



1764

Specimen algorithmi singularis

Leonhard Euler

Follow this and additional works at: <https://scholarlycommons.pacific.edu/euler-works>

 Part of the [Mathematics Commons](#)

Record Created:

2018-09-25

Recommended Citation

Euler, Leonhard, "Specimen algorithmi singularis" (1764). *Euler Archive - All Works*. 281.
<https://scholarlycommons.pacific.edu/euler-works/281>

This Article is brought to you for free and open access by the Euler Archive at Scholarly Commons. It has been accepted for inclusion in Euler Archive - All Works by an authorized administrator of Scholarly Commons. For more information, please contact mgibney@pacific.edu.

S P E C I M E N ALGORITHMI SINGULARIS.

Auctore
L. E V L E R O.

I.

Consideratio fractionum continuarum, quarum usus uberrimum per totam Analysin iam aliquoties ostendi, deduxit me ad quantitates certo quodam modo ex indicibus formatas, quarum natura ita est comparata, ut singularem algorithmum requirat. Cum igitur summa Analyseos inuenta maximam partem algorithmo ad certas quasdam quantitates accommodato innitantur, non immerito suspicari licet, et hunc algorithmum singularem non exigui usus in Analyfi esse futurum, si quidem diligentius excolatur: etiamsi ei tantum non tribuendum censeam, ut cum receptis algorithmis comparari mereatur.

2. Sequenti autem modo ad eas quantitates, de quibus hic agere constitui, sum deductus: si habeatur fractio continua $a + \frac{1}{b + \frac{1}{c + \frac{1}{d}}}$ cuius valor sit in-

$$\frac{a + \frac{1}{b + \frac{1}{c + \frac{1}{d}}}}$$

vestigandus, ex numeris a, b, c, d , tanquam indicibus assumtis, sequenti modo fractiones formantur:

$$\frac{a}{1}, \frac{ab+1}{b}, \frac{abc+c+a}{bc+1}, \frac{abcd+cd+ad+ab+1}{bcd+d+b} \quad \text{G 3} \quad \text{Pit}$$

Primum scilicet locum obtinet semper fractio $\frac{2}{3}$, secundum $\frac{a}{7}$, cuius numerator est primus indicum a , denominator vero unitas. Sequentis cuiusque fractionis tam numerator, quam denominator, inuenitur, si praecedentium ultimus per indicem supra scriptum multiplicetur, et ad productum penultimus addatur.

3. Constat autem harum fractionum postremam ipsi fractioni continuae propositae esse aequalem, praecedentes autem tam prope ad hunc ipsum valorem accedere, ut nulla fractio numeris non maioribus contenta exhiberi queat, quae ad illum propius accedat. Atque ex hoc fonte problema illud a Wallisio olim tractatum facile resoluitur, quo proposita quacunque fractione ex ingentibus numeris constante, aliae quaeruntur fractiones ex minoribus numeris constantes, quae tam parum a proposita discrepent, ut minus discrepantes exhiberi plane nequeant, nisi maiores numeros adhibere velimus.

4. Hoc autem aliisque vsibus, quos fractiones continuae suppeditant, praetermissis, hic imprimis obseruo, in serie illa fractionum ex indicibus formatarum, tam numeratores, quam denominatores, eandem, progressionis legem sequi, et seorsim efformari posse. In utraque enim serie, siue numeratorum, siue denominatorum, quilibet terminus per indicem supra scriptum multiplicatus, et termino antecedente auctus, praebet terminum sequentem. Ultimus autem numerus superioris seriei componitur ex omnibus quatuor indicibus a, b, c, d , penultimus tantum ex tribus a, b, c , antepenultimus tantum

tantum ex duobus a , et b . Inferiores autem numeri primum indicem a plane non inuoluunt, sed ex reliquis b, c, d aequali lege formantur.

