1764

De resolutione aequationis $dy + ayy\,dx = bx^m\,dx$

Leonhard Euler

DE RESOLVTIONE AEQVATIONIS

$dy + ayy\,dx = bx^m\,dx.$

Auctore

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Problema I.

I.

Inuenire numeros loco exponentis indefiniti m substituendos, vt valor ipsius y algebraice per x definiri queat.

Solutio.

Ponatur $y = cx^{n-1} + \frac{dz}{azdx}$, ac posito dx constante, erit $dy = (n-1)cx^{n-2}dx + \frac{ddz}{azdx} - \frac{dz^2}{azzdx^2}$. Cum vero sit $yy = ccx^{2n-2} + \frac{2cx^{n-1}dz}{azdx} + \frac{dz^2}{a^2z^2dx^2}$ facta substitutione transibit aequatio proposita in hanc:

$$\frac{ddz}{azdx} + (n-1)cx^{n-2}dx + accx^{2n-2}dx + \frac{2cx^{n-1}dz}{z} = bx^m dx.$$

Fiat $m = 2n-2$ et $b = acc$, habebiturque:

$$ddz + (n-1)acx^{n-2}zdx^2 + 2acx^{n-1}dxdz = 0$$

quae

quae ergo resultat ex hac aequatione propositae aequivalente

$$dy + ayydx = accx^{2n-2}dx$$

facta substitutione $y = cx^{n-1} + \frac{dz}{azdx}$. Fingatur iam haec aequatio:

$$z = Ax^{\frac{-n+1}{2}} + Bx^{\frac{-3n+1}{2}} + Cx^{\frac{-5n+1}{2}} + Dx^{\frac{-7n+1}{2}} + \text{etc.}$$

eritque differentiando:

$$\frac{dz}{dx} = -\frac{(n-1)}{2}Ax^{\frac{-n-1}{2}} - \frac{(3n-1)}{2}Bx^{\frac{-3n-1}{2}} - \frac{(5n-1)}{2}Cx^{\frac{-5n-1}{2}} - \text{etc.}$$

$$\frac{ddz}{dx^2} = +\frac{(nn-1)}{4}Ax^{\frac{-n-3}{2}} + \frac{(9nn-1)}{4}Bx^{\frac{-3n-3}{2}} + \frac{(25nn-1)}{4}Cx^{\frac{-5n-3}{2}} - \text{etc.}$$

Cum vero ex superiori aequatione per dx^2 diuisa sit:

$$\frac{ddz}{dx^2} + \frac{2acx^{n-1}dz}{dx} + (n-1)acx^{n-2}z = 0$$

si series assumta substituatur, prodibit sequens aequatio:

$$0 = \begin{cases} +\frac{(nn-1)}{4}Ax^{\frac{-n-3}{2}} + \frac{(9nn-1)}{4}Bx^{\frac{-3n-3}{2}} + \frac{(25nn-1)}{4}Cx^{\frac{-5n-3}{2}} \\ \qquad + \frac{(49nn-1)}{4}Dx^{\frac{-7n-3}{2}} + \text{etc.} \\ -(n-1)acAx^{\frac{n-3}{2}} - (3n-1)acBx^{\frac{-n-3}{2}} - (5n-1)acCx^{\frac{-3n-3}{2}} \\ \qquad -(7n-1)acDx^{\frac{-5n-3}{2}} - (9n-1)acEx^{\frac{-7n-3}{2}} - \text{etc.} \\ +(n-1)acAx^{\frac{n-3}{2}} + (n-1)acBx^{\frac{-n-3}{2}} + (n-1)acCx^{\frac{-3n-3}{2}} \\ \qquad + (n-1)acDx^{\frac{-5n-3}{2}} + (n-1)acEx^{\frac{-7n-3}{2}} - \text{etc.} \end{cases}$$

 Ponan-

Ponantur termini homogenei iunctim sumti nihilo aequales, vt determinentur coefficientes A, B, C, D, E, etc. eritque

$$B = \frac{(nn-1)A}{2n.4ac} = \frac{(nn-1)}{2}.\frac{A}{4nac}$$

$$C = \frac{(9nn-1)}{4n}.\frac{B}{4ac} = \frac{(nn-1)}{2}\frac{(9nn-1)}{4}.\frac{A}{4^2n^2a^2c^2}$$

$$D = \frac{(25nn-1)}{6n}.\frac{C}{4ac} = \frac{(nn-1)}{2}\frac{(9nn-1)}{4}\frac{(25nn-1)}{6}.\frac{A}{4^3.n^3a^3c^3}$$

$$E = \frac{(49nn-1)}{8n}.\frac{D}{4ac} = \frac{(nn-1)}{2}\frac{(9nn-1)}{4}\frac{(25nn-1)}{6}\frac{(49nn-1)}{8}.\frac{A}{4^4n^4a^4c^4}$$

etc.

