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De resolutione aequationum cuiusvis gradus

Leonhard Euler

DE RESOLVTIONE AEQVATIONVM CVIVSVIS GRADVS.

Auctore

L. EVLERO.

1.

Quae in Algebra adhuc de resolutione aequationum sunt tradita, ea, si ad regulas generales spectemus, tantum ad aequationes, quae quartum gradum non superant, patent, neque etiamnum regulae sunt inuentae, quarum ope aequationes quinti altiorisue cuiuspiam gradus resolui queant: ita vt vniuersa Algebra ad aequationes quatuor primorum ordinum restringatur. Hoc autem de regulis generalibus est tenendum, quae ad omnes aequationes eiusdem gradus sint accommodatae; nam in quouis gradu dantur infinitae aequationes, quae per diuisionem in duas pluresue aequationes graduum inferiorum resolui possunt, quarum idcirco radices iunctim sumtae praebent omnes radices illarum aequationum altiorum graduum. Tum vero a Cel. *Moiuraeo* obseruatae sunt in quouis gradu quaedam aequationes speciales, quae etsi per diuisionem in factores resolui nequeunt, tamen earum radices assignare liceat.

2. Ex cognita autem resolutione generali aequationum primi, secundi, tertii et quarti gradus, constat

ſtat quidem, aequationes primi gradus ſine vlla radicis extractione reſolui poſſe: at aequationum ſecundi gradus reſolutio iam extractionem radicis quadratae poſtulat. Reſolutio autem aequationum tertii gradus tam extractionem radicis quadratae, quam cubicae, implicat, et quarti gradus reſolutio inſuper extractionem radicis biquadratae exigit. Ex his autem tuto concludere licet, reſolutionem aequationis quinti gradus generalem extractionem radicis ſurdeſolidae praeter omnes radices inferiores poſtulare, atque in genere radix aequationis cuiusuis gradus n exprimetur per formam, quae ex omnibus ſignis radicalibus, tam gradus n, quam graduum inferiorum, erit compoſita.

3. Haec perpendens olim in Comment. Acad. Imper. Petrop. Tomi VI. coniecturam auſus ſum proferre circa formas radicum cuiuscunque aequationis. Propoſita namque aequatione gradus cuiusuis

$$x^n + A x^{n-2} + B x^{n-3} + C x^{n-4} + \text{etc.} = 0.$$

in qua ſecundum terminum deeſſe aſſumſi, quod quidem ſemper ponere licet, ſuſpicatus ſum, ſemper dari aequationem vno gradu inferioris, veluti

$$y^{n-1} + \mathfrak{A} y^{n-2} + \mathfrak{C} y^{n-3} + \mathfrak{B} y^{n-4} + \text{etc.} = 0$$

quam illius reſoluentem appellaui, ita vt, ſi huius conſtent omnes radices, quae ſint α, β, γ, δ, ε etc. quarum numerus eſt $n-1$; ex iis radix illius aequationis ita exprimatur, vt ſit:

$$x = \sqrt[n]{\alpha} + \sqrt[n]{\beta} + \sqrt[n]{\gamma} + \sqrt[n]{\delta} + \text{etc.}$$

Quam

Quam coniecturam confirmaui, ostendens, resolutionem aequationum inferiorum reuera ex hac forma generali deduci: neque etiam nunc dubito, quin haec coniectura veritati sit consentanea.

4. Praeterquam autem quod inuentio aequationis resoluentis, si proposita quartum gradum transcendit, fit difficillima, atque adeo in genere vires nostras aeque superare videtur, atque ipsa propositae aequationis resolutio; ita vt praeter formas speciales casibus Moivreanis similes nobis nihil admodum suppeditet: alia insuper incommoda in illa forma obseruaui, quae me eo induxerunt, vt arbitrarer, aliam forte dari formam illi non admodum dissimilem, quae istis incommodis non esset subiecta, ideoque maiorem spem nobis faceret, in hoc arduo Algebrae opere tandem vlterius penetrandi Non parum autem in hoc negotio proderit, veram formam radicum cuiusque aequationis accuratius perspexisse.

5. In forma autem per superiorem coniecturam eruta hoc imprimis desidero, quod omnes aequationis propositae radices non satis distincte exprimantur. Etsi enim quodvis signum radicale $\sqrt[n]{\alpha}$ tot valores diuersos complectitur, quot numerus n continet vnitates, ita vt, si $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ etc. omnes valores formulae $\sqrt[n]{1}$ denotent, pro $\sqrt[n]{\alpha}$ scribere liceat quamlibet harum formularum $\mathfrak{a}\sqrt[n]{\alpha}, \mathfrak{b}\sqrt[n]{\alpha}, \mathfrak{c}\sqrt[n]{\alpha}, \mathfrak{d}\sqrt[n]{\alpha}$ etc. tamen manife-

stum

ſtum, hanc variationem in ſingulis terminis $\sqrt[n]{\alpha}$, $\sqrt[n]{\beta}$, $\sqrt[n]{\gamma}$, $\sqrt[n]{\delta}$ etc. non pro lubitu conſtitui poſſe. Si enim combinatio horum terminorum cum litteris $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, $\mathfrak{e}$ etc. arbitrio noſtro relinqueretur, tum multo plures combinationes reſultarent, quam aequatio continet radices, quarum numerus eſt $= n$.

6. Quo igitur forma radicis x ſupra exhibita omnes aequationis radices ſimul complectatur, neceſſe eſt, vt combinationes terminorum $\sqrt[n]{\alpha}, \sqrt[n]{\beta}, \sqrt[n]{\gamma}, \sqrt[n]{\delta}$ etc. cum litteris $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ etc. certo quodam modo circumſcribantur, atque combinationes, quae ad aequationes radices repraeſentandas ſunt ineptae, excludantur. Ex reſolutione quidem aequationum tertii et quarti gradus vidimus inter radices vnitatis eiusdem nominis $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, certum quendam ordinem conſtitui debere, ſecundum quem etiam combinationes ſint perficiendae. Hunc in finem autem ſimilis ordo in ipſis radicis membris $\sqrt[n]{\alpha}$, $\sqrt[n]{\beta}$, $\sqrt[n]{\gamma}$, $\sqrt[n]{\delta}$ etc. erit tenendus, quo combinatio dirigatur. Verum quia non conſtat quemadmodum in radicibus ſuperiorum graduum talis ordo ſit conſtituendus, hoc ſine dubio inſigne eſt incommodum, quo forma coniecturae meae innixa laborat, quod igitur remouere in hac diſſertatione mihi eſt propoſitum.

7. Primum autem conueniet, ordinem certum in radicibus cuiusuis poteſtatis ex vnitate conſtituere, quo ſumma plerumque varietas combinationum reſtringatur.

 Quem

Quem in finem obferuo, fi praeter vnitatem alius quicunque valor ipfius $\sqrt[n]{1}$ fit $= \mathfrak{a}$, tum etiam $\mathfrak{a}^2$, $\mathfrak{a}^3$, $\mathfrak{a}^4$, etc. idoneos valores ipfius $\sqrt[n]{1}$ exhibere: nam fi fit $\mathfrak{a}^n = 1$, erit quoque $(\mathfrak{a}^2)^n = 1$, $(\mathfrak{a}^3)^n = 1$, $(\mathfrak{a}^4)^n = 1$, etc. Hinc fi reliquae radices ponantur $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, etc. quoniam in iis reperiuntur $\mathfrak{a}^2$, $\mathfrak{a}^3$, $\mathfrak{a}^4$, etc. iam certus quidam ordo perfpicitur, quo hae litterae inter fe difponi debent. Ita fi poft vnitatem, quae femper primum locum tenere cenfenda eft, a littera $\mathfrak{a}$ incipiamus, valores formulae $\sqrt[n]{1}$ erunt $1, \mathfrak{a}, \mathfrak{a}^2, \mathfrak{a}^3, \mathfrak{a}^4 \ldots \mathfrak{a}^{n-1}$ quorum numerus eft n; plures enim occurrere nequevnt, cum fit $\mathfrak{a}^n = 1$, $\mathfrak{a}^{n+1} = \mathfrak{a}$, $\mathfrak{a}^{n+2} = \mathfrak{a}^2$ etc. fimilique modo res fe habebit, fi poft vnitatem a quauis alia littera $\mathfrak{b}$, vel $\mathfrak{c}$, vel $\mathfrak{d}$ etc. incipiamus.