5. Quoniam igitur ratio formationis ex indicibus, tam pro numeratoribus, quam pro denominatoribus, est eadem, ac datis indicibus numerus inde formatus innotescit, hos ipsos numeros, quatenus ex indicibus sunt formati, hic sum contemplaturus, eorumque algorithmum traditurus. Propositis autem indicibus quibuscunque et quotcunque a, b, c, d , numerum ex iis formatum hoc modo (a, b, c, d) denotabo, eritque ergo euolutione instituta:

$$(a, b, c, d) = abcd + cd + ad + ab + x$$

similique modo pro denominatoribus indicem primum a omittingo

$$(b, c, d) = bcd + d + b.$$

6. Haec ergo teneatur definitio signorum (), inter quae indices ordine a sinistra ad dextram scribere constitui; atque indices hoc modo clausulis inclusi in posterum denotabunt numerum ex istis indicibus formatum. Ita a simplicissimis casibus inchoando, habebimus:

$$(a) = a$$

$$(a, b) = ab + x$$

$$(a, b, c) = abc + c + a$$

$$(a, b, c, d) = abcd + cd + ad + ab + x$$

$$(a, b, c, d, e) = abcde + cde + ade + abe + abc + e + c + a$$

etc.

ex qua progressionem patet, unitatem tenere locum huius signi () si scilicet nullus adfit index.

7. Quemadmodum hae expressiones crescente indicum numero ulterius sint continuandae ex formationis lege, qua quilibet ex duobus antecedentibus componitur, sponte liquet. Est scilicet:

$$(a, b) = b(a) + 1 = b(a) + ()$$

$$(a, b, c) = c(a, b) + (a)$$

$$(a, b, c, d) = d(a, b, c) + (a, b)$$

$$(a, b, c, d, e) = e(a, b, c, d) + (a, b, c)$$

In genere ergo habebitur:

$$(a, b, c, \dots, p, q, r) = r(a, b, c, \dots, p, q) + (a, b, c, \dots, p)$$

quae connexio, tanquam corollarium definitionis numerorum, quos hic contemplamur, spectari debet.

8. In evolutione horum valorum, uti ante §. 6 sunt exhibiti, difficulter ratio compositionis cernitur. Possunt autem ii quoque hoc modo representari:

$$(a) = a(1)$$

$$(a, b) = ab(1 + \frac{1}{ab})$$

$$(a, b, c) = abc(1 + \frac{1}{ab} + \frac{1}{bc})$$

$$(a, b, c, d) = abcd(1 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{cd} + \frac{1}{abcd})$$

$$(a, b, c, d, e) = abcde(1 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{cd} + \frac{1}{de} + \frac{1}{abcd} + \frac{1}{abcde} + \frac{1}{bcde})$$

In his autem denominatoribus occurrunt primo facta ex binis indicibus contiguis, tum vero producta ex binis illorum factorum, qui nullum indicem communem inuoluunt, tum sequentur producta ex ternis, quater-

nis

nis etc. combinationibus, quae nullum implicant indicem communem; unde ratio compositionis iam fit perspicua.

9. Ex hac evolutione iam manifestum est, si indices ordine retrogrado disponantur, eosdem plane prodire numeros inde formatos. Erit scilicet

$$\begin{aligned}(a, b) &= (b, a) \\(a, b, c) &= (c, b, a) \\(a, b, c, d) &= (d, c, b, a) \\(a, b, c, d, e) &= (e, d, c, b, a).\end{aligned}$$

Dummodo ergo ordo indicum detur, siue sit directus, siue retrogradus, perinde est; utroque enim modo idem numerus inde formatus obtinetur.

10. Hinc ergo sequitur, fore formulas §. 7 hoc modo inuertendo:

$$\begin{aligned}(a, b) &= a(b) + 1 \\(a, b, c) &= a(b, c) + (c) \\(a, b, c, d) &= a(b, c, d) + (c, d) \\(a, b, c, d, e) &= a(b, c, d, e) + (c, d, e)\end{aligned}$$

atque in genere erit pro quocunque indicibus:

$$(a, b, c, d, \text{etc.}) = a(b, c, d, \text{etc.}) + (c, d, \text{etc.})$$

11. Si ergo ponatur:

$$\begin{aligned}(a, b, c, d, e, \text{etc.}) &= A \\(b, c, d, e, \text{etc.}) &= B \\(c, d, e, \text{etc.}) &= C \\(d, e, \text{etc.}) &= D \\(e, \text{etc.}) &= E\end{aligned}$$

habebimus has aequalitates :

$$A = aB + C \text{ seu } \frac{A}{B} = a + \frac{C}{B}$$

$$B = bC + D \text{ seu } \frac{B}{C} = b + \frac{D}{C}$$

$$C = cD + E \text{ seu } \frac{C}{D} = c + \frac{E}{D}$$

etc.

etc.