Determinabitur ergo z per x sequenti modo: $z =$

$$Ax^{\frac{-n+1}{2}} + \frac{(nn-1)}{8}\frac{A}{nac}x^{\frac{-3n+1}{2}} + \frac{(nn-1)}{8}\frac{(9nn-1)}{16}\frac{A}{n^2a^2c^2}x^{\frac{-5n+1}{2}}$$

$$+ \frac{(nn-1)}{8}\frac{(9nn-1)}{16}\frac{(25nn-1)}{24}\frac{A}{n^3a^3c^3}x^{\frac{-7n+1}{2}} + \text{etc.}$$

Valore hoc substituto resultabit valor quaesitus: $y = cx^{n-1}$

$$-\frac{1}{a}\left\{\frac{\frac{(n-1)}{2}Ax^{\frac{-n-1}{2}} + \frac{(3n-1)(nn-1)}{2\cdot 8}\frac{A}{nac}x^{\frac{-3n-1}{2}} + \frac{(5n-1)}{2}.\frac{(nn-1)(9nn-1)}{8\cdot 16}\frac{A}{n^2a^2c^2}x^{\frac{-5n-1}{2}} + \text{etc.}}{Ax^{\frac{-n+1}{2}} + \frac{(nn-1)}{8}.\frac{A}{nac}x^{\frac{-3n+1}{2}} + \frac{(nn-1)}{8}.\frac{(9nn-1)}{16}.\frac{A}{n^2a^2c^2}x^{\frac{-5n+1}{2}} + \text{etc.}}\right\}$$

siue numeratore ac denominatore per $Ax^{\frac{-n-1}{2}}$ diuiso:

$y = cx^{n-1}$

$$-\frac{1}{ax}\left\{\frac{\frac{(n-1)}{2} + \frac{(3n-1)(nn-1)}{2\cdot 8}\frac{x^{-n}}{nac} + \frac{(5n-1)}{2}.\frac{(nn-1)(9nn-1)}{8\cdot 16}\frac{x^{-2n}}{n^2a^2c^2} + \frac{(7n-1)}{2}.\frac{(nn-1)(nn-1)(9nn-1)(25nn-1)}{8\cdot 8\cdot 16\cdot 24}\frac{x^{-3n}}{n^3a^3c^3} + \text{etc.}}{1 + \frac{(nn-1)}{8}.\frac{x^{-n}}{nac} + \frac{(nn-1)(9nn-1)}{8\cdot 16}.\frac{x^{-2n}}{n^2a^2c^2} + \frac{(nn-1)(9nn-1)(25nn-1)}{8\cdot 16\cdot 24}.\frac{x^{-3n}}{n^3a^3c^3} + \text{etc.}}\right\}$$

Haec

Haec ergo expreſſio generaliter in infinitum excurrens fit finita, ſi fuerit $(2i+1)^2 nn - 1 = 0$, denotante i numerum quemcunque integrum, hoc eſt, ſi fuerit $n = \frac{\pm 1}{2i+1}$; et $m = 2n - 2 = \frac{-4i-2\pm 2}{2i+1}$. Huius ergo aequationis, quoties i fuerit numerus integer:

$$dy + ayydx = accx^{\frac{-4i-2\pm 2}{2i+1}}dx$$

integrale ſemper in terminis finitis poterit exhiberi, ſeu valor ipſius y per x algebraice exponi.

Sit primo $n = \frac{+1}{2i+1}$, vt ſit $m = 2n - 2 = \frac{-4i}{2i+1}$, erit huius aequationis:

$$dy + ayydx = accx^{\frac{-4i}{2i+1}}dx$$

integrale in terminis algebraicis expreſſum:

$$ayx = acx^{\frac{1}{2i+1}}$$

$$\frac{+\frac{i}{2i+1} - \frac{i(i^2-1)}{2(2i+1)^2}\cdot\frac{x^{\frac{-1}{2i+1}}}{ac} + \frac{i(i^2-1)(i^2-4)}{2\cdot 4(2i+1)^3}\cdot\frac{x^{\frac{-2}{2i+1}}}{a^2c^2} - \frac{i(i^2-1)(i^2-4)(i^2-9)}{2\cdot 4\cdot 6(2i+1)^4}\cdot\frac{x^{\frac{-3}{2i+1}}}{a^3c^3} + \text{etc.}}{1 - \frac{i(i+1)}{2(2i+1)}\cdot\frac{x^{\frac{-1}{2i+1}}}{ac} + \frac{i(i^2-1)(i+2)}{2\cdot 4(2i+1)^2}\cdot\frac{x^{\frac{-2}{2i+1}}}{a^2c^2} - \frac{i(i^2-1)(i^2-4)(i+3)}{2\cdot 4\cdot 6(2i+1)^3}\cdot\frac{x^{\frac{-3}{2i+1}}}{a^3c^3} + \text{etc.}}$$

ſeu facta ad communem denominatorem reductione erit: $ayx =$

$$\frac{acx^{\frac{+1}{2i+1}} - \frac{ii-i}{2(2i+1)} + \frac{i(i^2-1)(i-2)}{2\cdot 4(2i+1)^2}\cdot\frac{x^{\frac{-1}{2i+1}}}{ac} - \frac{i(i^2-1)(i^2-4)(i-3)}{2\cdot 4\cdot 6(2i+1)^3}\cdot\frac{x^{\frac{-2}{2i+1}}}{a^2c^2} + \text{etc.}}{1 - \frac{i(i+1)}{2(2i+1)}\cdot\frac{x^{\frac{-1}{2i+1}}}{ac} + \frac{i(i^2-1)(i+2)}{2\cdot 4(2i+1)^2}\cdot\frac{x^{\frac{-2}{2i+1}}}{a^2c^2} - \frac{i(i^2-1)(i^2-4)(i+3)}{2\cdot 4\cdot 6(2i+1)^3}\cdot\frac{x^{\frac{-3}{2i+1}}}{a^3c^3} + \text{etc.}}$$