8. Hinc ergo merito fufpicor, talem quoque ordinem in ipfis terminis radicem aequationis x exprimentibus ineffe; feu fingula membra radicalia ita effe comparata, vt refpectu vniuscuiusque reliquae fint eius poteftates: fingulis autem membris nunc neceffe erit, coefficientes indefinitos tribuere. Quare fi aequatio, termino fecundo deftituta, fuerit:

$$x^n + Ax^{n-2} + Bx^{n-3} + Cx^{n-4} + Dx^{n-5} \ldots = 0$$

maxime probabile videtur radicem quamlibet huius aequationis ita exprimi, vt fit:

$$x = \mathfrak{A}\sqrt[n]{v} + \mathfrak{B}\sqrt[n]{v^2} + \mathfrak{C}\sqrt[n]{v^3} + \mathfrak{D}\sqrt[n]{v^4} + \ldots + \mathfrak{Q}\sqrt[n]{v^{n-1}}$$

vbi $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. fint quantitates, vel rationales, vel faltem non fignum radicale $\sqrt[n]{}$ inuoluant, quippe pe quod tantum quantitatem v eiusque poteſtates afficiat, multo minus ipſa quantitas v tale ſignam inuoluat.

9 Ex hac forma primum patet, eam non plura membra, quam quorum numerus ſit $n-1$, continere poſſe: nam etiam ſi ſeriem illam ex ſua indole vlterius continuemus, termini ſequentes iam in praecedentibus contenti deprehendentur: erit enim $\sqrt[n]{v^{n+1}} = v\sqrt[n]{v}$; $\sqrt[n]{v^{n+2}} = v\sqrt[n]{v^2}$ etc. ita vt irrationalitas ſignum radicale $\sqrt[n]{\ }$ inuoluens, plures diuerſas ſpecies non admittat, quam quarum numerus eſt $= n-1$. Etiamſi ergo illa ſeries in infinitum continuetur, tamen terminos eiusdem ſpeciei ratione irrationalitatis addendo omnes ad terminos numero $n-1$ redigentur. Cum igitur iam ante vidimus, plures terminos in radicis expreſſionem non ingredi; hinc non leue argumentum habetur, hanc nouam formam veritati plane eſſe conſentaneam: eius autem veritas per ſequentia argumenta multo magis confirmabitur.

10. Haec expreſſio quoque ſponte ſe extendit ad aequationes, in quibus ſecundus terminus non deeſt, dum ſuperior remotionem ſecundi termini exigebat, ex quo ipſo haec noua magis naturalis eſt aeſtimanda. Continuatio enim terminorum irrationalium $\sqrt[n]{v}$, $\sqrt[n]{v^2}$, $\sqrt[n]{v^3}$ etc. etiam terminos rationales $\sqrt[n]{v^0}$, $\sqrt[n]{v^n}$ inuoluit, qui ob aequationis terminum ſecundum adiici debent. Hinc generalius pronunciare poterimus, ſi aequatio completa ordinis cuiusque n fuerit propoſita:

$$x^n + \Delta x^{n-1} + A x^{n-2} + B x^{n-3} + C x^{n-4} + \text{etc.} = 0$$

eius radicem exprimi huiusmodi forma:

$$x = \omega + \mathfrak{A}\sqrt[n]{v} + \mathfrak{B}\sqrt[n]{v^2} + \mathfrak{C}\sqrt[n]{v^3} + \mathfrak{D}\sqrt[n]{v^4} \ldots\ldots + \mathfrak{O}\sqrt[n]{v^{n-1}}$$

vbi ω partem radicis rationalem exhibet, quam constat esse $= -\frac{1}{n}\Delta$. Reliqui autem termini continent partes irrationales radicem potestatis n inuoluentes, quarum, quatenus sunt diuersae, numerus excedere nequit $n-1$, omnino vti per formam superiorem intelligitur.

11. Hinc porro videmus, si v fuerit eiusmodi quantitas, vt ex ea radix potestatis n actu extrahi, seu $\sqrt[n]{v}$ vel rationaliter, vel per signa radicalia inferiorum potestatum exprimi queat, tum irrationalitatem gradus n prorsus ex forma radicis egredi. Hoc autem necessario vsu venire debet, quoties aequatio proposita in factores est resolubilis, tum enim nulla radix signum radicale $\sqrt[n]{}$ continebit. Quare cum natura rei postulet, vt his casibus omnia signa radicalia $\sqrt[n]{}$ euanescant, et ad signa simpliciora reducantur: ex forma autem superiori non pateat, quomodo euanescente vno huiusmodi signo $\sqrt[n]{\alpha}$ reliqua $\sqrt[n]{\beta}$, $\sqrt[n]{\gamma}$, etc. euanescant, ista expressio ob hanc rationem multo magis ad aequationum naturam accommodata est censenda.

12. Praeterea vero haec forma, in quo cardo totius negotii versatur, etiam omnes aequationis radices sine vlla ambiguitate ostendit: neque enim amplius haeremus, quomodo cum omnibus signis radicalibus $\sqrt[n]{}$ totidem valores radicis $\sqrt[n]{1}$ combinandi sint. Si enim omnes

omnes radices poteſtatis n ex vnitate ſint $1, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$, etc. ac $\sqrt[n]{v}$ cum earum quacunque $\mathfrak{a}$ combinauerimus, propterea quod $\sqrt[n]{v}$ vtique eſt $\mathfrak{a}\sqrt[n]{v}$, tum pro $\sqrt[n]{v^2}$, $\sqrt[n]{v^3}$, $\sqrt[n]{v^4}$ etc. ſcribere oportebit $\mathfrak{a}^2\sqrt[n]{v^2}$, $\mathfrak{a}^3\sqrt[n]{v^3}$, $\mathfrak{a}^4\sqrt[n]{v^4}$, etc. Terminus autem conſtans ω, quia formam $\omega\sqrt[n]{v^0}$ repraeſentat, abibit in $\mathfrak{a}^0\omega\sqrt[n]{v^0} = 1$ ob $\mathfrak{a}^0 = 1$; ideoque in omnibus radicibus nullam mutationem ſubit, quemadmodum reliqua membra. Quod cum ex reſolutione omnium aequationum per ſe ſit manifeſtum, hinc nouum ac ſatis luculentum habemus criterium veritatis huius nouae formae, quae omnium aequationum radices in ſe complecti videtur.

13. Hinc autem porro manifeſtum eſt, quomodo vna cuiusque aequationis radice cognita, reliquae radices omnes exhiberi queant: ad hoc tantum noſſe oportet omnes radices eiusdem poteſtatis ex vnitate, ſeu omnes valores ipſius $\sqrt[n]{1}$, quorum numerus $= n$. Ac ſi iſtae vnitatis radices fuerint $1, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$, etc. aequationisque vna radix inuenta ſit

$$x = \omega + \mathfrak{A}\sqrt[n]{v} + \mathfrak{B}\sqrt[n]{v^2} + \mathfrak{C}\sqrt[n]{v^3} + \dots \mathfrak{D}\sqrt[n]{v^{n-1}}$$

radices reliquae erunt:

$$x = \omega + \mathfrak{A}\mathfrak{a}\sqrt[n]{v} + \mathfrak{B}\mathfrak{a}^2\sqrt[n]{v^2} + \mathfrak{C}\mathfrak{a}^3\sqrt[n]{v^3} + \dots \mathfrak{D}\mathfrak{a}^{n-1}\sqrt[n]{v^{n-1}}$$

$$x = \omega + \mathfrak{A}\mathfrak{b}\sqrt[n]{v} + \mathfrak{B}\mathfrak{b}^2\sqrt[n]{v^2} + \mathfrak{C}\mathfrak{b}^3\sqrt[n]{v^3} + \dots \mathfrak{D}\mathfrak{b}^{n-1}\sqrt[n]{v^{n-1}}$$

$$x = \omega + \mathfrak{A}\mathfrak{c}\sqrt[n]{v} + \mathfrak{B}\mathfrak{c}^2\sqrt[n]{v^2} + \mathfrak{C}\mathfrak{c}^3\sqrt[n]{v^3} + \dots \mathfrak{D}\mathfrak{c}^{n-1}\sqrt[n]{v^{n-1}}$$

ſicque ſemper tot obtinentur radices, quot exponens n, qui aequationis gradum deſignat, continet vnitates.

