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Considerationes de motu corporum coelestium

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CONSIDERATIONES DE MOTV CORPORVM COELESTIVM.

Auctore
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1.

Etsi nullum est dubium, quin leges motus corporum coelestium a *Keplero* obseruatae atque a *Neutono* confirmatae, Astronomiae maxima incrementa attulerint, tamen nunc quidem certissimum est, nullum in coelo reperiri corpus, quod leges istas in motu suo perfecte sequatur, cum potius in omnibus haud leues aberrationes ab istis legibus deprehendantur. Vera scilicet omnium motuum coelestium causa in mutua horum corporum attractione est posita, qua vnumquodque ad singula reliqua vrgetur viribus rationem compositam ex directa simplici massarum, et inuersa duplicata distantiarum tenentibus. Semper autem commode vsu venit, vt inter has vires vna prae reliquis maxime emineat, ideoque motus proxime regulis *Keplerianis* conformis euadat; sicque effectus a reliquis oriundus veluti minimus per methodos appropinquandi definiri possit. Quod nisi eueniret, in maxima adhuc ignoratione motuum coelestium versaremur, cum nulla methodus adhuc sit inuenta, cuius ope trium saltem corporum se mutuo attrahentium motus assignari queat; nisi forte vna vis caeteras plurimum superet.

2. Ve-

2. Verum etiam hic casus, in quo solo Geometrae operam suam non omnino frustra consumserunt, neutiquam pro confecto haberi potest, cum ipsa methodus appropinquandi, qua Geometrae vti solent, plurimis difficultatibus adhuc sit inuoluta, atque infinita minorum perturbationum multitudo negligatur, quo fit vt haec ipsa approximatio negotium minime conficiat, sed ad eam perficiendam plurima adhuc adminicula desiderentur. Quare etsi motus Lunae ex hac Theoria satis accurate est definitus, id tamen potius singularibus circumstantiis, quae in Luna locum inueniunt, est tribuendum, quam cuipiam perfectioni, ad quam Theoria euecta censeri queat; si enim Luna bis vel ter longius a terra abesset, vel eius orbita magis esset excentrica, omnes labores adhuc exantlati omni fructu caruissent, ac ne nunc quidem eius motum obiter saltem ad certam quandam regulam reuocare liceret.

3. Plurimum igitur is in Theoria Astronomiae praestitisse esset censendus, qui in hypothesi ficta, qua Luna multo longius a terra abesset, eius motum assignare valuerit, cum inde maxima adiumenta in hanc scientiam certo essent redundatura. Si quidem Luna centies longius a terra esset remota, nullum est dubium, quin leges motus planetae principalis esset secutura, neque amplius, tanquam satelles terrae, spectari posset. Sin autem decies tantum magis distaret, eius motus ita foret comparatus, vt in dubio relinqueretur, vtrum planetis primariis, an secundariis, esset accensenda. Tantopere certe ab omnibus motibus in coelo obseruatis

discreparet, vt vix intelligi possit, quemadmodum saltem ideam motus medii constitui conueniat. Innumerabiles forsitan obseruationes legem quandam aperuissent, ex qua in posterum eius loca quodammodo praedicere licuisset; nequaquam autem patet, quomodo Theoria ad huiusmodi motum explicandum accommodari potuisset. Imbecillitati nostrae sapientissimus creator consuluisse videtur, quod nulla corpora in coelo ita collocauerit, vt eorum motus, neque ad legem planetarum principalium, neque satellitum, referri posset.

4. Huiusmodi inuestigationem, quae vires ingenii humani tantum non transcendere videtur, certe non subito suscipi conueniet, sed potius conatus nostros pedetentim eo dirigi oportebit. Generale ergo problema trium corporum se mutuo attrahentium ita commodissime restringetur, vt vnius massa prae binis reliquis quasi euanescat, quo pacto id commodi assequemur, vt duo corpora, maiora scilicet, secundum leges *Keplerianas* moueantur, omnisque perturbatio in motu tertii consumatur, cuius situs et motus si ab initio ita fuerit comparatus, vt ad ambo maiorum aequa vi quasi attrahatur, habebimus eiusmodi casum, cuius inuestigatio nouam plane methodum postulat. Plurimum abest, vt hoc problema aggredi ausim, vt potius, frustra in eo euoluendo desudasse, fateri cogar; verum tamen casum obseruaui omnino singularem, ac simplicitate memorabilem, quo Lunae eiusmodi motus imprimi potuisset, vt perpetuo Soli, vel coniuncta, vel opposita, appariturа fuisset, cuius casus consideratio, cum forte vsu in hoc

hoc difficillimo negotio non destituatur, haud displicitura videtur.