 Sit

Sit deinde $n = \frac{-1}{2i+1}$, vt fit $m = \frac{-4i-4}{2i+1}$, erit huius aequationis

$$dy + ayydx = accx^{\frac{-4i-4}{2i+1}}dx$$

integrale in terminis algebraicis expressum:

$$ayx = acx^{\frac{-1}{2i+1}} + \frac{\frac{i+1}{2i+1} + \frac{i(i+1)(i+2)}{2(2i+1)^2}\frac{x^{\frac{1}{2i+1}}}{ac} + \frac{i(i^2-1)(i+2)(i+3)}{2\cdot4(2i+1)^3}\cdot\frac{x^{\frac{2}{2i+1}}}{a^2c^2} + \frac{i(i^2-1)(i^2-4)(i+3)(i+4)}{2\cdot4\cdot6(2i+1)^4}\cdot\frac{x^{\frac{3}{2i+1}}}{a^3c^3} + \text{etc.}}{1 + \frac{i(i+1)}{2(2i+1)}\cdot\frac{x^{\frac{1}{2i+1}}}{ac} + \frac{i(i^2-1)(i+2)}{2\cdot4\cdot(2i+1)^2}\cdot\frac{x^{\frac{2}{2i+1}}}{a^2c^2} + \frac{i(i^2-1)(i^2-4)(i+3)}{2\cdot4\cdot6(2i+1)^3}\cdot\frac{x^{\frac{3}{2i+1}}}{a^3c^3} + \text{etc.}}$$

seu facta ad communem denominatorem reductione, erit $ayx =$

$$\frac{acx^{\frac{-1}{2i+1}} + \frac{(i+1)(i+2)}{2(2i+1)} + \frac{i(i+1)(i+2)(i+3)}{2\cdot4(2i+1)^2}\cdot\frac{x^{\frac{1}{2i+1}}}{ac} + \frac{i(i^2-1)(i+2)(i+3)(i+4)}{2\cdot4\cdot6(2i+1)^3}\cdot\frac{x^{\frac{2}{2i+1}}}{a^2c^2} + \text{etc.}}{1 + \frac{i(i+1)}{2(2i+1)}\cdot\frac{x^{\frac{1}{2i+1}}}{ac} + \frac{i(i^2-1)(i+2)}{2\cdot4(2i+1)^2}\cdot\frac{x^{\frac{2}{2i+1}}}{a^2c^2} + \frac{i(i^2-1)(i^2-4)(i+3)}{2\cdot4\cdot6(2i+1)^3}\cdot\frac{x^{\frac{3}{2i+1}}}{a^3c^3} + \text{etc.}}$$

Quotiescunque igitur fuerit i numerus integer, toties huius aequationis:

$$dy + ayydx = accx^{\frac{-4i-2\pm2}{2i+1}}dx$$

integrale in terminis algebraicis potest exprimi. Q. E. I.

Coroll. 1.

Coroll. 1.

2. Aequatio ergo propoſita $dy + ayydx = accx^m dx$ integrationem algebraicam admittit, ſi fuerit exponens m, vel terminus huius ſeriei:

$-0;\ -\frac{4}{3};\ -\frac{8}{5};\ -\frac{12}{7};\ -\frac{16}{9};\ -\frac{20}{11};\ -\frac{24}{13};$ etc.

vel ſi fuerit m terminus ex hac fractionum ſerie:

$-\frac{4}{1};\ -\frac{8}{3};\ -\frac{12}{5};\ -\frac{16}{7};\ -\frac{20}{9};\ -\frac{24}{11};\ -\frac{28}{13};$ etc.

Coroll. 2.

3. Subſtituamus in priori integrabilitatis claſſe loco i ſucceſſiue numeros 0, 1, 2, 3, 4, etc. atque reperietur, vt ſequitur.

Si $i = 0$; huius aequationis:

I. $dy + ayydx = accdx$, integrale erit:

$ayx = acx$; ſiue $y = c$.

Si $i = 1$; huius aequationis:

II. $dy + ayydx = accx^{-\frac{4}{3}}dx$, integrale erit:

$$ayx = \frac{acx^{\frac{1}{3}}}{1 - \frac{1\cdot 2}{2\cdot 3}\frac{x^{-\frac{1}{3}}}{ac}} \text{ ſeu } y = \frac{cx^{-\frac{2}{3}}}{1 - \frac{1}{3}\frac{x^{-\frac{1}{3}}}{ac}} = \frac{3acc}{3acx^{\frac{2}{3}} - x^{\frac{1}{3}}}$$

Si $i = 2$; huius aequationis:

III. $dy + ayydx = accx^{-\frac{8}{5}}dx$, integrale erit:

$$ayx = \frac{acx^{\frac{1}{5}} - \frac{2\cdot 1}{1\cdot 5}}{1 - \frac{2\cdot 3}{2\cdot 5}\cdot\frac{x^{-\frac{1}{5}}}{ac} + \frac{2\cdot 3\cdot 4}{2\cdot 4\cdot 5^2}\cdot\frac{x^{-\frac{2}{5}}}{a^2c^2}} = \frac{acx^{\frac{1}{5}} - \frac{1}{5}}{1 - \frac{3x^{-\frac{1}{5}}}{5ac} + \frac{3x^{-\frac{2}{5}}}{5^2a^2c^2}}$$