14. His igitur argumentis noua haec radicum forma iam ad ſummum probabilitatis eſt euecta; atque ad plenam certitudinem oſtendendam nihil aliud requiritur, niſi vt regula inueniatur, cuius ope pro quauis aequatione propoſita iſta forma definiri, et coefficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. cum quantitate v aſſignari queant, quod ſi praeſtare poſſemus, haberemus ſine dubio generalem omnium aequationum reſolutionem, irrito adhuc omnium Geometrarum labore requiſitam. Neque igitur equidem tantum mihi tribuo, vt hanc regulam me inuenire poſſe credam; ſed contentus ero plene demonſtraſſe, omnium aequationum radices certo in hac forma eſſe contentas. Hoc autem ſine dubio plurimum luminis foenerabitur ad reſolutionem aequationum, cum cognita radicum vera forma via inueſtigationis non mediocriter facilior reddatur, quam ne ingredi quidem licet, quam diu forma radicum fuerit incognita.

15. Quanquam autem ex ipſa aequatione propoſita nobis adhuc non licet radicem eius, ſeu coefficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. cum quantitate v aſſignare, tamen demonſtratio veritatis aeque ſuccedet, ſi viciſſim ex aſſumta radice illam aequationem, cuius eſt radix, eliciamus. Haec autem aequatio libera eſſe debet a ſignis radicalibus $\sqrt[n]{}$, quoniam aequationes, quarum radices inueſtigantur, ex terminis rationalibus conſtare aſſumi ſolent.

ſent. Quaeſtio ergo huc reducitur, vt huiusmodi aequatio $x = \omega + \mathfrak{A}\sqrt[n]{v} + \mathfrak{B}\sqrt[n]{v^2} + \mathfrak{C}\sqrt[n]{v^3} + \ldots\ldots + \mathfrak{D}\sqrt[n]{v^{n-1}}$ ab irrationalitate, ſeu ſignis radicalibus $\sqrt[n]{\ }$, liberetur, atque aequatio rationalis inde deducatur, de qua deinceps certo affirmare poterimus, eius radicem eſſe ipſam expreſſionem aſſumtam; ſimulque inde reliquas radices, quae eidem aequationi aeque conueniunt, aſſignare valebimus. Hoc ergo modo ſaltem infinitas aequationes exhibere poterimus, quarum radices nobis erunt cognitae, atque ſi hae aequationes in ſe complectantur omnium graduum aequationes generales, etiam harum reſolutio in noſtra erit poteſtate.

16. Parum quidem a nobis praeſtitum iri videbitur, ſi tantum plures aequationes, quarum radices aſſignari queant, exhibuerimus; cum ex primis elementis conſtet, quomodo cuiusuis gradus aequatio formari debeat, quae datas habeat radices: ſi enim quotcunque huiusmodi formulae $x-a$, $x-b$, $x-c$, etc. in ſe inuicem multiplicentur, obtinebitur vtique aequatio, cuius radices futurae ſunt: $x=a$, $x=b$, $x=c$, etc. ſed talis aequationis formatio parum lucri affert ad reſolutionem aequationum. Primum autem obſeruo, hoc modo alias aequationes non naſci, niſi quae ſint habiturae factores; aequationum autem, quae in factores reſolui poſſunt, reſolutio, nulla laborat difficultate. Haud maioris quoque momenti ſunt in hoc negotio aequationes, quae ex multiplicatione duarum pluriumue inferiorum aequationum producuntur, quarum reſolutio nihil plane prodeſt ad reſolutionem generalem perficiendam.

17.

17. Quodsi autem ex nostra forma $x = \omega + \mathfrak{A}\sqrt[n]{v} + \mathfrak{B}\sqrt[n]{v^2} +$ etc. ad aequationem rationalem perueniamus, ea certo factores rationales non habebit: si enim haberet, eius radices, quae simul essent radices aequationum inferiorum graduum, signum radicale $\sqrt[n]{}$ non implicarent. Plurimum is praestare censendus est, qui aequationis cuiuspiam altioris gradus, quae in factores resolui nequeat, radices assignauerit: quam ob rem etiam Cel. *Moivreo* ingentes debentur gratiae, quod ex singulis aequationum gradibus vnam exhibuerit in factores irresolubilem, cuius radices assignari possunt; atque si eius formulae latius paterent, multo maiorem sine dubio essent habiturae vtilitatem, dum contra aequationibus in factores resolubilibus in hoc negotio nihil plane emolumenti attribui potest.

18. Verum reuertamur ad illam formam ab irrationalitate signi $\sqrt[n]{}$ liberandam, ac si consuetas methodos signa radicalia eliminandi consulamus, aequatio resultans ad plurimas dimensiones plerumque ascendere videatur. Si enim vnicum adesset signum radicale, puta $x = \omega + \mathfrak{A}\sqrt[n]{v}$, aequatio rationalis ad n dimensiones ipsius x ascenderet, vnde ea ad multo plures dimensiones ascensura videtur, si plura eiusmodi adsint signa radicalia; id quod sine dubio euenire deberet, si illa signa radicalia a se inuicem prorsus non penderent. Sed quia omnia sunt potestates primi, ostendam, perfectam rationalitatem obtineri posse, non vltra potestatem exponentis n ascendendo. Ita scilicet docebo formam

$x =$

$$x = \omega + \mathfrak{A}\sqrt[n]{v} + \mathfrak{B}\sqrt[n]{v^2} + \mathfrak{C}\sqrt[n]{v^3} + \ldots + \mathfrak{D}\sqrt[n]{v^{n-1}}$$

ita ab irrationalitate liberari posse, vt aequatio rationalis inde resultans potestatem x^n non superet. Prodibit ergo aequatio huius formae:

$$x^n + \Delta x^{n-1} + A x^{n-2} + B x^{n-3} + \text{etc.} = 0$$

cuius radix erit illa forma assumta: et quia radicum huius aequationis numerus est $= n$ ex eadem forma omnes huius aequationis radices assignare poterimus.

19. Cum hoc iam sit eximium criterium veritatis huius formae, tum etiam annotasse iuuabit, quoniam forma radicis $n-1$ quantitates arbitrarias continet, totidem quoque quantitates arbitrarias in aequationem rationalem ingredi, vnde perspicuum est, istas quantitates ita determinari posse, vt aequatio rationalis inde datos coefficientes Δ, A, B, C etc. obtineat, hoc est: vt aequatio generalis huius gradus obtineatur. Quae determinatio si actu institui queat, nanciscemur inde resolutionem generalem aequationum cuiuscunque gradus; ex quo saltem possibilitas resolutionis hoc modo perficiendae elucet. Difficultates quidem insignes in hoc negotio occurrent, quas eo clarius agnoscemus, si nostram formam ad quemuis gradum a simplicissimis incipiendo, accommodemus. Simplicitati autem et concinnitati calculi consulentes, partem radicis rationalem ω omittamus, vt in quouis gradu ad eiusmodi aequationes rationales pertingamus, in quibus secundus terminus desit, quo ipso amplitudo resolutionis non restringi est censenda.

I. Resolutio aequationum secundi gradus.

20. Vt igitur ab aequationibus secundi gradus incipiamus, sit $n = 2$, et posito $\omega = 0$, forma nostra radicis erit:

$$x = \mathfrak{A} \sqrt{v}$$

quae rationalis facta dat $xx = \mathfrak{A}\mathfrak{A}v$. Comparetur haec aequatio cum forma generali secundi gradus $xx = \mathrm{A}$, deficiente secundo termino, sitque $\mathfrak{A}\mathfrak{A}v = \mathrm{A}$: cui vt satisfiat, statuatur $\mathfrak{A} = 1$, eritque $v = \mathrm{A}$; vnde proposita aequatione $xx = \mathrm{A}$, si sumatur $\mathfrak{A} = 1$, et $v = \mathrm{A}$, eius radix vna erit $x = \mathfrak{A}\sqrt{v} = \sqrt{\mathrm{A}}$, et quia $\sqrt{1}$ duos habet valores 1 et -1, altera radix erit $x = -\mathfrak{A}\sqrt{v} = -\sqrt{\mathrm{A}}$: quod quidem per se est perspicuum.

