1764

De motu vibratorio fili flexilis, corpusculis quotcunque onusti

Leonhard Euler

E286

DE MOTV VIBRATORIO FILI FLEXILIS, CORPVSCVLIS QVOTCVNQVE ONVSTI,

Auctore

L. EVLERO.

1.

Confidero hic filum perfecte flexile, fimulque omni inertia deſtitutum, quod in datis interuallis ſit oneratum ponduscul is quibuscunque A, B, C, D, etc. concipio autem filum hoc in terminis I et O firmiter fixum, et extenſum data quadam vi; ita vt in ſtatu naturali ſitum teneat rectilineum IO, in quo acquieſcat. Quodſi autem a cauſa quacunque de hoc ſitu deturbetur, ita vt ſingula ponduſcula A, B, C, D etc. ad datas diſtantias a recto IO depellantur, ſubitoque dimittantur, totum filum certo quodam motu agitabitur, quem hic inueſtigare conſtitui. Ne autem ſolutio huius quaeſtionis vires analyſeos penitus ſuperet, tam ponduſcula, quam interualla, ad quae a recta IO fuerint depulſa, tanquam infinite parua ſpectabo, vnde hoc commodum ſim aſſecuturus, vt viae, quas ſingula ponduſcula motu ſuo percurrent, ſint rectae ad IO normales, ac tenſio in omnibus fili partibus manſura ſit perpetuo eadem. Tab. I. Fig. 1.

2.

2. Facta hac hypothesi, interualla ponduſculorum etiam durante motu nullam mutationem recipient, quae cum ſint data et conſtantia vocentur:

$IA = a$; $AB = b$; $BC = c$; $CD = d$; $DE = e$; $EF = f$ etc.

eritque etiam

$IP = a$; $PQ = b$; $QR = c$; $RS = d$; $ST = e$; $TV = f$; etc.

maiusculae autem litterae A, B, C, D, etc. ipſas maſſas ſingulorum corpuſculorum exprimant. Deinde ſit vis, qua filum tenditur $= K$; et graecae litterae α, β, γ, δ, etc. denotent interualla, ad quae initio ſingula corpuscula A, B, C, D etc. a linea recta IO fuerint diducta. Quibus poſitis, quaeſtio huc redit, vt elapſo ab iſto initio tempore quocunque, quod ſit ω min. ſec. ſtatus et motus fili determinetur.

3. Ponamus ergo, hoc tempore filum cum corpuſculis in eum ſitum perueniſſe, quem figura oſtendit; et deſignemus iam ſingulorum corpuſculorum ab axe IO diſtantias:

$AP = p$; $BQ = q$; $CR = r$; $DS = s$; $ET = t$; etc.

quas prae interuallis a, b, c, d, tanquam minimas ſpectare licebit. Cum igitur tenſio in ſingulis fili partibus ſit eadem $= K$, quodlibet corpuſculum a tanta vi vtrinque ſollicitabitur, et quatenus hae vires ſibi non ſunt e diametro oppoſitae, eatenus inde vis naſcetur, qua vnumquodque corpuſculum recta ad axem pelletur, vel ab eo repelletur. In has ergo ſingulas vires ante omnia erit inquirendum, quoniam ab iis corpuſcula motus ſui determinationem, hoc eſt, ſiue accele-

accelerationem, ſiue retardationem, nanciſcuntur, quandoquidem per hypotheſin certum eſt, ſingula corpuſcula A, B, C, D, etc. perpetuo per rectas AP, BQ, CR etc. ad axem normales agitari.

4. Si igitur ſecundum regulas cognitas has vires colligamus, deprehendemus:

Corpuſculum	Vrgeri in directione	Vi
A	AP	$K\left(\frac{p}{a}+\frac{p-q}{b}\right)$
B	BQ	$K\left(\frac{q-p}{b}+\frac{q-r}{c}\right)$
C	CR	$K\left(\frac{r-q}{c}+\frac{r-s}{d}\right)$
D	DS	$K\left(\frac{s-r}{d}+\frac{s-t}{e}\right)$
E	ET	$K\left(\frac{t-s}{e}+\frac{t-v}{f}\right)$
F	FV	$K\left(\frac{v-t}{f}+\frac{v-x}{g}\right)$
G	GX	$K\left(\frac{v-x}{g}+\frac{x}{h}\right)$.

Hic ſcilicet poſui, corpuſculum ſeptimum G eſſe vltimum; manifeſtum autem eſt, quotcunque fuerint ponduſcula, quomodo has formulas conſtrui oporteat.

5. Exprimunt autem hae formulae vires motrices, quibus ſingula corpuſcula axem IO verſus incitantur; earum ergo quaelibet per maſſam ponduſculi diuiſa praebebit accelerationem eius. Verum ex diſtantia cuiusque corpuſculi ab axe, quae in genere ſit $=z$, cum tempore generatim expreſſo t collata, oritur quoque per regulas mechanicas acceleratio $=-\frac{2ddz}{dt^2}$, ſumto elemento temporis conſtante. Sed haec formula non eſt ad menſuram temporis in minutis ſecundis expri-

mendi, quam hic assumsimus, accommodata; sed petita est ex ea ratione, qua tempus per spatium ad celeritatem applicatum, celeritas autem per radicem quadratam altitudinis debitae, exhiberi solet. Quare si k denotet altitudinem, ex qua graue vno minuto secundo libere descendit, referet expressio $2\sqrt{k}$ vnum minutum secundum, eritque propterea $t:\omega = 2\sqrt{k}:1$, sicque $t = 2\omega\sqrt{k}$ et $dt^2 = 4kd\omega^2$, vnde acceleratio ad nostrum scopum accommodata prodit $= \frac{-ddz}{2kd\omega^2}$.

6. Quodsi iam has singulas accelerationes cum iis, quae ex sollicitationibus sunt erutae, conferamus, obtinebimus sequentes aequationes: siue

$$\begin{array}{l|l}
\frac{K}{A}\left(\frac{p}{a}+\frac{p-q}{b}\right)=\frac{-ddp}{2kd\omega^2} & \frac{p}{a}+\frac{p-q}{b}+\frac{A\,ddp}{2Kkd\omega^2}=0 \\
\frac{K}{B}\left(\frac{q-p}{b}+\frac{q-r}{c}\right)=\frac{-ddq}{2kd\omega^2} & \frac{q-p}{b}+\frac{q-r}{c}+\frac{B\,ddq}{2Kkd\omega^2}=0 \\
\frac{K}{C}\left(\frac{r-q}{c}+\frac{r-s}{d}\right)=\frac{-ddr}{2kd\omega^2} & \frac{r-q}{c}+\frac{r-s}{d}+\frac{C\,ddr}{2Kkd\omega^2}=0 \\
\frac{K}{D}\left(\frac{s-r}{d}+\frac{s-t}{e}\right)=\frac{-dds}{2kd\omega^2} & \frac{s-r}{d}+\frac{s-t}{e}+\frac{D\,dds}{2Kkd\omega^2}=0 \\
\frac{K}{E}\left(\frac{t-s}{e}+\frac{t-v}{f}\right)=\frac{-ddt}{2kd\omega^2} & \frac{t-s}{e}+\frac{t-v}{f}+\frac{E\,ddt}{2Kkd\omega^2}=0 \\
\frac{K}{F}\left(\frac{v-t}{f}+\frac{v-x}{g}\right)=\frac{-ddv}{2kd\omega^2} & \frac{v-t}{f}+\frac{v-x}{g}+\frac{F\,ddv}{2Kkd\omega^2}=0 \\
\frac{K}{G}\left(\frac{x-v}{g}+\frac{x}{h}\right)=\frac{-ddx}{2kd\omega^2} & \frac{x-v}{g}+\frac{x}{h}+\frac{G\,ddx}{2Kkd\omega^2}=0.
\end{array}$$

6. Totidem igitur quouis casu impetramus huiusmodi aequationes differentio-differentiales, quod ponduscvlis filum intra terminos I et O fuerit oneratum, quarum resolutio, ob variabilium permixtionem, summopere difficilis primo intuitu videatur. Quoniam vero in omnibus his aequationibus variabiles vnicam tantum dimensionem obtinent, manifestum est, singulas istas aequa-

aequationes per eiusmodi constantes multiplicari posse, vt si omnes in vnam summam colligantur, prodeat huiusmodi aequatio:

$$\mathfrak{A}p+\mathfrak{B}q+\mathfrak{C}r+\mathfrak{D}s+\mathfrak{E}t+\mathfrak{F}v+\mathfrak{G}x+ \tfrac{1}{nn}(\mathfrak{A}ddp+\mathfrak{B}ddq+\mathfrak{C}ddr+\mathfrak{D}dds+\mathfrak{E}ddt+\mathfrak{F}ddv+\mathfrak{G}ddx)=0,$$

cuius integratio iam nulli amplius difficultati est obnoxia, cum sit:

$$\mathfrak{A}p+\mathfrak{B}q+\mathfrak{C}r+\mathfrak{D}s+\mathfrak{E}t+\mathfrak{F}v+\mathfrak{G}x=\text{Const.}\cos.\omega n.$$

8. At si hos multiplicatores, qui ad huiusmodi aequationem integrabilem perducant, inuestigemus, eos non vno modo, sed adeo semper tot modis, quot fuerint corpuscula, definiri deprehendemus; sicque tandem etiam totidem aequationes integrales diuersas adipiscemur. Ex tot autem aequationibus deinceps valores singularum applicatarum p, q, r, s etc. elicere poterimus, quorum quilibet huiusmodi formam sortietur:

$$\mathfrak{A}\cos\mathfrak{a}\omega+\mathfrak{B}\cos\mathfrak{b}\omega+\mathfrak{C}\cos.\mathfrak{c}\omega+\mathfrak{D}\cos.\mathfrak{d}\omega+\text{etc.}$$

vbi $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. sunt constantes arbitrariae, ex statu fili initiali, quando ponitur tempus $\omega=0$, definiendae, et pro singulis applicatis p, q, r etc. peculiares obtinebunt valores. At vero litterae $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, etc. in omnibus erunt eaedem, ac per totidem radices aequationis cuiuspiam tot dimensionum, quot fuerint pondufcula, exhibebuntur.

