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De progressionibus arcuum circularium, quorum tangentes secundum certam legem procedunt

Leonhard Euler

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D E

PROGRESSIONIBVS ARCVVM
CIRCVLARIVM QVORVM TANGENTES SE-
CVNDVM CERTAM LEGEM
PROCEDVNT.

Auctore

L. E V L E R O.

I.

Infinitas huiusmodi progressionēs exhiberi posse vel ex his exemplis liquet, quae olim proposui, scilicet denotante π arcum duos angulos rectos metientem inueni, esse $\frac{\pi}{4} = A \operatorname{tang} \frac{1}{2} + A \operatorname{tang} \frac{1}{3} + A \operatorname{tang} \frac{1}{4} + A \operatorname{tang} \frac{1}{5} + A \operatorname{tang} \frac{1}{6} + \text{etc.}$ quae series arcuum in infinitum progreditur, tangente cuiusque indefinite existente $= \frac{1}{2xx}$, simili modo est $\frac{\pi}{4} = A \operatorname{tang} \frac{1}{3} + A \operatorname{tang} \frac{1}{7} + A \operatorname{tang} \frac{1}{13} + A \operatorname{tang} \frac{1}{21} + A \operatorname{tang} \frac{1}{31} + \text{etc.}$ hac arcuum serie pariter in infinitum continuata, cuius quisque terminus indefinite est $A \operatorname{tang} \frac{1}{xx + \gamma + 1}$. Tales autem series eo magis videntur omni attentione dignae, quod nulla adhuc constat methodus earum summam a priori inueniendi, atque etiam ipsi arcus omnes inter se sint incommensurabiles. Quin etiam ne expectare quidem licet methodum, cuius ope in genere huiusmodi serierum, quaecunque legem tangentes sequantur, summa inuestigari queat; sed potius, nisi haec

haec lex certis conditionibus sit adstricta, nullo modo eae ad summam reuocari posse videntur, quae quidem arcu circulari exprimitur. Quam ob rem in hoc negotio alia via non patet, nisi vt à posteriori huiusmodi series inuestigemus, quantum deinceps contemplatio fortasse viam quandam directam patefaciet; hincque modum exponam facilem ad quocunque huiusmodi series perueniendi, qui cum, simplicissimis principiis innixus, ad tam ardua perducatur, omnino mereri videtur, vt diligentius euoluatur.

2. Non solum autem hoc modo ad series infinitas deducimur, sed pro lubitu progressionem dato terminorum numero constantes consequi possumus. Fundamentum enim totius inuestigationis in eo consistit, vt pro lubitu numeros quocunque assumamus, qui sint:

$$\alpha, \beta, \gamma, \delta, \varepsilon,$$

qui vt tangentes angulorum spectentur. Cum enim manifesto sit

$$\left. \begin{aligned} &+A \operatorname{tang} . \alpha + A \operatorname{tang} . \beta + A \operatorname{tang} . \gamma + A \operatorname{tang} . \delta \\ &-A \operatorname{tang} . \beta - A \operatorname{tang} . \gamma - A \operatorname{tang} . \delta - A \operatorname{tang} . \varepsilon \end{aligned} \right\} = A \operatorname{tang} . \alpha - A \operatorname{tang} . \varepsilon$$

binis arcubus subscriptis colligendis ob $A \operatorname{tang} . p - A \operatorname{tang} . q$

$= A \operatorname{tang} . \frac{p-q}{p+q}$ habebimus

$$\begin{aligned} A \operatorname{tang} . \alpha - A \operatorname{tang} . \varepsilon &= A \operatorname{tang} . \frac{\alpha - \beta}{\alpha + \beta} + A \operatorname{tang} . \frac{\beta - \gamma}{\beta + \gamma} \\ &+ A \operatorname{tang} . \frac{\gamma - \delta}{\gamma + \delta} + A \operatorname{tang} . \frac{\delta - \varepsilon}{\delta + \varepsilon} = A \operatorname{tang} . \frac{\alpha - \varepsilon}{\alpha + \varepsilon} . \end{aligned}$$