5. Motum igitur tam Solis, quam Lunae, ex terra visum in plano ecliptico fieri assumens, terram quiescentem in T, et post aliquod tempus elapsum Solem in S, Lunam vero in L, versari pono, et ducta recta fixa TA, ad principium arietis directa, statuo, angulos ATS $= \theta$, ATL $= \Phi$, et STL $= \Phi - \theta = \eta$, tum vero distantias TS $= u$, TL $= v$ et LS $= \sqrt{(uu - 2uv \cos.\eta + vv)} = z$. Tab. XX. Fig. 13.
Sit porro longitudo Solis media $= \zeta$, eiusque distantia media a terra $= a$, hisque positis pro motu Solis vtpote regulari habebimus:

$$\frac{2\,du\,d\theta + u\,dd\theta}{d\zeta^2} = 0 \quad \text{et} \quad \frac{ddu - u\,d\theta^2}{d\zeta^2} + \frac{a^3}{uu} = 0$$

pro motu autem Lunae:

$$\frac{2\,dv\,d\Phi + v\,dd\Phi}{d\zeta^2} - \frac{a^3}{uu}\left(1 - \frac{u^3}{z^3}\right)\sin\eta = 0$$

$$\frac{ddv - v\,d\Phi^2}{d\zeta^2} + \frac{nnc^3}{vv} + \frac{a^3 v}{z^3} + \frac{a^3}{uu}\left(1 - \frac{u^3}{z^3}\right)\cos.\eta = 0,$$

vbi c est distantia media, ad quam Luna, a sola vi terrae sollicitata, pari motu medio reuolueretur, existente $n:1$ ratione motus medii Lunae ad motum medium Solis. Caeterum circa differentialia secundi gradus hic est monendum, elementum $d\zeta$ constans esse sumtum.

6. Tota ergo difficultas in resolutione harum duarum aequationum consistit, vt scilicet inde ad quoduis tempus, seu longitudinem Solis mediam ζ, tam distantia v, quam angulus Φ, definiatur. Quod cum in genere fieri nequeat, Geometrae adhuc in eo laborarunt, vt saltem pro casu, quo distantia v, prae u,

est vehementer parua, simulque n numerus mediocriter magnus, idoneas approximationes eruerent, in quo tamen negotio plurimum adhuc iure desideratur. Hic autem binas istas aequationes in genere specto, sine vllo respectu ad Lunam habito, et quosdam casus sum euoluturus, quibus iis absolute satisfieri queat. Eiusmodi scilicet motus in coelo locum habere posse ostendam, quos perfecte cognoscere in nostra sit potestate, etiamsi eorum ratio maxime a motu regulari abhorreat.

7. Primum igitur obseruo, has duas aequationes absolutam resolutionem admittere casu $\eta=0$, seu $\Phi=\theta$, ita vt tum Luna perpetuo in coniunctione cum *Sole* esset apparitura. Cum enim sit $\sin.\eta=0$, et $\cos.\eta=1$, erit $z=u-v$, nostrae aequationes has induent formas:

$$\frac{2dvd\theta+vdd\theta}{d\zeta^2}=0\text{, et }\frac{ddv-vd\theta^2}{d\zeta^2}+\frac{nnc^3}{vv}+\frac{a^3v}{(u-v)^3}+\frac{a^3}{uu}\cdot\frac{-3uuv+3uv^2-v^3}{(u-v)^3}=0$$

seu
$$\frac{ddv-vd\theta^2}{d\zeta^2}+\frac{nnc^3}{vv}-\frac{a^3v(3uu-3uv+vv)}{uu(u-v)^3}=0$$

quae cum formulis, pro motu Solis datis, comparatae statim dant $v=\alpha u$, quippe quo pacto prioribus aequationibus satisfit. Hinc altera aequatio pro Luna erit

$$\frac{\alpha(ddu-ud\theta^2)}{d\zeta^2}+\frac{nnc^3}{\alpha\alpha uu}-\frac{\alpha a^3(2-3\alpha+\alpha\alpha)}{(1-\alpha)^3uu}=0.$$