9 Hinc aliam eumque multo faciliorem nanciscimur methodum, cunctas superiores aequationes differentia-

rentiales secundi gradus quasi vno actu resoluendi. Cum enim, si omnes coefficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. praeter vnum in vna forma integrali euanescant, iidem in reliquis omnibus euanescere debeant, statuamus statim mutata harum litterarum significatione:

$$p = \mathfrak{A} \operatorname{cos.} n\omega; \quad q = \mathfrak{B} \operatorname{cos.} n\omega; \quad r = \mathfrak{C} \operatorname{cos.} n\omega; \quad s = \mathfrak{D} \operatorname{cos.} n\omega; \text{ etc.}$$

quibus valoribus substitutis, non solum relatio inter hos coefficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. determinabitur, sed etiam valor litterae n per aequationem tot dimensionum, quot fuerint corpuscula, definietur, vnde etiam totidem valores diuersos recipiet. His autem inuentis singulae expressiones completae reddentur, et huiusmodi formas induent:

$$p = \mathfrak{A} \operatorname{cos.} n\omega + \mathfrak{A}' \operatorname{cos.} n'\omega + \mathfrak{A}'' \operatorname{cos.} n''\omega + \mathfrak{A}''' \operatorname{cos.} n'''\omega + \text{ect.}$$

$$q = \mathfrak{B} \operatorname{cos.} n\omega + \mathfrak{B}' \operatorname{cos.} n'\omega + \mathfrak{B}'' \operatorname{cos.} n''\omega + \mathfrak{B}''' \operatorname{cos.} n'''\omega + \text{etc.}$$

etc.

10. Pro quouis enim alio valore litterae n, alios quoque valores litterae $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. sortientur, qui si debito modo in has aequationes introducantur, obtinebimus valores integrales completos pro singulis applicatis p, q, r, s, etc. qui propterea ad quoduis tempus statum fili praebebunt, ex cuius variatione instantanea simul eius motus innotescet. Praeterea vero totidem adhuc manebunt coefficientes arbitrarii, quot fuerint corpuscula, quos denique ita definire licebit, vt initio $\omega = 0$ distantiae singulorum corpusculorum

lorum ab axe fint, statui, filo inducto consentaneae: tum vero hae formulae iam ita sunt comparatae, vt initio motus singulorum corpusculorum euanescat; seu motus tum a quiete incipiat, alioquin enim etiam sinus angulorum $n\omega$, $n'\omega$ etc. introduci potuissent. Exposita autem methodo solutionis in genere, conueniet eam pro quolibet corpusculorum numero accuratius euolui.

Problema I.

11. Si filum in terminis I et O fixum, et a data vi $=K$ tensum, vnico corpusculo A fit oneratum, determinare eius motum, postquam de statu naturali vtcunque fuerit deturbatum. Fig. 2.

Solutio.

Cum igitur fit $IA = IP = a$; $AO = PO = b$; et elapso tempore ω min. sec. ponatur distantia $PA = p$, quae initio fuerat $= \alpha$; habebimus hanc vnicam aequationem differentio-differentialem:

$$\frac{p}{a} + \frac{p}{b} + \frac{A\,ddp}{2Kk\,d\omega^2} = 0$$

Statuamus ergo $p = \mathfrak{A}\,\text{cof.}\,n\omega$, eritque $\frac{ddp}{d\omega^2} = -nn\mathfrak{A}\,\text{cof.}\,np$ vnde fit $\frac{1}{a} + \frac{1}{b} = \frac{Ann}{2Kk}$, et $n = \sqrt{\frac{2Kk}{A}(\frac{1}{a} + \frac{1}{b})}$. Quare aequatio integralis quaesita habebitur:

$$p = \mathfrak{A}\,\text{cof.}\,\omega\sqrt{\frac{2Kk}{A}\left(\frac{1}{a} + \frac{1}{b}\right)}$$

quae vt praebeat $p = \alpha$, posito $\omega = 0$, poni oportet $\mathfrak{A} = \alpha$, sicque erit pro casu proposito:

$$p = \alpha\,\text{cof.}\,\omega\sqrt{\frac{2Kk}{A}\left(\frac{1}{a} + \frac{1}{b}\right)}$$

Ee 3 vnde

vnde ad quoduis tempus ω min. ſec. ab initio elapſum locus corpuſculi A cognoſcitur.

Coroll. 1.

12. Hinc ſtatim innoteſcit tempus, quo corpuſculum A ab initio motus primum in rectam I O perueniet, quod eueniet, quando fit $p = 0 = \alpha \operatorname{coſ.} \frac{1}{2}\pi$, denotante π ſemiperipheriam circuli, cuius radius eſt 1, vt $\frac{1}{2}\pi$ ſit menſura anguli recti: erit ergo hoc tempus:

$$= \frac{\pi}{2\sqrt{\frac{2Kk}{A}\left(\frac{1}{a} + \frac{1}{b}\right)}} \text{ min. ſec.}$$

quod ſimul eſt tempus dimidiae vibrationis fili.

Coroll. 2.

13. Si enim tempus capiatur duplo maius, corpus A perueniet ad parem diſtantiam α ab axe in altera parte, integramque vibrationem confeciſſe eſt cenſendum. Quare tempus ſingularum vibrationum, quae inter ſe erunt iſochronae, erit:

$$\frac{\pi}{\sqrt{\left(\frac{2Kk}{A}\left(\frac{1}{c} + \frac{1}{b}\right)\right)}} = \frac{\pi\sqrt{Aab}}{\sqrt{2Kk(a+b)}} \text{ min. ſecund.}$$

Coroll. 3.

14. Hoc ergo tempus erit maximum, ſi corpuſculum A medium locum in filo I O tenuerit. Si enim ponamus totam longitudinem $IO = a + b = l$, et $a = \frac{l+u}{2}$; $b = \frac{l-u}{2}$, erit tempus vibrationis $\frac{\pi\sqrt{A(ll-uu)}}{2\sqrt{2Kkl}}$, vnde

vnde patet, quo magis corpusculum a fili puncto medio remoueatur, eo rapidiores fore vibrationes; ipsum autem tempus maximum, quo $u = 0$ fit $= \frac{\pi}{1}\sqrt{\frac{Al}{2Kk}}$ min. secund.

Problema 2.

15. Si filum in terminis I et O fixum, et data vi K tensum, duobus pondusculis A et B fuerit oneratum, determinare eius motum, postquam de statu suo naturali recto IO vtcunque fuerit deturbatum. Fig. 3.

Solutio.

Hic igitur habemus $IA = a$; $AB = b$; $BO = c$, et si elapso tempore ω min. secund. ponamus distantias $PA = p$, et $QB = q$, quae initio fuerant α et β, sequentes duae aequationes differentio-differentiales resoluendae occurrunt:

$$\frac{p}{a} + \frac{p-q}{b} + \frac{A\,ddp}{2Kk\,d\omega^2} = 0$$

$$\frac{q-p}{b} + \frac{q}{c} + \frac{B\,ddq}{2Kk\,d\omega^2} = 0$$

Statuamus ergo:

$$p = \mathfrak{A}\,\text{cos.}\,n\omega \quad \text{et} \quad q = \mathfrak{B}\,\text{cos.}\,n\omega, \text{ eritque}$$

$$\frac{ddp}{d\omega^2} = -nn\mathfrak{A}\,\text{cos.}\,n\omega \quad \text{et} \quad \frac{ddq}{d\omega^2} = -nn\mathfrak{B}\,\text{cos.}\,n\omega.$$

Hi valores substituti praebent:

$$\frac{\mathfrak{A}}{a} + \frac{\mathfrak{A}-\mathfrak{B}}{b} = \frac{nnA\mathfrak{A}}{2Kk} \quad \text{et} \quad \frac{\mathfrak{B}-\mathfrak{A}}{b} + \frac{\mathfrak{B}}{c} = \frac{nnB\mathfrak{B}}{2Kk}$$

ideoque

$$\frac{\mathfrak{B}}{\mathfrak{A}} = 1 + \frac{b}{a} - \frac{nnAb}{2Kk} \quad \text{et} \quad \frac{\mathfrak{A}}{\mathfrak{B}} = 1 + \frac{b}{c} - \frac{nnBb}{2Kk}$$

vnde

vnde per multiplicationem oritur, ponendo $\frac{nn}{2Kk}=z$:

$$1=(1+\frac{b}{a}-Abz)(1+\frac{b}{c}-Bbz) \text{ seu}$$

$$0=\frac{1}{a}+\frac{1}{c}+\frac{b}{ac}-A\,z)\,(1+\frac{b}{c})-Bz(1+\frac{b}{a})+ABbzz,$$

quae reducitur ad hanc formam:

$$zz-\frac{z}{A}(\frac{1}{a}+\frac{1}{b})-\frac{z}{B}(\frac{1}{b}+\frac{1}{c})+\frac{1}{AB}(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc})=0,$$

vnde elicitur:

$$z=\frac{1}{2A}(\frac{1}{a}+\frac{1}{b})+\frac{1}{2B}(\frac{1}{b}+\frac{1}{c}))\pm$$

$$\surd(\frac{1}{4AA})\frac{1}{a}+\frac{1}{b})^2+\frac{1}{4BB}(\frac{1}{b}+\frac{1}{c})^2+\frac{1}{2AB}(\frac{1}{bb}-\frac{1}{ab}-\frac{1}{ac}-\frac{1}{bc})),$$

quo valore inuento, est $n=\surd 2Kkz$. Tum vero habebitur

$$\frac{\mathfrak{B}}{\mathfrak{A}}=\frac{1}{2}+\frac{b}{2a}-\frac{Ab}{2B}(\frac{1}{b}+\frac{1}{c})\mp$$

$$Ab\surd(\frac{1}{4AA}(\frac{1}{a}+\frac{1}{b})^2+\frac{1}{4BB}(\frac{1}{b}+\frac{1}{c})^2-\frac{1}{2Ab}(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}-\frac{1}{bb})).$$

Cum igitur hoc modo duo inueniantur valores ipsius n, qui sint n et n', et pro vtroque relatio inter $\mathfrak{A}$ et $\mathfrak{B}$ definiatur, vnde prodeant valores $\mathfrak{A}$, $\mathfrak{B}$ et $\mathfrak{A}'$, $\mathfrak{B}'$, obtinebimus hinc sequentes valores completos pro applicatis p et q:

$$p=\mathfrak{A}\,\text{cos.}\,n\omega+\mathfrak{A}'\,\text{cos.}\,n'\omega$$

$$q=\mathfrak{B}\,\text{cos.}\,n\omega+\mathfrak{B}'\,\text{cos.}\,n'\omega$$

ob statum autem initialem esse oportet:

$$\mathfrak{A}+\mathfrak{A}'=\alpha, \text{ et } \mathfrak{B}+\mathfrak{B}'=\beta$$

alter autem tam valorum $\mathfrak{A}$ et $\mathfrak{B}$, quam $\mathfrak{A}'$ et $\mathfrak{B}'$, erat indefinitus, vnde ii hinc determinabuntur.

