \frac{p}{m}\right)\pi}{(mq-np) \text{ fin. } \frac{p}{m}\pi \cdot \text{ fin. } \frac{q}{n}\pi}$$

in qua fi ponatur $n = 2q$ erit:

$$\int \frac{x^{p-1}\,dx}{\sqrt{(1-x^m)}} \cdot \int \frac{x^{q-1}\,dx}{\sqrt[m]{(1-x^{2q})^p}} = \frac{\pi \text{ cof. } \frac{p}{m}\pi}{q(m-2p) \text{ fin. } \frac{p}{m}\pi}$$

At in pofteriori formula integrali fi ponatur $x^{2q} = 1 - y^m$ erit: $\int \frac{x^{q-1}\,dx}{\sqrt[m]{(1-x^{2q})^p}} = \frac{m}{2q} \int \frac{y^{m-p-1}\,dy}{\sqrt{(1-y^m)}}$, unde fcripto x pro y

$$\int \frac{x^{p-1}\,dx}{\sqrt{(1-x^m)}} \cdot \int \frac{x^{m-p-1}\,dx}{\sqrt{(1-x^m)}} = \frac{2\pi \text{ cof. } \frac{p}{m}\pi}{m(m-2p) \text{ fin. } \frac{p}{m}\pi}$$

Simili modo si in genere ponatur, pro altera formula integrali, $1 - x^n = y^m$, fiet: $\int \frac{x^{q-1}\,dx}{\sqrt[m]{(1-x^n)^p}} = \frac{m}{n} \int \frac{y^{m-p-1}\,dy}{\sqrt[n]{(1-y^m)^{n-q}}}$; unde, scripto iterum x pro y, obtinebitur:

$$\int \frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^m)^q}} \cdot \int \frac{x^{m-p-1}\,dx}{\sqrt[n]{(1-x^m)^{n-q}}} = \frac{n\pi\,\text{sin.}\,(\frac{q}{n} - \frac{p}{m})\,\pi}{m(mq-np)\,\text{sin.}\,\frac{p}{m}\pi \cdot \text{sin.}\,\frac{q}{n}\pi}$$

qui valor reducitur ad: $\frac{n\pi}{m\,(mq-np)}\,(\text{cot.}\,\frac{p}{m}\pi - \text{cot.}\,\frac{q}{n}\pi)$.

Atque hinc ista forma concinnior resultat:

$$\int \frac{x^{\frac{m-r}{2}-1}\,dx}{\sqrt[n]{(1-x^m)^{\frac{n-s}{2}}}} \cdot \int \frac{x^{\frac{m+r}{2}-1}\,dx}{\sqrt[n]{(1-x^m)^{\frac{n+s}{2}}}} = \frac{2\,n\,\pi\,(\text{tang.}\,\frac{r\pi}{2m} - \text{tang.}\,\frac{s\pi}{2n})}{m\,(nr-ms)}$$

Cum fundamentum harum reductionum situm sit in hac aequalitate: $(\frac{n-p}{q})\,(\frac{n-q}{p}) = \frac{(\frac{n-p}{p})\,(\frac{n-q}{q})}{(q-p)\,(\frac{n-q+p}{q-p})}$,

quae ad hanc formam reducitur:

$$(\frac{n-p}{q})\,(\frac{n-q}{p})\,(\frac{n-q+p}{q-p}) = (\frac{n}{q-p})\,(\frac{n-p}{p})\,(\frac{n-q}{q})$$

ejus veritas hoc modo directe ostendi potest.

Sumtis in reductione §. VIII. tradita pro numeris ternis p, q, r his $n-q$, $q-p$, q habebimus:

$$(\frac{n-q}{q-p})\,(\frac{n-p}{q}) = (\frac{n-q}{q})\,(\frac{n}{q-p})$$

tum vero sumtis eorum loco $n-q$, $q-p$, p erit

$$(\frac{n-q}{p})\,(\frac{n-q+p}{q-p}) = (\frac{n-q}{q-p})\,(\frac{n-p}{p})$$

quibus

quibus æquationibus in se [illegible]

$(\frac{n-[illegible]}{q})$ ut utrinque communi per [illegible]

$(\frac{n-[illegible]}{q})(\frac{n-q}{[illegible]})(\frac{n-q-[illegible]}{r-p}) = (\frac{n}{q})(\frac{n-[illegible]}{p})$ [illegible]

Quin etiam [illegible] ternarum æqualitas ab [illegible] non pendens exhiberi poterit, scilicet:

$$(\frac{s}{p})(\frac{r+s}{q})(\frac{p+s}{r}) = (\frac{r+s}{p})(\frac{s}{q})(\frac{q+s}{r}) = (\frac{s}{r})(\frac{r+s}{q})$$

$$(\frac{q+r}{s}) = (\frac{r+s}{p})(\frac{r}{q})(\frac{q+r}{s}),$$ quæ quatuor a se [illegible] litteras ab n non pendentes involvit, ac simili est æqualitas inter binarum formularum producta:

$$(\frac{r}{p})(\frac{p+r}{q}) = (\frac{q+r}{p})(\frac{r}{q}) = (\frac{q}{p})(\frac{p+q}{r}).$$

Æqualitas autem inter ternarum formularum producta habetur etiam ista:

$$(\frac{p}{q})(\frac{r}{s})(\frac{p+q}{r+s}) = (\frac{q}{r})(\frac{s}{p})(\frac{q+r}{p+s}) = (\frac{p}{r})(\frac{q}{s})(\frac{p+r}{q+s}) =$$

$$(\frac{p}{q})(\frac{p+q}{r})(\frac{p+q+r}{s}) = (\frac{p}{q})(\frac{p+q}{s})(\frac{p+q+s}{r})$$ &c.

In his enim litteræ p, q, r, s utcunque inter se permutari possunt.