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Elementa calculi variationum

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ELEMENTA
CALCVLI VARIATIONVM.

Auctore
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Mox post inuenta calculi differentialis principia eiusmodi problemata tractari sunt coepta, quae plane singularem huius calculi applicationem requirebant. Cum enim munus calculi differentialis in hoc potissimum consistat, vt proposita functione quacunque quantitatis variabilis x eius incrementum inuestigetur, dum quantitas x suo differentiali dx crescere assumitur: translatione ad Geometriam facta in promtu erat inde linearum curuarum tangentes et ipsas curuaturas definire, quarum rerum inuestigatio immediate ex natura differentialium deriuatur. Aliter vero comparata est ratio eiusmodi problematum, quibus innumerabiles lineae curuae sub quapiam generali aequatione contentae proponuntur, a quibus arcus longitudine aequales, siue qui a graui descendente eodem tempore percurrantur, abscindi oporteat, ex quo posteriori genere problema curuarum synchronarum est natum. In his enim quaestionibus non tam id est spectandum, quantum incrementum cuiusque curuae applicata accipiat, dum abscissa suo differentiali augetur; quam, quantum siue tempus descensus per eundem varietur, si ipse arcus in alia curua capiatur. Talia problemata per differentia-

 tionem

tionem parametrorum resolui dicuntur, quandoquidem variabilitas parametri omnes illas innumerabiles curuas propositas complectitur. Quo autem clarius perspiciatur principium, ex quo huiusmodi problematum solutio est petenda, proposita sit aequatio quaecunque inter abscissam x et applicatam y, quae insuper quantitatem constantem a parametrum vocandam contineat; quae quamdiu eundem valorem retinet, aequatio praebebit vnam quandam lineam curuam, verum si ipsi a successiue alii atque alii valores tribuantur, aliae continuo lineae curuae orientur. Quodsi iam quaestio circa arcus harum curuarum versetur, quoniam cuiusque curuae arcus per $\int V(dx^2 + dy^2)$ exprimitur, in qua integratione parameter a pro constanti assumitur; totum negotium huc redit, vt formulae integralis $\int V(dx^2 + dy^2)$ incrementum definiatur, quod accipit, dum in ea loco quantitatis a eadem suo incremento da aucta substituatur. Generatim igitur si loco arcus alia quaecunque expressio integralis $\int Z dx$ tractetur, quae integratio ex data inter x et y aequatione sit conficienda, parametro a pro constanti habita; quaeritur quantam variationem eadem expressio $\int Z dx$ iam integrata sit passura, si in aequatione inter x et y data parameter a elemento da augeatur. Eodem referendum est celebre illud traiectoriarum orthogonalium problema, in quo etiam infinitae lineae curuae sub data aequatione inter abscissam x, applicatam y, et parametrum variabilem a, contentae proponuntur, atque eiusmodi linea curua quaeritur, quae illas omnes ad angulos rectos traiiciat. Ad quod problema soluendum applicata y vt functio ipsarum

ipſarum x et a ſpectari ſolet, ex cuius differentiatione talis forma $dy = p\,dx + q\,da$ emergere concipitur, tum vero peruenitur ad hanc aequationem differentialem: $dx(1+pp) + pq\,da = 0$, ſiue ad hanc: $dx + p\,dy = 0$, ex qua cum illa coniuncta parametrum a eliminari oportet, vt eliciatur aequatio inter x et y, naturam curuae quaeſitae exprimens. Quando quidem curuae ſecandae per aequationem algebraicam inter x et y dantur, res nullam habet difficultatem, cum inde valor ipſius y abſolute per x et a definiri, indeque per differentiationem valores litterarum p et q aſſignari queant, vnde aequatio differentialis inter duas tantum variabiles x et a obtinetur; verum ſi aequatio pro curuis ſecandis ipſa iam ſit differentialis, parametrum a vt quantitatem conſtantem inuoluens; quae idcirco erit huius formae $dy = p\,dx$, ſeu $y = \int p\,dx$ ante omnia inueſtigari oportet, cuius modi aequatio differentialis proditura eſſet, ſi praeter x etiam parameter a vt variabilis ſtatuatur, vt inde quantitas q innoteſcat; quae inueſtigatio ſaepe maxime ſit difficilis, atque adeo vires Analyſeos ſuperare videtur. Etiamſi autem ratio huius inueſtigationis ex ſolis calculi differentialis principiis ſit petenda, tamen in ipſa tractatione ingens ſtatim cernitur diſcrimen; propterea quod, cum differentiatio ordinaria nulli difficultati ſoleat eſſe obnoxia, hic tota difficultas in inuentione differentialium ex variabilitate parametri oriundorum reſideat, haecque ipſa inventio ſingulares regulas requirat. Quam ob rem non praeter neceſſitatem partes Analyſeos infinitorum multiplicari videbuntur, ſi inueſtigationem huiusmodi differentia-

 rentia-

rentialium, quae ex variabilitate parametri nascuntur, ad peculiarem calculum referamus, quem distinctionis causa *Calculum variationum* appellare liceat. Cuius necessitas adhuc clarius perspicietur, si perpendamus, eius vim multo latius patere, quam ad solam parametrorum variabilitatem, qua etsi lineae curuae in infinitum multiplicantur, omnes tamen semper sub certo quodam genere, quod scilicet in data aequatione contineatur, comprehenduntur. Nostrum autem calculum variationum non solum ad huiusmodi genera curuarum determinata extendi conueniet, sed etiam ad omnes omnino curuas, quae quidem concipi queant, veluti si inter omnes plane curuas ea sit definienda, quae data quapiam maximi minimiue proprietate gaudeat. Atque huc referendum erit celebratissimum illud problema isoperimetricum latissimo sensu acceptum, prout id quidem in libro singulari pertractaui; quem qui attente legerit, non dubitabit, quin huius generis inuestigationes calculi speciem prorsus singularem postulent, a consuetis Analyseos regulis non parum diuersam. Haec enim problemata ad talem quaestionem reducuntur, vt eiusmodi aequatio inter binas variabiles x et y determinetur, ex qua expressio quaepiam integralis $\int Z dx$, quomodocunque Z per x et y componatur, maximum siue minimum valorem consequatur. Ad hoc autem efficiendum necesse est, vt proposita huiusmodi formula $\int Z dx$ quacunque, quae quidem ex assumta quauis relatione inter x et y determinatum valorem accipiat, in genere definiatur, quantam mutationem ea formula sit subitura, si ipsa relatio inter x et et y infinite parum quomodocunque varietur; haecque quaestio iam infinities latius patet, quam superior, vbi tantum mutatio ex variatione parametri oriunda assignari debebat. Potest etiam loco simplicis formulae integralis $\int Z dx$ expressio quaecunque, vtcunque ex x, y, harumque differentialibus atque formulis integralibus composita, considerari, quo longius haec tractatio extendatur; tum vero calculus variationum regulas suppeditabit, mutationem huiusmodi expressionum definiendi, quae ob relationem variabilium x et y, vtcunque infinite parum mutatam, ipsis inducitur. Methodus quidem *in solutione problematum isoperimetricorum* adhiberi solita iam eximia huius calculi specimina suggerit, quae autem cum sint ex alieno quasi fonte, Geometria scilicet, hausta, non ad constitutionem principiorum istius calculi desiderati referri possunt. Deinde vero etiam haec specimina ad scopum nimis particularem sunt adstricta, quam vt amplitudinem nostri calculi complecti possint. Quam ob rem constitui eius elementa ex primis Analyseos principiis repetere, eaque ita euoluere, vt non solum problematibus supra commemoratis facile et concinne soluendis inseruire possint, sed etiam nouum quasi campum aperiant, sese ad plurima alia talium quaestionum genera extendentem, in quo Geometrae non sine insigni finium Analyseos promotione vires suas exercere queant.

Elemen-

Elementa Calculi Variationum.

Hypothesis 1.

1.

Detur inter variabiles binas x et y aequatio quaecunque, qua earum relatio mutua exprimatur, ita vt inde quicunque valor determinatus ipsi x tribuatur, valor quoque determinatus pro y definiatur.

Coroll. 1.

2. Proposita ergo aequatione inter binas variabiles x et y, singulis valoribus ipsius x, quicunque concipi possunt, determinati valores ipsius y respondebunt.

Coroll. 2.

3. Vi ergo istius aequationis propositae erit y certa quaedam functio ipsius x, et quemadmodum ipsi x respondet y, ita illius valori sequenti $x' = x + dx$, respondebit $y' = y + dy$, cuius differentia a praecedente y, differentiale nempe dy, per vulgares differentiandi regulas assignari poterit.

Coroll. 3.

4. Cum y sit functio ipsius x, etiam $\frac{dy}{dx}$ erit functio ipsius x, per relationem inter x et y datam assignabilis; ac si ponatur $\frac{dy}{dx} = p$, simili modo $\frac{dp}{dx}$ erit certa

certa functio ipsius x: ac si porro ponamus $\frac{dp}{dx} = q$, $\frac{dq}{dx} = r$, $\frac{dr}{dx} = s$ etc. etiam hae quantitates q, r, s etc. erunt certae functiones ipsius x itidem per relationem inter x et y datam assignabiles.

Coroll. 4.

5. Si deinde V sit expressio quomodocunque ex x et y conflata, ea quoque ope relationis inter x et y datae ita erit comparata, vt pro omnibus valoribus ipsius x valores determinatos adipiscatur. Ac si V' designet valorem sequentem, seu ipsi $x + dx$ conuenientem; erit $V' = V + dV$, siue $dV = V' - V$, secundum prima calculi differentialis principia.

Hypothesis 2.

6. Quaecunque proponatur relatio inter x et y, quia inde simul relatio differentialium dx et dy innotescit, ponam in sequentibus perpetuo:

$$\frac{dy}{dx} = p;\ \frac{dp}{dx} = q;\ \frac{dq}{dx} = r;\ \frac{dr}{dx} = s \text{ etc.}$$

eruntque p, q, r, s etc. perinde ac y, functiones ipsius x, per illam relationem datam assignabiles.

Coroll. 1.

7. Quemadmodum littera p relationem differentialium dx et dy continet, ita q complectetur relationem differentialium secundi gradus; r vero differentialium tertii gradus, s quarti gradus etc.

Coroll. 2.

8. Viciſſim igitur etiam, ſi qua in expreſſione V differentialia, ſiue primi, ſiue ſecundi, ſiue altioris ordinis, inſint, ea introducendis his qnantitatibus p, q, r, s, etc. ex calculo tolli poterunt.

Axioma.

9. Si inter variabiles x et y alia relatio a propoſita infinite parum diſcrepans conſtituatur, valores ipſius y, ſingulis valoribus ipſius x reſpondentes, ab iis, quos propoſita relatio praebet, infinite parum diſcrepabunt.

Coroll. 1.