Coroll. 1.

Coroll. 1.

16. Si ponamus breuitatis gratia:

$\frac{1}{2A}\left(\frac{1}{a}+\frac{1}{b}\right)=P$; $\frac{1}{2B}\left(\frac{1}{b}+\frac{1}{c}\right)=Q$; $\frac{1}{AB}\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right)=R$ vt fit $zz-2(P+Q)z+R=0$; et geminus ipſius z valor:

$$z=P+Q\pm\sqrt{((P+Q)^2-R)},$$

erit $\mathfrak{B}=\mathfrak{A}Ab(2P-z)$ ſeu $\mathfrak{A}=\mathfrak{B}Bb(2Q-z)$.

Coroll. 2.

17. Verum cum ſit $4PQ=\frac{1}{AB}\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}+\frac{1}{bb}\right)$ $=R+\frac{1}{ABbb}$, ideoque $R=4PQ-\frac{1}{ABbb}$, habebitur:

$$z=P+Q\pm\sqrt{((P-Q)^2+\tfrac{1}{ABbb})},$$

ex qua forma patet, ambos valores ipſius z ſemper eſſe reales, ex priori autem eſſe poſitiuos; vnde vterque valor ipſius $n=\sqrt{2Kkz}$ erit realis.

Coroll. 3.

18. Ponamus porro ad abbreuiandum $\sqrt{((P-Q)^2+\frac{1}{ABbb})}=S$, ac diſtinguendo geminos valores, obtinebimus:

$$z=P+Q+S;\qquad z'=P+Q-S$$
$$n=\sqrt{2Kk(P+Q+S)};\qquad n'=\sqrt{2Kk(P+Q-S)}$$
$$\mathfrak{B}=\mathfrak{A}Ab(P-Q-S);\qquad \mathfrak{B}'=\mathfrak{A}'Ab(P-Q+S)$$

ac motus fili his duabus aequationibus continebitur:

$$p=\mathfrak{A}\cos.n\omega+\mathfrak{A}'\cos.n'\omega$$
$$q=\mathfrak{B}\cos.n\omega+\mathfrak{B}'\cos.n'\omega.$$

Coroll. 4.

19. Vt autem motus ad ſtatum initialem datum accommodetur; fieri debet $\mathfrak{A}+\mathfrak{A}'=\alpha$ et $\alpha Ab(P-Q)+(\mathfrak{A}'-\mathfrak{A})AbS=\beta$, vnde erit $\mathfrak{A}'-\mathfrak{A}=\frac{\beta}{A\beta S}-\frac{\alpha(P-Q)}{S}$. Quare hinc nanciſcimur vtriusque conſtantis $\mathfrak{A}$ et $\mathfrak{A}'$ determinationem:

$$\mathfrak{A}=\tfrac{1}{2}\alpha+\frac{\alpha(P-Q)}{2S}-\frac{\beta}{2AbS} \text{ et } \mathfrak{A}'=\tfrac{1}{2}\alpha-\frac{\alpha(P-Q)}{2S}+\frac{\beta}{2AbS}, \text{ item}$$

$$\mathfrak{B}=\tfrac{1}{2}\beta-\frac{\beta(P-Q)}{2S}-\frac{\alpha}{2BbS} \text{ et } \mathfrak{B}'=\tfrac{1}{2}\beta-\frac{\beta(P-Q)}{2S}+\frac{\alpha}{2BbS}.$$

Coroll. 5.

20. Si ſtatus initialis ita fuerit comparatus, vt vel $\mathfrak{A}$ vel $\mathfrak{A}'$ fuerit $=0$, tum etiam vel $\mathfrak{B}$ vel $\mathfrak{B}'$ euaneſcet, motusque continebitur

vel in his formulis:	vel in his formulis:
$p=\mathfrak{A}\operatorname{cof.} n\omega$	$p=\mathfrak{A}'\operatorname{cof.} n'\omega$
$q=\mathfrak{B}\operatorname{cof.} n\omega$	$q=\mathfrak{B}'\operatorname{cof.} n'\omega$

vtroque ergo caſu vibrationes orientur regulares oſcillationibus penduli ſimplicis conformes, ac tempus vnius vibrationis erit

Caſu priori $=\frac{\pi}{n}$ min. ſecund.; poſteriori $=\frac{\pi}{n'}$ min. ſec.

Coroll. 6.

21. Sin autem ſtatus initialis fuerit eiusmodi, vt neque $\mathfrak{A}$ neque $\mathfrak{A}'$ euaneſcat, vibrationes orientur irregulares, et quaſi ex vtroque genere ſimplici mixtae; neque filum vnquam ad eundem ſitum reuertetur, niſi numeri n et n' rationem inter ſe teneant ratio-

tionalem. Ponatur huiusmodi ratio $n:n'=\mu:\nu$, ac fiet $\frac{P+Q+S}{P+Q-S}=\frac{\mu\mu}{\nu\nu}$; seu $(\mu\mu-\nu\nu)(P+Q)=(\mu\mu+\nu\nu)S$, ideoque $4(\mu^4+\nu^4)PQ-4\mu\mu\nu\nu(PP+QQ)=\frac{(\mu\mu+\nu\nu)^2}{ABbb}$, hincque

$$P=\frac{\mu^4+\nu^4}{2\mu\mu\nu\nu}Q\pm\frac{(\mu\mu+\nu\nu)}{2\mu\mu\nu\nu}\sqrt{\left((\mu\mu-\nu\nu)^2QQ-\frac{\mu\mu\nu\nu}{ABbb}\right)}.$$

Coroll. 7.

22. Vt ergo vibrationes euadant regulares, status initialis ita debet esse comparatus, vt sit

$$\text{vel } \frac{\alpha}{\beta}=\frac{-1}{Ab(S-P+Q)}=-Bb(S+P-Q)$$

$$\text{vel } \frac{\alpha}{\beta}=\frac{1}{Ab(S+P-Q)}=+Bb(S-P-Q)$$

est enim $SS-(P-Q)^2=\frac{1}{ABbb}$, ideoque $S>(P-Q)$.

Coroll. 8.

23. Si omnia interualla corpusculorum fuerint inter se aequalia, seu $a=b=c$; erit $P=\frac{1}{Aa}$, $Q=\frac{1}{Ba}$, et $R=\frac{1}{ABaa}$, vnde fit $z=\frac{1}{a}\left(\frac{1}{A}+\frac{1}{B}\right)\pm\frac{1}{a}\sqrt{\left(\frac{1}{AA}-\frac{1}{AB}+\frac{1}{BB}\right)}$, seu $z=\frac{A+B\pm\sqrt{(AA-AB+BB)}}{ABa}$, hincque

$$n=\sqrt{\frac{2Kk}{ABa}(A+B+\sqrt{(AA-AB+BB)})};\quad n'=\sqrt{\frac{2Kk}{ABa}(A+B-\sqrt{(AA-AB+BB)})}$$

$$\mathfrak{B}=\mathfrak{A}\cdot\frac{B-A-\sqrt{(AA-AB+BB)}}{B};\quad \mathfrak{B}'=\mathfrak{A}'\cdot\frac{B-A+\sqrt{(AA-AB+BB)}}{B}$$

et pro statu initiali adimplendo

$$\mathfrak{A}=\tfrac{1}{2}\alpha+\frac{\alpha(B-A)-\beta B}{2\sqrt{(AA-AB+BB)}};\quad \mathfrak{A}'=\tfrac{1}{2}\alpha-\frac{\alpha(B-A)+\beta B}{2\sqrt{(AA-AB+BB)}}$$

$$\mathfrak{B}=\tfrac{1}{2}\beta-\frac{\beta(B-A)-\alpha A}{2\sqrt{(AA-AB+BB)}};\quad \mathfrak{B}'=\tfrac{1}{2}\beta-\frac{\beta(B-A)+\alpha A}{2\sqrt{(AA-AB+BB)}}.$$

Motus vero his aequationibus exprimetur:

$$p = \mathfrak{A}\cos.\omega + \mathfrak{A}'\cos.n'\omega$$
$$q = \mathfrak{B}\cos.\omega + \mathfrak{B}'\cos.n'\omega$$

Coroll. 9.

24. Vt fiat hoc casu $n : n' = \mu : \nu$, oportet esse:

$$\frac{4(\mu^4 + \nu^4)}{AB} - 4\mu\mu\nu\nu\left(\frac{1}{AA} + \frac{1}{BB}\right) = \frac{(\mu\mu + \nu\nu)^2}{AB}, \text{ seu}$$
$$\frac{B}{A} = \frac{3\mu^4 - 2\mu\mu\nu\nu + 3\nu^4 \pm \surd(9\mu^8 - 12\mu^6\nu\nu - 42\mu^4\nu^4 - 12\mu\mu\nu^6 + 9\nu^8)}{8\mu\mu\nu\nu}$$

vnde, si sit $\mu = 2$, et $\nu = 1$, seu $n : n' = 2 : 1$, fiet

$$\frac{B}{A} = \frac{43 \pm \surd 825}{32} = \frac{43 \pm 5\surd 33}{32}.$$

Coroll. 10.