12. Cum igitur fit

$$\frac{C}{B} = \frac{1}{b + \frac{D}{C}}; \quad \frac{D}{C} = \frac{1}{c + \frac{E}{D}}; \quad \frac{E}{D} = \frac{1}{d + \frac{F}{E}} \text{ etc.}$$

erit his valoribus substituendis :

$$\frac{A}{B} = a + \frac{C}{B} = a + \frac{1}{b + \frac{D}{C}} = a + \frac{1}{b + \frac{1}{c + \frac{E}{D}}} = a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \frac{F}{E}}}}$$

Vnde si e fit indicum ultimus, ita ut fit $E = e$ etc. $F = 1$, erit

$$\frac{A}{B} = \frac{(a, b, c, d, e)}{(b, c, d, e)} = a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \frac{1}{e}}}}$$

ficque patet, quemadmodum per huiusmodi numeros valores fractionum continuarum commode exprimi queant.

13. Si ergo indicum numerus fuerit infinitus, etiam fractio continua in infinitum excurreret, eiusque valor erit $= \frac{(a, b, c, d, e, \text{etc.})}{(b, c, d, e, \text{etc.})}$. Vicissim autem fractionum continuarum proprietates cognitae nobis insignes affectiones huiusmodi numerorum ex indicibus formatorum manifesta-

manifestabunt, quas diligentius euoluere operae erit pretium. Sit igitur fractio continua, siue in infinitum excurrent, siue secus proposita:

$$a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \frac{1}{e + \frac{1}{f \text{ etc.}}}}}}$$

cuius valor indicetur littera V

et sumendis omnibus indicibus a, b, c, d, e, f etc. erit, uti demonstrauimus:

$$V = \frac{(a, b, c, d, e, f \text{ etc.})}{(b, c, d, e, f, \text{ etc.})}$$

14. Si quispiam horum indicum fiat infinite magnus, is in hac expressione cum omnibus sequentibus poterit omitti, et valor fractionis continuæ tantum per indices, qui infinitum præcedunt, exprimetur.

Ita si sit $b = \infty$ erit $V = \frac{(a)}{(b)}$

si sit $c = \infty$ erit $V = \frac{(a, b)}{(b)}$

si sit $d = \infty$ erit $V = \frac{(a, b, c)}{(b, c)}$

si sit $e = \infty$ erit $V = \frac{(a, b, c, d)}{(b, c, d)}$

Vti ergo his casibus fractio continua abrumpitur, ita etiam valor V alios indices non implicat, nisi qui indicem infinitum antecedunt.

15. Sin autem nullus indicum in infinitum excrefcit, hi ipsi valores continuo propius ad veram va-

lorem V accedunt. Scilicet si fuerit $V = \frac{(a, b, c, d, e \text{ etc.})}{(b, c, d, e, \text{ etc.})}$
 fractiones in sequenti serie expositae :

$$\frac{(a)}{1}; \frac{(a, b)}{(b)}; \frac{(a, b, c)}{(b, c)}; \frac{(a, b, c, d)}{(b, c, d)}; \frac{(a, b, c, d, e)}{(b, c, d, e)}; \text{ etc.}$$

continuo proprius ad valorem V accedunt, earumque
 vltima demum eius valorem verum exhibebit; siqui-
 dem indices a, b, c, d etc. fuerint numeri vnitatem
 maiores. Prima quidem $\frac{a}{1}$ notabiliter ab V discrepare
 poterit, secunda autem propius accedet, tertia adhuc
 propius, et ita porro, donec tandem vltima verum va-
 lorem V sit praebitura.

16. Necessesse ergo est, vt differentiae inter binas
 huiusmodi fractiones contiguas continuo fiant minores;
 quod quo clarius perspiciatur, has differentias inuestige-
 mus, quae erunt:

$$\begin{aligned} \frac{(a)}{1} - \frac{(a, b)}{(b)} &= \frac{(a)(b) - 1(a, b)}{1(b)} \\ \frac{(a, b)}{(b)} - \frac{(a, b, c)}{(b, c)} &= \frac{(a, b)(b, c) - (b)(a, b, c)}{(b)(b, c)} \\ \frac{(a, b, c)}{(b, c)} - \frac{(a, b, c, d)}{(b, c, d)} &= \frac{(a, b, c)(b, c, d) - (b, c)(a, b, c, d)}{(b, c)(b, c, d)} \\ \frac{(a, b, c, d)}{(b, c, d)} - \frac{(a, b, c, d, e)}{(b, c, d, e)} &= \frac{(a, b, c, d)(b, c, d, e) - (b, c, d)(a, b, c, d, e)}{(b, c, d)(b, c, d, e)}. \end{aligned}$$