10. Cum huiusmodi relatio variata infinitis modis a relatione propoſita diſcrepare poſſit, ita vt diſcrepantia ſit infinite parua, euenire poteſt, vt vnus pluresue valores ipſius y, qui certis valoribus ipſius x reſpondent, nullam inde mutationem patiantur.

Coroll. 2.

11. Iſta variatio relationis ita generalis concipi poteſt, vt inde omnes valores ipſius y mutationes quascunque patiantur, quo nullo modo a ſe inuicem pendeant. Quo igitur haec tractatio latiſſime pateat, huiusmodi variationem relationis generaliſſime conceptam intelligi conueniet.

Hypotheſis 3.

12. Si relatio inter x et y propoſita parum mutetur, valorem ipſius y, qui inde ipſi x reſpondet,

per

per $y + \delta y$ defignemus, ita vt δy variationem denotet, quam y ob variatam relationem patitur.

Coroll. 1.

13. Simili modo cum y' fit valor ipfi $x + dx$ vi relationis propofitae refpondens, eius valorem, qui eidem $x + dx$ vi relationis variatae conuenit, per $y' + \delta y'$ exprimamus, ita vt $\delta y'$ variationem ipfius y' denotet, quae ex variatione relationis oritur.

Coroll. 2.

14. Cum igitur fit $y' = y + dy$, erit $\delta y' = \delta (y + dy) = \delta y + \delta dy$, et $\delta dy = \delta y' - \delta y$. Denotabit autem δdy variationem ipfius dy, ex variatione inter x et y propofitae ortam.

Coroll. 3.

15. Quemadmodum autem y' ftatum fequentem ipfius y denotat, ftatu fcilicet fequente ad $x + dx$ relato; ita $\delta y'$ ftatum fequentem ipfius δy denotat, ex quo $\delta y' - \delta y$ exprimet differentiale ipfius δy, quod eft $d \delta y$. Cum ergo fit $\delta dy = \delta y' - \delta y$, erit $\delta dy = d \delta y$.

Coroll. 4.

16. Hinc ergo confequimur iftam infignem proprietatem: quod variatio differentialis ipfius y aequalis fit differentiali variationis ipfius y. Eft enim δdy variatio ipfius dy, hoc eft differentialis ipfius y, et $d \delta y$ eft differentiale ipfius δy, hoc eft variationis ipfius y.

Definitio 1.

17. Si V ſit expreſſio vtcunque ex x et y conflata, propoſita quadam relatione inter x et y, eius variatio, quam per δV indicabo, eſt incrementum, quod quantitas V capit, ſi relatio inter x et y propoſita infinite parum varietur.

Coroll. 1.

18. Probe ergo diſtingui oportet differentiale dV, et variatio δV; differentiale enim denotat incrementum ipſius V, dum x ſuo elemento dx augetur, manente relatione inter x et y propoſita; variatio autem denotat incrementum ipſius V, dum ipſa relatio variatur manente x.

Coroll. 2.

19. Cum per variationem relationis, inter x et y propoſitae, quantitas y incrementum capiat δy, manente x eadem; quomodocunque quantitas V ex x et y fuerit conflata, eius variatio reperietur, ſi loco y vbique ſcribatur $y + \delta y$, et a valore hinc pro V oriundo ipſe valor V ſubtrahatur.

Coroll. 3.

20. Scilicet ſi in V vbique pro y ſcribatur $y + \delta y$, prodibit valor variatus ipſius V, qui eſt $V + \delta V$; ipſa autem variatio reperitur, ſi a valore variato $V + \delta V$ valor primitiuus V ſubtrahatur.

Definitio.

Definitio 2.

21. Calculus variationum eſt methodus inueniendi variationes quantitatum vtcunque ex binis variabilibus x et y conflatarum, quas patiuntur, ſi relatio inter x et y propoſita infinite parum quomodocunque immutetur.

Coroll. 1.

22. Propoſita ergo relatione inter x et y, ſi V denotet quantitatem quomodocunque ab x et y pendentem, hic calculus docet inuenire variationem ipſius V, ſeu valorem ipſius δV.

Coroll. 2.

23. Quia relationem inter x et y datam vtcunque immutari aſſumimus, vt y pro ſingulis valoribus ipſius x variationes quascunque, quae etiam a ſe inuicem non pendeant, accipiat, hic calculus latiſſime patet, atque ad quasuis conditiones variationum datas accommodari poterit.

Scholion 1.

24. Quo vſus huius calculi clarius perſpici queat, exemplum afferamus. Propoſita ergo ſit haec relatio inter x et y:

$$aayy - bbxx = aabb$$

quae ſcribendo $b + db$ loco b infinite parum immutetur. Iam ſi proponatur quantitas quaepiam ab x et y pendens, veluti $\int \frac{\surd(dx^2 + dy^2)}{\surd y}$, huius variatio ex illa immutatione relationis oriunda ope iſtius calculi exhiberi

beri poterit; cum enim sit $y = \frac{b}{a} \sqrt{(aa - xx)}$ erit $\delta y = \frac{db}{a} \sqrt{(aa - xx)}$ quae est variatio ipsius y. Quemadmodum autem ex cognita variatione ipsius y quantitatum vtcunque ab y et x pendentium, ideoque etiam huius: $\int \frac{\sqrt{(dx^2 + dy^2)}}{\sqrt{y}}$, variationes inde natae determinari debeant, in hoc calculo est ostendendum; vnde patet, omnia, quae de variabilitate parametrorum passim sunt tradita, hic contineri. Deinde vero etiam quaestiones inuerti possunt, veluti si proposita huiusmodi formula $\int \frac{\sqrt{(dx^2 + dy^2)}}{\sqrt{y}}$ ea relatio inter x et y quaeratur, vnde variatio istius formulae datae prodeat magnitudinis, vel etiam nulla, quo posteriori casu relatio inuenta formulae propositae maximum minimumue valorem comparabit; atque huc referenda erunt omnia problemata, quae circa curuas maximi minimiue proprietate gaudentes adhuc sunt tractata.

Scholion 2.

25. Praecepta huius calculi ad diuersitatem rationis, qua formula quaepiam proposita V a binis variabilibus x et y pendet, sunt accommodanda, quae diuersitas cum sit infinita, eam ad aliquot genera praecipua reuocari conueniet. Primum ergo genus complectatur eas formulas, quae ex ipsis quantitatibus x et y earumque deriuatis $p = \frac{dy}{dx}$; $q = \frac{dp}{dx}$; $r = \frac{dq}{dx}$ etc. vtcunque sunt compositae, ita tamen, vt nullas formulas integrales inuoluant. Ad secundum genus refero eas formulas, quae integralia huiusmodi $\int Z dx$ contineant, ita tamen, vt ipsa

ipsa formula Z ad primum genus pertineat. Tertium genus comprehendet eiusmodi formulas, in quibus non solum integralia $\int Z\,dx$ insunt, sed vbi quantitas Z ipsa insuper integralia inuoluit. Tandem sequetur quartum genus, in quo formula varianda V non absolute, sed demum per aequationem differentialem, vel primi, vel adeo altioris gradus, definitur, quod genus vtique latissime patet, ac praecedentia omnia in se complectitur. Quod autem ad aequationem, qua relatio inter x et y exprimitur, attinet, etsi eam vt datam specto, tamen non definio, ne praecepta tradenda vllo modo limitentur.

Theorema 1.

26. Variatio differentialis cuiusuis quantitatis V aequalis est differentiali variationis eiusdem, seu est $\delta dV = d\delta V$.

Demonstratio.

Cum sit $dV = V' - V$ denotante V' valorem *sequentem* ipsius V, qui ipsi $x + dx$ conuenit, vti V ipsi x respondet, erit $\delta dV = \delta V' - \delta V$; verum $d\delta V$ exprimit differentiam inter δV eiusque valorem sequentem, qui est $\delta V'$, ita vt sit $d\delta V = \delta V' - \delta V$, vnde perspicuum est, esse $\delta dV = d\delta V$.

Coroll. 1.

27. Eodem modo, si loco V scribamus dV, patet esse $\delta ddV = d\delta dV$; sed $\delta dV = d\delta V$, vnde fit $d\delta dV = dd\delta V$, sicque aequales inter se erunt hae tres formae:

$$\delta ddV = d\delta dV = dd\delta V.$$

Coroll.

Coroll. 2.

28. Porro vero si et hic pro V scribamus dV, obtinebimus aequalitatem inter has quatuor formas:

$$\delta dddV = d\delta ddV = dd\delta dV = ddd\delta V$$

tum vero inter has quinque:

$$\delta d^4V = d\delta d^3V = d^2\delta d^2V = d^3\delta dV = d^4\delta V.$$

Coroll. 3.

29. Si habeatur differentiale cuiuscunque ordinis ipsius V, nempe $d^n V$, cuius variatio sit inuestiganda, erit:

$$\delta d^n V = d^m \delta d^{n-m} V = d^n \delta V$$

aequatur scilicet differentiali ordinis n ipsius variationis δV. Hinc ergo reducitur variatio differentialium ad differentiationem variationis.

Problema 1.

30. Determinare variationes quantitatum p, q, r, s etc. rationem differentialium ipsarum x et y in se continentium.

Solutio.

Quia variatio non ad x pertinere censetur, erit $\delta x = 0$, et variatio ipsius y, nempe δy, tanquam cognita spectatur. Hinc cum sit $p = \frac{dy}{dx}$, erit $\delta p = \frac{\delta dy}{dx} = \frac{d\delta y}{dx}$. Deinde ob $q = \frac{dp}{dx}$ erit $\delta q = \frac{\delta dp}{dx} = \frac{d\delta p}{dx}$; sumto autem elemento dx constante, est $d\delta p = \frac{dd\delta y}{dx}$, hincque $\delta q = \frac{dd\delta y}{dx^2}$; et $d\delta q = \frac{d^3 dy}{dx^2}$. Porro autem cum sit $r = \frac{dq}{dx}$, erit $\delta r = \frac{\delta dq}{dx} = \frac{d\delta q}{dx}$, ideoque $\delta r = \frac{d^3\delta y}{dx^3}$, vnde variationes quanti-

quantitatum ex x et y deriuatarum p, q, r, s, etc. ita se habebunt:

$$\delta p = \frac{d\delta y}{dx}; \ \delta q = \frac{d^2\delta y}{dx^2}; \ \delta r = \frac{d^3\delta y}{dx^3}; \ \delta s = \frac{d^4\delta y}{dx^4}; \text{ etc.}$$

si quidem elementum dx pro constante assumatur.

Coroll. 1.