Si

Si $i = 3$ huius aequationis:

IV. $dy + ayydx = accx^{-\frac{12}{7}}dx$, integrale erit:

$$ayx = \frac{acx^{\frac{1}{7}} - \frac{3\cdot 2}{2\cdot 7} + \frac{3\cdot 2\cdot 1\cdot 4}{2\cdot 4\cdot 7^2}\cdot\frac{x^{-\frac{1}{7}}}{ac}}{1 - \frac{3\cdot 4}{2\cdot 7}\frac{x^{-\frac{1}{7}}}{ac} + \frac{3\cdot 4\cdot 5\cdot 2}{2\cdot 4\cdot 7^2}\cdot\frac{x^{-\frac{2}{7}}}{a^2c^2} - \frac{3\cdot 4\cdot 5\cdot 6\cdot 2\cdot 1}{2\cdot 4\cdot 6\cdot 7^3}\cdot\frac{x^{-\frac{3}{7}}}{a^3c^3}} \text{ siue}$$

$$ayx = \frac{acx^{\frac{1}{7}} - \frac{3}{7} + \frac{3\cdot 1}{7^2}\cdot\frac{x^{-\frac{1}{7}}}{ac}}{1 - \frac{6}{7}\cdot\frac{x^{-\frac{1}{7}}}{ac} + \frac{3\cdot 5}{7^2}\cdot\frac{x^{-\frac{2}{7}}}{a^2c^2} - \frac{1\cdot 2\cdot 5}{7^3}\cdot\frac{x^{-\frac{3}{7}}}{a^3c^3}}.$$

Si $i = 4$, huius aequationis:

V. $dy + ayydx = accx^{-\frac{16}{9}}dx$, integrale erit:

$$ayx = \frac{acx^{\frac{1}{9}} - \frac{4\cdot 3}{2\cdot 9} + \frac{4\cdot 3\cdot 2\cdot 5}{2\cdot 4\cdot 9^2}\cdot\frac{x^{-\frac{1}{9}}}{ac} - \frac{4\cdot 3\cdot 2\cdot 1\cdot 5\cdot 6}{2\cdot 4\cdot 6\cdot 9^3}\cdot\frac{x^{-\frac{2}{9}}}{a^2c^2}}{1 - \frac{4\cdot 5}{2\cdot 9}\frac{x^{-\frac{1}{9}}}{ac} + \frac{4\cdot 5\cdot 6\cdot 3}{2\cdot 4\cdot 9^2}\cdot\frac{x^{-\frac{2}{9}}}{a^2c^2} - \frac{4\cdot 5\cdot 6\cdot 7\cdot 3\cdot 2}{2\cdot 4\cdot 6\cdot 9^3}\cdot\frac{x^{-\frac{3}{9}}}{a^3c^3} + \frac{4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 3\cdot 2\cdot 1}{2\cdot 4\cdot 6\cdot 8\cdot 9^4}\cdot\frac{x^{-\frac{4}{9}}}{a^4c^4}}$$

Si $i = 5$; huius aequationis

VI. $dy + ayydx = accx^{-\frac{20}{11}}dx$, integrale erit:

$$ayx = \frac{acx^{\frac{1}{11}} - \frac{5\cdot 4}{2\cdot 11} + \frac{5\cdot 4\cdot 3\cdot 6}{2\cdot 4\cdot 11^2}\cdot\frac{x^{-\frac{1}{11}}}{ac} - \frac{5\cdot 4\cdot 3\cdot 2\cdot 6\cdot 7}{2\cdot 4\cdot 6\cdot 11^3}\frac{x^{-\frac{2}{11}}}{a^2c^2} + \frac{5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 6\cdot 7\cdot 8}{2\cdot 4\cdot 6\cdot 8\cdot 11^4}\cdot\frac{x^{-\frac{3}{11}}}{a^3c^3}}{1 - \frac{5\cdot 6}{2\cdot 11}\frac{x^{-\frac{1}{11}}}{ac} + \frac{5\cdot 6\cdot 7\cdot 4}{2\cdot 4\cdot 11^2}\frac{x^{-\frac{2}{11}}}{a^2c^2} - \frac{5\cdot 6\cdot 7\cdot 8\cdot 4\cdot 3}{2\cdot 4\cdot 6\cdot 11^3}\frac{x^{-\frac{3}{11}}}{a^3c^3} + \frac{5\cdot 6\cdot 7\cdot 8\cdot 5\cdot 4\cdot 3\cdot 2}{2\cdot 4\cdot 6\cdot 8\cdot 11^4}\cdot\frac{x^{-\frac{4}{11}}}{a^4c^4} - \frac{5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 4\cdot 3\cdot 2}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 11^5}\cdot\frac{x^{-\frac{5}{11}}}{a^5c^5}}.$$

Coroll. 3.

3. In posteriori integrabilitatis ordine substituamus pariter loco i numeros 0, 1, 2, 3, 4, etc. ac reperietur, vt sequitur.