17. De harum differentiarum denominatoribus,
 qui ex binis factoribus sunt conflati, primum ob-
 seruo, hos factores inter se esse numeros primos,
 quod quidem ex antecedentibus est satis manifestum.
 Cum enim pro denominatore $(b, c, d)(b, c, d, e)$
 fit $(b, c, d, e) = e(b, c, d) + (b, c)$ erit $\frac{(b, c, d, e)}{(b, c, d)}$
 $= e + \frac{(b, c)}{(b, c, d)}$, vnde factores (b, c, d) et (b, c, d, e)
 communem diuisorem non habebunt, nisi qui simul sit
 communis diuisor numerorum (b, c) et (b, c, d) ; ve-
 rum ob eandem rationem horum numerorum commu-
 nis

nis diuisor non datur, nisi qui simul sit communis diuisor horum (b) et (b, c) , ac denique horum 1 et b ; qui cum nullum habeant communem diuisorem, neque illi habebunt, eruntque propterea numeri primi. Hinc vero etiam intelligitur, numeros $(a, b, c, d, \text{etc.})$ et $(b, c, d, \text{etc.})$ esse inter se primos.

18. Differentiae ergo illae minores esse nequeunt, quam si numeratores in unitatem, siue affirmatiuam, siue negatiuam, abeant, id quod re vera euenire exempla declarant. Conueniet ergo, idem ex natura istorum numerorum per indices formatorum demonstrari. Pro primo quidem numeratore cum sit $(a, b) = b(a) + 1$ per §. 7. erit:

$$(a)(b) - 1(a, b) = ab - b(a) - 1 = -1.$$

Tum vero pro secundo, ob $(b, c) = c(b) + 1$ et $(a, b, c) = c(a, b) + (a)$ erit

$$(a, b)(b, c) - (b)(a, b, c) = (a, b)c(b) + (a, b) - (b)c(a, b) - (b)(a)$$

quae propter terminos $(a, b)c(b) - (b)c(a, b)$ se tollentes abit in

$$(a, b) - (b)(a) = -1, \text{ ita vt sit secundus numerator}$$

$$(a, b)(b, c) - (b)(a, b, c) = +1.$$

19. Quemadmodum hic numerator secundus ad primum negatiue sumtum est reductus, ita tertius ostendi potest secundo negatiue sumto esse aequalis.

Nam quia $(b, c, d) = d(b, c) + (b)$ et

$$(a, b, c, d) = d(a, b, c) + (a, b) \text{ erit}$$

$$(a, b, c)(b, c, d) - (b, c)(a, b, c, d) = (a, b, c)d(b, c) + (a, b, c)(b) - (b, c)d(a, b, c) - (b, c)(a, b)$$

Haec ergo expressio transit in $-(a,b)(b,c)+(b)(a,b,c)=-1$, quia est denominator secundus negative sumtus. Eodem autem modo numerator quartus aequabitur tertio negative sumto, et in genere quilibet sequens praecedenti negative sumto.

20. Hinc ergo consequimur sequentes reductiones non parum notatu dignas:

$$\begin{aligned} (a)(b) & \quad -1(a,b) = -1 \\ (a,b)(b,c) & \quad -(b)(a,b,c) = +1 \\ (a,b,c)(b,c,d) & \quad -(b,c)(a,b,c,d) = -1 \\ (a,b,c,d)(b,c,d,e) & \quad -(b,c,d)(a,b,c,d,e) = +1 \\ & \quad \text{et in genere} \\ (a,b,c,d,\dots,m)(b,c,d,\dots,m,n) & \quad -(b,c,d,\dots,m) \\ & \quad (a,b,c,d,\dots,m,n) = +1 \end{aligned}$$

vbi $+1$ valet, si in primis vinculis numerus indicum fuerit par, contra vero -1 .