31. Haec differentialia primi altiorumque graduum variationis δy determinantur per variationes valorum ipsius y, qui conueniunt sequentibus valoribus ipsius x, scilicet $x + dx$; $x + 2dx$; $x + 3dx$; etc. Si enim sequentes valores ipsius y ita exhibeantur: y', y''; y''', y'''' etc. eorumque variationes ita: $\delta y'$; $\delta y''$; $\delta y'''$; $\delta y''''$ ex natura differentialium nouimus esse:

$$d\delta y = \delta y' - \delta y; \ dd\delta y = \delta y'' - 2\delta y' + \delta y;$$
$$d^3\delta y = \delta y''' - 3\delta y'' + 3\delta y' - \delta y \text{ etc.}$$

Coroll. 2.

32. Si ergo solus valor y variationem pateretur, sequentes vero y', y'', y''' nulli essent obnoxiae, vt esset $\delta y' = 0$, $\delta y'' = 0$, $\delta y''' = 0$ etc. foret

$$d\delta y = -\delta y; \ dd\delta y = +\delta y; \ d^3\delta y = -\delta y; \ d^4\delta y = +\delta y \text{ etc.}$$

ideoque:

$$\delta p = -\frac{\delta y}{dx}; \ \delta q = +\frac{\delta y}{dx^2}; \ \delta r = -\frac{\delta y}{dx^3}; \ \delta s = +\frac{\delta y}{dx^4} \text{ etc.}$$

Problema 2.

33. Si V fuerit quantitas quomodocunque ex variabilibus x et y earumque differentialibus cuiuscunque

 ordinis

ordinis conflata, seu si fuerit functio quaecunque quantitatum x, y, p, q, r, s etc. determinare eius variationem δV.

Solutio.

Differentietur more consueto haec functio V, prodeatque

$$dV = M\,dx + N\,dy + P\,dp + Q\,dq + R\,dr + S\,ds + \text{etc.}$$

quod differentiale nil aliud est, nisi incrementum quod functio V capit, si loco quantitatum x, y, p, q, r, s etc. substituantur istae $x+dx$; $y+dy$; $p+dp$; $q+dq$; $r+dr$ etc. Simili ergo modo si pro x, y, p, q, r, s etc. substituantur

$$x+0;\ y+\delta y;\ p+\delta p,\ q+\delta q,\ r+\delta r,\ s+\delta s \text{ etc.}$$

incrementum, quod inde functio V capit, erit eius variatio:

$$\delta V = N\delta y + P\delta p + Q\delta q + R\delta r + S\delta s + \text{etc.}$$

Quare, si pro δp, δq, δr etc. valores supra inuenti scribantur, prodibit variatio quaesita:

$$\delta V = N\delta y + \frac{P\,d\delta y}{dx} + \frac{Q\,dd\delta y}{dx^2} + \frac{R\,d^3\delta y}{dx^3} + \frac{S\,d^4\delta y}{dx^4} \text{ etc.}$$

Theorema 2.

34. Proposita formula integrali quacunque $\int Z\,dx$ eius variatio aequalis est integrali variationis differentialis $Z\,dx$, seu erit $\delta \int Z\,dx = \int \delta Z\,dx$.

Demonstratio.

Cum $\int Z\,dx$ exprimat summam omnium $Z\,dx$, eius variatio $\delta \int Z\,dx$ comprehendet summam omnium varia-

variationum ipsius Zdx, seu erit $\delta\int Zdx = \int\delta Zdx$. Quod etiam hoc modo distinctius ostendi potest: sit $\int Zdx = V$, ita vt definiri oporteat δV; cum igitur sit $dV = Zdx$, erit $\delta dV = \delta Zdx = d\delta V$; vnde sumtis integralibus fiet $\delta V = \int\delta Zdx$.

Problema 3.

35. Proposita formula integrali $\int Zdx$, in qua Z quantitas quomodocunque ex ipsis x et y, earumque differentialibus cuiuscunque ordinis conflata, inuestigare eius variationem $\delta\int Zdx$.

Solutio.

Cum ergo Z sit functio ipsarum x, y, p, q, r, s etc. eius differentiale more consueto sumtum huiusmodi formam habebit:

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + Sds \text{ etc.}$$

vnde eiusdem quantitatis Z variatio erit:

$$\delta Z = N\delta y + \frac{Pd\delta y}{dx} + \frac{Qdd\delta y}{dx^2} + \frac{Rd^3\delta y}{dx^3} + \frac{Sd^4\delta y}{dx^4} \text{ etc.}$$

Cum nunc sit $\delta\int Zdx = \int\delta Zdx$, erit:

$$\delta\int Zdx = \int N\delta y dx + \int Pd\delta y + \int\frac{Qdd\delta y}{dx} + \int\frac{Rd^3\delta y}{dx^2} + \text{etc.}$$

ne iam in vlteriori reductione expressio δy turbet, ponamus tantisper $\delta y = \omega$, et reductiones ita se habebunt:

$$\int Pd\omega = P\omega - \int\omega dP$$

$$\int\frac{Qdd\omega}{dx} = \frac{Qd\omega}{dx} - \int\frac{dQ}{dx}d\omega = \frac{Qd\omega}{dx} - \frac{\omega dQ}{dx} + \int\frac{\omega ddQ}{dx}$$

$$\int\frac{Rd^3\omega}{dx^2} = \frac{Rdd\omega}{dx^2} - \frac{dRd\omega}{dx^2} + \frac{\omega ddR}{dx^2} - \int\frac{\omega d^3R}{dx^2}$$

etc.

 Colli-

Colligantur omnes isti valores, et pro ω restituatur δy, sicque obtinebitur:

$$\delta\int Z dx = \int\delta y\, dx\left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.}\right)$$
$$+ \delta y\left(P - \frac{dQ}{dx} + \frac{ddR}{dx^2} - \frac{d^3S}{dx^3} + \text{etc.}\right)$$
$$+ \frac{d\delta y}{dx}\left(Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \text{etc.}\right)$$
$$+ \frac{dd\delta y}{dx^2}\left(R - \frac{dS}{dx} + \text{etc.}\right)$$
$$+ \frac{d^3\delta y}{dx^3}\left(S - \text{etc.}\right)$$

in qua expressione differentiale dx sumtum est constans.

Coroll. 1.

36. Constat ergo variatio formulae integralis $\int Z dx$ ex parte integrali $\int\delta y\, dx\left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.}\right)$ et partibus absolutis, quae praeter ipsam variationem δy etiam eius differentialia $d\delta y$, $dd\delta y$, $d^3\delta y$ etc. complectuntur.

Coroll. 2.

37. Partem autem integralem per reductiones adhibitas ita instruximus, vt tantum ipsam variationem δy complecteretur, ab eiusque differentialibus immunis exhiberetur, quae forma in applicatione calculi variationum maximam praestat vtilitatem.

Problema 4.

38. Si in formula integrali $\int Z dx$, quantitas Z non solum litteras x et y cum relationibus differentialium p, q, r, s etc. sed etiam formulam integralem $\Pi = \int \mathfrak{Z} dx$ vtcun-

vtcunque complectatur, in qua autem sit $\mathfrak{Z}$ functio ipsarum x, y, p, q, r, s etc. definire variationem formulae illius integralis $\int Z\,dx$.

Solutio.

Cum quantitas Z praeter quantitates x, y, p, q, r, s etc. etiam formulam integralem $\Pi = \int \mathfrak{Z}\,dx$ involuat, spectari poterit tanquam functio quantitatum Π, x, y, p, q, r, s etc. vnde si more solito differentietur, prodibit talis forma:

$$dZ = L\,d\Pi + M\,dx + N\,dy + P\,dp + Q\,dq + R\,dr + S\,ds \text{ etc.}$$

vnde colligitur variatio ipsius Z:

$$\delta Z = L\,\delta\Pi + N\,\delta y + P\,\delta p + Q\,\delta q + R\,\delta r + S\,\delta s + \text{etc.}$$

Cum deinde sit $\mathfrak{Z}$ functio ipsarum x, y, p, q, r, s etc. ponatur:

$$d\mathfrak{Z} = \mathfrak{M}\,dx + \mathfrak{N}\,dy + \mathfrak{P}\,dp + \mathfrak{Q}\,dq + \mathfrak{R}\,dr + \mathfrak{S}\,ds + \text{etc.}$$

atque ex praecedente problemate erit $\delta\Pi$, seu

$$\begin{aligned}\delta\int\mathfrak{Z}\,dx = {} & \int\delta y\,dx\left(\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \frac{d^4\mathfrak{S}}{dx^4} - \text{etc.}\right)\\ & + \delta y\left(\mathfrak{P} - \frac{d\mathfrak{Q}}{dx} + \frac{dd\mathfrak{R}}{dx^2} - \frac{d^3\mathfrak{S}}{dx^3} + \text{etc.}\right)\\ & + \frac{d\delta y}{dx}\left(\mathfrak{Q} - \frac{d\mathfrak{R}}{dx} + \frac{dd\mathfrak{S}}{dx^2} - \text{etc.}\right)\\ & + \frac{dd\delta y}{dx^2}\left(\mathfrak{R} - \frac{d\mathfrak{S}}{dx} + \text{etc.}\right)\\ & + \frac{d^3\delta y}{dx^3}\left(\mathfrak{S} - \text{etc.}\right)\end{aligned}$$

Vel sumatur potius prior forma:

$$\delta\int\mathfrak{Z}\,dx = \int\mathfrak{N}\,\delta y\,dx + \int\mathfrak{P}\,d\delta y + \int\frac{\mathfrak{Q}\,dd\delta y}{dx} + \int\frac{\mathfrak{R}\,d^3\delta y}{dx^2} + \int\frac{\mathfrak{S}\,d^4\delta y}{dx^3} + \text{etc.}$$

 eritque

eritque ob $\delta\Pi = \delta\int \mathfrak{Z}dx$:

$$\delta Z = \mathfrak{L}\int\mathfrak{N}\delta y\,dx + \mathfrak{L}\int\mathfrak{P}\,d\delta y + \mathfrak{L}\int\frac{\mathfrak{Q}\,dd\delta y}{dx} + \mathfrak{L}\int\frac{\mathfrak{R}\,d^3\delta y}{dx^2} + \mathfrak{L}\int\frac{\mathfrak{S}\,d^4\delta y}{dx^3} + \text{etc.}$$
$$+ N\delta y + \frac{P\,d\delta y}{dx} + \frac{Q\,dd\delta y}{dx^2} + \frac{R\,d^3\delta y}{dx^3} + \frac{S\,d^4\delta y}{dx^4}.$$

Cum igitur sit $\delta\int Z dx = \int\delta Z dx$, habebimus:

$$\delta\int Z dx = \int\mathfrak{L}dx\int\mathfrak{N}\delta y\,dx + \int\mathfrak{L}dx\int\mathfrak{P}\,d\delta y + \int\mathfrak{L}dx\int\frac{\mathfrak{Q}\,dd\delta y}{dx} + \int\mathfrak{L}dx\int\frac{\mathfrak{R}\,d^3\delta y}{dx^2} + \text{etc.}$$
$$+ \int N\delta y\,dx + \int P\,d\delta y + \int\frac{Q\,dd\delta y}{dx} + \int\frac{R\,d^3\delta y}{dx^2} + \text{etc.}$$