Si

Si $i=0$; huius aequationis:

I. $dy+ayydx=accx^{-4}dx$, integrale erit:

$$ayx=\frac{acx^{-1}+\frac{1\cdot 2}{2\cdot 1}}{1}=1+\frac{ac}{x} \text{ seu } y=\frac{1}{ax}+\frac{c}{xx}$$

Si $i=1$; huius aequationis:

II. $dy+ayydx=accx^{-\frac{8}{3}}dx$, integrale erit:

$$ayx=\frac{acx^{-\frac{1}{3}}+\frac{2\cdot 3}{2\cdot 3}+\frac{2\cdot 3\cdot 4\cdot 1}{2\cdot 4\cdot 3^2}\cdot\frac{x^{\frac{1}{3}}}{ac}}{1+\frac{1\cdot 2}{2\cdot 3}\cdot\frac{x^{\frac{1}{3}}}{ac}}=\frac{acx^{-\frac{1}{3}}+1+\frac{x^{\frac{1}{3}}}{3ac}}{1+\frac{x^{\frac{1}{3}}}{3ac}}$$

Si $i=2$ huius aequationis:

III. $dy+ayydx=accx^{-\frac{12}{5}}dx$, integrale erit:

$$ayx=\frac{acx^{-\frac{1}{5}}+\frac{3\cdot 4}{2\cdot 5}+\frac{2\cdot 3\cdot 4\cdot 5}{2\cdot 4\cdot 5^2}\cdot\frac{x^{\frac{1}{5}}}{ac}+\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}{2\cdot 4\cdot 6\cdot 5^3}\cdot\frac{x^{\frac{2}{5}}}{a^2c^2}}{1+\frac{2\cdot 3}{2\cdot 5}\cdot\frac{x^{\frac{1}{5}}}{ac}+\frac{1\cdot 2\cdot 3\cdot 4}{2\cdot 4\cdot 5^2}\cdot\frac{x^{\frac{2}{5}}}{a^2c^2}}$$

Si $i=3$; huius aequationis:

IV. $dy+ayydx=accx^{-\frac{16}{7}}dx$, integrale erit:

$$ayx=\frac{acx^{-\frac{1}{7}}+\frac{4\cdot 5}{2\cdot 7}+\frac{3\cdot 4\cdot 5\cdot 6}{2\cdot 4\cdot 7^2}\cdot\frac{x^{\frac{1}{7}}}{ac}+\frac{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}{2\cdot 4\cdot 6\cdot 7^3}\cdot\frac{x^{\frac{2}{7}}}{a^2c^2}+\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}{2\cdot 4\cdot 6\cdot 8\cdot 7^4}\cdot\frac{x^{\frac{3}{7}}}{a^3c^3}}{1+\frac{3\cdot 4}{2\cdot 7}\cdot\frac{x^{\frac{1}{7}}}{ac}+\frac{2\cdot 3\cdot 4\cdot 5}{2\cdot 4\cdot 7^2}\cdot\frac{x^{\frac{2}{7}}}{a^2c^2}+\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}{2\cdot 4\cdot 6\cdot 7^3}\cdot\frac{x^{\frac{3}{7}}}{a^3c^3}}$$

Atque ex his casibus analogia patet, cuius ope omnium casuum, qui quidem integrationem admittunt, integralia algebraica expedite formari poterunt.

Scholion.

5. De his integralibus autem probe notandum est, ea non esse completa, neque ideo aeque late patere, ac

ac aequationem differentialem; id quod vel ex primo casu $dy + ayydx = accdx$ patet, cui etsi satisfacit $y = c$, tamen facile intelligitur, logarithmos insuper in ea comprehendi. Manifestum autem hoc est quoque hinc, quod in his integralibus non contineatur noua constans arbitraria, quae in differentiali non inerat; in quo criterium integrationis completae versatur. Caeterum vero hinc duplicia integralia cuiusuis casus obtinentur, eo quod c tam affirmatiue, quam negatiue, accipere licet, aequatione differentiali, quae tantum cc continet non mutata.

Problema 2.

6. Inuento ope praecedentis methodi integrali particulari pro casibus assignatis aequationis $dy + ayydx = accx^m dx$, inuenire integrale completum pro iisdem casibus.

Solutio.

Posito $m = 2n - 2$, integrale particulare aequationis propositae inuentum est esse $ayx = acx^n -$

$$\frac{\frac{(n-1)}{2} - \frac{(3n-1)(nn-1)}{2\cdot 8n}\cdot\frac{x^{-n}}{ac} - \frac{(5n-1)(nn-1)(9nn-1)}{2\cdot 8n\cdot 16n}\cdot\frac{x^{-2n}}{a^2c^2} - \frac{(7n-1)(nn-1)(9nn-1)(25nn-1)}{2\cdot 8n\cdot 16n\cdot 24n}\cdot\frac{x^{-3n}}{a^3c^3} - \text{etc.}}{1 + \frac{(nn-1)}{8n}\cdot\frac{x^{-n}}{ac} + \frac{(nn-1)(9nn-1)}{8n\cdot 16n}\cdot\frac{x^{-2n}}{a^2c^2} + \frac{(nn-1)(9nn-1)(25nn-1)}{8n\cdot 16n\cdot 24n}\cdot\frac{x^{-3n}}{a^3c^3} + \text{etc.}}$$

cuius loco scribamus breuitatis gratia $y = P$. Cum igitur P sit eiusmodi valor, per variabilem x datus, qui satisfaciat aequationi $dy + ayydx = accx^{2n-2}dx$, erit vtique $dP + aP^2dx = accx^{2n-2}dx$. Ponamus iam, integrale completum aequationis propositae $dy + ayydx =$