25. Si praeterea ambo corpora A et B sint aequalia, erit $z = \frac{2 \pm 1}{A\alpha}$, vnde sequentes obtinentur determinationes:

$$n = \surd\frac{6Kk}{A\alpha}\ ; \quad n' = \surd\frac{2Kk}{A\alpha}$$
$$\mathfrak{B} = -\mathfrak{A}\ ; \quad \mathfrak{B}' = +\mathfrak{A}'$$
$$\mathfrak{A} = \tfrac{1}{2}(\alpha - \beta)\ ; \quad \mathfrak{A}' = \tfrac{1}{2}(\alpha + \beta)$$
$$\mathfrak{B} = \tfrac{1}{2}(\beta - \alpha)\ ; \quad \mathfrak{B}' = \tfrac{1}{2}(\alpha + \beta)$$

et pro motu:

$$p = \tfrac{1}{2}(\alpha - \beta)\cos.\omega\surd\frac{6Kk}{A\alpha} + \tfrac{1}{2}(\alpha + \beta)\cos.\omega\surd\frac{2Kk}{A\alpha}$$
$$q = -\tfrac{1}{2}(\alpha - \beta)\cos.\omega\surd\frac{6Kk}{A\alpha} + \tfrac{1}{2}(\alpha + \beta)\cos.\omega\surd\frac{2Kk}{A\alpha}.$$

Problema 3.

Fig. 4. 26. Si filum in terminis I et O fixum et data vi $= K$ tensum tribus ponduscul is A, B, et C fuerit onera-

oneratum, determinare eius motum, postquam de statu suo naturali recto IO vtcunque fuerit deturbatum.

Solutio.

Hic igitur habemus $IA = a$; $AB = b$; $BC = c$ et $CO = d$, ac si, elapso tempore ω min. sec., ponamus distantias:

$PA = p$; $QB = q$ et $RC = r$

quae initio fuerant respectiue α, β, γ, sequentes tres resoluendae sunt aequationes:

$$\frac{p}{A}\left(\frac{1}{a}+\frac{1}{b}\right)-\frac{q}{A}\cdot\frac{1}{b}+\frac{ddp}{2Kkd\omega^2}=0$$

$$\frac{q}{B}\left(\frac{1}{b}+\frac{1}{c}\right)-\frac{p}{B}\cdot\frac{1}{b}-\frac{r}{B}\cdot\frac{1}{c}+\frac{ddq}{2Kkd\omega^2}=0$$

$$\frac{r}{C}\left(\frac{1}{c}+\frac{1}{d}\right)-\frac{q}{C}\cdot\frac{1}{c}+\frac{ddr}{2Kkd\omega^2}=0.$$

Quodsi ergo statuamus $p=\mathfrak{A}\operatorname{cos}.n\omega$; $q=\mathfrak{B}\operatorname{cos}.n\omega$; $r=\mathfrak{C}\operatorname{cos}.n\omega$; habebimus, ponendo $\frac{nn}{2Kk}=z$, has aequationes:

$$\frac{\mathfrak{A}}{A}\left(\frac{1}{a}+\frac{1}{b}\right)-\frac{\mathfrak{B}}{Ab}=\mathfrak{A}z$$

$$\frac{\mathfrak{B}}{B}\left(\frac{1}{b}+\frac{1}{c}\right)-\frac{\mathfrak{A}}{Bb}-\frac{\mathfrak{C}}{Bc}=\mathfrak{B}z$$

$$\frac{\mathfrak{C}}{C}\left(\frac{1}{c}+\frac{1}{d}\right)-\frac{\mathfrak{B}}{Cc}=\mathfrak{C}z$$

vnde eruimus:

$$\mathfrak{A}=\frac{\mathfrak{B}}{b\left(\frac{1}{a}+\frac{1}{b}\right)-Abz};\quad \mathfrak{C}=\frac{\mathfrak{B}}{c\left(\frac{1}{c}+\frac{1}{d}\right)-Cez}$$

qui in secunda substituti praebent:

$$\frac{1}{b}+\frac{1}{c}-\frac{1}{bb\left(\frac{1}{a}+\frac{1}{b}\right)-Abbz}-\frac{1}{cc\left(\frac{1}{c}+\frac{1}{d}\right)-Cccz}=Bz$$

qua aequatione ordinata oritur:

$$z^3 \left.\begin{matrix} -\frac{1}{A}\left(\frac{1}{a}+\frac{1}{b}\right) \\ -\frac{1}{B}\left(\frac{1}{b}+\frac{1}{c}\right) \\ -\frac{1}{C}\left(\frac{1}{c}+\frac{1}{d}\right) \end{matrix}\right\} zz \left.\begin{matrix} +\frac{1}{AB}\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right) \\ +\frac{1}{AC}\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{c}+\frac{1}{d}\right) \\ +\frac{1}{Bc}\left(\frac{1}{bc}+\frac{1}{bd}+\frac{1}{cd}\right) \end{matrix}\right\} z - \frac{1}{ABC}\left(\frac{1}{abc}+\frac{1}{abd}+\frac{1}{acd}+\frac{1}{bcd}\right)=0$$

quae aequatio, ſi ponatur breuitatis gratia:

$$\tfrac{1}{A}\left(\tfrac{1}{a}+\tfrac{1}{b}\right)=P;\ \tfrac{1}{B}\left(\tfrac{1}{b}+\tfrac{1}{c}\right)=Q;\ \tfrac{1}{C}\left(\tfrac{1}{c}+\tfrac{1}{d}\right)=R$$

transmutatur in ſequentem formam:

$$(z-P)(z-Q)(z-R)=\frac{z-R}{ABbb}+\frac{z-P}{BCc}$$

vnde terni eliciuntur valores ipſius z, iique ſemper reales et poſitiui, qui ſint z, z', et z'', ex quibus ſequentes terni valores porro eruuntur:

$$n=\sqrt{2Kkz};\quad n'=\sqrt{2Kkz'}\quad n''=\sqrt{2Kkz''}$$

$$\mathfrak{A}=\frac{\mathfrak{B}}{Ab(P-z)};\quad \mathfrak{A}'=\frac{\mathfrak{B}'}{Ab(P-z')}\quad \mathfrak{A}''=\frac{\mathfrak{B}''}{Ab(P-z'')}$$

$$\mathfrak{C}=\frac{\mathfrak{B}}{Cc(R-z)};\quad \mathfrak{C}'=\frac{\mathfrak{B}'}{Cc(R-z')}\quad \mathfrak{C}''=\frac{\mathfrak{B}''}{Cc(R-z'')}$$

quibus valoribus inuentis, motus his formulis definietur:

$$p=\mathfrak{A}\operatorname{coſ}.n\omega+\mathfrak{A}'\operatorname{coſ}.n'\omega+\mathfrak{A}''\operatorname{coſ}.n''\omega$$

$$q=\mathfrak{B}\operatorname{coſ}.n\omega+\mathfrak{B}'\operatorname{coſ}.n'\omega+\mathfrak{B}''\operatorname{coſ}.n''\omega$$

$$r=\mathfrak{C}\operatorname{coſ}.n\omega+\mathfrak{C}'\operatorname{coſ}.n'\omega+\mathfrak{C}''\operatorname{coſ}.n''\omega$$

quae, vt ad ſtatum propoſitum initialem accommodentur, fiat:

$$\mathfrak{A}+\mathfrak{A}'+\mathfrak{A}''=\alpha;\ \mathfrak{B}+\mathfrak{B}'+\mathfrak{B}''=\beta \text{ et } \mathfrak{C}+\mathfrak{C}'+\mathfrak{C}''=\gamma$$

ſicque motus quaeſitus erit determinatus.

Coroll.

Coroll. 1.

27. Tota ergo ſolutio reducitur ad reſolutionem huius aequationis cubicae

$$z^3 \left.\begin{array}{l} -\frac{1}{A}\left(\frac{1}{a}+\frac{1}{b}\right) \\ -\frac{1}{B}\left(\frac{1}{b}+\frac{1}{c}\right) \\ -\frac{1}{C}\left(\frac{1}{c}+\frac{1}{d}\right) \end{array}\right\} z^2 \left.\begin{array}{l} +\frac{1}{AB}\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right) \\ +\frac{1}{AC}\left(\frac{1}{ac}+\frac{1}{bc}+\frac{1}{ad}+\frac{1}{bd}\right) \\ +\frac{1}{BC}\left(\frac{1}{bc}+\frac{1}{bd}+\frac{1}{cd}\right) \end{array}\right\} z - \frac{1}{ABC}\left(\frac{1}{abc}+\frac{1}{abd}+\frac{1}{add}+\frac{1}{bcd}\right)=0$$

dum caſu problematis praecedentis, vbi filum duobus tantum ponduſculis erat onuſtum, haec aequatio quadratica ſolutionem continebat:

$$zz \left.\begin{array}{l} -\frac{1}{A}\left(\frac{1}{a}+\frac{1}{b}\right) \\ -\frac{1}{B}\left(\frac{1}{b}+\frac{1}{c}\right) \end{array}\right\} z + \frac{1}{AB}\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right)=0.$$

Coroll. 2.

28. Tres ſcilicet huius aequationis cubicae radices, generaliter modo expoſitq coniunctae, generalem problematis ſuppeditant ſolutionem, quae ad omnes caſus, Vnquicunque flatus filo fuerit inductus initio, pateat. Vnde patet, ſi terni numeri n, n', n'' fuerint incommenſurabiles inter ſe, motum fili admodum fore irregularem, neque certas periodos eſſe habiturum.

Coroll. 3.

29. Tres autem dantur caſus, quibus filum ad vibrationes regulares et iſochronas excitari poteſt, qui conditionibus his locum habebunt:

Caſus I

	Casus I	Casus II	Casus III
si sit	$\alpha = \mathfrak{A}$	$\alpha = \mathfrak{A}'$	$\alpha = \mathfrak{A}''$
	$\beta = \mathfrak{B}$	$\beta = \mathfrak{B}'$	$\beta = \mathfrak{B}''$
	$\gamma = \mathfrak{C}$	$\gamma = \mathfrak{C}'$	$\gamma = \mathfrak{C}''$
tum erit	$p = \mathfrak{A} \operatorname{cos.} n\omega$	$p = \mathfrak{A}' \operatorname{cos} n'\omega$	$p = \mathfrak{A}'' \operatorname{cos.} n''\omega$
	$q = \mathfrak{B} \operatorname{cos} n\omega$	$q = \mathfrak{B}' \operatorname{cos.} n'\omega$	$q = \mathfrak{B}'' \operatorname{cos.} n''\omega$
	$r = \mathfrak{C} \operatorname{cos} n\omega$	$r = \mathfrak{C}' \operatorname{cos.} n'\omega$	$r = \mathfrak{C}'' \operatorname{cos.} n''\omega$
temp. vibrat.	$= \frac{\pi}{n}$ min. sec.	$= \frac{\pi}{n'}$ min. sec.	$= \frac{\pi}{n''}$ min. sec.