Ponatur $\int\mathfrak{L}dx = W$, seu $\mathfrak{L}dx = dW$, et ob

$$\int\mathfrak{L}dx\int\mathfrak{N}\delta y\,dx = W\int\mathfrak{N}\delta y\,dx - \int\mathfrak{N}W\delta y\,dx$$
$$\int\mathfrak{L}dx\int\mathfrak{P}\,d\delta y = W\int\mathfrak{P}\,d\delta y - \int\mathfrak{P}W\,d\delta y$$
$$\int\mathfrak{L}dx\int\frac{\mathfrak{Q}\,dd\delta y}{dx} = W\int\frac{\mathfrak{Q}\,dd\delta y}{dx} - \int\frac{\mathfrak{Q}W\,dd\delta y}{dx}$$

obtinebimus:

$$\delta\int Z dx = W\int\mathfrak{N}\delta y\,dx + W\int\mathfrak{P}\,d\delta y + W\int\frac{\mathfrak{Q}\,dd\delta y}{dx} + W\int\frac{\mathfrak{R}\,d^3\delta y}{dx^2} + \text{etc.}$$
$$+ \int(N-\mathfrak{N}W)\delta y\,dx + \int(P-\mathfrak{P}W)\,d\delta y$$
$$+ \int(Q-\mathfrak{Q}W)\frac{dd\delta y}{dx} + \int(R-\mathfrak{R}W)\frac{d^3\delta y}{dx^2} + \text{etc.}$$

Hae formulae eodem modo vt supra reductae dabunt:

$$\delta\int Z dx = W\int\delta y\,dx\left(\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \text{etc.}\right)$$
$$+ W\delta y\left(\mathfrak{P} - \frac{d\mathfrak{Q}}{dx} + \frac{dd\mathfrak{R}}{dx^2} - \text{etc.}\right)$$
$$+ \frac{W\,d\delta y}{dx}\left(\mathfrak{Q} - \frac{d\mathfrak{R}}{dx} + \text{etc.}\right)$$
$$+ \frac{W\,dd\delta y}{dx^2}\left(\mathfrak{R} - \text{etc.}\right)$$
$$+ \int\delta y\,dx\left((N-\mathfrak{N}W) - \frac{d(P-\mathfrak{P}W)}{dx} + \frac{dd(Q-\mathfrak{Q}W)}{dx^2} - \frac{d^3(R-\mathfrak{R}W)}{dx^3} + \text{etc.}\right)$$
$$+ \delta y\left((P-\mathfrak{P}W) - \frac{d(Q-\mathfrak{Q}W)}{dx} - \frac{dd(R-\mathfrak{R}W)}{dx^2} - \text{etc.}\right)$$
$$+ \frac{d\delta y}{dx}\left((Q-\mathfrak{Q}W) - \frac{d(R-\mathfrak{R}W)}{dx} + \text{etc.}\right)$$
$$+ \frac{dd\delta y}{dx^2}\left((R-\mathfrak{R}W) - \text{etc.}\right)$$

Coroll.

Coroll. 1.

39. Quia reductiones adhibitae quouis casu facile expediri possunt, iis praetermissis variatio quaesita hoc modo succinctius exhibitur, posito $W = \int L\,dx$:

$$\delta\int Z\,dx = W\int dx\left(\mathfrak{N}\delta y + \frac{\mathfrak{P}\,d\delta y}{dx} + \frac{\mathfrak{Q}\,dd\delta y}{dx^2} + \frac{\mathfrak{R}\,d^3\delta y}{dx^3} + \text{etc.}\right) + \int dx\left((N - \mathfrak{N}W)\delta y + (P - \mathfrak{P}W)\frac{d\delta y}{dx} + (Q - \mathfrak{Q}W)\frac{dd\delta y}{dx^2} + (R - \mathfrak{R}W)\frac{d^3\delta y}{dx^3} + \text{etc.}\right)$$

Coroll. 2.

40. Ac si quantitas Z inuoluat insuper aliam formulam integralem $\Pi' = \int \mathfrak{Z}'\,dx$, vt sit:

$$dZ = L\,d\Pi + L'\,d\Pi' + M\,dx + N\,dy + P\,dp + Q\,dq + \text{etc.}$$

tum vero:

$$d\mathfrak{Z}' = \mathfrak{M}'\,dx + \mathfrak{N}'\,dy + \mathfrak{P}'\,dp + \mathfrak{Q}'\,dq + \mathfrak{R}'\,dr + \text{etc.}$$

Si ponatur $\int L\,dx = W$, $\int L'\,dx = W'$, insuperque ad abbreuiandum:

$$N - \mathfrak{N}W - \mathfrak{N}'W' = (N);\quad P - \mathfrak{P}W - \mathfrak{P}'W' = (P)$$
$$Q - \mathfrak{Q}W - \mathfrak{Q}'W' = (Q);\quad R - \mathfrak{R}W - \mathfrak{R}'W' = (R)\ \text{etc.}$$

erit variatio quaesita:

$$\delta\int Z\,dx = W\int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \mathfrak{R}\frac{d^3\delta y}{dx^3} + \text{etc.}\right) + W'\int dx\left(\mathfrak{N}'\delta y + \mathfrak{P}'\frac{d\delta y}{dx} + \mathfrak{Q}'\frac{dd\delta y}{dx^2} + \mathfrak{R}'\frac{d^3\delta y}{dx^3} + \text{etc.}\right) + \int dx\left((N)\delta y + (P)\frac{d\delta y}{dx} + (Q)\frac{dd\delta y}{dx^2} + (R)\frac{d^3\delta y}{dx^3} + \text{etc.}\right)$$

Problema 5.

41. Si in formula $\int Z\,dx$ quantitas Z praeter litteras x, y, p, q, r etc. inuoluat formulam integralem

lem $\Pi = \int \mathfrak{Z}\, dx$, in qua quantitas $\mathfrak{Z}$ praeter litteras x, y, p, q, r etc. insuper complectatur formulam integralem $\pi = \int \mathfrak{z}\, dx$, vbi $\mathfrak{z}$ autem sit functio solarum litterarum x, y, p, q, r etc. inuenire variationem formulae $\int Z\, dx$.

Solutio.

Cum Z sit functio quantitatum x, y, p, q, r etc. et $\Pi = \int \mathfrak{Z}\, dx$, eius differentiale more consueto sumtum erit huius formae:

$$dZ = L\, d\Pi + M\, dx + N\, dy + P\, dp + Q\, dq + R\, dr + \text{etc.}$$

ideoque eius variatio

$$\delta Z = L\, \delta\Pi + N\, \delta y + P \frac{d\,\delta y}{dx} + Q \frac{dd\,\delta y}{dx^2} + R \frac{d^3\,\delta y}{dx^3} + \text{etc.}$$

vnde variatio quaesita erit $\delta \int Z\, dx = \int \delta Z\, dx =$

$$\int L\, dx\, \delta\Pi + \int dx \left(N\, \delta y + P \frac{d\,\delta y}{dx} + Q \frac{dd\,\delta y}{dx^2} + R \frac{d^3\,\delta y}{dx^3} + \text{etc.}\right)$$

At cum sit $\mathfrak{Z}$ functio quantitatum x, y, p, q, r etc. et $\pi = \int \mathfrak{z}\, dx$, erit differentiando:

$$d\mathfrak{Z} = \mathfrak{L}\, d\pi + \mathfrak{M}\, dx + \mathfrak{N}\, dy + \mathfrak{P}\, dp + \mathfrak{Q}\, dq + \mathfrak{R}\, dr + \text{etc.}$$

hincque variatio eius

$$\delta \mathfrak{Z} = \mathfrak{L}\, \delta\pi + \mathfrak{N}\, \delta y + \mathfrak{P} \frac{d\,\delta y}{dx} + \mathfrak{Q} \frac{dd\,\delta y}{dx^2} + \mathfrak{R} \frac{d^3\,\delta y}{dx^3} + \text{etc.}$$

quare cum sit $\Pi = \int \mathfrak{Z}\, dx$, erit $\delta\Pi = \delta \int \mathfrak{Z}\, dx = \int \delta \mathfrak{Z}\, dx$, ac propterea:

$$\delta\Pi = \int \mathfrak{L}\, dx\, \delta\pi + \int dx \left(\mathfrak{N}\, \delta y + \mathfrak{P} \frac{d\,\delta y}{dx} + \mathfrak{Q} \frac{dd\,\delta y}{dx^2} + \mathfrak{R} \frac{d^3\,\delta y}{dx^3} + \text{etc.}\right)$$

vnde reperitur:

$$\int L\, dx\, \delta\Pi = \int L\, dx \int \mathfrak{L}\, dx\, \delta\pi + \int L\, dx \int dx \left(\mathfrak{N}\, \delta y + \mathfrak{P} \frac{d\,\delta y}{dx} + \mathfrak{Q} \frac{dd\,\delta y}{dx^2} + \text{etc.}\right)$$

Supereſt

Superest ergo vt definiamus $\delta\pi$, est autem $\pi = \int z\,dx$, et quia z est functio litterarum x, y, p, q, r etc. tantum, fiat differentiando:

$$dz = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.}$$

ex quo concluditur eius variatio:

$$\delta z = \mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \mathfrak{R}\frac{d^3\delta y}{dx^3} + \text{etc.}$$

tum vero, ob $\delta\pi = \delta\int z\,dx = \int\delta z\,dx$, erit:

$$\delta\pi = \int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \mathfrak{R}\frac{d^3\delta y}{dx^3} + \text{etc.}\right)$$

Quamobrem habebimus $\int L\,dx\int\mathfrak{L}\,dx\,\delta\pi =$

$$\int L\,dx\int\mathfrak{L}\,dx\int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \mathfrak{R}\frac{d^3\delta y}{dx^3} + \text{etc.}\right)$$

Vt iam hanc formulam a signis integralibus multiplicatis liberemus, ponamus $\int L\,dx = W$, eritque:

$$\int L\,dx\,\delta\Pi = W\delta\Pi - \int W\,d\delta\Pi,$$

verum $d\delta\Pi = \delta\mathfrak{Z}\,dx$, vnde $\int L\,dx\,\delta\Pi = W\delta\Pi - \int W\delta\mathfrak{Z}\,dx$ ideoque:

$$\int L\,dx\,\delta\Pi = W\int\mathfrak{L}\,dx\,\delta\pi + W\int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.}\right)$$
$$-\int\mathfrak{L}W\,dx\,\delta\pi \quad \int W\,dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.}\right)$$

sit $\int\mathfrak{L}\,dx = \mathfrak{W}$, erit $\int\mathfrak{L}\,dx\,\delta\pi = \mathfrak{W}\delta\pi - \int\mathfrak{W}\delta z\,x$ hincque:

$$\int\mathfrak{L}\,dx\,\delta\pi = \mathfrak{W}\int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \mathfrak{R}\frac{d^3\delta y}{dx^3} + \text{etc.}\right)$$
$$-\int\mathfrak{W}\,dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \mathfrak{R}\frac{d^3\delta y}{dx^3} + \text{etc.}\right)$$