Quare cum altera aequatio pro Sole sit

$$\frac{ddu-ud\theta^2}{d^2}+\frac{a^3}{uu}=0$$

necesse est sit: $\alpha a^3=\frac{nnc^3}{\alpha\alpha}-\frac{\alpha a^3(2-3\alpha+\alpha\alpha)}{(1-\alpha)^3}$

seu $\frac{nnc^3}{\alpha\alpha a^3}=\frac{3\alpha-3\alpha\alpha+\alpha^3}{(1-\alpha)^2}$

vbi

vbi cum sit $\frac{nnc^3}{a^3}$ quantitas constans, ponatur breuitatis causa: $\frac{nnc^3}{a^3}=m$, eritque $m(1-\alpha)^2=\alpha\alpha(3\alpha-3\alpha^2+\alpha^3)$ seu $m(1-\alpha)^2=\alpha\alpha-\alpha\alpha(1-\alpha)^3$. Posito ergo $1-\alpha=x$, fit $mxx=(1-x^3)(1-x)^2$, seu

$$1-2x+xx-mxx-x^3+2x^4-x^5=0.$$

8. Pendet ergo determinatio numeri α vel x ab aequatione quinti gradus, pro cuius resolutione notari oportet, esse m fractionem quam minimam; quare cum sit

$$m(1-\alpha)^2=3\alpha^3-3\alpha^4+\alpha^5$$

euidens est quoque, α minimum esse habiturum valorem, et quam proxime fore $\alpha=\sqrt[3]{\frac{m}{3}}=\frac{c}{a}\sqrt[3]{\frac{nn}{3}}$, accuratius autem $\alpha=\sqrt[3]{\frac{m}{3}}-\frac{1}{3}\sqrt[3]{\frac{mm}{9}}-\frac{1}{27}m+\frac{1}{81}m\sqrt[3]{\frac{m}{3}}$. Primus autem terminus sufficit, sicque est $v=\frac{cu}{a}\sqrt[3]{\frac{nn}{3}}$ vnde cum sit proxime $u=a$, et $nn=175$, erit circiter $v=4c$; seu si Luna fere quater longius a nobis esset remota, eiusmodi motum habere posset, vt Soli perpetuo iuncta appareret. Talis Luna aequo iure tanquam Satelles terrae ac planeta primarius spectari posset, et vterque motus maxime foret regularis, hoc tantum a regulis *Keplerianis* recedens, quod Soli propior, quam terra, pari tamen tempore reuoluatur, ob vim scilicet terrae vis Solis tantum imminuitur, vt cum maiori tempore periodico consistere possit. Hinc distantiam a terra quasi quadruplo maiorem, quam Luna reuera inde distat, tanquam limitem spectare licet, vt corpora magis remota pro planetis primariis, propiora vero pro satellitibus terrae ſint habenda. Similes limites circa reliquos planetas conſtitui poterunt.

9. Quemadmodum caſus euolutus in perpetua coniunctione cum Sole conſtat, ita etiam perpetua oppoſitio ſimilem caſum ſuppeditat. Pro quo ponamus $\eta = 180°$, vt ſit ſin. $\eta = 0$, coſ. $\eta = -1$ et $\Phi = 180° + \theta$, ideoque $d\Phi = d\theta$, atque $z = u + v$. Aequationes ergo pro motu Lunae ſequentes induent formas:

$$\frac{2dv\,d\theta + v\,dd\theta}{d\zeta^2} = 0, \text{ et } \frac{ddv - v\,d\theta^2}{d\zeta^2} + \frac{nnc^3}{v^2} + \frac{a^3 v}{(u+v)^3}$$
$$- \frac{a^3}{uu}\left(1 - \frac{u^3}{(u+v)^3}\right) = 0$$

quae poſterior reducitur ad hanc:

$$\frac{ddv - v\,d\theta^2}{d\zeta^2} + \frac{nnc^3}{vv} - \frac{a^3}{uu} + \frac{a^3}{(u+v)^2} = 0.$$

Prior cum motu Solis collata praebet ſtatim $v = \alpha u$, vnde poſterior fit

$$\frac{\alpha(ddu - u\,d\theta^2)}{d\zeta^2} + \frac{nnc^3}{\alpha\alpha uu} - \frac{a^3}{uu} + \frac{a^3}{(1+\alpha)^2 uu} = 0.$$