Coroll. 4.

30. Inuentis autem his tribus casibus, quibus vibrationes isochronae euadunt, ex iis omnes reliqui casus motuum irregularium per compositionem definiri poterunt; vbi notari meretur, vtcunque hi motus appareant irregulares, eos tamen ex combinatione vibrationum isochronarum oriri.

Coroll. 5.

31. Quando autem tria corpora A, B, C, tam ratione massae, quam distantiarum, ita fuerint comparata, vt numeri n, n', n'' inde resultent commensurabiles, irregularitas motus eatenus euanescit, quod in motu percipientur periodi, quibus filum in eundem statum restituitur.

Coroll. 6.

32. Si corpusculorum interualla a, b, c, d inter se fuerint aequalia, aequatio cubica resoluenda abibit in hanc formam:

$$z^3 - \left(\frac{2}{A} + \frac{2}{B} + \frac{2}{C}\right)\frac{zz}{a} + \left(\frac{3}{AB} + \frac{4}{AC} + \frac{3}{BC}\right)\frac{z}{aa} - \frac{4}{ABCa^3} = 0$$

quae

quae si corpuscula insuper sint inter se aequalia, fit:

$$z^3 - \frac{6zz}{Aa} + \frac{10z}{A^2a^2} - \frac{4}{A^3a^3} = 0$$

cuius tres radices sunt:

$$\text{I. } z = \frac{2}{Aa}; \quad \text{II. } z' = \frac{2-\sqrt{2}}{Aa}; \quad \text{III. } z'' = \frac{2+\sqrt{2}}{Aa}.$$

Coroll. 7.

33. In hoc ergo postremo casu, quo $A=B=C$ et $a=b=c=d$ erit pro motu fili generatim determinando ob $P=\frac{2}{Aa}=Q=R$:

$$n = \sqrt{\frac{4Kk}{Aa}}; \quad n' = \sqrt{\frac{2(2-\sqrt{2})Kk}{Aa}}; \quad n'' = \sqrt{\frac{2(2+\sqrt{2})Kk}{Aa}}$$

$$\mathfrak{A} = \frac{\mathfrak{B}}{0Aa} \qquad \mathfrak{A}' = \frac{\mathfrak{B}'}{\sqrt{2}} \qquad \mathfrak{A}'' = -\frac{\mathfrak{B}''}{\sqrt{2}}$$

$$\mathfrak{C} = \frac{\mathfrak{B}}{0Aa} \qquad \mathfrak{C}' = \frac{\mathfrak{B}'}{\sqrt{2}} \qquad \mathfrak{C}'' = -\frac{\mathfrak{B}''}{\sqrt{2}}$$

seu $\mathfrak{B} = 0$ $\qquad \mathfrak{B}' = \mathfrak{A}'\sqrt{2} \qquad \mathfrak{B}'' = -\mathfrak{A}''\sqrt{2}$

et $\mathfrak{C} = -\mathfrak{A}$ $\qquad \mathfrak{C}' = \mathfrak{A}' \qquad \mathfrak{C}'' = \mathfrak{A}''.$

Ac pro statu initiali dato habebitur:

$$\mathfrak{A}+\mathfrak{A}'+\mathfrak{A}''=\alpha; \quad 0+\mathfrak{A}'\sqrt{2}-\mathfrak{A}''\sqrt{2}=\beta; \quad -\mathfrak{A}+\mathfrak{A}'+\mathfrak{A}''=\gamma$$

unde obtinetur:

$$\mathfrak{A} = \frac{\alpha-\gamma}{2}; \quad \mathfrak{A}' = \frac{\alpha+\beta\sqrt{2}+\gamma}{4}; \quad \mathfrak{A}'' = \frac{\alpha-\beta\sqrt{2}+\gamma}{4}$$

$$\mathfrak{B} = 0 \quad \mathfrak{B}' = \frac{\alpha+\beta\sqrt{2}+\gamma}{2\sqrt{2}}; \quad \mathfrak{B}'' = \frac{-\alpha+\beta\sqrt{2}-\gamma}{2\sqrt{2}}$$

$$\mathfrak{C} = \frac{-\alpha+\gamma}{2}; \quad \mathfrak{C}' = \frac{\alpha+\beta\sqrt{2}+\gamma}{4}; \quad \mathfrak{C}'' = \frac{\alpha-\beta\sqrt{2}+\gamma}{4}$$

Consequenter motus definietur per has formulas:

$$p = \frac{\alpha-\gamma}{2}\cos.\omega\sqrt{\frac{4Kk}{Aa}} + \frac{\alpha+\beta\sqrt{2}+\gamma}{4}\cos.\omega\sqrt{\frac{2(2-\sqrt{2})Kk}{Aa}} + \frac{\alpha-\beta\sqrt{2}+\gamma}{4}\cos\omega\sqrt{\frac{2(2+\sqrt{2})Kk}{Aa}}$$

$$q = +\frac{\alpha+\beta\sqrt{2}+\gamma}{2\sqrt{2}}\cos\omega\sqrt{\frac{2(2-\sqrt{2})Kk}{Aa}} - \frac{\alpha+\beta\sqrt{2}-\gamma}{2\sqrt{2}}\cos\omega\sqrt{\frac{2(2+\sqrt{2})Kk}{Aa}}$$

$$r = \frac{-\alpha+\gamma}{2}\cos.\omega\sqrt{\frac{4Kk}{Aa}} + \frac{\alpha+\beta\sqrt{2}+\gamma}{4}\cos.\omega\sqrt{\frac{2(2-\sqrt{2})Kk}{Aa}} + \frac{\alpha-\beta\sqrt{2}+\gamma}{4}\cos.\omega\sqrt{\frac{2(2+\sqrt{2})Kk}{Aa}}.$$

Problema 4.

Fig. 5. 34. Si filum, in terminis I et O fixum et data vi $=K$ tenſum, quatuor corpuſculis A, B, C, D fuerit oneratum, determinare eius motum, poſtquam de ſtatu ſuo naturali recto IO vtcunque fuerit depulſum.

Solutio.

Habemus ergo $IA=a$, $AB=b$, $BC=c$, $CD=d$ et $DO=e$, ac ſi elapſo tempore ω min. ſec. ponamus diſtantias

$PA=p$; $QB=q$; $RC=r$ et $SD=s$

quae initio fuerant reſpectiue α, β, γ, δ, ſequentes quatuor reſolui debent aequationes:

$$\frac{p}{A}\left(\frac{1}{a}+\frac{1}{b}\right)-\frac{q}{A}\cdot\frac{1}{b}+\frac{ddp}{2Kkd\omega^2}=0$$
$$\frac{q}{B}\left(\frac{1}{b}+\frac{1}{c}\right)-\frac{p}{B}\cdot\frac{1}{b}-\frac{r}{B}\cdot\frac{1}{c}+\frac{ddq}{2Kkd\omega^2}=0$$
$$\frac{r}{C}\left(\frac{1}{c}+\frac{1}{d}\right)-\frac{q}{C}\cdot\frac{1}{c}-\frac{s}{C}\cdot\frac{1}{d}+\frac{ddr}{2Kkd\omega^2}=0$$
$$\frac{s}{D}\left(\frac{1}{d}+\frac{1}{e}\right)-\frac{r}{D}\cdot\frac{1}{d}+\frac{dds}{2Kkd\omega^2}=0.$$

Quodſi iam ſtatuamus

$p=\mathfrak{A}\,\mathrm{coſ}.n\omega$; $q=\mathfrak{B}\,\mathrm{coſ}.n\omega$; $r=\mathfrak{C}\,\mathrm{coſ}.n\omega$; $s=\mathfrak{D}\,\mathrm{coſ}.n\omega$

ac ad abbreuiationem ponamus $\frac{nn}{2Kk}=z$, nec non $\frac{1}{A}\left(\frac{1}{a}+\frac{1}{b}\right)=P$; $\frac{1}{B}\left(\frac{1}{b}+\frac{1}{c}\right)=Q$; $\frac{1}{C}\left(\frac{1}{c}+\frac{1}{d}\right)=R$ et $\frac{1}{D}\left(\frac{1}{d}+\frac{1}{e}\right)=S$ orientur ſequentes aequationes:

$$\mathfrak{A}P-\frac{\mathfrak{B}}{Ab}=\mathfrak{A}z$$
$$\mathfrak{B}Q-\frac{\mathfrak{A}}{Bb}-\frac{\mathfrak{C}}{Bc}=\mathfrak{B}z$$
$$\mathfrak{C}R-\frac{\mathfrak{B}}{Cc}-\frac{\mathfrak{D}}{Cd}=\mathfrak{C}z$$
$$\mathfrak{D}S-\frac{\mathfrak{C}}{Dd}=\mathfrak{D}z$$

ex quibus elicitur:

$$\mathfrak{B} = \mathfrak{A} A b (P - z);$$

$$\mathfrak{C} = \mathfrak{A} A B b c (P - z)(Q - z) - \frac{\mathfrak{A} c}{b}$$

$$\mathfrak{D} = \mathfrak{A} A B C b c d (P - z)(Q - z)(R - z) - \frac{\mathfrak{A} C c d}{b}(R - z) - \frac{\mathfrak{A} A b d}{c}(P - z),$$

qui valores in vltima aequatione ſubſtituti praebent:

$$A B C b c d (P - z)(Q - z)(R - z)(S - z) - \frac{C c d}{b}(R - z)(S - z)$$
$$- \frac{A b d}{c}(P - z)(S - z) - \frac{A B b c}{P d}(P - z)(Q - z) + \frac{c}{D b d} = 0$$

quae per $ABCbcd$ diuiſa abit in hanc formam:

$$(P - z)(Q - z)(R - z)(S - z) - \frac{(R - z)(S - z)}{A B b b} - \frac{(P - z)(S - z)}{B C c c} - \frac{(P - z)(Q - z)}{C D d d}$$
$$+ \frac{1}{A B C D b b d d} = 0,$$

cuius indoles clarius perſpicietur ex hac forma:

$$1 - \frac{1}{A B b b (P - z)(Q - z)} - \frac{1}{B C c c (Q - z)(R - z)} - \frac{1}{C D d d (R - z)(S - z)}$$
$$+ \frac{1}{A B C D b b d d (P - z)(Q - z)(R - z)(S - z)} = 0.$$