Porro ponatur $\int\mathfrak{L}W\,dx = \int W\,d\mathfrak{W} = \mathfrak{V}$, vt sit:

$$\int\mathfrak{L}W\,dx\,\delta\pi = \mathfrak{V}\int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \mathfrak{R}\frac{d^3\delta y}{dx^3} + \text{etc.}\right)$$
$$-\int\mathfrak{V}\,dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \mathfrak{R}\frac{d^3\delta y}{dx^3} + \text{etc.}\right)$$

Ex his omnibus colligetur variatio quaesita $\delta\int Z dx =$

$$(W\mathfrak{W}-\mathfrak{V})\int dx\left(\mathfrak{n}\delta y+\mathfrak{p}\frac{d\delta y}{dx}+\mathfrak{q}\frac{dd\delta y}{dx^2}+\text{etc.}\right)$$
$$-W\int\mathfrak{W}dx\left(\mathfrak{n}\delta y+\mathfrak{p}\frac{d\delta y}{dx}+\mathfrak{q}\frac{dd\delta y}{dx^2}+\text{etc}\right)$$
$$+\int\mathfrak{V}dx\left(\mathfrak{n}\delta y+\mathfrak{p}\frac{d\delta y}{dx}+\mathfrak{q}\frac{dd\delta y}{dx^2}+\text{etc.}\right)$$
$$+W\int dx\left(\mathfrak{N}\delta y+\mathfrak{P}\frac{d\delta y}{dx}+\mathfrak{Q}\frac{dd\delta y}{dx^2}+\text{etc.}\right)$$
$$-\int W dx\left(\mathfrak{N}\delta y+\mathfrak{P}\frac{d\delta y}{dx}+\mathfrak{Q}\frac{dd\delta y}{dx^2}+\text{etc.}\right)$$
$$+\int dx\left(N\delta y+P\frac{d\delta y}{dx}+Q\frac{dd\delta y}{dx^2}+\text{etc.}\right)$$

Coroll. 1.

42. Si quaeratur variatio formulae $\int Z dx$ a valore $x=0$ vsque ad valorem determinatum $x=a$, sumantur integralia $W=\int L dx$, $\mathfrak{W}=\int\mathfrak{L}dx$ et $\mathfrak{V}=\int W d\mathfrak{W}$, ita vt euanescant posito $x=0$, tum vero facto $x=a$ fiat $W=A$, $\mathfrak{W}=\mathfrak{A}$ et $\mathfrak{V}=\mathfrak{B}$, quos valores in formula inuenta loco litterarum W, $\mathfrak{W}$ et $\mathfrak{V}$, vbi extra signum integrale occurrunt, ponere licebit.

Coroll. 2.

43. Ponatur ergo ad abbreuiandum:

$$N+(A-W)\mathfrak{N}+(A\mathfrak{A}-\mathfrak{B}-A\mathfrak{W}+\mathfrak{V})\mathfrak{n}=(N)$$
$$P+(A-W)\mathfrak{P}+(A\mathfrak{A}-\mathfrak{B}-A\mathfrak{W}+\mathfrak{V})\mathfrak{p}=(P)$$
$$Q+(A-W)\mathfrak{Q}+(A\mathfrak{A}-\mathfrak{B}-A\mathfrak{W}+\mathfrak{V})\mathfrak{q}=(Q)$$
$$R+(A-W)\mathfrak{R}+(A\mathfrak{A}-\mathfrak{B}-A\mathfrak{W}+\mathfrak{V})\mathfrak{r}=(R)$$
etc.

et variatio quaesita formulae $\int Z dx$ vsque ad valorem determinatum $x=a$ erit:

$$\int dx\left((N)\delta y+(P)\frac{d\delta y}{dx}+(Q)\frac{dd\delta y}{dx^2}+(R)\frac{d^3\delta y}{dx^3}+\text{etc.}\right)$$

Coroll.

Coroll. 3.

44. Quodsi iam hic reductiones superiores adhibeantur, reperietur eadem variatio ita expressa:

$$\begin{aligned}\delta\int Z\,dx = \int dx\,\delta y\left((N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \text{etc.}\right) \\ + \delta y\left((P) - \frac{d(Q)}{dx} + \frac{dd(R)}{dx^2} - \text{etc.}\right) \\ + \frac{d\delta y}{dx}\left((Q) - \frac{d(R)}{dx} + \text{etc.}\right) \\ + \frac{dd\delta y}{dx^2}\left((R) - \text{etc.}\right)\end{aligned}$$

Coroll. 4.

45. Cum sit $\mathfrak{V} = \int W\,d\mathfrak{W}$, erit $A\mathfrak{W} - \mathfrak{V} = \int(A-W)\mathfrak{L}\,dx$; quare si ponatur integrale $\int(A-W)\mathfrak{L}\,dx = X$, ita sumtum, vt euanescat, posito $x=0$, tum vero facto $x=a$, fiat $X=B$, ita vt sit:

$\int \mathfrak{L}\,dx = W$, et posito $x=a$, fiat $W=A$,

$\int(A-W)\mathfrak{L}\,dx = X$, et posito $x=a$, fiat $X=B$

superiores valores Coroll. 2. exhibiti ita se habebunt:

$$\begin{aligned}N + (A-W)\mathfrak{N} + (B-X)\mathfrak{n} &= (N)\\ P + (A-W)\mathfrak{P} + (B-X)\mathfrak{p} &= (P)\\ Q + (A-W)\mathfrak{Q} + (B-X)\mathfrak{q} &= (Q)\\ R + (A-W)\mathfrak{R} + (B-X)\mathfrak{r} &= (R)\\ &\text{etc.}\end{aligned}$$

Problema 6.

46. Si in formula integrali $\Phi = \int Z\,dx$, quantitas Z praeter litteras x, y, p, q, r etc. etiam ipsam formulam integralem Φ inuoluat, determinare eius variationem $\delta\Phi = \delta\int Z\,dx$.

Solutio.

Solutio.

Cum Z sit functio quantitatum x, y, p, q, r etc. insuperque ipsam formulam integralem $\Phi = \int Z dx$ involuat, differentietur more solito ac prodeat

$$dZ = L d\Phi + M dx + N dy + P dp + Q dq + R dr + \text{etc.}$$

Hinc igitur erit variatio ipsius Z scilicet:

$$\delta Z = L \delta \Phi + N \delta y + P \frac{d \delta y}{dx} + Q \frac{dd \delta y}{dx^2} + R \frac{d^3 \delta y}{dx^3} + \text{etc.}$$

ideoque ob $\delta \Phi = \delta \int Z dx = \int \delta Z dx$

$$\delta \Phi = \int L dx \delta \Phi + \int dx \left(N \delta y + P \frac{d\delta y}{dx} + Q \frac{dd\delta y}{dx^2} + R \frac{d^3\delta y}{dx^3} + \text{etc.}\right)$$

Ponamus $\delta \Phi = z$, cum sit id ipsum quod quaeritur, et breuitatis gratia $\int dx \left(N dy + P \frac{d\delta y}{dx} + Q \frac{dd\delta y}{dx^2} + \text{etc.}\right) = u$, vt habeatur $z = \int L z dx + u$, et differentiando:

$$dz = L z dx + du, \text{ eritque integrando}$$

$$z = e^{\int L dx} \int e^{-\int L dx} du,$$

statuatur breuitatis gratia $\int L dx = W$, et habebitur variatio quaesita:

$$\delta \int Z dx = e^{W} \int e^{-W} dx \left(N \delta y + P \frac{d\delta y}{dx} + Q \frac{dd\delta y}{dx^2} + \text{etc.}\right)$$

si desideretur variatio vsque ad datum terminum $x = a$, fiatque tum $W = A$; ponatur ad abbreuiandum:

$$e^{A-W} N = (N);\ e^{A-W} P = (P);\ e^{A-W} Q = (Q) \text{ etc.}$$

eritque reductionibus vt supra factis variatio:

$$\begin{aligned} \delta \int Z dx = \int dx \delta y & \left((N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \text{etc.}\right) \\ & + \delta y \left((P) - \frac{d(Q)}{dx} + \frac{dd(R)}{dx^2} - \text{etc.}\right) \\ & + \frac{d\delta y}{dx} \left((Q) - \frac{d(R)}{dx} + \text{etc.}\right) \\ & + \frac{dd\delta y}{dx^2} \left((R) - \text{etc.}\right) \end{aligned}$$

Corol-

Corollarium.

47. Si ergo quantitas varianda Φ definiatur per hanc aequationem differentialem $d\Phi = Z\,dx$, in qua Z inuoluat vtcunque ipsam quantitatem Φ et insuper litteras x, y, p, q, r etc. eius variatio $\delta\Phi$ per hoc problema assignari poterit.

Problema 7.

48. Si in formula integrali $\Phi = \int Z\,dx$ quantitas Z praeter litteras x, y, p, q, r etc. non solum ipsam quantitatem Φ, sed insuper adhuc aliam formulam integralem $\Pi = \int \mathfrak{Z}\,dx$ quomodocunque implicet, in qua autem quantitas $\mathfrak{Z}$ tantum per litteras x, y, p, q, r etc. detur; inuestigare variationem huius formulae $\delta\Phi = \delta\int Z\,dx = \int \delta Z\,dx$.

Solutio.