At ex motu Solis eſt $\frac{ddu - u\,d\theta^2}{d\zeta^2} = -\frac{a^3}{uu}$, ex quo fit $-\alpha a^3 + \frac{nnc^3}{\alpha\alpha} - a^3 + \frac{a^3}{(1+\alpha)^2} = 0$, ſeu

$$\frac{nnc^3}{a^3} - \alpha\alpha(1+\alpha) + \frac{\alpha\alpha}{(1+\alpha)^2} = 0$$

et poſito breuitatis gratia $\frac{nnc^3}{a^3} = m$, erit

$$m(1+\alpha)^2 = \alpha\alpha(1+\alpha)^3 - \alpha\alpha$$

qu1e ex ſuperiori naſcitur, ſumendis m et α negatiuis. Quamobrem hinc colligitur

$$\alpha = \sqrt[3]{\frac{m}{3}} + \frac{1}{3}\sqrt[3]{\frac{mm}{9}} - \frac{1}{27}m - \frac{1}{81}m\sqrt[3]{\frac{m}{3}}$$

fatis

satis autem exacte est $a = \sqrt[3]{\frac{m}{s}}$ et $v = \frac{ca}{a}\sqrt[3]{\frac{nn}{s}}$ vt ante.

10. Casus hi eo magis sunt notatu digni, quod sine vlla approximatione absolute expediri possunt, etiamsi ambae vires Solis et terrae ad motum producendum concurrant, id quod nullo alio casu praestare licet. Tali autem motu simplici corpus re vera moueretur, si ipsi in distantia assignata, dum Soli, vel coniunctum, vel oppositum, ex terra appareret, eiusmodi motus imprimeretur, vt cum terra pari passu in plano eclipticae ingredi inciperet. Sin autem motus impressus tantillum ab hac lege discrepet, non quidem perpetuo Soli, vel coniunctum, vel oppositum, maneret, sed exiguas excursiones hinc inde quasi oscillando conficeret. Quo casu cum motus minime ab inuenta ratione esset discrepaturus, more solito, approximando etiam, eiusmodi motum definire licebit; in quo cum quasi initium motuum irregularium, quos nullo etiamnum modo ad calculum reuocare licet, conspiciatur, vsu certe non carebit, si in naturam istiusmodi motuum accuratius inquisiuero.

11. Cum autem haec inuestigatio haud leuibus difficultatibus implicetur, statim ab initio aequationes nostras tractatu faciliores reddi conueniet, quod, cum distantia v prae u vehementer sit exigua, commode per approximationem fieri potest. Scilicet ob $z = \sqrt{(uu - 2uv\cos.\eta + vv)}$ eliciemus proxime $\frac{1}{z^3} = \frac{1}{u^3} + \frac{3v\cos.\eta}{u^4} - \frac{3vv}{2u^5} + \frac{15vv\cos.\eta^2}{2u^5}$, ideoque $1 - \frac{u^3}{z^3} = -\frac{3v}{u}\cos.\eta + \frac{3vv}{2uu}(1 - 5\cos.\eta^2)$, ex quo aequationes no-

noſtrae, pro motu Lunae inuentae, in ſequentes formas tranſibunt:

I. $\frac{2dvd\Phi + vdd\Phi}{d\zeta^2} + \frac{3a^3v}{u^3}\sin.\eta\cos.\eta - \frac{3a^3vv}{2u^4}\sin.\eta(1-5\cos.\eta^2) = 0$

II. $\frac{ddv - vd\Phi^2}{d\zeta^2} + \frac{nnc^3}{vv} + \frac{a^3v}{u^3}(1-3\cos.\eta^2 + \frac{3a^3vv}{2u^4}(3\cos.\eta - 5\cos.\eta^3) = 0.$

Deinde etiam calculus non parum ſubleuabitur, ſi motum Solis, vt vniformem, ſpectemus, vt ſit $u = a$, et $\theta = \zeta$, ideoque $\eta = \Phi - \zeta$, ſeu $\Phi = \eta + \zeta$. vnde ſequentes emergunt aequationes:

I. $\frac{2dvd\eta + vdd\eta}{d\zeta^2} + \frac{2dv}{d\zeta} + 3v\sin.\eta\cos.\eta - \frac{3vv}{2a}\sin.\eta(1-5\cos.\eta^2) = 0$

II. $\frac{ddv}{d\zeta^2} - v(1+\frac{d\eta}{d\zeta})^2 + v(1\;3\cos.\eta^2) + \frac{nnc^3}{vv} + \frac{3vv}{2a}\cos.\eta(3-5\cos.\eta^2) = 0,$

vbi etiam poſtrema membra facile omitti poſſunt, quia fractio $\frac{v}{a}$ eſt vehementer parua, etiamſi diſtantia Lunae quadruplo maior ſtatuatur.