II. Resolutio aequationum tertii gradus.

21. Posito iam $n = 3$, forma radicis pro hoc casu erit:

$$x = \mathfrak{A}\sqrt[3]{v} + \mathfrak{B}\sqrt[3]{v^2}$$

vnde vt aequatio rationalis eruatur, sumatur primo cubus:

$$x^3 = \mathfrak{A}^3 v + 3\mathfrak{A}\mathfrak{A}\mathfrak{B} v\sqrt[3]{v} + 3\mathfrak{A}\mathfrak{A}\mathfrak{B} v\sqrt[3]{v^2} + \mathfrak{B}^3 v^2$$

Fingatur iam haec aequatio cubica:

$$x^3 = \mathrm{A}x + \mathrm{B}$$

vnde

vnde, pro x valorem aſſumtum ſubſtituendo, orietur quoque

$$x^3 = A\mathfrak{A}\sqrt[3]{v} + A\mathfrak{B}\sqrt[3]{v^2} + B$$

quae forma illi aequalis eſt reddenda, aequandis inter ſe tam partibus rationalibus, quam irrationalibus, vtriusque ſpeciei $\sqrt[3]{v}$ et $\sqrt[3]{v^2}$.

22. Comparatio autem terminorum rationalium praebet:

$$B = \mathfrak{A}^3 v + \mathfrak{B}^3 v^2$$

et ex collatione irrationalium fit:

$$A\mathfrak{A} = 3\mathfrak{A}\mathfrak{A}\mathfrak{B}v \text{ et } A\mathfrak{B} = 3\mathfrak{A}\mathfrak{B}\mathfrak{B}v$$

quarum vtraque dat $A = 3\mathfrak{A}\mathfrak{B}v$.

Hinc ſi iſta aequatio cubica fuerit propoſita:

$$x^3 = 3\mathfrak{A}\mathfrak{B}vx + \mathfrak{A}^3 v^2 + \mathfrak{B}^3 v^2$$

eius radix vna erit:

$$x = \mathfrak{A}\sqrt[3]{v} + \mathfrak{B}\sqrt[3]{v^2}$$

et ſi 1, $\mathfrak{a}$, $\mathfrak{b}$ ſint tres radices cubicae vnitatis, duae reliquae radices erunt:

$$x = \mathfrak{A}\mathfrak{a}\sqrt[3]{v} + \mathfrak{B}\mathfrak{a}^2\sqrt[3]{v^2}; \quad x = \mathfrak{A}\mathfrak{b}\sqrt[3]{v} + \mathfrak{B}\mathfrak{b}^2\sqrt[3]{v^2}$$

eſt autem $\mathfrak{a} = \mathfrak{b}^2 = \frac{-1+\sqrt{-3}}{2}$ et $\mathfrak{b} = \mathfrak{a}^2 = \frac{-1-\sqrt{-3}}{2}$

23. Poſſunt autem viciſſim, ſi aequatio cubica proponatur

$$x^3 = Ax + B$$

ex coefficientibus A et B quantitates $\mathfrak{A}$, $\mathfrak{B}$ et v determinari, vt inde omnes tres huius aequationis radices obtineantur. Hunc autem in finem, quia tantum duae aequationes adimplendae habentur, vna litterarum

$\mathfrak{A}$ et $\mathfrak{B}$ pro lubitu aſſumi poteſt. Sit igitur $\mathfrak{A}=1$, et aequatio
$A=3\mathfrak{A}\mathfrak{B}v=3\mathfrak{B}v$ praebet $\mathfrak{B}=\frac{A}{3v}$; vnde fit $\mathfrak{B}^3=\frac{A^3}{27v^3}$ qui valor in prima aquatione $B=v+\mathfrak{B}^3v^2$ ſubſtitutus dat:

$$B=v+\frac{A^3}{27v} \text{ ſeu } vv=Bv-\frac{1}{27}A^3$$

vnde fit $v=\frac{1}{2}B\pm\sqrt{(\frac{1}{4}BB-\frac{1}{27}A^3)}$: perinde autem eſt vter horum duorum valorum aſſumatur.

24. Inuento autem valore ipſius $v=\frac{1}{2}B\pm\sqrt{(\frac{1}{4}BB-\frac{1}{27}A^3)}$ erit $\mathfrak{B}=\frac{A}{3v}$ et $\mathfrak{B}\sqrt[3]{v^2}=\frac{A}{3\sqrt[3]{v}}$ hincque tres aequationes propoſitae:

$$x^3=Ax+B$$

erunt radices:

$$\text{I. } x=\sqrt[3]{v}+\frac{A}{3\sqrt[3]{v}}:\ \text{II. } x=\alpha\sqrt[3]{v}+\frac{\beta A}{3\sqrt[3]{v}}:\ \text{III. } x=\beta\sqrt[3]{v}+\frac{\alpha A}{3\sqrt[3]{v}}$$

Cum autem ſit $\frac{1}{v}=\frac{\frac{1}{2}B\mp\sqrt{(\frac{1}{4}BB-\frac{1}{27}A^3)}}{\frac{1}{27}A^3}$ erit

$$\sqrt[3]{v}=\sqrt[3]{(\tfrac{1}{2}B\pm\sqrt{(\tfrac{1}{4}BB-\tfrac{1}{27}A^3)})} \text{ et}$$

$$\frac{A}{3\sqrt[3]{v}}=\sqrt[3]{(\tfrac{1}{2}B\mp\sqrt{(\tfrac{1}{4}BB-\tfrac{1}{27}A^3)})}$$

hincque naſcuntur formulae vulgares pro reſolutione aequationum cubicarum.

III. Reſolutio aequationum quarti gradus.

25. Poſito $n=4$, conſideremus hanc radicis formam:

$$x=\mathfrak{A}\sqrt[4]{v}+\mathfrak{B}\sqrt[4]{v^2}+\mathfrak{C}\sqrt[4]{v^3}$$

eſt

et quaeramus aequationem quarti gradus, cuius haec forma fit radix. Atque hoc quidem casu calculus facile instituitur, quo irrationalitates tolluntur; nam ob $\sqrt[4]{v^2}=\sqrt{v}$, sumatur haec aequatio:

$$x-\mathfrak{B}\sqrt{v}=\mathfrak{A}\sqrt[4]{v}+\mathfrak{C}\sqrt[4]{v^3}$$

quae quadrata dat:

$$xx-2\mathfrak{B}x\sqrt{v}+\mathfrak{B}\mathfrak{B}v=\mathfrak{A}\mathfrak{A}\sqrt{v}+2\mathfrak{A}\mathfrak{C}v+\mathfrak{C}\mathfrak{C}v\sqrt{v}$$

quae partibus irrationalibus ad eandem partem translatis fit:

$$xx+(\mathfrak{B}\mathfrak{B}-2\mathfrak{A}\mathfrak{C})v=2\mathfrak{B}x\sqrt{v}+(\mathfrak{A}\mathfrak{A}+\mathfrak{C}\mathfrak{C}v)\sqrt{v}$$

et sumtis denuo quadratis prodibit haec aequatio rationalis:

$$x^4+2(\mathfrak{B}\mathfrak{B}-2\mathfrak{A}\mathfrak{C})vxx+(\mathfrak{B}\mathfrak{B}-2\mathfrak{A}\mathfrak{C})^2vv=4\mathfrak{B}\mathfrak{B}vxx$$
$$+4(\mathfrak{A}\mathfrak{A}+\mathfrak{C}\mathfrak{C}v)\mathfrak{B}vx+(\mathfrak{A}\mathfrak{A}+\mathfrak{C}\mathfrak{C}v)^2v$$

quae ordinata abit in hanc formam;

$$x^4=2(\mathfrak{B}\mathfrak{B}+2\mathfrak{A}\mathfrak{C})vxx+4(\mathfrak{A}\mathfrak{A}+\mathfrak{C}\mathfrak{C}v)\mathfrak{B}vx+\mathfrak{A}^4v$$
$$-\mathfrak{B}^4vv+\mathfrak{C}^4v^3+4\mathfrak{A}\mathfrak{B}\mathfrak{B}\mathfrak{C}vv-\mathfrak{A}\mathfrak{A}\mathfrak{C}\mathfrak{C}vv$$

26. Huius igitur aequationis biquadraticae radix vna est:

$$x=\mathfrak{A}\sqrt[4]{v}+\mathfrak{B}\sqrt[4]{v^2}+\mathfrak{C}\sqrt[4]{v^3}$$

ac si radices biquadratae vnitatis ponantur 1, $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, ita vt fit:

$$\mathfrak{a}=+\sqrt{-1};\quad \mathfrak{b}=-1;\quad \text{et}\quad \mathfrak{c}=-\sqrt{-1}$$

erit $\mathfrak{a}^2=-1=\mathfrak{b}$; $\quad\mathfrak{a}^3=-\sqrt{-1}=\mathfrak{c}$;