Cum Z sit functio quantitatum x, y, p, q, r etc. insuperque formularum $\Phi = \int Z\,dx$, et $\Pi = \int \mathfrak{Z}\,dx$, praebeat ea differentiando:

$$dZ = K\,d\Phi + L\,d\Pi + M\,dx + N\,dy + P\,dp + Q\,dq + \text{etc.}$$

vnde eius variatio erit:

$$\delta Z = K\,\delta\Phi + L\,\delta\Pi + N\,\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{etc.}$$

Porro autem cum $\mathfrak{Z}$ sit functio litterarum x, y, p, q, r etc. tantum ponatur:

$$d\mathfrak{Z} = \mathfrak{M}\,dx + \mathfrak{N}\,dy + \mathfrak{P}\,dp + \mathfrak{Q}\,dq + \mathfrak{R}\,dr + \text{etc.}$$

eritque ob $\delta\Pi = \int \delta\mathfrak{Z}\,dx$

$$\delta\Pi = \int dx\left(\mathfrak{N}\,\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \mathfrak{R}\frac{d^3\delta y}{dx^3} + \text{etc.}\right)$$

Ponatur vt ante $\delta\Phi = z$ et $L\delta\Pi + N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{etc.} = u$; ob $\delta\Phi = \int\delta Z dx = z$ erit $\delta Z = \frac{dz}{dx}$ ideoque $\frac{dz}{dx} = Kz + u$; vnde oritur

$$z = e^{\int K dx}\int e^{-\int K dx} u dx = \delta\Phi$$

sit $\int K dx = V$, eritque

$$e^{-\int K dx} u dx = e^{-V}\mathfrak{L} dx \int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.}\right)$$
$$+ e^{-V} dx\left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{etc.}\right)$$

statuatur porro $\int e^{-V} L dx = W$, eritque integrando variatio quaesita:

$$\delta\Phi = e^{V} W \int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.}\right)$$
$$- e^{V}\int W dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.}\right)$$
$$+ e^{V}\int e^{-V} dx\left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{etc.}\right)$$

Si variatio vsque ad datum terminum $x = a$ desideretur, ac posito $x = a$, fiat $V = A$ et $W = B$, tum statuatur breuitatis gratia:

$$e^{A-V}N + e^{A}(B-W)\mathfrak{N} = (N)$$
$$e^{A-V}P + e^{A}(B-W)\mathfrak{P} = (P)$$
$$e^{A-V}Q + e^{A}(B-W)\mathfrak{Q} = (Q)$$
$$e^{A-V}R + e^{A}(B-W)\mathfrak{R} = (R)$$
etc.

quo facto erit variatio formulae $\Phi = \int Z dx$ vsque ad terminum $x = a$ extensa:

$$\delta\Phi = \int dx\,\delta y\left((N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \text{etc.}\right.$$
$$+ \delta y\left((P) - \frac{d(Q)}{dx} + \frac{dd(R)}{dx^2} - \text{etc.}\right)$$
$$+ \frac{d\delta y}{dx}\left((Q) - \frac{d(R)}{dx} + \text{etc.}\right)$$
$$+ \frac{dd\delta y}{dx^2}\left((R) - \text{etc.}\right)$$

Corol-

Corollarium.

49. Sic ergo variatio definitur quantitatis Φ per aequationem differentialem $d\Phi = Z\,dx$ datae, in qua Z non solum praeter litteras x, y, p, q, r etc. ipsam Φ, sed insuper formulam integralem $\int \mathfrak{Z}\,dx = \Pi$ utcunque involuit, dummodo $\mathfrak{Z}$ per solas litteras x, y, p, q, r etc. determinetur.

Problema 8.

50. Si in formula integrali $\Phi = \int Z\,dx$ quantitas Z praeter litteras x, y, p, q, r etc. formulam integralem $\Pi = \int \mathfrak{Z}\,dx$ involuat, hic autem quantitas $\mathfrak{Z}$ praeter litteras x, y, p, q, r etc. ipsam formulam integralem $\Pi = \int \mathfrak{Z}\,dx$ contineat, definire variationem formulae propositae $\Phi = \int Z\,dx$.

Solutio.

Cum Z sit functio quantitatum x, y, p, q, r etc. et ipsius $\Pi = \int \mathfrak{Z}\,dx$, eius differentiale erit huiusmodi:

$$dZ = L\,d\Pi + M\,dx + N\,dy + P\,dp + Q\,dq + R\,dr + \text{etc.}$$

hinc eius variatio erit

$$\delta Z = L\,\delta\Pi + N\,\delta y + P\frac{d\,\delta y}{dx} + Q\frac{dd\,\delta y}{dx^2} + R\frac{d^3\,\delta y}{dx^3} + \text{etc.}$$

ex quo ob $\delta\Phi = \int \delta Z\,dx$ habebitur:

$$\delta\Phi = \int L\,dx\,\delta\Pi + \int dx\left(N\,\delta y + P\frac{d\,\delta y}{dx} + Q\frac{dd\,\delta y}{dx^2} + \text{etc.}\right)$$

At quia $\mathfrak{Z}$ est functio ipsarum x, y, p, q, r etc. et $\Pi = \int \mathfrak{Z}\,dx$, sit eius differentiale:

$$d\mathfrak{Z} = \mathfrak{L}\,d\Pi + \mathfrak{M}\,dx + \mathfrak{N}\,dy + \mathfrak{P}\,dp + \mathfrak{Q}\,dq + \text{etc.}$$

eritque

$$\delta\mathfrak{Z} = \frac{d\,\delta\Pi}{dx} = \mathfrak{L}\,\delta\Pi + \mathfrak{N}\,\delta y + \mathfrak{P}\frac{d\,\delta y}{dx} + \mathfrak{Q}\frac{dd\,\delta y}{dx^2} + \text{etc.}$$

Ponatur

Ponatur $\int \mathfrak{L}\, dx = \mathfrak{W}$, eritque:

$$\delta \Pi = e^{\mathfrak{W}} \int e^{-\mathfrak{W}} dx \left(\mathfrak{N} \delta y + \mathfrak{P} \frac{d\delta y}{dx} + \mathfrak{Q} \frac{dd\delta y}{dx^2} + \text{etc.} \right)$$

Fiat $\int e^{\mathfrak{W}} L\, dx = W$ et obtinebitur:

$$\delta \Phi = W \int e^{-\mathfrak{W}} dx \left(\mathfrak{N} \delta y + \mathfrak{P} \frac{d\delta y}{dx} + \mathfrak{Q} \frac{dd\delta y}{dx^2} + \text{etc.} \right)$$
$$- \int e^{-\mathfrak{W}} W\, dx \left(\mathfrak{N} \delta y + \mathfrak{P} \frac{d\delta y}{dx} + \mathfrak{Q} \frac{dd\delta y}{dx^2} + \text{etc.} \right)$$
$$+ \int dx \left(N\, dy + P \frac{d\delta y}{dx} + Q \frac{dd\delta y}{dx^2} + \text{etc.} \right)$$

Si hanc variationem ad terminum $x = a$ vsque extendi oporteat, ac posito $x = a$ fiat $W = A$, vocetur breuitatis gratia

$$N + e^{-\mathfrak{W}} (A - W) \mathfrak{N} = (N)$$
$$P + e^{-\mathfrak{W}} (A - W) \mathfrak{P} = (P)$$
$$Q + e^{-\mathfrak{W}} (A - W) \mathfrak{Q} = (Q)$$
$$\text{etc.}$$

eritque reductiones supra expositas introducendo variatio formulae integralis $\Phi = \int Z dx$ ad terminum $x - a$ extensa:

$$\delta \int Z dx = \int dx\, \delta y \left((N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \text{etc.} \right)$$
$$+ \delta y \left((P) - \frac{d(Q)}{dx} + \frac{dd(R)}{dx^2} - \text{etc.} \right)$$
$$+ \frac{d\delta y}{dx} \left((Q) - \frac{(dR)}{dx} + \text{etc.} \right)$$
$$\frac{dd\delta y}{dx^2} \left((R) - \text{etc.} \right)$$

Scholion.

51. Vsus huius problematis cernitur in descensu corporum super lineis curuis in medio quocunque resistente, dum corpora a viribus quibuscunque sollicitantur, si variationem temporis descensus definire velimus, dum curua quomodocunque variatur. Denotet hoc

casu

casu Φ tempus descensus per arcum, qui abscissae x respondeat, sitque applicata $= y$, et Π altitudo celeritati acquisitae debita; ac tempus descensus erit $\Phi = \int \frac{dx\sqrt{(1+pp)}}{\sqrt{\Pi}}$, posito $dy = p\,dx$, vt $dx\sqrt{(1+pp)}$ elementum arcus designet. Verum ex sollicitationibus erit:

$$d\Pi = X\,dx + Y\,dy - V\sqrt{(dx^2 + dy^2)}$$

vbi X et Y significant functiones ipsarum x et y, et V functionem ipsius Π, cui resistentia est proportionalis. Erit ergo ob $dy = p\,dx$

$$\Pi = \int (X + Yp - V\sqrt{(1+pp)})\,dx$$

ideoque $\mathfrak{Z} = X + Yp - V\sqrt{(1+pp)}$, existente $Z = \frac{\sqrt{(1+pp)}}{\sqrt{\Pi}}$.

Corollarium.

52. *Si ad similitudinem valorum* (N), (P), (Q) *ponatur:*

$$M + e^{-\mathfrak{W}}(A - W)\mathfrak{M} = (M)$$

erit $(M)dx + (N)dy + (P)dp + (Q)dq + (R)dr +$ *etc. differentiale verum huius formulae:*

$$Z + e^{-\mathfrak{W}}(A - W)\mathfrak{Z}.$$

Conclusio.

53. Quaecunque ergo formula integralis $\Phi = \int Z\,dx$ proponatur, cuius variationem inuestigari oporteat, eius

variatio, vsque ad terminum $x=a$ extensa, semper exprimetur hoc modo:

$$\delta\Phi = \int dx\,\delta y\left((N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \frac{d^4(S)}{dx^4} - \text{etc.}\right)$$
$$+ \delta y\left((P) - \frac{d(Q)}{dx} + \frac{dd(R)}{dx^2} - \frac{d^3(S)}{dx^3} + \text{etc.}\right)$$
$$+ \frac{d\delta y}{dx}\left((Q) - \frac{d(R)}{dx} + \frac{dd(S)}{dx^2} - \text{etc.}\right)$$
$$+ \frac{dd\delta y}{dx^2}\left((R) - \frac{d(S)}{dx} + \text{etc.}\right)$$
$$+ \frac{d^3\delta y}{dx^3}\left((S) - \text{etc.}\right)$$

sumto elemento dx constante. Quemadmodum autem litterae (N), (P), (Q), (R), (S) etc. se habeant, id quouis casu patebit.

Casus I.

54. Si $dZ = Mdx + Ndy + Pdp + Qdq + Rdr + Sds$ etc. erit

$$(N) = N;\ (P) = P;\ (Q) = Q;\ (R) = R;\ (S) = S \text{ etc.}$$

Casus II.

55. Si $dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr +$ etc. existente $\Pi = \int \mathfrak{Z}\,dx$, et

$$d\mathfrak{Z} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.}$$

sit $\int L\,dx = W$, ac posito $x=a$, fiat $W=A$, quo facto erit:

$$(N) = N + (A-W)\mathfrak{N};\quad (P) = P + (A-W)\mathfrak{P}$$
$$(Q) = Q + (A-W)\mathfrak{Q};\quad (R) = R + (A-W)\mathfrak{R}$$
$$(S) = S + (A-W)\mathfrak{S};\quad \text{etc.}$$

Casus

Casus III.