12. Vt iam hinc caſum memoratum, quo Luna circa Solem motu quaſi oſcillatiorio nutare videbitur, eliciamus, angulum η quam minimum concipiamus, vt ſit $\sin.\eta = \eta$, et $\cos.\eta = 1 - \frac{1}{2}\eta\eta$, et habebimus:

I. $\frac{2dvd\eta + vdd\eta}{d\zeta^2} + \frac{2dv}{d\zeta} + 3v\eta\eta = 0$

II. $\frac{ddv}{d\zeta^2} - v(1 + \frac{d\eta}{d\zeta})^2 + \frac{nnc^3}{vv} - 2v + 3v\eta\eta = 0.$

Deinde quia diſtantia v parum immutatur, ponamus $v = b(1+x)$, vt x ſit quantitas minima, tum vero ſit breuitatis gratia $\frac{nnc^3}{b^3} = m$, eritque:

I. $\frac{2dxd\eta + xdd\eta}{d\zeta^2} + \frac{dd\eta}{d\zeta^2} + \frac{2dx}{d\zeta} + 3\eta + 3x\eta = 0$

II. $\frac{ddx}{d\zeta^2} - 3 - 3x - \frac{2d\eta}{d\zeta} - \frac{2xd\eta}{d\zeta} - \frac{d\eta^2}{d\zeta^2} - \frac{xd\eta^2}{d\zeta^2} + 3\eta\eta$
$+ 3x\eta\eta + m - 2mx + 3mxx = 0,$

vnde

vnde pro quouis angulo ζ valores quantitatum x et η definiri oportet.

13. Cum angulus η sit minimus, alternatimque positiuus euadat et negatiuus; quoniam Luna vltro citroque a Sole digredi conspicietur: facile colligere licet, eum per quempiam angulum ω ad ζ datam rationem tenente ita definiri, vt sit

$$\eta = A \sin. \omega + B \sin. 2\omega + C \sin. 3\omega \text{ etc.}$$

atque $d\omega = \alpha d\zeta$. Quo posito erit

$$\frac{d\eta}{d\zeta} = \alpha A \cos. \omega + 2\alpha B \cos. 2\omega + 3\alpha C \cos. 3\omega \text{ et}$$

$$\frac{dd\eta}{d\zeta^2} = -\alpha^2 A \sin \omega - 4\alpha^2 B \sin. 2\omega - 9\alpha\alpha C \sin. 3\omega.$$

Quare cum aequatio prima in hanc formam transfundatur

$$\frac{2dx}{1+x} + \frac{dd\eta + 3\eta d\zeta^2}{d\eta + d\zeta} = 0$$

fiet $2l(1+x) + l(1+\frac{d\eta}{d\zeta}) + 3\int \frac{\eta d\zeta}{1+\frac{d\eta}{d\zeta}} = \text{Const.}$

seu ob x et $\frac{d\eta}{d\zeta}$ minima:

$$2x - xx + \tfrac{2}{3}x^3 + \frac{d\eta}{d\zeta} - \frac{d\eta^2}{2d\zeta^2} + \frac{d\eta^3}{3d\zeta^3} + 3\int \eta d\zeta$$
$$- \tfrac{3}{2}\eta\eta + 3\int \frac{\eta d\eta^2}{d\zeta} - 3\int \frac{\eta d\eta^3}{d\zeta^2} = \text{Const.}$$