21. Ipsae ergo differentiae supra expositae erunt:

$$\begin{aligned} \frac{(a)}{1} - \frac{(a,b)}{(b)} & = \frac{-1}{(b)} \\ \frac{(a,b)}{(b)} - \frac{(a,b,c)}{(b,c)} & = \frac{+1}{(b)(b,c)} \\ \frac{(a,b,c)}{(b,c)} - \frac{(a,b,c,d)}{(b,c,d)} & = \frac{-1}{(b,c)(b,c,d)} \\ \frac{(a,b,c,d)}{(b,c,d)} - \frac{(a,b,c,d,e)}{(b,c,d,e)} & = \frac{+1}{(b,c,d)(b,c,d,e)} \\ \frac{(a,b,c,d,e)}{(b,c,d,e)} - \frac{(a,b,c,d,e,f)}{(b,c,d,e,f)} & = \frac{-1}{(b,c,d,e)(b,c,d,e,f)} \\ & \text{etc.} \end{aligned}$$

vnde, cum hae differentiae minores esse nequeant, ipsae fractiones tam prope ad se invicem accedunt, quam fieri potest.

22. Cum fit ex §. 7. $(b, c) - 1 = c(b)$; $(b, c, d) - (b) = d(b, c)$; $(b, c, d, e) - (b, c) = e(b, c, d)$ etc. erit binis illarum differentiarum addendis.

$$\begin{aligned} \frac{(a)}{1} - \frac{(a, b, c)}{(b, c)} &= -\frac{c}{1(b, c)} \\ \frac{(a, b)}{(b)} - \frac{(a, b, c, d)}{(b, c, d)} &= +\frac{d}{(b)(b, c, d)} \\ \frac{(a, b, c)}{(b, c)} - \frac{(a, b, c, d, e)}{(b, c, d, e)} &= -\frac{e}{(b, c)(b, c, d, e)} \\ \frac{(a, b, c, d)}{(b, c, d)} - \frac{(a, b, c, d, e, f)}{(b, c, d, e, f)} &= +\frac{f}{(b, c, d)(b, c, d, e, f)} \\ &\text{etc.} \end{aligned}$$

eritque hic $\frac{(a)}{1} = a$, et $\frac{(a, b)}{(b)} = a + \frac{1}{b}$; vnde reliquae formulae concinne poterunt exhiberi.

23. Ex formulis ergo §. 21. habebimus sequentes fractionum continuarum valores:

$$\begin{aligned} \frac{(a)}{1} &= a \\ \frac{(a, b)}{(b)} &= a + \frac{1}{1(b)} \\ \frac{(a, b, c)}{(b, c)} &= a + \frac{1}{1(b)} - \frac{1}{(b)(b, c)} \\ \frac{(a, b, c, d)}{(b, c, d)} &= a + \frac{1}{1(b)} - \frac{1}{(b)(b, c)} + \frac{1}{(b, c)(b, c, d)} \\ &\text{etc.} \end{aligned}$$

vnde in genere erit, si etiam indices in infinitum excurrant,

$$\begin{aligned} \frac{(a, b, c, d, e, \text{etc.})}{(b, c, d, e, \text{etc.})} &= a + \frac{1}{1(b)} - \frac{1}{(b)(b, c)} + \frac{1}{(b, c)(b, c, d)} \\ &\quad - \frac{1}{(b, c, d)(b, c, d, e)} + \text{etc.} \end{aligned}$$

24. Ex formulis autem §. 22. obtinebimus:

$$\begin{aligned} \frac{(a, b, c)}{(b, c)} &= a + \frac{c}{1(b, c)} \\ \frac{(a, b, c, d, e)}{(b, c, d, e)} &= a + \frac{c}{1(b, c)} + \frac{e}{(b, c)(b, c, d, e)} \end{aligned}$$

vnde

vnde generaliter :

$$\frac{(a, b, c, d, e, \text{etc.})}{(b, c, d, e, \text{etc.})} = a + \frac{1}{(b, c)} + \frac{e}{(b, c)(b, c, d, e)} + \frac{g}{(b, c, d, e)(b, c, d, e, f, g)} \text{ etc.}$$

Tum vero etiam :

$$\frac{(a, b, c, d)}{(b, c, d)} = a + \frac{1}{b} - \frac{d}{(b)(b, c, d)}$$

$$\frac{(a, b, c, d, e, f)}{(b, c, d, e, f)} = a + \frac{1}{b} - \frac{d}{(b)(b, c, d)} - \frac{f}{(b, c, d)(b, c, d, e, f)}$$

ideoque generaliter :

$$\frac{(a, b, c, d, e, \text{etc.})}{(b, c, d, e, \text{etc.})} = a + \frac{1}{b} - \frac{d}{(b)(b, c, d)} - \frac{f}{(b, c, d)(b, c, d, e, f)} - \frac{g}{(b, c, d, e, f)(b, c, d, e, f, g, b)} - \text{etc.}$$