Verum ſi illa aequatio, reſtituendis pro P, Q, R, S valoribus, penitus euoluatur, obtinebitur ſequens aequatio biquadratica:

$$z^4 \left.\begin{matrix} -\frac{1}{A}\left(\frac{1}{a}+\frac{1}{b}\right) \\ -\frac{1}{B}\left(\frac{1}{b}+\frac{1}{c}\right) \\ -\frac{1}{C}\left(\frac{1}{c}+\frac{1}{d}\right) \\ -\frac{1}{D}\left(\frac{1}{d}+\frac{1}{e}\right) \end{matrix}\right\} z^3 \left.\begin{matrix} +\frac{1}{AB}\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right) \\ +\frac{1}{AC}\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{c}+\frac{1}{d}\right) \\ +\frac{1}{AD}\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{d}+\frac{1}{e}\right) \\ +\frac{1}{BC}\left(\frac{1}{bc}+\frac{1}{bd}+\frac{1}{cd}\right) \\ +\frac{1}{BD}\left(\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{d}+\frac{1}{e}\right) \\ +\frac{1}{CD}\left(\frac{1}{cd}+\frac{1}{ce}+\frac{1}{de}\right) \end{matrix}\right\} zz \left.\begin{matrix} -\frac{1}{ABC}\left(\frac{1}{abc}+\frac{1}{abd}+\frac{1}{acd}+\frac{1}{bcd}\right) \\ -\frac{1}{ABD}\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right)\left(\frac{1}{d}+\frac{1}{e}\right) \\ -\frac{1}{ACD}\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{cd}+\frac{1}{ce}+\frac{1}{de}\right) \\ -\frac{1}{BCD}\left(\frac{1}{bcd}+\frac{1}{bce}+\frac{1}{bde}+\frac{1}{cde}\right) \end{matrix}\right\} z$$
$$+ \frac{1}{ABCD}\left(\frac{1}{abcd}+\frac{1}{abce}+\frac{1}{abde}+\frac{1}{acde}+\frac{1}{bcde}\right) = 0.$$

Inuentis autem huius aequationis quaternis radicibus z, z', z'' et z''', ex illis totidem valores numeri n habebunbebuntur per formulam $n = \sqrt{2Kkz}$; ac sumtis quoque quaternis arbitrariis $\mathfrak{A}$, $\mathfrak{A}'$, $\mathfrak{A}''$, $\mathfrak{A}'''$, ex vnoquoque reliqui $\mathfrak{B}$, $\mathfrak{C}$ et $\mathfrak{D}$ respondentes reperientur ope formularum :

$$\mathfrak{B} = \mathfrak{A}Ab(P-z)$$

$$\mathfrak{C} = \mathfrak{A}ABbc((P-z)(Q-z) - \tfrac{1}{AB\,bb})$$

$$\mathfrak{D} = \mathfrak{A}ABCbcd((P-z)(Q-z)(R-z) - \tfrac{1}{ABbb}(R-z) - \tfrac{1}{BC\,cc}(P-z))$$

ac tandem formulae pro motu fili erunt :

$$p = \mathfrak{A}\cos.n\omega + \mathfrak{A}'\cos.n'\omega + \mathfrak{A}''\cos n''\omega + \mathfrak{A}'''\cos n'''\omega$$

$$q = \mathfrak{B}\cos.n\omega + \mathfrak{B}'\cos n'\omega + \mathfrak{B}''\cos.n''\omega + \mathfrak{B}'''\cos.n'''\omega$$

$$r = \mathfrak{C}\cos.n\omega + \mathfrak{C}'\cos.n'\omega + \mathfrak{C}''\cos.n''\omega + \mathfrak{C}'''\cos.u'''\omega$$

$$s = \mathfrak{D}\cos.n\omega + \mathfrak{D}'\cos.n'\omega + \mathfrak{D}''\cos.n''\omega + \mathfrak{D}'''\cos.n'''\omega.$$

quatuor autem constantes arbitrariae $\mathfrak{A}$, $\mathfrak{A}'$, $\mathfrak{A}''$, $\mathfrak{A}'''$ ex statu initiali ita definiri debent, vt fiat :

$$\mathfrak{A} + \mathfrak{A}' + \mathfrak{A}'' + \mathfrak{A}''' = \alpha$$

$$\mathfrak{B} + \mathfrak{B}' + \mathfrak{B}'' + \mathfrak{B}''' = \beta$$

$$\mathfrak{C} + \mathfrak{C}' + \mathfrak{C}'' + \mathfrak{C}''' = \gamma$$

$$\mathfrak{D} + \mathfrak{D}' + \mathfrak{D}'' + \mathfrak{D}''' = \delta.$$

Coroll. 1.

35. Iam igitur quatuor existunt casus, quibus vibrationes erunt isochronae, quarum tempora erunt:

$\frac{\pi}{n}$; $\frac{\pi}{n'}$; $\frac{\pi}{n''}$; $\frac{\pi}{n'''}$ min. sec.

atque ex his casibus, tanquam motibus simplicibus, reliqui omnes per compositionem oriuntur.

Coroll.

Coroll. 2.

36. Ex his iam lex istarum formularum, si filum pluribus corpusculis fuerit onustum, non difficulter perspicitur. Si enim quinque habeantur corpuscula, adiecto valore $\frac{1}{E}(\frac{1}{e}+\frac{1}{f})=\Gamma$, aequatio principalis resoluenda ita se habebit:

$$1-\frac{1}{ABbb(P-z)(Q-z)}-\frac{1}{BCcc(Q-z)(R-z)}-\frac{1}{CDdd(R-z)(S-z)}-\frac{1}{DEee(S-z)(T-z)}$$
$$+\frac{1}{ABCDb^2d^2(P-z)(Q-z)(R-z)(S-z)}+\frac{1}{BCDEccee(Q-z)(R-z)(S-z)(T-z)}=0$$

Ac si sex fuerint corpuscula, posito $\frac{1}{F}(\frac{1}{f}+\frac{1}{g})=V$, erit

$$1-\frac{1:ABbb}{(P-z)(Q-z)}-\frac{1:BCcc}{(Q-z)(R-z)}-\frac{1:CDdd}{(R-z)(S-z)}-\frac{1:DEee}{(S-z)(T-z)}-\frac{1:EFff}{(T-z)(V-z)}$$
$$+\frac{1:ABCDbbdd}{(P-z)(Q-z)(R-z)(S-z)}+\frac{1:BCDEccee}{(Q-z)(R-z)(S-z)(T-z)}+\frac{1:CDEFddff}{(R-z)(S-z)(T-z)(V-z)}$$
$$-\frac{1:ABCDEFbbddff}{(P-z)(Q-z)(R-z)(S-z)(T-z)(V-z)}=0.$$

Coroll. 3.

37. Hae autem formulae in genere nimis sunt complicatae, quam vt quidquam ad cognitionem motus inde concludi queat. Concipiamus ergo interualla a, b, c, d; etc. inter se aequalia, ac pro quouis corpusculorum numero aequationes, ex quibus valores ipsius z elici oportet, ita se habebunt:

Pro vno corpusculo

$$z-\frac{2}{Aa}=0.$$

Pro duobus corpusculis

$$zz-\frac{1}{a}(\frac{2}{A}+\frac{2}{B})z+\frac{1}{aa}\cdot\frac{3}{AB}=0$$

Pro tribus corpusculis

$$z^3-\frac{1}{a}(\frac{2}{A}+\frac{2}{B}+\frac{2}{C})zz+\frac{1}{aa}(\frac{3}{AB}+\frac{4}{AC}+\frac{3}{BC})z-\frac{1}{a^3}\cdot\frac{4}{ABC}=0$$

 Pro

Pro quatuor corpusculis

$$z^4 - \frac{1}{a}\left(\frac{2}{A} + \frac{2}{B} + \frac{2}{C} + \frac{2}{D}\right)z^3 + \frac{1}{aa}\left(\frac{3}{AB} + \frac{4}{AC} + \frac{4}{AD} + \frac{3}{BC} + \frac{4}{BD} + \frac{3}{CD}\right)z^2$$
$$- \frac{1}{a^3}\left(\frac{4}{ABC} + \frac{6}{ABD} + \frac{6}{ACD} + \frac{4}{BCD}\right)z + \frac{1}{a^4} \cdot \frac{5}{ABCD} = 0$$

Pro quinque corpusculis

$$z^5 - \frac{1}{a}\left(\frac{2}{A} + \frac{2}{B} + \frac{2}{C} + \frac{2}{D} + \frac{2}{E}\right)z^4$$
$$+ \frac{1}{aa}\left(\frac{3}{AB} + \frac{4}{AC} + \frac{4}{AD} + \frac{4}{AE} + \frac{3}{BC} + \frac{4}{BD} + \frac{4}{BE} + \frac{3}{CD} + \frac{4}{CE} + \frac{3}{DE}\right) z^3$$
$$- \frac{1}{a^3}\left(\frac{4}{ABC} + \frac{6}{ABD} + \frac{6}{ABE} + \frac{6}{ACD} + \frac{8}{ACE} + \frac{6}{ADE} + \frac{4}{BCD} + \frac{6}{BCE} + \frac{6}{BDE} + \frac{4}{CDE}\right)z^2$$
$$+ \frac{1}{a^4}\left(\frac{5}{ABCD} + \frac{8}{ABCE} + \frac{9}{ABDE} + \frac{8}{ACDE} + \frac{5}{BCDE}\right)z - \frac{1}{a^5} \cdot \frac{6}{ABCDE} = 0.$$

Coroll. 4.