Nunc vero ob $d\zeta = \frac{d\omega}{\alpha}$ est

$$\int \eta d\zeta = -\frac{A}{\alpha}\cos. \omega - \frac{B}{2\alpha}\cos. 2\omega - \frac{C}{3\alpha}\cos. 3\omega$$

$$\eta\eta = \tfrac{1}{2}AA + AB\cos. \omega - \tfrac{1}{2}AA\cos. 2\omega - AB\cos. 3\omega$$
$$+ \tfrac{1}{2}BB \qquad\qquad + AC$$

$$\frac{d\eta^2}{d\zeta^2} = \tfrac{1}{2}\alpha\alpha AA + 2\alpha\alpha AB\cos.\omega + \tfrac{1}{2}\alpha\alpha AA\cos.2\omega + 2\alpha\alpha AB\cos.3\omega$$
$$+ 2\alpha\alpha BB \qquad\qquad + 3\alpha\alpha AC$$

$$\frac{d\eta^3}{d\zeta^2} = + \alpha^3 AAB + \tfrac{3}{4}\alpha^3 A^3\cos.\omega + \alpha^3 AAB\cos.2\omega$$
$$+ 4\alpha^3 ABB$$

vbi ob literas A, B, C, minimas altiores potestates merito negligimus.

14. Cum ergo sit:

$$\frac{\eta\, d\eta^2}{d\zeta^2} = \tfrac{1}{4}\alpha\alpha A^3\sin.\omega + \tfrac{3}{2}\alpha\alpha AAB\sin.2\omega + \tfrac{1}{4}\alpha\alpha A^3\sin.3\omega$$
$$+ 3\alpha\alpha ABB + 2\alpha\alpha BBB \qquad + \tfrac{3}{8}\alpha\alpha A^2 C$$
$$- \tfrac{3}{8}\alpha\alpha A^2 C \qquad\qquad + \alpha\alpha ABB$$

ob $d\zeta = \frac{d\omega}{\alpha}$ habebimus integrando:

$$\int \frac{\eta\, d\eta^2}{d\zeta} = - \tfrac{1}{4}\alpha A^3\cos.\omega - \tfrac{3}{4}\alpha AAB\cos.2\omega - \tfrac{1}{12}\alpha A^3\cos.3\omega$$
$$- 3\alpha ABB - \alpha B^3 \qquad - \tfrac{1}{8}\alpha A^2 C$$
$$+ \tfrac{3}{8}\alpha A^2 C \qquad\qquad - \tfrac{1}{3}\alpha ABB$$

vbi cum series A, B, C maxime decrescat, plura membra omitti possunt. Deinde cum sit:

$$\frac{\eta\, d\eta^3}{d\zeta^3} = \tfrac{7}{8}\alpha^4 A^3 B\sin.\omega + \tfrac{5}{8}\alpha^4 A^4\sin.2\omega + \tfrac{7}{8}\alpha^4 A^3 B\sin.3\omega$$

erit integrando,

$$\int \frac{\eta\, d\eta^3}{d\zeta^2} = - \tfrac{7}{8}\alpha^3 A^3 B\cos.\omega - \tfrac{5}{16}\alpha^3 A^4\cos.2\omega - \tfrac{7}{24}\alpha^3 A^3 B\cos.3\omega$$

atque ex his tandem conficitur haec aequatio, omissis partibus constantibus:

$$2x - xx + \tfrac{2}{3}x^3 \;+ \alpha A \cos.\omega \;+ 2\alpha B \cos.2\omega \;+ 3\alpha C \cos.3\omega = 0$$
$$\begin{array}{lll} - \alpha\alpha AB & - \tfrac{1}{4}\alpha\alpha AA & - \alpha\alpha AB \\ + \tfrac{1}{4}\alpha^3 A^3 & - \tfrac{3}{2}\alpha\alpha AC & - \tfrac{3C}{3\alpha} \\ - \tfrac{3\alpha}{\alpha} & + \tfrac{1}{3}\alpha^3 AAB & + \tfrac{3}{2} AB \\ - \tfrac{3}{2} AB & - \tfrac{3B}{2\alpha} & - \tfrac{1}{4}\alpha A^3 \\ - \tfrac{3}{4}\alpha A^3 & + \tfrac{3}{4} AA & + \tfrac{1}{4}\alpha^3 A^3 \\ & + \tfrac{3}{2} AC & \\ & - \tfrac{9}{8}\alpha AAB. & \end{array}$$