$\mathfrak{b}^2=+1$; $\quad\mathfrak{b}^3=-1=\mathfrak{b}$;

$\mathfrak{c}^2=-1=\mathfrak{b}$; $\quad\mathfrak{c}^3=+\sqrt{-1}=\mathfrak{a}$;

 vnde

vnde tres reliquae radices eiusdem aequationis erunt:

$$x = \mathfrak{A}\mathfrak{a}\sqrt[4]{v} + \mathfrak{B}\mathfrak{b}\sqrt[4]{v^2} + \mathfrak{C}\mathfrak{c}\sqrt[4]{v^3}$$

$$x = \mathfrak{A}\mathfrak{b}\sqrt[4]{v} + \mathfrak{B}\sqrt[4]{v^2} + \mathfrak{C}\mathfrak{b}\sqrt[4]{v^3}$$

$$x = \mathfrak{A}\mathfrak{c}\sqrt[4]{v} + \mathfrak{B}\mathfrak{b}\sqrt[4]{v^2} + \mathfrak{C}\mathfrak{a}\sqrt[4]{v^3}$$

27. Hinc autem viciſſim aequatio biquadrata quaecunque ad illam formam reduci, eiusque radices aſſignari poterunt. Sit enim propoſita haec aequatio:

$$x^4 = Axx + Bx + C$$

et quaeri opportet valores coefficientium $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ cum quantitate v, quibus inuentis ſimul huius aequationis radices innoteſcent. Erit autem:

$A = 2(\mathfrak{B}\mathfrak{B} + 2\mathfrak{A}\mathfrak{C})v$; $\quad B = 4(\mathfrak{A}\mathfrak{A} + \mathfrak{C}\mathfrak{C}v)\mathfrak{B}v$

$C = \mathfrak{A}^4 v - \mathfrak{B}^4 vv + \mathfrak{C}^4 v^3 + 4\mathfrak{A}\mathfrak{B}\mathfrak{B}\mathfrak{C}vv - 2\mathfrak{A}\mathfrak{A}\mathfrak{C}\mathfrak{C}vv$ ſeu

$C = (\mathfrak{A}\mathfrak{A} + \mathfrak{C}\mathfrak{C}v)^2 v - (\mathfrak{B}\mathfrak{B} + 2\mathfrak{A}\mathfrak{C})^2 vv + 8\mathfrak{A}\mathfrak{B}\mathfrak{B}\mathfrak{C}vv$

Illinc autem eſt $(\mathfrak{B}\mathfrak{B} + 2\mathfrak{A}\mathfrak{C})v = \frac{1}{2}A$; et $\mathfrak{A}\mathfrak{A} + \mathfrak{C}\mathfrak{C}v = \frac{B}{4\mathfrak{B}v}$; qui valores hic ſubſtituti dant:

$C = \frac{BB}{16\mathfrak{B}\mathfrak{B}v} - \frac{1}{4}AA + 8\mathfrak{A}\mathfrak{B}\mathfrak{B}\mathfrak{C}vv$

Prima autem formula praebet $4\mathfrak{A}\mathfrak{C}v = A - 2\mathfrak{B}\mathfrak{B}v$, qui valor denuo ſubſtitutus dat:

$C = \frac{BB}{16\mathfrak{B}\mathfrak{B}v} - \frac{1}{4}AA + 2A\mathfrak{B}\mathfrak{B}v - 4\mathfrak{B}^4 vv$

ita vt iam duae litterae $\mathfrak{A}$ et $\mathfrak{C}$ ſint eliminatae.

28. Quia hic adhuc duae incognitae $\mathfrak{B}$ et v ſuperſunt, valor ipſius $\mathfrak{B}$ arbitrio noſtro relinquitur. Sit igitur $\mathfrak{B} = 1$, et quantitas v ex ſequenti aequatione cubica determinari debebit:

$$v^3 - \frac{1}{2}Av^2 + \frac{1}{4}(C + \frac{1}{4}AA)v - \frac{1}{64}BB = 0$$

In-

Inuenta autem hinc radice v, ex prioribus aequationibus quaeri debent litterae $\mathfrak{A}$ et $\mathfrak{C}$. Cum igitur sit:

$$\mathfrak{A}\mathfrak{A}+\mathfrak{C}\mathfrak{C}v=\frac{B}{4v} \text{ et } 2\mathfrak{A}\mathfrak{C}\sqrt{v}=\frac{A-2v}{2\sqrt{v}}$$

erit tam addendo, quam subtrahendo, et radicem quadratam extrahendo

$$\mathfrak{A}+\mathfrak{C}\sqrt{v}=\sqrt{\left(\frac{B}{4v}+\frac{A}{2\sqrt{v}}-\sqrt{v}\right)} \text{ et}$$

$$\mathfrak{A}-\mathfrak{C}\sqrt{v}=\sqrt{\left(\frac{B}{4v}-\frac{A}{2\sqrt{v}}+\sqrt{v}\right)} \text{ vnde reperietur;}$$

$$\mathfrak{A}=\frac{1}{4\sqrt{v}}\sqrt{(B+2A\sqrt{v}-4v\sqrt{v})}+\frac{1}{4\sqrt{v}}\sqrt{(B-2A\sqrt{v}+4v\sqrt{v})} \text{ et}$$

$$\mathfrak{C}=\frac{1}{4v}\sqrt{(B+2A\sqrt{v}\;\;4v\sqrt{v})}-\frac{1}{4v}\sqrt{(3-2A\sqrt{v}+4v\sqrt{v})}.$$

29 Cum sit $\mathfrak{A}\sqrt[4]{v}\pm\mathfrak{C}\sqrt[4]{v^3}=(\mathfrak{A}\pm\mathfrak{C}\sqrt{v})\sqrt[4]{v}$ erunt aequationis propositae:

$$x^4=Axx+Bx+C$$

postquam valor v inuentus fuerit ex aequatione:

$$v^3-\tfrac{1}{2}Av^2+\tfrac{1}{4}(C+\tfrac{1}{4}AA)v-\tfrac{1}{64}BB=0$$

quatuor radices:

I. $x=\sqrt{v}+\frac{1}{2\sqrt{v}}\sqrt{(B\sqrt{v}+2Av-4vv)}$

II. $x=\sqrt{v}-\frac{1}{2\sqrt{v}}\sqrt{(B\sqrt{v}+2Av-4vv)}$

III. $x=-\sqrt{v}+\frac{1}{2\sqrt{v}}\sqrt{(-B\sqrt{v}+2Av-4vv)}$

IV. $x=-\sqrt{v}-\frac{1}{2\sqrt{v}}\sqrt{(-B\sqrt{v}+2Av-4vv)}$

Hocque modo, vt constat, resolutio aequationis biquadraticae ad resolutionem aequationis cubicae reducitur.

IV. Resolutio aequationum quinti gradus.

30. Posito $n=5$, erit forma nostra radicis:

$$x=\mathfrak{A}\sqrt[5]{v}+\mathfrak{B}\sqrt[5]{vv}+\mathfrak{C}\sqrt[5]{v^3}+\mathfrak{D}\sqrt[5]{v^4}$$

ac

ac primo quaeri debet aequatio quinti gradus, cuius haec futura ſit radix, ſeu quod eodem redit, ex hac forma ſigna radicalia eliminari oportet. In hoc autem ipſo ſumma occurrit difficultas, cum operatio haec eliminationis neutiquam eo modo, quo in aequationibus quarti gradus ſum vſus, inſtitui queat. Manifeſtum quidem eſt, quia omnes poteſtates ipſius x eadem ſigna radicalia inuoluunt, ſi aequatio quaeſita fingatur:

$$x^5 = Ax^3 + Bx^2 + Cx + D$$

tum ſubſtituendo pro x valorem aſſumtum, quatuor obtineri aequationes, quarum ope quaterna ſigna radicalia eliminari liceat; ſed tum litterae hae aſſumtae A, B, C, et D ſingulae difficillime determinabuntur.