$= accx^{2n-2}dx$ esse $y = P + v$, quo valore loco y substituto habebimus hanc aequationem $dP + dv + aP^2dx + 2aPvdx + avvdx = accx^{2n-2}dx$. Cum vero sit $dP + aP^2dx = accx^{2n-2}dx$, erit $dv + 2aPvdx + avvdx = 0$. Sit $v = \frac{1}{u}$, erit $du - 2aPudx = + adx$, quae multiplicata per $e^{-2a\int Pdx}$ denotante e numerum, cuius logarithmus hyperbolicus est $= 1$, fit integrabilis; erit scilicet aequationis $e^{-2a\int Pdx}(du - 2aPudx) = e^{-2a\int Pdx}adx$, integrale $e^{-2a\int Pdx}u = \int e^{-2a\int Pdx}adx$: ideoque $u = e^{2a\int Pdx}\int e^{-2a\int Pdx}adx$. Quo valore cum sit $v = \frac{1}{u}$ substituto, erit integrale completum aequationis propositae $y = P + \frac{e^{-2a\int Pdx}}{\int e^{-2a\int Pdx}adx}$. At ex problemate primo est valor ipsius y particularis, quem hic ponimus $P = cx^{n-1} + \frac{dz}{azdx}$; existente

$$z = x^{\frac{-n+1}{2}} + \frac{(nn-1)}{8n} \cdot \frac{x^{\frac{-3n+3}{2}}}{ac} + \frac{(nn-1)(9nn-1)}{8n \cdot 16n} \cdot \frac{x^{\frac{-5n+5}{2}}}{a^2c^2} + \frac{(nn-1)(9nn-1)(25nn-1)}{8n \cdot 16n \cdot 24n} \cdot \frac{x^{\frac{-7n+7}{2}}}{a^3c^3} + \text{etc.}$$

Hinc erit $\int Pdx = \frac{cx^n}{n} + \frac{1}{a}lz$, et $e^{-2a\int Pdx} = e^{\frac{-2acx^n}{n}} : zz$. Quo valore substituto habebitur integrale completum:

$$y = cx^{n-1} + \frac{dz}{azdx} + \frac{e^{\frac{-2acx^n}{n}}}{zz\int e^{\frac{-2acx^n}{n}}adx : zz}. \quad \text{Q. E. I.}$$

Aliter.

Quemadmodum hac ratione ex vno integrali particulari inuenitur integrale completum, ita ex duobus integralibus particularibus expeditius integrale comple-

 tum

tum indagabitur, neque in hoc modo peruenitur ad formulam integralem, cuiusmodi eſt ea $\int e^{\frac{-2acx^n}{n}} adx : zz$, quae integrali completo, quod inuenimus, inuoluitur. Cum enim aequatio $dy + ayydx = accx^{2n-2}dx$ maneat inuariata, ſiue c affirmatiue, ſiue negatiue, accipiatur, habemus vtique duo integralia particularia, quorum prius eſt $y = P = cx^{n-1} + \frac{dz}{azdx}$, exiſtente $z = x^{\frac{-n+1}{2}} + \frac{(nn-1)}{8n} \cdot \frac{x^{\frac{-3n+1}{2}}}{ac} + \frac{(nn-1)(9nn-1)}{8n \cdot 16n} \cdot \frac{x^{\frac{-5n+1}{2}}}{a^2c^2} +$ etc. Poſterius vero ſimili modo inueſtigandum erit $y = Q = -cx^{n-1} + \frac{du}{audx}$; fietque $u = x^{\frac{-n+1}{2}} - \frac{(nn-1)}{8n} \cdot \frac{x^{\frac{-3n+1}{2}}}{ac} + \frac{(nn-1)(9nn-1)}{8n \cdot 16n} \cdot \frac{x^{\frac{-5n+1}{2}}}{a^2c^2} -$ etc. qui duo valores z et u tantum ſignis inter ſe differunt. Erit ergo tam $dP + aP^2dx = accx^{2n-2}dx$, quam $dQ + aQ^2dx = accx^{2n-2}dx$. Ponamus iam $R = \frac{P-y}{Q-y}$, quae aequatio ſit integralis completa propoſitae differentialis; quam formam ideo aſſumimus, quia in ea vtraque particularium $y = P$ et $y = Q$ continetur, illa nempe ſi fiat $R = 0$, haec ſi $R = \infty$. Fiet ergo $QR - Ry = P - y$, hincque $y = \frac{QR-P}{R-1}$, quae dat $dy = \frac{RRdQ - QdR - RdQ - RdP + dP + PdR}{(R-1)^2}$ ſubſtituantur hic valores ſupra inuenti $dP = -aP^2dx + accx^{2n-2}dx$ et $dQ = -aQQdx + accx^{2n-2}dx$, eritque $dy = accx^{2n-2}dx + \frac{aP^2dx}{R-1} - \frac{aQ^2Rdx}{R-1} + \frac{(P-Q)dR}{(R-1)^2} = -a\frac{(QR-P)^2dx}{(R-1)^2} + accx^{2n-2}dx$. Ex hac aequatione reſultat haec $(P-Q)dR = -aRdx(P-Q)^2$, quae diviſa per $R(P-Q)$ dat $\frac{dR}{R} = a(Q-P)dx = -2acx^{n-1}dx$