56. Si fuerit

$$dZ = L\,d\Pi + L'\,d\Pi' + M\,dx + N\,dy + P\,dp + Q\,dq + R\,dr + \text{etc.}$$

existente $\Pi = \int \mathfrak{Z}\,dx$ et $\Pi' = \int \mathfrak{Z}'\,dx$, tum vero:

$$d\mathfrak{Z} = \mathfrak{M}\,dx + \mathfrak{N}\,dy + \mathfrak{P}\,dp + \mathfrak{Q}\,dq + \mathfrak{R}\,dr + \text{etc.}$$
$$d\mathfrak{Z}' = \mathfrak{M}'\,dx + \mathfrak{N}'\,dy + \mathfrak{P}'\,dp + \mathfrak{Q}'\,dq + \mathfrak{R}'\,dr + \text{etc.}$$

ponatur $\int L\,dx = W$, et $\int L'\,dx = W'$, ac facto $x = a$, fiat

$$W = A \quad \text{et} \quad W' = A'$$

quo facto erit:

$$(N) = N + (A - W)\mathfrak{N} + (A' - W')\mathfrak{N}'$$
$$(P) = P + (A - W)\mathfrak{P} + (A' - W')\mathfrak{P}'$$
$$(Q) = Q + (A - W)\mathfrak{Q} + (A' - W')\mathfrak{Q}'$$
$$(R) = R + (A - W)\mathfrak{R} + (A' - W')\mathfrak{R}'$$

etc.

Casus IV.

57. Si Z contineat formulam integralem $\Pi = \int \mathfrak{Z}\,dx$, vt sit:

$$dZ = L\,d\Pi + M\,dx + N\,dy + P\,dp + Q\,dq + R\,dr + \text{etc.}$$

quantitas $\mathfrak{Z}$ vero formulam integralem $\pi = \int \mathfrak{z}\,dx$, vt sit:

$$d\mathfrak{Z} = \mathfrak{L}\,d\pi + \mathfrak{M}\,dx + \mathfrak{N}\,dy + \mathfrak{P}\,dp + \mathfrak{Q}\,dq + \mathfrak{R}\,dr + \text{etc.}$$

at $\mathfrak{z}$ nullum porro integrale inuoluat, ita vt sit:

$$d\mathfrak{z} = \mathfrak{m}\,dx + \mathfrak{n}\,dy + \mathfrak{p}\,dp + \mathfrak{q}\,dq + \mathfrak{r}\,dr + \text{etc.}$$

Ponatur $\int L\,dx = W$, et posito $x = a$, fiat $W = A$;

 tum

tum vero ponatur $\int(A-W)\mathfrak{L}\,dx=\mathfrak{W}$, casuque $x=a$ fiat $\mathfrak{W}=\mathfrak{A}$, quo facto erit:

$$(N)=N+(A-W)\mathfrak{N}+(\mathfrak{A}-\mathfrak{W})\mathfrak{n}$$
$$(P)=P+(A-W)\mathfrak{P}+(\mathfrak{A}-\mathfrak{W})\mathfrak{p}$$
$$(Q)=Q+(A-W)\mathfrak{Q}+(\mathfrak{A}-\mathfrak{W})\mathfrak{q}$$
$$(R)=R+(A-W)\mathfrak{R}+(\mathfrak{A}-\mathfrak{W})\mathfrak{r}$$
etc.

Casus V.

58. Si Z contineat ipsam formulam $\Phi=\int Z\,dx$, vt sit:

$$dZ=K\,d\Phi+M\,dx+N\,dy+P\,dp+Q\,dq+R\,dr+\text{etc.}$$

ponatur $\int K\,dx=V$, et facto $x=a$, fit $V=C$, erit:

$(N)=e^{C-V}N$; $(P)=e^{C-V}P$; $(Q)=e^{C-V}Q$; $(R)=e^{C-V}R$ etc.

Casus VI.

59. Si Z praeter formulam $\Phi=\int Z\,dx$ contineat aliam formulam integralem $\Pi=\int\mathfrak{Z}\,dx$, sitque:

$$dZ=K\,d\Phi+L\,d\Pi+M\,dx+N\,dy+P\,dp+Q\,dq+R\,dr+\text{etc.}$$

tum vero $\mathfrak{Z}$ nullam formulam integralem inuoluat:

$$d\mathfrak{Z}=\mathfrak{M}\,dx+\mathfrak{N}\,dy+\mathfrak{P}\,dp+\mathfrak{Q}\,dq+\mathfrak{R}\,dr+\text{etc.}$$

sit $\int K\,dx=V$, et posito $x=a$, fiat $V=C$. Deinde sit $\int e^{C-V}L\,dx=W$, et posito $x=a$, fiat $W=A$, eritque:

$$(N)=e^{C-V}N+(A-W)\mathfrak{N}$$
$$(P)=e^{C-V}P+(A-W)\mathfrak{P}$$
$$(Q)=e^{C-V}Q+(A-W)\mathfrak{Q}$$
$$(R)=e^{C-V}R+(A-W)\mathfrak{R}$$
etc.

Casus

Casus VII.

60. Si Z contineat formulam $\Pi = \int \mathfrak{Z} dx$, vt sit:
$dZ = L d\Pi + M dx + N dy + P dp + Q dq + R dr +$ etc.
tum vero $\mathfrak{Z}$ denuo eandem formulam $\Pi = \int \mathfrak{Z} dx$ inuoluat, vt sit:
$d\mathfrak{Z} = \mathfrak{L} d\Pi + \mathfrak{M} dx + \mathfrak{N} dy + \mathfrak{P} dp + \mathfrak{Q} dq + \mathfrak{R} dr +$ etc.
Ponatur $\int \mathfrak{L} dx = \mathfrak{W}$, et posito $x = a$, fiat $\mathfrak{W} = \mathfrak{A}$; deinde ponatur $\int e^{-\mathfrak{A} + \mathfrak{W}} L dx = W$, et posito $x = a$, fiat $W = A$, eritque

$$(N) = N + e^{\mathfrak{A} - \mathfrak{W}} (A - W) \mathfrak{N}$$
$$(P) = P + e^{\mathfrak{A} - \mathfrak{W}} (A - W) \mathfrak{P}$$
$$(Q) = Q + e^{\mathfrak{A} - \mathfrak{W}} (A - W) \mathfrak{Q}$$
$$(R) = R + e^{\mathfrak{A} - \mathfrak{W}} (A - W) \mathfrak{R}$$

etc.

61. Simili modo hanc inuestigationem ad alias formulas complicatas extendere licet, verum cum tales vix vnquam occurrere soleant, labor superfluus foret. *Cum igitur* formularum integralium tam simpliciorum, quam magis compositarum, variationes definire docuerim, calculus variationum fere penitus absolutus videtur; quomodocunque enim quantitas varianda fuerit, tam ex formulis absolutis, quam integralibus, conflata, ope differentiationis ordinariae eius variatio reperiri poterit. Veluti si quantitas varianda U contineat formulas integrales quascunque:

$$\Phi = \int Z dx; \quad \Phi' = \int Z' dx; \quad \Phi'' = \int Z'' dx \text{ etc.}$$

differentietur ea more solito, prodeatque:

$$\delta U = K d\Phi + K' d\Phi' + K'' d\Phi'' \text{ etc.}$$

 tum

tum euidens eſt, fore eius variationem:

$$\delta U = K\delta\Phi + K'\delta\Phi' + K''\delta\Phi'' \text{ etc.}$$

at variationes $\delta\Phi$, $\delta\Phi'$, $\delta\Phi''$ etc. per praecepta modo expoſita aſſignabuntur. Simul vero patet, variationem δU ſemper huiusmodi forma expreſſum iri, vt ſit:

$$\delta V = \int (A)\,dx\,\delta y + (B)\delta y + (C)\frac{d\delta y}{dx} + (D)\frac{dd\delta y}{dx^2} + \text{etc.}$$

vbi (A), (B), (C), etc. ſunt functiones ex regulis ſupra traditis inueniendae. Iſtius autem calculi variationum vſum in ſolutione celeberrimi problematis iſoperimetrici, latiſſima ſignificatione accepti, breuiter indicari conueniet.

Applicatio Calculi variationum ad ſolutionem problematis iſoperimetrici latiſſima ſignificatione accepti.

62. Problema primarium huc ſpectans ita enunciari poteſt, vt inter omnes curuas ſuper eadem data baſi $x = a$ extruendas, ea definiatur, pro qua formula quaepiam U maximum minimumue valorem obtineat. Etſi enim enunciatio problematis curuas tantum eiusdem longitudinis complectitur, tamen haec conditio commode, vt eius ambitus latius pateat, omittitur, neque etiam commemoratione vnicae formulae U, cuius valor maximus minimusue euadere debet, eius vis reſtringi eſt cenſenda; poſtquam in genere demonſtraui: ſi inter omnes curuas ſuper eadem baſi $x = a$ extruendas, pro quibus formula V eundem nanciſcatur valorem, definiri

niri debeat ea, in qua valor formulae U maximus euadat minimusue; quaestionem huc reduci, vt inter omnes plane curuas, super basi $x = a$ extruendas, ea definiatur, pro qua haec formula composita $\alpha V + \beta U$ consequatur maximum minimum ve valorem. Interim tamen et huius reductionis ratio ex ipsis calculi variationum principiis dilucide explicari potest.

63. Quaestio autem haec a consideratione linearum curuarum remota hoc modo proponi potest:

Proposita formula quacunque U *definire eam relationem inter binas variabiles* x *et* y, *per quam si valor ipsius* U *determinetur, atque a valore* $x = 0$ *vsque ad valorem* $x = a$ *extendatur, is prodituros sit siue maximus siue minimus.*

Spectemus ergo relationem inter x et y tanquam iam inuentam, ita vt inde oriatur valor ipsius U maximus vel minimus; atque manifestum est, si relatio inter x et y infinite parum immutetur, nullam inde mutationem in valore ipsius U nasci debere; seu quod eodem redit variationem ipsius U seu δU nihilo aequalem esse oportere; sicque aequatio $\delta U = 0$ relationem quaesitam inter x et y complectetur.

64. Variationem autem δU inde definire docuimus, quod pro quouis valore ipsius x valorem ipsius y, qui ipsi vi relationis quaesitae competeret, particula quapiam δy augeri assumsimus. Cum igitur relatio quaesita inter omnes plane possibiles hac praerogatiua gaudere debeat, variatio δU semper esse debet nihilo aequalis, quomodocunque singuli valores ipsius y tali-

bus

bus particulis δy augeantur; et quomodocunque haec augmenta fuerint comparata, quoniam prorsus sunt arbitraria, neque vllo modo a se inuicem pendentia. Neque etiam opus est, vt omnibus valoribus ipsius y huiusmodi variationes tribuantur, sed siue vnicus quispiam, siue duo, siue quotcunque pro lubitu varientur, semper aeque necesse est, vt variatio, quae inde in totum valorem ipsius U, quatenus is a termino $x = o$ vsque ad terminum $x = a$ extenditur, redundat, in nihilum abeat.