25. Sed missis his, quae ad series spectant, quoniam ea iam fusius sum persecutus, perpendamus ea, quae ad singularem harum quantitatum algorithmum pertinent. Et formulas quidem iis similes, quae in §. 20. sunt inuentae, suppeditabit nobis §. 22. ex quo patet esse :

$$(a)(b, c) \quad -1(a, b, c) = -c$$

$$(a, b)(b, c, d) \quad -(b)(a, b, c, d) = +d$$

$$(a, b, c)(b, c, d, e) \quad -(b, c)(a, b, c, d, e) = -e$$

$$(a, b, c, d)(b, c, d, e, f) - (b, c, d)(a, b, c, d, e, f) = +1$$

ideoque generaliter :

$$(a, b, \dots, l)(b, \dots, l, m, n) - (b, \dots, l)(a, b, \dots, l, m, n) = \pm n$$

vbi signum \pm valet, si in primo vinculo numerus indicum sit par; contra signum $-$.

26. Per similes autem reductiones intelligitur fore,

$$(a)(b, c, d) \quad -1(a, b, c, d) = -(c, d)$$

$$(a, b)(b, c, d, e) \quad -(b)(a, b, c, d, e) = +(d, e)$$

$$(a, b, c)(b, c, d, e, f) - (b, c)(a, b, c, d, e, f) = -(e, f)$$

et

et generaliter :

$$(a, b, \dots, k)(b, \dots, k, l, m, n) - (b, \dots, k)(a, b, \dots, k, l, m, n) = \pm (m, n)$$

vbi signorum, vel superiorum, vel inferiorum, valet, prout in primo vinculo numerus indicum fuerit, vel par, vel impar.

27. Ratio autem huius formulae ex supra reper- tis facile derivatur. Si enim ponatur :

$$(a, b, \dots, k, l, m)(b, \dots, k, l, m, n) - (b, \dots, k, l, m)(a, b, \dots, k, l, m, n) = A$$

$$(a, b, \dots, k, l)(b, \dots, k, l, m, n) - (b, \dots, k, l)(a, b, \dots, k, l, m, n) = B$$

$$(a, b, \dots, k)(b, \dots, k)(a, b, \dots, k, l, m, n) = C$$

manifestum est, esse $A = mB + C$. At est $A = \pm 1$; et $B = \mp n$; ideoque $C = \pm 1 + mn = \pm (m, n)$, vbi de ambiguitate signorum tenenda sunt praecepta superiora.

28 Si ordo indicum in his formulis inuertatur, eae fient :

$$(a, \dots, y)(a, b, \dots, y, z) - (a, b, \dots, y, z)(a, b, \dots, y) = 0$$

$$(a, b, \dots, y)(b, c, \dots, y, z) - (a, b, \dots, y, z)(b, c, \dots, y) = \pm 1$$

$$(a, b, c, \dots, y)(c, d, \dots, y, z) - (a, b, \dots, y, z)(c, d, \dots, y) = \pm (a)$$

$$(a, b, c, d, \dots, y)(d, e, \dots, y, z) - (a, b, \dots, y, z)(d, e, \dots, y) = \pm (a, b)$$

$$(a, b, c, d, e, \dots, y)(e, f, \dots, y, z) - (a, b, \dots, y, z)(e, f, \dots, y) = \pm (a, b, c)$$

$$(a, b, \dots, y)(f, g, \dots, y, z) - (a, b, \dots, y, z)(f, g, \dots, y) = \pm (a, b, c, d)$$

vbi signa valent superiora, si numerus indicum in secundo vinculo fuerit par, contra autem valent inferiora.