15. Ad valorem ipsius x hinc definiendum ponamus breuitatis gratia

$$(\alpha - \tfrac{3}{\alpha})A - (\alpha\alpha + \tfrac{3}{2})AB + \tfrac{1}{4}\alpha(\alpha\alpha - 3)A^3 = \mathfrak{A}$$
$$\frac{4\alpha\alpha - 3}{2\alpha}B - \frac{(\alpha\alpha - 3)}{4}AA = \mathfrak{B}$$
$$\frac{3\alpha\alpha - 1}{\alpha}C - \frac{(2\alpha\alpha - 3)}{2}AB + \tfrac{1}{4}\alpha(\alpha\alpha - 1)A^3 = \mathfrak{C}$$

vt sit $2l(1+x) + \mathfrak{A}\cos.\omega + \mathfrak{B}\cos.2\omega + \mathfrak{C}\cos.3\omega = 0$

et $1 + x = e^{-\frac{1}{2}\mathfrak{A}\cos.\omega - \frac{1}{2}\mathfrak{B}\cos.2\omega - \frac{1}{2}\mathfrak{C}\cos.3\omega}$

vnde concludimus fore

$$x = -\tfrac{1}{2}\mathfrak{A}\cos.\omega - \tfrac{1}{2}\mathfrak{B}\cos.2\omega - \tfrac{1}{2}\mathfrak{C}\cos.3\omega$$
$$\begin{array}{lll} + \tfrac{1}{8}\mathfrak{A}\mathfrak{B} & + \tfrac{1}{16}\mathfrak{A}\mathfrak{A} & + \tfrac{1}{8}\mathfrak{A}\mathfrak{B} \\ + \tfrac{1}{64}\mathfrak{A}^3 & & - \tfrac{1}{192}\mathfrak{A}^3. \end{array}$$

Verum ne in calculos nimis taediosos immergamur, rem aliquanto minus curate expediamus, neglectoque angulo triplo, vt sit $\eta = A\sin.\omega + B\sin 2\omega$, habebimus

$$x = -\frac{(\alpha\alpha - 3)}{2\alpha}A\cos.\omega - \frac{(4\alpha\alpha - 3)}{4\alpha}B\cos.2\omega$$
$$+ \frac{3(\alpha\alpha - 1)(\alpha\alpha - 3)}{16\alpha\alpha}AA$$

 vbi

vbi breuitatis gratia ſcribamus:

$$x = E \cos\omega + F \cos 2\omega, \text{ vt ſit}$$

$$E = \frac{3-\alpha\alpha}{2\alpha}A \text{ et } F = \frac{3-4\alpha\alpha}{4\alpha}B + \frac{3(\alpha\alpha-1)(\alpha\alpha-3)}{16\alpha\alpha}AA.$$

16. Hi iam valores in aequatione ſecunda ſubſtituantur, atque reperiemus:

$$\begin{array}{llll}
\frac{ddx}{d\zeta^2} & = & -\alpha\alpha E\cos\omega & -4\alpha\alpha F\cos 2\omega \\
-3-3x & = -3 & -3E & -3F \\
-\frac{2d\eta}{d\zeta} & = -\alpha AE & -2\alpha A & -4\alpha B \\
 & & -2\alpha BE & -\alpha AE \\
-\frac{2xd\eta}{d\zeta} & = & -\alpha AF & \\
-\frac{d\eta^2}{d\zeta^2} & = -\frac{1}{2}\alpha\alpha AA & +2\alpha\alpha AB & -\frac{1}{2}\alpha\alpha AA \\
+3\eta\eta & = +\frac{3}{2}AA & +3AB\cos\omega & -\frac{3}{2}AA\cos 2\omega \\
+m & = +m & & \\
=2mx & = & -2mE & -2mF \\
+3mxx & = +\frac{3}{2}mEE & +3mEF & +\frac{3}{2}mEE
\end{array}$$

vnde primo concludimus:

$$m(1+\tfrac{3}{2}EE) = 3 + \alpha AE + \tfrac{1}{2}\alpha\alpha AA - \tfrac{3}{2}AA = 3$$

$$\text{ideoque } m = 3 - \tfrac{9}{2}EE$$

pro determinatione numeri m indeque diſtantia b. Manifeſtum autem eſt, eſſe proxime $m = 3$, ideoque $b = c\sqrt[3]{\frac{nn}{3}}$. Porro autem fit

$$-\alpha\alpha E - 9E - 2\alpha A - 2\alpha BE - \alpha AF + (2\alpha\alpha+3)AB + 9EF = 0$$

vnde neglectis terminis minimis ob $\frac{E}{A} = \frac{3-\alpha\alpha}{2\alpha}$, erit $(\alpha\alpha+9)(3-\alpha\alpha) + 4\alpha\alpha = 0$, ſeu

$$\alpha^4 + 2\alpha\alpha - 27 = 0, \text{ hincque } \alpha\alpha = \sqrt{28} - 1.$$