38. Si non solum interualla corpusculorum a, b, c, d etc. sed etiam ipsa corpuscula A, B, C, D etc. inter se aequalia assumamus, aequationes sequentes prodibunt:

num. corp.	
I	$z - \frac{2}{Aa} = 0$
II	$z^2 - \frac{4z}{Aa} + \frac{3}{AAaa} = 0$
III	$z^3 - \frac{6zz}{Aa} + \frac{10z}{AAaa} - \frac{4}{A^3a^3} = 0$
IV	$z^4 - \frac{8z^3}{Aa} + \frac{21z^2}{A^2a^2} - \frac{20z}{A^3a^3} + \frac{5}{A^4a^4} = 0$
V	$z^5 - \frac{10z^4}{Aa} + \frac{36z^3}{A^2a^2} - \frac{56z^3}{A^3a^3} + \frac{35z^2}{A^4a^4} - \frac{6}{A^5a^5} = 0$
VI	$z^6 - \frac{12z^5}{Aa} + \frac{55z^4}{A^2a^2} - \frac{120z^3}{A^3a^3} + \frac{126z^2}{A^4a^4} - \frac{56z}{A^5a^5} + \frac{7}{A^6a^6} = 0$

vnde pro corpusculorum numero quocunque m concluditur, fore:

z^{2m}

$$z^{2m} - \frac{2m}{1}\cdot\frac{z^{2m-1}}{Aa} + \frac{(2m-1)(2m-2)}{1.\ 2}\cdot\frac{z^{2m-2}}{A^2a^2} - \frac{(2m-2)(2m-3)(2m-4)}{1.\ 2.\ 3}\cdot\frac{z^{2m-3}}{A^3a^3}$$
$$+ \frac{(2m-3)(2m-4)(2m-5)(2m-6)}{1.\ 2.\ 3.\ 4}\cdot\frac{z^{2m-4}}{A^4a^4} - \frac{(2m-4)(2m-5)(2m-6)(2m-7)(2m-8)}{1.\ 2.\ 3.\ 4.\ 5}\cdot\frac{z^{2m-5}}{A^5a^5}$$
$$+ \text{etc.} = 0$$

Coroll. 5.

39. Hoc autem casu coefficientes $\mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. ex primo $\mathfrak{A}$ arbitrario pro quouis valore ipsius z ita definientur, vt sit:

$$\frac{\mathfrak{B}}{\mathfrak{A}} = Aa\left(\frac{2}{Aa} - z\right)$$
$$\frac{\mathfrak{C}}{\mathfrak{A}} = A^2a^2\left(\frac{2}{Aa} - z\right)^2 - 1$$
$$\frac{\mathfrak{D}}{\mathfrak{A}} = A^3a^3\left(\frac{2}{Aa} - z\right)^3 - 2Aa\left(\frac{2}{Ab} - z\right)$$
$$\frac{\mathfrak{E}}{\mathfrak{A}} = A^4a^4\left(\frac{2}{Aa} - z\right)^4 - 3A^2a^2\left(\frac{2}{Aa} - z\right)^2 + 1$$

siue

$$\frac{\mathfrak{B}}{\mathfrak{A}} = 2 - Aaz$$
$$\frac{\mathfrak{C}}{\mathfrak{A}} = 3 - 4Aaz + A^2a^2zz$$
$$\frac{\mathfrak{D}}{\mathfrak{A}} = 4 - 10Aaz + 6A^2a^2zz - A^3a^3z^3$$
$$\frac{\mathfrak{E}}{\mathfrak{A}} = 5 - 20Aaz + 21A^2a^2zz - 8A^3a^3z^3 + A^4a^4z^4$$

etc.

quarum formularum progreſſio ex ſuperioribus facillime colligitur.

Coroll. 6.

40. Ponamus pro eodem caſu breuitatis gratia $Aaz = y$, ac pro quouis corpuſculorum numero aequationes reſoluendae ita ſe habebunt:

Pro

Pro vno corpuſculo

$y - 2 = 0$, cuius radix eſt $y = 2$

Pro duobus corpuſculis

$yy - 4y + 3 = 0$, cuius radices ſunt $y = 1$; $y' = 3$

Pro tribus corpuſculis

$y^3 - 6yy + 10y - 4 = 0$, cuius radices ſunt

$y = 2$; $y' = 2 + \sqrt{2}$; $y'' = 2 - \sqrt{2}$

Pro quatuor corpuſculis

$y^4 - 8y^3 + 21y^2 - 20y + 5 = 0$, cuius radices ſunt

$y = \frac{3 + \sqrt{5}}{2}$; $y' = \frac{3 - \sqrt{5}}{2}$; $y'' = \frac{5 + \sqrt{5}}{2}$; $y''' = \frac{5 - \sqrt{5}}{2}$.

Coroll. 7.

41. Si has formulas bene perpendamus, eas per quadrata ſinuum, denotante ϱ angulum rectum, ſequenti modo exhiberi poſſe deprehendemus:

Pro vno corpuſculo

$y = 4(\text{ſin.}\, \tfrac{1}{2}\varrho)^2$

Pro duobus corpuſculis

$y = 4(\text{ſin.}\, \tfrac{1}{3}\varrho)^2$; $y' = 4(\text{ſin.}\, \tfrac{2}{3}\varrho)^2$

Pro tribus corpuſculis

$y = 4(\text{ſin.}\, \tfrac{1}{4}\varrho)^2$; $y' = 4(\text{ſin.}\, \tfrac{2}{4}\varrho)^2$; $y'' = 4(\text{ſin.}\, \tfrac{3}{4}\varrho)^2$

Pro quatuor corpuſculis

$y = 4(\text{ſin.}\, \tfrac{1}{5}\varrho)^2$; $y' = 4(\text{ſin.}\, \tfrac{2}{5}\varrho)^2$; $y'' = 4(\text{ſin.}\, \tfrac{3}{5}\varrho)^2$;
$y''' = 4(\text{ſin.}\, \tfrac{4}{5}\varrho)^2$

quarum formularum progreſſio per ſe eſt manifeſta.

Coroll.

Coroll. 8.

42. Inuentis autem pro quouis casu valoribus ipsius y, ob $z = \frac{y}{Aa}$, erit $n = \sqrt{\frac{2Kk}{Aa}}y$, et pro reliquis coefficientibus:

$$\mathfrak{B} = \mathfrak{A}(2 - y)$$
$$\mathfrak{C} = \mathfrak{A}(3 - 4y + yy)$$
$$\mathfrak{D} = \mathfrak{A}(4 - 10y + 6yy - y^3)$$
$$\mathfrak{E} = \mathfrak{A}(5 - 20y + 21yy - 8y^3 + y^4)$$
$$\mathfrak{F} = \mathfrak{A}(6 - 35y + 56yy - 36y^3 + 10y^4 - y^5)$$
etc.

Cum autem y habeat huiusmodi formam $y = 4(\sin.\Phi)^2$, erit:

$$\mathfrak{B} = \mathfrak{A}.\, 2\cos.\, 2\Phi = \mathfrak{A}.\frac{\sin.\, 4\Phi}{\sin.\, 2\Phi}$$
$$\mathfrak{C} = \mathfrak{A}.(2\cos.\, 4\Phi + 1) = \mathfrak{A}.\frac{\sin.\, 6\Phi}{\sin.\, 2\Phi}$$
$$\mathfrak{D} = \mathfrak{A}.(2\cos.\, 6\Phi + 2\cos.\, 2\Phi) = \mathfrak{A}.\frac{\sin.\, 8\Phi}{\sin.\, 2\Phi}$$
$$\mathfrak{E} = \mathfrak{A}.(2\cos.\, 8\Phi + 2\cos.\, 4\Phi + 1) = \mathfrak{A}.\frac{\sin.\, 10\Phi}{\sin.\, 2\Phi}$$
$$\mathfrak{F} = \mathfrak{A}.(2\cos.\, 10\Phi + 2\cos.\, 6\Phi + 2\cos.\, 2\Phi) = \mathfrak{A}.\frac{\sin.\, 12\Phi}{\sin.\, 2\Phi}$$
$$\mathfrak{G} = \mathfrak{A}.(\cos.\, 12\Phi + 2\cos.\, 8\Phi + 2\cos.\, 4\Phi + 1) = \mathfrak{A}.\frac{\sin.\, 14\Phi}{\sin.\, 2\Phi}$$

vnde sequens problema poterit in genere resolui.

Problema 5.

43. Si filum, terminis I et O fixum et data vi K tensum, onustum sit quotcunque corpusculis A, B, C etc. aequalibus et paribus interuallis a se inuicem distinctis, definire motum eius, postquam de statu suo naturali vtcunque fuerit depulsum.

 Solutio.

Solutio.

Sit numerus corpusculorum $= m$; massa vniuscuiusque $= A$, et binorum interuallum $= a$, erit totius fili massa $= mA$, et longitudo $IO = (m+1)a$. Reducta sint initio corpuscula A, B, C etc. ad distantias ab axe α, β, γ, etc. elapso autem tempore ω min. secund. peruenerint ad distantias $PA = p$; $QB = q$; $RC = r$, etc. His positis, si angulus rectus denotetur signo ϱ, et i sumatur pro numero quocunque integro positiuo; valor quilibet ipsius y erit $y = A(\text{sin.}\frac{i}{m+1}\varrho)^2$, vnde fit $n = 2\,\text{sin.}\frac{i}{m+1}\varrho\sqrt{\frac{2Kk}{Aa}}$; et ob $\Phi = \frac{i}{m+1}\varrho$, erit:

$$\mathfrak{B} = \mathfrak{A}\,\text{sin.}\tfrac{4i}{m+1}\varrho : \text{sin.}\tfrac{2i}{m+1}\varrho$$
$$\mathfrak{C} = \mathfrak{A}\,\text{sin.}\tfrac{6i}{m+1}\varrho : \text{sin.}\tfrac{2i}{m+1}\varrho$$
$$\mathfrak{D} = \mathfrak{A}\,\text{sin.}\tfrac{8i}{m+1}\varrho : \text{sin.}\tfrac{2i}{m+1}\varrho$$
etc.

Ponatur iam $\mathfrak{A} = \mathfrak{a}\,\text{sin.}\frac{2i}{m+1}\varrho$, ac pro motu habebuntur hae formulae:

$$p = \mathfrak{a}\,\text{sin.}\tfrac{2i}{m+1}\varrho.\ \text{cos.}(2\omega\,\text{sin.}\tfrac{i}{m+1}\varrho\sqrt{\tfrac{2Kk}{Aa}}) + \text{etc.}$$
$$q = \mathfrak{a}\,\text{sin.}\tfrac{4i}{m+1}\varrho.\ \text{cos.}(2\omega\,\text{sin.}\tfrac{i}{m+1}\varrho\sqrt{\tfrac{2Kk}{Aa}}) + \text{etc.}$$
$$r = \mathfrak{a}\,\text{sin.}\tfrac{6i}{m+1}\varrho.\ \text{cos.}(2\omega\,\text{sin.}\tfrac{i}{m+1}\varrho\sqrt{\tfrac{2Kk}{Aa}}) + \text{etc.}$$
etc.