29. Si haec indicum series in fine duobus truncetur, orietur simili modo :

$$\begin{aligned} (a \dots x)(a \dots z) - (a \dots z)(a \dots x) &= 0 \\ (a \dots x)(b \dots z) - (a \dots z)(b \dots x) &= \pm(z) \\ (a \dots x)(c \dots z) - (a \dots z)(c \dots x) &= \pm(a)(z) \\ (a \dots x)(d \dots z) - (a \dots z)(d \dots x) &= \pm(a, b)(z) \\ (a \dots x)(e \dots z) - (a \dots z)(e \dots x) &= \pm(a, b, c)(z) \end{aligned}$$

atque hinc tandem colligitur fore generaliter :

$$\begin{aligned} (a \dots l, m, n \dots p) (n \dots p, q, r \dots z) \\ - (a \dots l, m, n \dots p, q, r \dots z) (n \dots p) \\ = \pm(a \dots l)(r \dots z). \end{aligned}$$

30. Quo ratio ambiguitatis signorum pateat, notandum est, si sit $m = a$, fore $(a \dots l) = 1$, et si sit $q = z$, fore $(r \dots z) = 1$, vnde casus speciales, in quibus ratio signorum est cognita, erunt

$$\begin{aligned} (a)(b) - (a, b) 1 &= -1 \\ (a)(b, c) - (a, b, c) 1 &= -(c) \\ (a, b)(c) - (a, b, c) 1 &= -(a) \\ (a, b)(b, c, d) - (a, b, c, d)(b) &= +(d) \\ (a)(b, c, d) - (a, b, c, d)(1) &= -(c, d) \end{aligned}$$

vnde concluditur, valorem fore affirmativum, si in extremo quaternorum vinculorum numerus indicum sit impar, sin autem fuerit par, valor erit negativus. Ita in exemplis subiunctis erit

$$\begin{aligned} (a, b, c, d)(e, f, g, h) - (a, b, c, d, e, f, g, h) 1 &= -(a, b, c)(f, g, h) \\ (a, b, c, d, e)(c, d, e, f, g, h) - (a, b, c, d, e, f, g, h)(c, d, e) \\ &= +(a)(g, h). \end{aligned}$$

31. Huiusmodi autem formulae, quot lubuerit, facile sequenti modo exhiberi possunt; sumatur tertium vinculum, quod est completum, et omnes indices continet, abscindantur ab initio superne ii indices, qui primum vinculum constituent, tum inferne a fine ii, qui vinculum secundum constituent; ita tamen, vt in duobus primis vinculis omnes indices occurrant. Tum qui locis abscissis vtrique sunt vicini puncto notentur, indeque facile huiusmodi formulae exhibentur:

$$\begin{aligned} &\text{vt } \overline{a, b, c, d} \overline{e, f} \text{ dabit} \\ &(a, b, c, d)(c, d, e, f) - (a, b, c, d, e, f)(c, d) = -(a)(f) \\ &\text{vt } \overline{a, b, c} \overline{d, e, f} \text{ dat} \\ &(a, b, c)(d, e, f) - (a, b, c, d, e, f) \text{I} = -(a, b)(e, f) \\ &\text{vt } \overline{a, b, c, d} \overline{e, f} \text{ dat} \\ &(a, b, c, d)(d, e, f) - (a, b, c, d, e, f)(d) = +(a, b)(f). \end{aligned}$$

32. Quodsi ergo in duobus vinculis nulla littera bis occurrat, quartum vinculum erit unitas, vnde sequentes formulae nascuntur:

$$\begin{aligned} \int (a, b, c) &= (a, b)(c) + (a) \\ \int (a, b, c) &= (a)(b, c) + (c) \\ \int (a, b, c, d) &= (a, b, c)(d) + (a, b) \\ \int (a, b, c, d) &= (a, b)(c, d) + (a, d) \\ \int (a, b, c, d) &= (a)(b, c, d) + (c, d) \\ \int (a, b, c, d, e) &= (a, b, c, d)(e) + (a, b, c) \\ \int (a, b, c, d, e) &= (a, b, c)(d, e) + (a, b)(e) \\ \int (a, b, c, d, e) &= (a, b)(c, d, e) + (a)(d, e) \\ \int (a, b, c, d, e) &= (a)(b, c, d, e) + (e, d, e) \end{aligned}$$

$$\begin{cases}
 (a, b, c, d, e, f) = (a, b, c, d, e)(f) + (a, b, c, d) \\
 (a, b, c, d, e, f) = (a, b, c, d)(e, f) + (a, b, c)(f) \\
 (a, b, c, d, e, f) = (a, b, c)(d, e, f) + (a, b)(e, f) \\
 (a, b, c, d, e, f) = (a, b)(c, d, e, f) + (a)(d, e, f) \\
 (a, b, c, d, e, f) = (a)(b, c, d, e, f) + (c, d, e, f) \\
 \text{etc.}
 \end{cases}$$