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$+\frac{du}{u}-\frac{dz}{z}$. Haec iam aequatio integrabilis exiſtit, eritque integrale $lR-lC=-\frac{2acx^n}{n}+lu-lz$. Cum vero ſit $R=\frac{P-y}{Q-y}$, erit $\frac{P-y}{Q-y}=\frac{(acx^{n-1}zdx+dz-ayzdx):z}{(-acx^{n-1}udx+du\ ayudx):u}$ $=\frac{Ce^{\frac{-2acx^n}{n}}u}{z}$. Hinc ita, quia valores ipſarum u et z per x conſtant, habebitur aequatio integralis completa $Ce^{\frac{-2acx^n}{n}}=\frac{dz+acx^{n-1}zdx-ayzdx}{du-acx^{n-1}udx-ayudx}=\frac{(P-y)z}{(Q-y)u}$ Q. E. I.

Coroll. 1.

7. Valor particularis, quem ſupra pro y inueni-mus, ita erat comparatus, vt eſſet $y=cx^{n-1}-\frac{(K+L)}{ax(M+N)}$; exiſtente

$$K=\frac{(n-1)}{2}+\frac{(5n-1)(nn-1)}{2\cdot 8n}\cdot\frac{(9nn-1)}{16n}\cdot\frac{x^{-2n}}{a^2c^2}+\frac{(9n-1)}{2}\cdot\frac{(n^2-1)(9n^2-1)(25n^2-1)(49n^2-1)}{8n\cdot 16n\cdot 24n\cdot 32n}\cdot\frac{x^{-4n}}{a^4c^4}+\text{etc.}$$

$$L=\frac{(3n-1)(nn-1)}{2\cdot 8n}\cdot\frac{x^{-n}}{ac}+\frac{(7n-1)(nn-1)(9nn-1)(25nn-1)}{2\cdot 8n\cdot 16n\cdot 24n}\cdot\frac{x^{-3n}}{a^3c^3}+\text{etc.}$$

$$M=1+\frac{(nn-1)(9nn-1)}{8n\cdot 16n}\cdot\frac{x^{-2n}}{a^2c^2}+\frac{(nn-1)(9nn-1)(25nn-1)(49nn-1)}{8n\cdot 16n\cdot 24n\cdot 32n}\cdot\frac{x^{-4n}}{a^4c^4}+\text{etc.}$$

$$N=\frac{(nn-1)}{8n}\cdot\frac{x^{-n}}{ac}+\frac{(nn-1)(9nn-1)(25nn-1)}{8n\cdot 16n\cdot 24n}\cdot\frac{x^{-3n}}{a^3c^3}+\text{etc.}$$

Facto autem c negatiuo, erit alter valor particularis $y=-cx^{n-1}-\frac{(K-L)}{ax(M-N)}$. Erit ergo $P=\frac{acx^n(M+N)\cdot K\text{-}L}{ax(M+N)}$; $Q=\frac{-acx^n(M-N)-K+L}{ax(M-N)}$; et $z:u=M+N:M\text{-}N$.

Ex quibus colligitur, aequationis propositae: $dy + ayydx = accx^{2n-2}dx$ integrale completum fore:

$$Ce^{\frac{-2acx^n}{n}} = \frac{(acx^n - axy)(M+N) - K - L}{-(acx^n + axy)(M-N) - K + L} \text{ siue } -C \text{ posito loco } C$$

$$Ce^{\frac{-2acx^n}{n}} = \frac{ax(cx^{n-1} - y)(M+N) - K - L}{ax(cx^{n-1} + y)(M-N) + K - L}.$$

Coroll. 2.

8. Si cc est numerus negatiuus, fiet c hincque L et N quantitates imaginariae, at $c\sqrt{-1}$; $L\sqrt{-1}$; et $N\sqrt{-1}$ quantitates reales: Tum autem integrale completum realiter expressum erit:

$$C + \frac{acx^n}{n}\sqrt{-1} = A \operatorname{tang.} \frac{acx^n N - axyM - K}{acx^n M\sqrt{-1} - axyN\sqrt{-1} - L\sqrt{-1}}.$$

Coroll. 3.

9. Sit $c = b\sqrt{-1}$, vt habeatur haec aequatio integranda:

$$dy + ayydx + abbx^{2n-2}dx = 0.$$

Huius ergo aequationis integrale completum erit:

$$C - \frac{abx^n}{n} = A \operatorname{tang.} \frac{abx^n N - axyM - K}{-abx^n M - axyN - L} \text{ siue}$$

$$C - \frac{abx^n}{n} = A \operatorname{tang.} \frac{K - abx^n N + axyM}{L + abx^n M + axyN}; \text{ existente}$$

$$K = \frac{(n-1)}{2} - \frac{(5n-1)(nn-1)(9nn-1)}{2 \cdot 8n \cdot 16n} \cdot \frac{x^{-2n}}{a^2 b^2} + \frac{(9n-1)(nn-1)(9nn-)(25nn-1)(49nn-1)}{2 \cdot 8n \cdot 16n \cdot 24n \cdot 32n} \cdot \frac{x^{-4n}}{a^4 b^4} - \text{etc.}$$

$$L = \frac{(5n-1)(nn-1)}{2 \cdot 8n} \cdot \frac{x^{-n}}{ab} - \frac{(7n-1)(nn-1)(9nn-1)(25nn-1)}{2 \cdot 8n \cdot 16n \cdot 24n} \cdot \frac{x^{-3n}}{a^3 b^3} + \text{etc.}$$