31. His difficultatibus perpenſis in alium incidi modum hanc operationem inſtituendi, qui ita eſt comparatus, vt ad omnes radicum formas, cuiuscunque ſint gradus, aeque pateat, et ex quo ſimul perſpicietur, aequationem rationalem nunquam vltra gradum, qui exponente n indicatur, eſſe aſcenſuram. Hic autem modus innititur ipſi naturae aequationum, qua ſingulorum terminorum coefficientes ex omnibus radicibus definiuntur. Cum igitur omnes quinque radices aequationis, quam quaerimus, conſtent, ex iis quoque coefficientes ſingulorum terminorum eius formari poſſunt per regulas cognitas. Sint igitur 1, $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, quinque radices ſurdeſolidae vnitatis, ſeu radices huius aequationis $z^5 - 1 = 0$, ac ponendo α, β, γ, δ, ε pro radicibus aequationis, quam quaerimus, erit:

$\alpha = \mathfrak{A}$

$$\alpha = \mathfrak{A}\sqrt[5]{v} + \mathfrak{B}\sqrt[5]{v^2} + \mathfrak{C}\sqrt[5]{v^3} + \mathfrak{D}\sqrt[5]{v^4}$$
$$\beta = \mathfrak{A}\mathfrak{a}\sqrt[5]{v} + \mathfrak{B}\mathfrak{a}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{a}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{a}^4\sqrt[5]{v^4}$$
$$\gamma = \mathfrak{A}\mathfrak{b}\sqrt[5]{v} + \mathfrak{B}\mathfrak{b}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{b}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{b}^4\sqrt[5]{v^4}$$
$$\delta = \mathfrak{A}\mathfrak{c}\sqrt[5]{v} + \mathfrak{B}\mathfrak{c}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{c}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{c}^4\sqrt[5]{v^4}$$
$$\varepsilon = \mathfrak{A}\mathfrak{d}\sqrt[5]{v} + \mathfrak{B}\mathfrak{d}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{d}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{d}^4\sqrt[5]{v^4}.$$

32. His quinque radicibus expoſitis, ſi aequatio quinti gradus has radices habens ſtatuatur:

$$x^5 - \Delta x^4 + A x^3 - B x^2 + C x - D = 0$$

hi coefficientes ex radicibus α, β, γ, δ, ε ita definiuntur, vt ſit

$\Delta =$ ſummae radicum

$A =$ ſummae productorum ex binis

$B =$ ſummae productorum ex ternis

$C =$ ſummae productorum ex quaternis

$D =$ producto ex omnibus quinis.

Quo autem hos valores facilius colligere queamus, eos ex ſummis poteſtatum radicum concludamus. Sit igitur:

$$P = \alpha + \beta + \gamma + \delta + \varepsilon$$
$$Q = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2$$
$$R = \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \varepsilon^3$$
$$S = \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \varepsilon^4$$
$$T = \alpha^5 + \beta^5 + \gamma^5 + \delta^5 + \varepsilon^5.$$

His enim valoribus definitis erit, vti nouimus:

$$\Delta = P$$
$$A = \frac{\Delta P - Q}{2}$$
$$B = \frac{AP - \Delta Q + R}{3}$$
$$C = \frac{BP - AQ + \Delta R - S}{4}$$
$$D = \frac{CP - BQ + AR - \Delta S + T}{5}.$$

33. Iam ad valores P, Q, R, S, T inuestigandos, debemus prius radicum vnitatis 1, $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ omnes potestates in vnam summam redigere; quae cum sint radices aequationis $z^5 - 1 = 0$, erit:

$$1 + \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} = 0$$
$$1 + \mathfrak{a}^2 + \mathfrak{b}^2 + \mathfrak{c}^2 + \mathfrak{d}^2 = 0$$
$$1 + \mathfrak{a}^3 + \mathfrak{b}^3 + \mathfrak{c}^3 + \mathfrak{d}^3 = 0$$
$$1 + \mathfrak{a}^4 + \mathfrak{b}^4 + \mathfrak{c}^4 + \mathfrak{d}^4 = 0$$
$$1 + \mathfrak{a}^5 + \mathfrak{b}^5 + \mathfrak{c}^5 + \mathfrak{d}^5 = 5.$$

Summae potestatum sextarum, septimarum, etc. vsque ad decimas, iterum euanescunt, at decimarum summa iterum fit $= 5$, cum sit $\mathfrak{a}^5 = 1$, $\mathfrak{b}^5 = 1$, $\mathfrak{c}^5 = 1$ et $\mathfrak{d}^5 = 1$. Breuitatis gratia in hoc calculo poterimus signa radicalia plane omittere, dummodo deinceps recordemur, cum literis $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ coniungenda esse $\sqrt[5]{v}$, $\sqrt[5]{v^2}$, $\sqrt[5]{v^3}$, $\sqrt[5]{v^4}$.

34. Nunc igitur addendis radicibus α, β, γ, δ, ε habebimus:

$P = \mathfrak{A}(1 + \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d}) + \mathfrak{B}(1 + \mathfrak{a}^2 + \mathfrak{b}^2 + \mathfrak{c}^2 + \mathfrak{d}^2) + \text{etc.} = 0,$
reliquis autem potestatibus sumendis, eliciemus insuper

$$P = 0$$

$$P = 0$$
$$Q = 10(\mathfrak{A}\mathfrak{D} + \mathfrak{B}\mathfrak{C})$$
$$R = 15(\mathfrak{A}\mathfrak{A}\mathfrak{C} + \mathfrak{A}\mathfrak{B}^2 + \mathfrak{B}\mathfrak{D}^2 + \mathfrak{C}^2\mathfrak{D})$$
$$S = 20(\mathfrak{A}^3\mathfrak{B} + \mathfrak{A}\mathfrak{C}^3 + \mathfrak{B}^3\mathfrak{D} + \mathfrak{C}\mathfrak{D}^3) + 30(\mathfrak{A}\mathfrak{A}\mathfrak{D}\mathfrak{D} + \mathfrak{B}\mathfrak{B}\mathfrak{C}\mathfrak{C}) + 120\,\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D}$$
$$T = 5(\mathfrak{A}^5 + \mathfrak{B}^5 + \mathfrak{C}^5 + \mathfrak{D}^5) + 100(\mathfrak{A}^3\mathfrak{C}\mathfrak{D} + \mathfrak{A}\mathfrak{B}^3\mathfrak{C} + \mathfrak{B}\mathfrak{C}^3\mathfrak{D} + \mathfrak{A}\mathfrak{B}\mathfrak{D}^3) + 150(\mathfrak{A}\mathfrak{C}^2\mathfrak{D}^2 + \mathfrak{A}^2\mathfrak{B}\mathfrak{C}^2 + \mathfrak{B}^2\mathfrak{C}\mathfrak{D}^2 + \mathfrak{A}^2\mathfrak{B}^2\mathfrak{D}).$$

Hic alia producta non occurrunt, nisi quae adiungendis signis radicalibus potestatem ipsius v rationalem producunt: seu si litterae $\mathfrak{A}$ vnam dimensionem tribuamus, litterae $\mathfrak{B}$ duas, litterae $\mathfrak{C}$ tres et litterae $\mathfrak{D}$ quatuor, in omnibus his productis numerus dimensionum est per 5 diuisibilis, coefficiens autem cuiusuis producti est quintuplum eius coefficientis, qui eidem producto ex lege conbinationum competit.

35. Cum igitur sit $P = 0$, erit quoque $\Delta = 0$, et pro reliquis coefficientibus habebimus:

$A = -\frac{1}{2}Q$; $B = \frac{1}{3}R$; $C = -\frac{1}{4}AQ - \frac{1}{4}S$; et $D = -\frac{1}{5}BQ + \frac{1}{5}AR + \frac{1}{5}T$.

Hinc ergo erit:

$$A = -5(\mathfrak{A}\mathfrak{D} + \mathfrak{B}\mathfrak{C})$$
$$B = 5(\mathfrak{A}\mathfrak{A}\mathfrak{C}^2 + \mathfrak{A}\mathfrak{B}^2 + \mathfrak{B}\mathfrak{D}^2 + \mathfrak{C}^2\mathfrak{D})$$
$$C = -5(\mathfrak{A}^3\mathfrak{B} + \mathfrak{B}^3\mathfrak{D} + \mathfrak{A}\mathfrak{C}^3 + \mathfrak{C}\mathfrak{D}^3) + 5(\mathfrak{A}^2\mathfrak{D}^2 + \mathfrak{B}^2\mathfrak{C}^2) - 5\,\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D}$$
$$D = \mathfrak{A}^5 + \mathfrak{B}^5 + \mathfrak{C}^5 + \mathfrak{D}^5) - 5(\mathfrak{A}^3\mathfrak{C}\mathfrak{D} + \mathfrak{A}\mathfrak{B}^3\mathfrak{C} + \mathfrak{B}\mathfrak{C}^3\mathfrak{D} + \mathfrak{A}\mathfrak{B}\mathfrak{D}^3) + 5(\mathfrak{A}\mathfrak{C}^2\mathfrak{D}^2 + \mathfrak{A}^2\mathfrak{B}\mathfrak{C}^2 + \mathfrak{B}^2\mathfrak{C}\mathfrak{D}^2 + \mathfrak{A}^2\mathfrak{B}^2\mathfrak{D})$$

cum quibus terminis iam debitae poteſtates ipſius v coniungi debent, vt obtineantur eorum iuſti valores.