65. Ex iis autem, quae supra sunt tradita, manifestum est, variationem ipsius U semper hoc modo exprimi, vt sit:

$$\delta U = \int (A) dx \delta y + (B) \delta y + (C) \frac{d \delta y}{dx} + (D) \frac{dd \delta y}{dx^2} \text{ etc.}$$

cuius formae singulas partes seorsim considerari conuenit. At praeter primum membrum integrale reliquae partes $(B) \delta y$, $(C) \frac{d \delta y}{dx}$ etc. tantum a variatione vltimi valoris y, qui valori $x = a$ conuenit, pendent, neque rationem variationis praecedentium implicant; vt enim tota variatio ipsius U obtineatur, in expressione inuenta vbique statui debet $x = a$, quod in singulis partibus praeter primam actu fieri potest, sicque in iis δy denotabit variationem, quae soli vltimo valori ipsius y tribuitur, et quae omnino est arbitraria, neque a praecedentibus pendet. Vnde perspicuum est, nisi membrum integrale adesset, ex reliquis partibus nihil plane pro relatione inter x et y concludi posse.

66. Verum membrum integrale $\int (A) dx \delta y$ etiam variationes, quae omnibus praecedentibus valoribus ipsius y tribuun-

tribuuntur inuoluit, dum continet summam omnium elementorum $(A)dx\delta y$ ex variatione singulorum valorum y oriundorum. Ita si vnicus eius valor ipsi x quasi determinatum valorem haberet, spectato respondens varietur, seu particula δy augeatur, membrum illud integrale tantum esset $=(A)dx\delta y$, nihilque summandum haberetur; sin autem insuper sequens valor y' ipsi $x+dx$ respondens, particula $\delta y'$ augeatur, positoque $x+dx$, loco x functio (A) abeat in $(A)'$ membrum integrale constabit his duabus partibus:

$$(A)dx\delta y+(A)'dx\delta y'$$

Simili modo si tres pluresue valores successiui y, y', y'', y''', y^{IV} etc. particulis δy, $\delta y'$, $\delta y''$, $\delta y'''$ etc. augeantur, membrum integrale aequiualebit huic expressioni:

$$(A)dx\delta y+(A)'dx\delta y'+(A)''dx\delta y''+(A)'''dx\delta y'''+\text{etc.}$$

quae series tam retrorsum vsque ad terminum $x=0$, quam antrorsum vsque ad terminum $x=a$, continuata concipi potest.

67. Etsi igitur variatio δU ad terminum determinatum $x=a$ adstringitur, tamen ob membrum integrale omnes variationes intermedias complectitur; vnde si pro reliquis partibus absolutis, quae tantum ad terminum vltimum $x=a$ referuntur, breuitatis gratia scribamus $\mathfrak{I}$, variatio δU ita erit expressa, vt sit:

$$\delta U=(A)dx\delta y+(A)'dx\delta y'+(A)''dx\delta y''+(A)'''dx\delta y''' + \text{etc.} + \mathfrak{I}$$

quae, vt problemati satisfiat, nihilo aequari debet. Cum autem variationes δy, $\delta y'$, $\delta y''$ etc. non a se inuicem pendeant, sed singulae mere sint arbitrariae, illa annihilatio locum habere nequit, nisi singulae partes sigillatim euanescant; ex quo necesse est, vt sit:

$$(A)=0, \ (A)'=0, \ (A)''=0, \ (A)'''=0, \text{ etc.}$$

quae aequatiunculae omnes in hac vna indefinita $(A)=0$ continentur, seu quicunque valor ipsi x tribuatur, perpetuo esse oportet $(A)=0$, hacque aequatione relatio quaesita inter x et y continetur.

68. En igitur solutionem facilem problematis propositi, quo ea relatio inter x et y requiritur, ex qua pro formula praescripta U, postquam eius valor a termino $x=0$ vsque ad $x=a$ fuerit extensus, maximus minimusue valor resultet. Quaeratur scilicet variatio formulae U pariter a termino $x=0$ vsque ad $x=a$ extensa, quae per praecepta supra tradita huiusmodi formam habere debet:

$$\delta U = \int (A)\, dx\, \delta y + (B)\,\delta y + (C)\frac{d\,\delta y}{dx} + (D)\frac{d\,d\,\delta y}{dx^2} + \text{etc.}$$

hincque ex solo membro integrali $\int (A)\, dx\, \delta y$ relatio inter x et y quaesita ita definietur, vt sit $(A)=0$, reliquae autem partes, quia vltimum tantum valorem ipsius y afficiunt, nihil conferunt ad relationem indefinitam inter x et y, quae desideratur.

69. Istae tamen partes posteriores relationi inventae magis determinandae inseruire possunt; eatenus enim tantum huiusmodi partes accedunt, quatenus in membro integrali $\int (A)\, dx\, \delta y$ functio (A) differentialium rationem $\frac{dy}{dx}=p$, vel etiam rationes differentialium superiorum, nempe $q=\frac{dp}{dx}$; $r=\frac{dq}{dx}$ etc. inuoluit. Quando autem hoc vsu venit, aequatio $(A)=0$ erit differentia-

rentialis vel primi vel etiam altioris gradus; ſicque relatio quaeſita inter x et y poſt vnam pluresue demum integrationes reperitur. Cum autem quaelibet integratio quantitatem conſtantem arbitrariam inuehat, hoc modo ad aequationem finitam vagam peruenietur, atque nunc noua quaeſtio exiſtet, quomodo has conſtantes arbitrarias determinari oporteat, vt valor ipſius U omnium maximus minimusue prodeat. Cum enim quaelibet illarum conſtantium determinatio iam per ſe maximi minimiue proprietate ſit praedita, hic porro vel maximum maximorum vel minimum minimorum inueſtigandum relinquitur.

70. Ad hoc igitur nouum problema acceſſorium reſoluendum partes illae a ſigno integrali immunes adhiberi poterunt. Conſtantes ſcilicet per integrationes inuectas ita determinari conueniet, vt poſito $x = a$ coefficientes ipſarum δy, $\frac{d\delta y}{dx}$, $\frac{dd\delta y}{dx^2}$ etc. ſinguli ſeorſim euaneſcant, ſiue vt hoc caſu ſatisfiat his conditionibus:

(B) $= 0$; (C) $= 0$; (D) $= 0$ etc.

Deinde quia ambos terminos $x = 0$, et $x = a$, inter ſe permutare licet, etiam, poſito $x = 0$, efficiendum erit, vt fiat (B) $= 0$, (C) $= 0$, (D) $= 0$ etc. Etſi enim partes, quae hoc exigant, in noſtra expreſſione non continentur, tamen eae in membro integrali contineri ſunt cenſendae.

71. Ex iisdem principiis etiam problemata, quae ad methodum relatiuam retuli, ſolui poſſunt; haec autem problemata ita generaliter enunciare licet:

Inter omnes relationes, quibus y *per* x *definitur, quae hac communi gaudent proprietate, vt pro formula* $\mathfrak{U}$ *posito* $x=a$ *eundem valorem exhibeant, determinare eam relationem, ex qua formula* U, *siquidem a termino* $x=0$ *vsque ad terminum* $x=a$ *extendatur, maximum vel minimum consequatur valorem.*

Hic igitur variationes, quae singulis valoribus ipsius y tribuuntur, non omnes sunt arbitrariae, sed ita statuendae sunt, vt fiat $\delta\mathfrak{U}=0$, si quidem eius valor a termino $x=0$ vsque ad $x=a$ extendatur. Tum vero etiam natura maximi minimiue postulat, vt secundum eandem extensionem sit, vt ante, $\delta U=0$.

72. Per methodum ergo ante expositam tam formulae $\mathfrak{U}$, quae debet esse communis, quam formulae U, quae maxima fieri debet vel minima, quaeratur variatio a termino $x=0$ vsque ad terminum $x=a$ extendenda; atque relatio quaesita inter x et y ex coniunctione harum duarum aequationum $\delta\mathfrak{U}=0$ et $\delta U=0$ erit inuestiganda. At hae variationes ita expressae reperientur:

$$\delta\mathfrak{U}=\int(\mathfrak{A})dx\delta y+(\mathfrak{B})\delta y+(\mathfrak{C})\frac{d\delta y}{dx}+(\mathfrak{D})\frac{dd\delta y}{dx^2}+\text{etc.}$$

$$\delta U=\int(A)dx\delta y+(B)\delta y+(C)\frac{d\delta y}{dx}+(D)\frac{dd\delta y}{dx^2}+\text{etc.}$$

vbi de membris a signo integrali liberis eadem sunt tenenda, quae supra iam obseruaui; ideoque relationem inter x et y quaesitam tantum ex membris integralibus deriuari oportebit.

73. Hinc itaque binas ſequentes conſequemur aequationes:

$$(\mathfrak{A})\delta y + (\mathfrak{A})'\delta y' + (\mathfrak{A})''\delta y'' + (\mathfrak{A})'''\delta y''' + \text{etc.} = 0$$

$$(A)\delta y + (A)'\delta y' + (A)''\delta y'' + (A)'''\delta y''' + \text{etc.} = 0$$

quarum priore aſſumtio variationum δy, $\delta y'$, $\delta y''$ etc. conditioni communi praeſcriptae conueniens definitur, quae deinde in alteram introducta relationem quaeſitam manifeſtabit. Omnes ergo variationes δy, $\delta y'$, $\delta y''$ etc. praeter vnam, vt arbitrariae, ſpectari poſſunt, quippe quae vna ex priori aequatione eſt definienda. Iam vero euidens eſt, *poſtquam vna ita fuerit ſumta, vt priori aequationi ſatisfiat, tum ſimul alteri ſatisfactum iri*, ſi ſtatuatur $(A) = n(\mathfrak{A})$, ſumendo pro n quantitatem quamcunque conſtantem.

74. Problema igitur propoſitum hac reſoluetur aequatione:

$$\alpha(A) + \beta(\mathfrak{A}) = 0$$

ſumtis pro α et β quantitatibus quibusuis conſtantibus. *Eadem* autem ſolutio prodiiſſet, ſi inter omnes omnino relationes inter x et y ea exquiri debuiſſet, vnde formula $\alpha U + \beta \mathfrak{U}$ maximum minimum ve valorem impetraret; ex quo ſimul intelligitur, binas formulas $\mathfrak{U}$ et U propoſitas inter ſe permutari poſſe, eaque omnia, quae in Tractatu meo annotaui hinc multo magis fiunt perſpicua. Simili enim modo res ſe habebit, ſi non vna formula $\mathfrak{U}$ ſed plures debeant eſſe communes; ſicque ſtabilito *Calculo variationum* omnia huius generis problemata facillime et breuiſſime expediuntur.