M =

$$M = 1 - \frac{(nn-1)(9nn-1)}{8n \cdot 16n} \cdot \frac{x^{-2n}}{a^2 b^2} + \frac{(nn-1)(9nn-1)(25nn-1)(49nn-1)}{8n \cdot 16n \cdot 24n \cdot 32n} \cdot \frac{x^{-4n}}{a^4 b^4} - \text{etc.}$$

$$N = \frac{(nn-1)}{8n} \cdot \frac{x^{-n}}{ab} - \frac{(nn-1)(9nn-1)(25nn-1)}{8n \cdot 16n \cdot 24n} \cdot \frac{x^{-3n}}{a^3 b^3} + \text{etc.}$$

His igitur caſibus integralia particularia, quae ſimul ſint algebraica, non dantur.

Coroll. 4.

10. Quoties ergo fuerit $n = \frac{\mp 1}{2i \pm 1}$, denotante i numerum quemcunque integrum, expreſſiones finitae algebraicae pro litteris K, L, M et N reperiuntur. His igitur caſibus integratio aequationis huius $dy + ayydx = accx^{2n-2}dx$ ope logarithmorum, huius vero aequationis $dy + ayydx + abbx^{2n-2}dx = 0$ ope quadraturae circuli abſoluitur.

Scholion.

11 Quoniam aequationis differentialis propoſitae $dy + ayydx = accx^{2n-2}dx$ integrale completum duplici modo expreſſimus, poterimus formulae integralis $\frac{\int e^{\frac{-2acx^n}{n}} dx}{zz}$, quae in priori ineſt, valorem ex poſteriori aſſignare, huiusque adeo integrationem, quae ſaepe numero maximopere difficilis videatur, exhibere. Poſteriori modo autem inuenimus $y = \frac{QR - P}{R - 1} = \frac{P - QR}{1 - R}$ $= P + \frac{(P - Q)R}{1 - R}$, at eſt $R = \frac{Ce^{\frac{-acx^n}{n}} u}{z}$; $P = cx^{n-1}$

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$+\frac{dz}{azdx}$ et $Q = -cx^{n-1} + \frac{du}{audx}$. Confequenter habebitur $y = cx^{n-1} + \frac{dz}{azdx} + \frac{(2cx^{n-1} + \frac{dz}{azdx} - \frac{du}{audx})Ce^{\frac{-2acx^n}{n}}u}{z - Ce^{\frac{-2acx^n}{n}}u}$.

Per priorem vero integrationem eft $y = cx^{n-1} + \frac{dz}{azdx}$ $+ \frac{e^{\frac{-2acx^n}{n}}}{zz\int e^{\frac{-2acx^n}{n}}adx : zz}$ ex quorum comparatione oritur $\frac{z - Ce^{\frac{-2acx^n}{n}}u}{Czzu(2cx^{n-1} + \frac{dz}{azdx} - \frac{du}{audx})} = \frac{\int e^{\frac{-2acx^n}{n}}adx}{zz}$.

Quae transmutatur in hanc aequationem:

$$\frac{zdx - Ce^{\frac{-2acx^n}{n}}udx}{Cz(2acx^{n-1}uzdx + udz - zdu)} = \int e^{\frac{-2acx^n}{n}} \frac{dx}{zz}.$$

Quodfi ergo fuerit:

$$z = x^{\frac{-n+1}{2}} + \frac{(nn-1)}{8n} \cdot \frac{x^{\frac{-3n+1}{2}}}{ac} + \frac{(nn-1)(9nn-1)}{8n \cdot 16n} \cdot \frac{x^{\frac{-5n+1}{2}}}{a^2c^2} + \text{etc.}$$

$$u = x^{\frac{-n+1}{2}} - \frac{(nn-1)}{8n} \cdot \frac{x^{\frac{-3n+1}{2}}}{ac} + \frac{(nn-1)(9nn-1)}{8n \cdot 16n} \cdot \frac{x^{\frac{-5n+1}{2}}}{a^2c^2} - \text{etc.}$$

haec formula differentialis $\frac{e^{\frac{-2acx^n}{n}}dx}{zz}$ integrari poterit eritque integrale $= \frac{zdx - Ce^{\frac{-2acx^n}{n}}udx}{Cz(2acx^{n-1}uzdx + udz - zdu)}$.

Simili

Simili vero modo facto c negatiuo, quo z et u inter se permutantur, erit formulae differentialis $\frac{e^{\frac{+2acx^n}{n}}dx}{uu}$

integrale $= \frac{udx - Ce^{\frac{+2acx^n}{n}}zdx}{Cu(-2acx^n - 1uzdx + zdu - udz)}$

$= \frac{Ce^{\frac{2acx^n}{n}}zdx - udx}{Cu(2acx^n - 1uzdx + udz - zdu)}$ in quibus integrationibus C denotat eam constantem arbitrariam, quae per integrationem more solito ingreditur.