36. Quodſi ergo mutatis ſignis coefficientium A et C proponatur haec aequatio:

$$x^5 = A x^3 + B x^2 + C x + D$$

cuius coefficientes hos teneant valores:

$$A = 5(\mathfrak{A}\mathfrak{D} + \mathfrak{B}\mathfrak{C})v$$

$$B = 5(\mathfrak{A}^2\mathfrak{C} + \mathfrak{A}\mathfrak{B}^2 + \mathfrak{B}\mathfrak{D}^2 v + \mathfrak{C}^2\mathfrak{D}v)v$$

$$C = 5(\mathfrak{A}^3\mathfrak{B} + \mathfrak{B}^3\mathfrak{D}v + \mathfrak{A}\mathfrak{C}^3 v + \mathfrak{C}\mathfrak{D}^3 vv)v - 5(\mathfrak{A}^2\mathfrak{D}^2 + \mathfrak{B}^2\mathfrak{C}^2)vv + 5\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D}v^2$$

$$D = \mathfrak{A}^5 v + \mathfrak{B}^5 v^2 + \mathfrak{C}^5 v^3 + \mathfrak{D}^5 v^4 - 5(\mathfrak{A}^3\mathfrak{C}\mathfrak{D} + \mathfrak{A}\mathfrak{B}^3\mathfrak{C} + \mathfrak{B}\mathfrak{C}^3\mathfrak{D}v + \mathfrak{A}\mathfrak{B}\mathfrak{D}^3 v)v^2 + 5(\mathfrak{A}^2\mathfrak{B}^2\mathfrak{D} + \mathfrak{A}^2\mathfrak{B}\mathfrak{C}^2 + \mathfrak{A}\mathfrak{C}^2\mathfrak{D}^2 v + \mathfrak{B}^2\mathfrak{C}\mathfrak{D}^2 v)v^2$$

erunt eius quinque radices:

I. $x = \mathfrak{A}\sqrt[5]{v} + \mathfrak{B}\sqrt[5]{v^2} + \mathfrak{C}\sqrt[5]{v^3} + \mathfrak{D}\sqrt[5]{v^4}$

II. $x = \mathfrak{A}\mathfrak{a}\sqrt[5]{v} + \mathfrak{B}\mathfrak{a}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{a}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{a}^4\sqrt[5]{v^4}$

III. $x = \mathfrak{A}\mathfrak{b}\sqrt[5]{v} + \mathfrak{B}\mathfrak{b}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{b}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{b}^4\sqrt[5]{v^4}$

IV. $x = \mathfrak{A}\mathfrak{c}\sqrt[5]{v} + \mathfrak{B}\mathfrak{c}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{c}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{c}^4\sqrt[5]{v^4}$

V. $x = \mathfrak{A}\mathfrak{d}\sqrt[5]{v} + \mathfrak{B}\mathfrak{d}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{d}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{d}^4\sqrt[5]{v^4}$

exiſtentibus $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ praeter vnitatem reliquis quatuor radicibus ſurdeſolidis vnitatis, quarum valores imaginarii conſtant.

37. Si nunc viciſſim ex datis coefficientibus A, B, C, D definiri poſſent quantitates $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ cum littera v, haberetur reſolutio generalis omnium aequationum

tionum quinti gradus. Verum in hoc ipso summa difficultas consistit, cum nulla via pateat, litteras 𝔄, 𝔅, ℭ, 𝔇, quarum quidem vnam pro lubitu assumere licet, successiue ita eliminandi, vt aequatio solum incognitam v cum datis A, B, C, D inuoluens resultet, quae quidem nullas radices superfluas complectatur. Satis tuto autem suspicari licet, si haec eliminatio rite administretur, tandem ad aequationem quarti gradus perueniri posse, qua valor ipsius v definiatur. Si enim aequatio altioris gradus prodiret, tum quoque valor ipsius v signa radicalia eiusdem gradus implicaret, quod absurdum videtur. Quoniam autem multitudo terminorum hunc laborem tam difficilem reddit, vt ne tentari quidem cum aliquo successu queat, haud abs re erit, casus quosdam minus generales euoluere, qui non ad formulas tantopere complicatas deducant.

38. Ad casus ergo particulares descensuri, tribuamus litteris 𝔄, 𝔅, ℭ, 𝔇 eiusmodi valores, quibus calculus in compendium reducatur; ac primo quidem sint $\mathfrak{B}=0$, $\mathfrak{C}=0$, et $\mathfrak{D}=0$, vnde nanciscemur:

$$A=0,\ B=0,\ C=0 \text{ et } D=\mathfrak{A}^5 v.$$

Hinc igitur fit $\mathfrak{A}\sqrt[5]{v}=\sqrt[5]{v}$. Quare si haec proposita fuerit aequatio:

$$x^5=D$$

erunt huius aequationis quinque radices:

$$\text{I. } x=\sqrt[5]{D};\ \text{II. } x=\mathfrak{a}\sqrt[5]{D};\ \text{III. } x=\mathfrak{b}\sqrt[5]{D};\ \text{IV. } x=\mathfrak{c}\sqrt[5]{D};\ \text{V } x=\mathfrak{d}\sqrt[5]{D}$$

 qui

qui casus cum per se sit manifestus, ab eo exordium capere visum est, vt pateat quomodo nostra methodus casus cognitos in se complectatur.

39. Euanescant iam duae litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ et $\mathfrak{D}$, si enim tres euanescentes ponantur, quaecunque eae sumantur, semper ad casum praecedentem deducimur. Sint igitur $\mathfrak{C}$ et $\mathfrak{D}$ nihilo aequales, seu aequatio quaeratur, cuius radix sit futura $x = \mathfrak{A}\sqrt[5]{v} + \mathfrak{B}\sqrt[5]{v^2}$, atque obtinebimus:

$$A = 0;\ B = 5\mathfrak{A}\mathfrak{B}^2 v;\ C = 5\mathfrak{A}^3\mathfrak{B}v;\ D = \mathfrak{A}^5 v + \mathfrak{B}^5 v^2$$

vnde proposita radix conueniet huic aequationi:

$$x^5 = 5\mathfrak{A}\mathfrak{B}^2 v x^2 + 5\mathfrak{A}^3\mathfrak{B} v x + \mathfrak{A}^5 v + \mathfrak{B}^5 v^2$$

Quae aequatio si comparetur cum hac forma:

$$x^5 = 5Pxx + 5Qx + R$$

erit $\mathfrak{A}\mathfrak{B}^2 v = P$; $\mathfrak{A}^3\mathfrak{B}v = Q$, vnde deducitur $\mathfrak{A}^5 v = \frac{QQ}{P}$ et $\mathfrak{B}^5 v^2 = \frac{P^3}{Q}$, ita vt sit $R = \frac{QQ}{P} + \frac{P^3}{Q}$.

40. Hinc ergo deducimur ad resolutionem huius aequationis specialis quinti gradus:

$$x^5 = 5Pxx + 5Qx + \frac{QQ}{P} + \frac{P^3}{Q}$$

cuius ob $\mathfrak{A}\sqrt[5]{v} = \sqrt[5]{\frac{QQ}{P}}$ et $\mathfrak{B}\sqrt[5]{vv} = \sqrt[5]{\frac{P^3}{Q}}$ quinque radices erunt:

I. $x = \sqrt[5]{\frac{QQ}{P}} + \sqrt[5]{\frac{P^3}{Q}}$

II. $x = \mathfrak{a}\sqrt[5]{\frac{QQ}{P}} + \mathfrak{a}^2\sqrt[5]{\frac{P^3}{Q}}$

III. $x = \mathfrak{b}\sqrt[5]{\frac{QQ}{P}} + \mathfrak{b}^2\sqrt[5]{\frac{P^3}{Q}}$

IV. $x = \mathfrak{c}\sqrt[5]{\frac{QQ}{P}} + \mathfrak{c}^2\sqrt[5]{\frac{P^3}{Q}}$

V. $x = \mathfrak{d}\sqrt[5]{\frac{QQ}{P}} + \mathfrak{d}^2\sqrt[5]{\frac{P^3}{Q}}$.

Aequa-

Aequatio autem haec non multum abſimilis eſt formulae Moivreanae, et quia ſe in factores reſolui non patitur, eius reſolutio hic tradita eo magis notari meretur.