33. Si ordo indicum inuertatur, sequentes formulæ hinc facile elicientur :

$$\begin{aligned}
 (a)(a, b, c, d, \dots) &= (a, a, b, c, d, \dots) - (b, c, d, \dots) \\
 (\alpha, \beta)(a, b, c, d, \dots) &= (\alpha, \beta, a, b, c, d, \dots) - (\alpha)(b, c, d, \dots) \\
 (\alpha, \beta, \gamma)(a, b, c, d, \dots) &= (\alpha, \beta, \gamma, a, b, c, d, \dots) - (\alpha, \beta)(b, c, d, \dots) \\
 &\text{etc.}
 \end{aligned}$$

unde productio ex duobus huius generis numeris ad eiusmodi numeros simplices reuocari poterunt :

$$\begin{aligned}
 (a)(a, b, c, d, \dots) &= (a, a, b, c, d, \dots) - (b, c, d, \dots) \\
 (\alpha, \beta)(a, b, c, d, \dots) &= (\alpha, \beta, a, b, c, d, \dots) - (\alpha, b, c, d, \dots) + (c, d, \dots) \\
 (\alpha, \beta, \gamma)(a, b, c, d, \dots) &= \begin{cases} +(\alpha, \beta, \gamma, a, b, c, d, \dots) \\ -(\alpha, \beta, b, c, d, \dots) \\ +(\alpha, c, d, \dots) \\ -(d, \dots) \end{cases} \\
 (\alpha, \beta, \gamma, \delta)(a, b, c, d, e, \dots) &= \begin{cases} +(\alpha, \beta, \gamma, \delta, a, b, c, d, e, \dots) \\ -(\alpha, \beta, \gamma, b, c, d, e, \dots) \\ +(\alpha, \beta, c, d, e, \dots) \\ -(\alpha, d, e, \dots) \\ +(e, \dots) \end{cases} \\
 &\text{etc.}
 \end{aligned}$$

quia ergo in utroque factore ordo indicum inuerti potest, hæc formæ pluribus modis variari poterunt.

34 Renertamur autem ad fractiones continuas, unde haec sunt nata, sitque valor huius $a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \frac{1}{e + \frac{1}{f + \text{etc.}}}}}}$ = S.

$$a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \frac{1}{e + \frac{1}{f + \text{etc.}}}}}}$$

atque supra iam inuenimus hos valores

$$A = \frac{(a)}{1}; B = \frac{(a, b)}{(b)}; C = \frac{(a, b, c)}{(b, c)}; D = \frac{(a, b, c, d)}{(b, c, d)}; E = \frac{(a, b, c, d, e)}{(b, c, d, e)} \text{ etc.}$$

continuo propius ad valorem S accedere. Horum terminorum igitur singulas differentias perpendamus:

$$\begin{array}{l} A - B = -\frac{1}{(b)} \\ A - C = -\frac{(c)}{1(b, c)} \\ A - D = -\frac{(c, d)}{1(b, c, d)} \\ A - E = -\frac{(c, d, e)}{1(b, c, d, e)} \end{array} \left| \begin{array}{l} B - C = +\frac{1}{(b)(b, c)} \\ B - D = +\frac{(d)}{(b)(b, c, d)} \\ B - E = +\frac{(d, e)}{(b)(b, c, d, e)} \\ B - F = +\frac{(d, e, f)}{(b)(b, c, d, e, f)} \end{array} \right. \left| \begin{array}{l} C - D = -\frac{1}{(b, c)(b, c, d)} \\ C - E = -\frac{(e)}{(b, c)(b, c, d, e)} \\ C - F = -\frac{(e, f)}{(b, c)(b, c, d, e, f)} \\ C - G = -\frac{(e, f, g)}{(b, c)(b, c, d, e, f, g)} \end{array} \right.$$

35. Quoniam igitur in doctrina de fractionibus continuis, cuius iam aliquot specimina edidi, huius generis numeri per indices, formati totum negotium conficiunt: algorithmi eorum species, quam hic exposui, nec non insignes comparationes inuentae, non exiguum praestabunt usum in hoc argumento vberius excolendo, unde has animaduersiones usu non carituras esse confido.