Scilicet ex quouis valore ipsius i formentur tales expressiones, eaeque coniunctae praebebunt valores generales pro applicatis p, q, r etc. At pro i successiue sumi debent numeri 1, 2, 3, 4 vsque ad m.

Coroll.

Coroll. 1.

44. Si breuitatis gratia ponatur angulus $\frac{1}{m+1}\varrho = \Phi$ et angulus $2\omega\sqrt{\frac{2Kk}{Aa}} = \psi$, habebuntur, substituendo pro i successiue numeros 1, 2, 3, 4 etc. sequentes expressiones pro applicatis:

$$p = \mathfrak{a}\,\text{sin.}\,2\Phi.\,\text{cos.}\,\psi\,\text{sin.}\,\Phi + \mathfrak{b}\,\text{sin.}\,4\Phi.\,\text{cos.}\,\psi\,\text{sin.}\,2\Phi + \mathfrak{c}\,\text{sin.}\,6\Phi.\,\text{cos.}\,\psi\,\text{sin.}\,3\Phi + \text{etc.}$$

$$q = \mathfrak{a}\,\text{sin.}\,4\Phi.\,\text{cos.}\,\psi\,\text{sin.}\,\Phi + \mathfrak{b}\,\text{sin.}\,8\Phi\,\text{cos.}\,\psi\,\text{sin.}\,2\Phi + \mathfrak{c}\,\text{sin.}\,12\Phi.\,\text{cos.}\,\psi\,\text{sin.}\,3\Phi + \text{etc.}$$

$$r = \mathfrak{a}\,\text{sin.}\,6\Phi.\,\text{cos.}\,\psi\,\text{sin.}\,\Phi + \mathfrak{b}\,\text{sin.}\,12\Phi.\,\text{cos.}\,\psi\,\text{sin.}\,2\Phi + \mathfrak{c}\,\text{sin.}\,18\Phi.\,\text{cos.}\,\psi\,\text{sin.}\,3\Phi + \text{etc.}$$

$$s = \mathfrak{a}\,\text{sin.}\,8\Phi.\,\text{cos.}\,\psi\,\text{sin.}\,\Phi + \mathfrak{b}\,\text{sin.}\,16\Phi.\,\text{cos.}\,\psi\,\text{sin.}\,2\Phi + \mathfrak{c}\,\text{sin.}\,24\Phi.\,\text{cos.}\,\psi\,\text{sin.}\,3\Phi + \text{etc.}$$

etc.

Coroll. 2.

45. Ratio autem harum formularum clarius apparebit, si eas ad quemuis corpusculorum numerum accommodemus. Maneat ergo breuitatis gratia angulus $2\omega\sqrt{\frac{2Kk}{Aa}} = \psi$, eritque pro casu vnius corpusculi, ob $\Phi = \frac{1}{2}\varrho$,

$$p = \mathfrak{a}\,\text{sin.}\,\varrho.\,\text{cos.}\,\psi\,\text{sin.}\,\tfrac{1}{2}\varrho.$$

Coroll. 3.

46. Pro casu autem duorum corpusculorum, vbi $\Phi = \frac{1}{3}\varrho$, habebimus:

$$p = \mathfrak{a}\,\text{sin.}\,\tfrac{2}{3}\varrho.\,\text{cos.}\,\psi\,\text{sin.}\,\tfrac{1}{3}\varrho + \mathfrak{b}\,\text{sin.}\,\tfrac{4}{3}\varrho.\,\text{cos.}\,\psi\,\text{sin.}\,\tfrac{2}{3}\varrho$$

$$q = \mathfrak{a}\,\text{sin.}\,\tfrac{4}{3}\varrho.\,\text{cos.}\,\psi\,\text{sin.}\,\tfrac{1}{3}\varrho - \mathfrak{b}\,\text{sin.}\,\tfrac{2}{3}\varrho.\,\text{cos.}\,\psi\,\text{sin.}\,\tfrac{2}{3}\varrho.$$

Coroll. 4.

47. Pro casu trium corpusculorum, ob $\Phi = \frac{1}{4}\varrho$ habebimus:

$$p = \mathfrak{a}\sin.\tfrac{2}{4}\varrho.\cos.\psi\sin.\tfrac{1}{4}\varrho + \mathfrak{b}\sin.\tfrac{4}{4}\varrho.\cos.\psi\sin.\tfrac{2}{4}\varrho + \mathfrak{c}\sin.\varrho.\cos.\psi\sin.\tfrac{3}{4}\varrho$$

$$q = \mathfrak{a}\sin.\tfrac{4}{4}\varrho.\cos.\psi\sin.\tfrac{1}{4}\varrho + \mathfrak{b}\sin.\tfrac{3}{4}\varrho.\cos.\psi\sin.\tfrac{2}{4}\varrho - \mathfrak{c}\sin.\tfrac{4}{4}\varrho.\cos.\psi\sin.\tfrac{3}{4}\varrho$$

$$r = \mathfrak{a}\sin.\tfrac{2}{4}\varrho.\cos.\psi\sin.\tfrac{1}{4}\varrho - \mathfrak{b}\sin.\tfrac{4}{4}\varrho.\cos.\psi\sin.\tfrac{2}{4}\varrho + \mathfrak{c}\sin.\tfrac{2}{4}\varrho.\cos.\psi\sin.\tfrac{3}{4}\varrho.$$

Coroll. 5.

48. Pro casu quatuor corpusculorum, ob $\Phi = \frac{1}{5}\varrho$, habebimus:

$$p = \mathfrak{a}\sin.\tfrac{2}{5}\varrho.\cos.\psi\sin.\tfrac{1}{5}\varrho + \mathfrak{b}\sin.\tfrac{4}{5}\varrho.\cos.\psi\sin.\tfrac{2}{5}\varrho + \mathfrak{c}\sin.\tfrac{4}{5}\varrho.\cos\psi\sin.\tfrac{3}{5}\varrho + \mathfrak{d}\sin.\tfrac{2}{5}\varrho.\cos.\psi\sin\tfrac{4}{5}\varrho$$

$$q = \mathfrak{a}\sin.\tfrac{4}{5}\varrho.\cos.\psi\sin\tfrac{1}{5}\varrho + \mathfrak{b}\sin.\tfrac{2}{5}\varrho.\cos.\psi\sin.\tfrac{2}{5}\varrho - \mathfrak{c}\sin.\tfrac{2}{5}\varrho.\cos.\psi\sin.\tfrac{3}{5}\varrho - \mathfrak{d}\sin.\tfrac{4}{5}\varrho.\cos.\psi\sin.\tfrac{4}{5}\varrho$$

$$r = \mathfrak{a}\sin.\tfrac{4}{5}\varrho.\cos.\psi\sin.\tfrac{1}{5}\varrho - \mathfrak{b}\sin.\tfrac{2}{5}\varrho.\cos.\psi\sin.\tfrac{2}{5}\varrho - \mathfrak{c}\sin.\tfrac{2}{5}\varrho.\cos.\psi\sin.\tfrac{3}{5}\varrho + \mathfrak{d}\sin.\tfrac{4}{5}\varrho.\cos.\psi\sin.\tfrac{4}{3}\varrho$$

$$s = \mathfrak{a}\sin\tfrac{2}{5}\varrho.\cos,\psi\sin.\tfrac{1}{5}\varrho - \mathfrak{b}\sin.\tfrac{4}{5}\varrho.\cos.\psi\sin.\tfrac{2}{5}\varrho + \mathfrak{c}\sin.\tfrac{4}{5}\varrho.\cos.\psi\sin\tfrac{3}{5}\varrho - \mathfrak{d}\sin.\tfrac{2}{5}\varrho.\cos.\psi\sin.\tfrac{4}{5}\varrho.$$

Coroll. 6.

49. Quod si vero numerus corpusculorum fuerit infinite magnus, ob $\sin.\Phi = \Phi = \frac{\varrho}{m}$, nanciscemur has formulas:

$$p = \frac{2\mathfrak{a}\varrho}{m}\cos.\frac{\psi\varrho}{m} + \frac{4\mathfrak{b}\varrho}{m}\cos.\frac{2\psi\varrho}{m} + \frac{6\mathfrak{c}\varrho}{m}\cos.\frac{3\psi\varrho}{m} + \text{etc.}$$

$$q = \frac{4\mathfrak{a}\varrho}{m}\cos.\frac{\psi\varrho}{m} + \frac{8\mathfrak{b}\varrho}{m}\cos.\frac{2\psi\varrho}{m} + \frac{12\mathfrak{c}\varrho}{m}\cos.\frac{3\psi\varrho}{m} + \text{etc.}$$

$$r = \frac{6\mathfrak{a}\varrho}{m}\cos.\frac{\psi\varrho}{m} + \frac{12\mathfrak{b}\varrho}{m}\cos.\frac{2\psi\varrho}{m} + \frac{18\mathfrak{c}\varrho}{m}\cos.\frac{3\psi\varrho}{m} + \text{etc.}$$

etc.

Coroll. 7.

50. Verum si huius cordae tota longitudo IO ponatur $= l$, et massa totius cordae $= M$ ob $a = \frac{l}{m}$, et $A = \frac{M}{m}$, erit $\psi = 2m\omega V \frac{2Kk}{Ml}$, vnde $\frac{\psi\varrho}{m} = 2\varrho\omega V \frac{2Kk}{Ml} = \pi\omega V \frac{2Kk}{Ml}$; in coefficientibus autem constantibus vtpote arbitrariis omitti poterunt litterae ϱ et m, ita vt sit:

$$p = \mathfrak{a}\cos.\pi\omega V\frac{2Kk}{Ml} + \mathfrak{b}\cos.2\pi\omega V\frac{2Kk}{Ml} + \mathfrak{c}\cos.3\pi\omega V\frac{2Kk}{Ml} + \text{etc.}$$

$q = 2p$; $r = 3p$; $s = 4p$ etc.

quae formula eundem exhibet motum, qui pro corda vniformiter crassa determinari solet.