Tertia

Tertia denique aequatio dat

$$-(4\alpha\alpha+9)F-4\alpha B-\alpha AE-\tfrac{3}{2}\alpha\alpha AA-\tfrac{3}{2}AA$$
$$+\tfrac{9}{2}EE=0$$

ideoque:

$$\left.\begin{array}{ll}\frac{(4\alpha\alpha-3)(4\alpha\alpha+9)}{4\alpha}B & -\frac{3(\alpha\alpha-1)(\alpha\alpha-3)(4\alpha\alpha+9)}{16\alpha\alpha}AA\\ -4\alpha B & +\frac{(\alpha\alpha-3)}{2}AA\\ & -\frac{(\alpha\alpha+3)}{2}AA\\ & +\frac{9(\alpha\alpha-3)^2}{2\alpha\alpha}AA\end{array}\right\}=0$$

vnde colligitur:

$$B(16\alpha^4+8\alpha\alpha-27)=\tfrac{3}{2}AA\alpha(13-7\alpha\alpha+2\alpha^4)$$

seu ob $27=\alpha^4+2\alpha\alpha$

$$3B(5\alpha\alpha+2)=\frac{3AA}{2\alpha}(13-7\alpha\alpha+2\alpha^4)$$

ideoque $B=\frac{13-7\alpha\alpha+2\alpha^4}{2\alpha(5\alpha\alpha+2)}AA=\frac{67-11\alpha\alpha}{2\alpha(5\alpha\alpha+2)}AA$

$$F=\frac{291-94\alpha\alpha-25\alpha^4}{2\alpha\alpha(5\alpha\alpha+2)}AA=-\frac{165-24\alpha\alpha}{4\alpha\alpha(5\alpha\alpha+2)}AA.$$

17. Quantitas ergo A arbitrio nostro relinquitur, a qua digressiones a linea syzygiarum pendent, pro ea autem valde paruam fractionem assumi oportet, quae si fuerit tam exigua, vt eius quadratum nullius sit momenti, primi termini sufficiunt. Pro distantia ergo $v=b(1+x)$ erit $b=c\sqrt[3]{\frac{nn}{2}}$, et angulus ω ita definitur, vt sit $\omega=\alpha\zeta+\beta$ existente $\alpha\alpha=\sqrt{28}-1$, hincque $\alpha\alpha=4,291502$ et $\alpha=2,071594$. Tum vero erit $\eta=A\sin.\omega$ et $v=b(1-\frac{(\alpha\alpha-3)}{2\alpha}A\cos.\omega)$ seu $v=b(1-0,311717A\cos.\omega)$.

Excurſiones fiunt maximae, ſi angulus ω ſit $90°$. $270°$. etc. ergo ab vna digreſſione maxima ad ſequentem eſt $\alpha\zeta = 180°$ et $\zeta = 86°, 53\frac{1}{2}'$: at in digreſſionibus maximis eſt $v = b$. Verum etiam huiusmodi librationis, ſi maior exiſteret, determinatio inſignibus laborat difficultatibus, vt, quo accuratius omnes variationes definire vellemus, eo minus certi de reliquis neglectis redderemur.

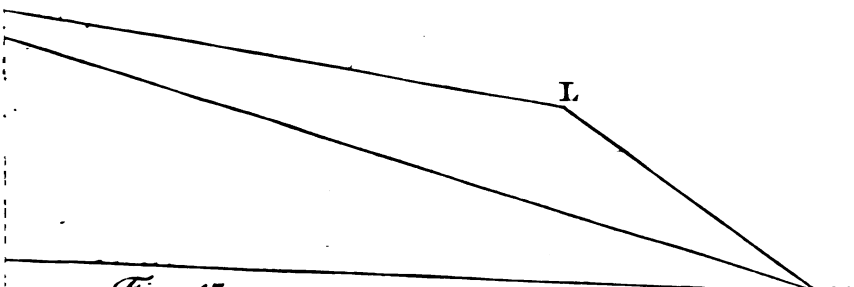

Fig. 13.

2.