41. Hanc aequationem a fractionibus liberare poterimus, ſi ponamus $P = MN$ et $Q = M^2N$, tum enim habebitur:

$$x^5 = 5MNxx + 5M^2Nx + M^2N + MN^2$$

cuius radix erit $x = \sqrt[5]{M^2N} + \sqrt[5]{MN^2}$, et ſi $\mathfrak{a}$ quamlibet aliam radicem ſurdeſolidam vnitatis denotet, erit huius aequationis quaelibet alia radix:

$$x = \mathfrak{a}\sqrt[5]{M^2N} + \mathfrak{a}^2\sqrt[5]{MN^2}.$$

Ita ſi exempli gratia ſtatuatur $M = 1$; et $N = 2$, huius aequationis:

$$x^5 = 10xx + 10x + 6$$

radix quaecunque eſt $x = \mathfrak{a}\sqrt[5]{2} + \mathfrak{a}^2\sqrt[5]{4}$; haecque aequatio ita eſt comparata, vt per nullam methodum cognitam reſolui poſſe videatur.

42. Si $\mathfrak{B}$ et $\mathfrak{D}$ ſint nihilo aequales, ad eundem caſum reuoluimur. Fiet enim

$A = 0$; $B = 5\mathfrak{A}^2\mathfrak{C}v$; $C = 5\mathfrak{A}\mathfrak{C}^2vv$ et $D = \mathfrak{A}^5v + \mathfrak{C}^5v^2$
vnde ſi ſtatuatur haec aequatio:

$$x^5 = 5Pxx + 5Qx + R$$

vt ſit $P = \mathfrak{A}^2\mathfrak{C}v$ et $Q = \mathfrak{A}\mathfrak{C}^2vv$, erit $\frac{QQ}{P} = \mathfrak{C}^5v^2$ et $\frac{P^2}{Q} = \mathfrak{A}^5v$: hincque fit, vt ante, $R = \frac{QQ}{P} + \frac{P^2}{Q}$, atque etiam eaedem reperiuntur radices. Eadem porro etiam aequatio reperitur, ſiue ponatur $\mathfrak{A} = 0$ et $\mathfrak{B} = 0$; ſiue $\mathfrak{A} = 0$ et $\mathfrak{C} = 0$. Sin autem vel $\mathfrak{A}$ et $\mathfrak{D}$, vel $\mathfrak{B}$ et $\mathfrak{C}$

euaneſcere

euanescere assumantur, vtrinque quidem eadem prodit aequatio, sed diuersa a praecedentibus casibus, quam ideo euoluere conueniet.

43. Sit igitur et $\mathfrak{B}=0$ et $\mathfrak{C}=0$, atque hinc consequemur sequentes valores:
$A=5\mathfrak{A}\mathfrak{D}v$; $B=0$, $C=-5\mathfrak{A}\mathfrak{A}\mathfrak{D}\mathfrak{D}vv$; et $D=\mathfrak{A}^5v+\mathfrak{D}^5v^4$.
Vnde si statuamus $\mathfrak{A}\mathfrak{D}v=P$; erit $A=5P$ et $C=-5PP$ tum vero erit:
$DD\ 4P^5=(\mathfrak{A}^5v-\mathfrak{D}^5v^4)^2$ et $\mathfrak{A}^5v-\mathfrak{D}^5v^4=\sqrt{(DD-4P^5)}$, ideoque

$$\mathfrak{A}^5v=\tfrac{1}{2}D+\tfrac{1}{2}\sqrt{(DD-4P^5)} \text{ et}$$
$$\mathfrak{D}^5v^4=\tfrac{1}{2}D-\tfrac{1}{2}\sqrt{(DD-4P^5)}$$

Hinc si proposita sit haec aequatio:

$$x^5=5Px^3-5PPx+D$$

quaelibet eius radicum est:

$$x=\mathfrak{a}\sqrt[5]{(\tfrac{1}{2}D+\tfrac{1}{2}\sqrt{(DD-4P^5)})}+\mathfrak{a}^4\sqrt[5]{(\tfrac{1}{2}D-\tfrac{1}{2}\sqrt{(DD-4P^5)})}$$

atque haec est ipsa illa aequatio cuius resolutionem Cel. Moivraeus docuit.

44. Possunt autem ex forma generali innumerabiles deduci aequationes quinti ordinis, quarum radices assignare licet, etiamsi ipsae illae aequationes in factores resolui nequeant. Proposita enim aequatione quinti gradus:

$$x^5=Ax^3+Bx^2+Cx+D$$

cuius coefficientes habeant sequentes valores:

A=

$$A = \frac{5}{gk}(g^3 + k^3)$$

$$B = \frac{5}{2mnrr}((m+n)(m^2g^3 - n^2k^3) - (m-n)rr)$$

$$C = \frac{5}{mnggkkrr}(g^3(m^2g^3 - n^2k^3)^2 - (m(m+n)g^6 - (m^2+mn-n^2)g^3k^3 + n(m-n)k^6)rr - k^3r^4)$$

$$D = \frac{gg}{mmnk^4g^3}((m^2g^3 - nnk^3)^3 - (m^2g^3 - n^2k^3)(m^2g^3 + rn^2k^3) - n^2k^3r^4) + \frac{kk}{mnng^4r^3}(m^2g^3r^2(m^2g^3 - n^2k^3) - (2m^2g^3 + n^2k^3)r^4 + r^6) + \frac{5(m-n)(g^3-k^3)(m^2g^3-n^2k^3)}{2mngkrr} - \frac{5(m+n)(g^3k^3)}{2mngk}$$

eius radices semper assignari possunt.

45. Ponatur enim breuitatis gratia:

$$T = (m^2g^3 - n^2k^3)^2 - 2(m^2g^3 + n^2k^3)rr + r^4$$

sitque:

$$\left.\begin{matrix} P \\ Q \end{matrix}\right\} = \frac{(m^2g^3-n^2k^3)^3 - (m^2g^3-n^2k^3)(m^2g^3+2n^2k^3)rr - n^2k^3r^4 \pm ((m^2g^3-n^2k^3)^2 - n^2k^3rr)\sqrt{T}}{2mnnr}$$

$$\left.\begin{matrix} R \\ S \end{matrix}\right\} = \frac{(m^2g^3-n^2k^3)m^2g^3 - (2m^2g^3+n^2k^3)rr + r^4 \pm (m^2g^3 - rr)\sqrt{T}}{2mnnr}$$

vbi signa superiora pro valoribus P et R, inferiora pro Q et S valent, ac quaelibet radix aequationis erit:

$$x = \mathfrak{a}\sqrt[5]{\tfrac{gg}{k^4}P} + \mathfrak{a}^2\sqrt[5]{\tfrac{kk}{g^4}R} + \mathfrak{a}^3\sqrt[5]{\tfrac{kk}{g^4}S} + \mathfrak{a}^4\sqrt[5]{\tfrac{gg}{k^4}Q}.$$

46. Vt rem exemplis illustremus, ex his formis sequentia formari possunt:

I. Aequationis $x^5 = 40x^3 + 70xx - 50x - 98$ radix est

$$x = \sqrt[5]{(-31-3\sqrt{-7})} + \sqrt[5]{(-18+10\sqrt{-7})} + \sqrt[5]{(-18-\sqrt{-7})} + \sqrt[5]{(-31-3\sqrt{-7})}$$

II. Aequationis $x^5 = 2625x + 16600$ radix eſt

$$x = \sqrt[5]{75(5+4\sqrt{10})} + \sqrt[5]{225(35+11\sqrt{10})} + \sqrt[5]{225(35-11\sqrt{10})} + \sqrt[5]{75(5-4\sqrt{10})}$$

quae eo magis ſunt notatu digna, quod hae aequationes nullo alio modo reſolui poſſunt. Simili autem modo huiusmodi inueſtigationes ad aequationes altiorum graduum extendi poſſunt: facileque erit, ex quouis gradu innumerabiles aequationes per alias methodos irreſolubiles exhibere, quarum huius methodi ope non ſolum vna, ſed omnes plane radices exhiberi queant.

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