En ergo formam maxime generalem, vnde omnes huiusmodi series arcuum originem ducunt, siue in in-

finitum excurrant, siue finito terminorum numero contentent :

$$A \operatorname{tang} \frac{\alpha - \beta}{\alpha\beta + 1} + A \operatorname{tang} \frac{\beta - \gamma}{\beta\gamma + 1} + A \operatorname{tang} \frac{\gamma - \delta}{\gamma\delta + 1} + A \operatorname{tang} \frac{\delta - \epsilon}{\delta\epsilon + 1} + \dots$$

$$\dots + A \operatorname{tang} \frac{\psi - \omega}{\psi\omega + 1} = A \operatorname{tang} \frac{\alpha - \omega}{\alpha\omega + 1}$$

3. Casu, quo numerus terminorum est finitus, hic non immorans, statim in series infinitas inquiram. Primo ergo pro α, β, γ etc. seriem harmonicam assumam in genere:

Hypothesis $\frac{1}{a}, \frac{1}{a+b}, \frac{1}{a+2b}, \frac{1}{a+3b}, \frac{1}{a+4b}, \frac{1}{a+5b}$ etc.

unde cum sit $\omega = 0$ habebimus :

$$A \operatorname{tang} \frac{1}{a} = A \operatorname{tang} \frac{b}{aa + ab + 1} + A \operatorname{tang} \frac{b}{aa + 3ab + 2bb + 1}$$

$$+ A \operatorname{tang} \frac{b}{aa + 5ab + 4bb + 1} + A \operatorname{tang} \frac{b}{aa + 7ab + 6bb + 1} + \dots$$

in infinitum.

Ac si singulis illis fractionibus communem tribuamus numeratorem c , erit simili modo

$$A \operatorname{tang} \frac{c}{a} = A \operatorname{tang} \frac{bc}{a(a+b) + cc} + A \operatorname{tang} \frac{bc}{(a+b)(a+2b) + cc}$$

$$+ A \operatorname{tang} \frac{bc}{(a+2b)(a+3b) + cc} + A \operatorname{tang} \frac{bc}{(a+3b)(a+4b) + cc} + \dots$$

in infinitum.

Hinc praecipue notari merentur casus, quibus numerator horum tangentium sit unitas, quod euenit, si fuerit vel $bc = 1$, vel si denominatores singuli per bc sicut diuisibiles.

4. Vtroque casu capi oportet vel $c = 1$, vel $c = 2$; ac si pro priori sumatur $b = 1$, prodibit

$$A \operatorname{tang} \frac{1}{a} = A \operatorname{tang} \frac{1}{aa + a + 1} + A \operatorname{tang} \frac{1}{aa + 3a + 3}$$

$$+ A \operatorname{tang} \frac{1}{aa + 5a + 7} + A \operatorname{tang} \frac{1}{aa + 7a + 13}$$

cuius

cujus terminus in genere est $A \operatorname{tang.} \frac{1}{aa + (2a-1)x + xx - x + 1}$
 vnde pro simplicioribus valoribus ipsius a nascuntur hae series :

$$A \operatorname{tang.} \frac{1}{2} = A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} \\
 + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + \text{etc.}$$

$$A \operatorname{tang.} 1 = A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} \\
 + A \operatorname{tang.} \frac{1}{2} + \text{etc.}$$

$$A \operatorname{tang.} \frac{3}{2} = A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} \\
 + A \operatorname{tang.} \frac{1}{2} + \text{etc.}$$

$$A \operatorname{tang.} \frac{5}{2} = A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} \\
 + A \operatorname{tang.} \frac{1}{2} + \text{etc.}$$

$$A \operatorname{tang.} \frac{7}{2} = A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} \\
 + A \operatorname{tang.} \frac{1}{2} + \text{etc.}$$

etc.

5. Pro b alios numeros capere non licet, nisi qui sint diuifores ipsius $aa + 1$, vnde si $b = 2$, necesse est sit a numerus impar: hincque sequentes nascuntur series:

$$A \operatorname{tang.} 1 = A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} \\
 + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + \text{etc.}$$

$$A \operatorname{tang.} \frac{1}{2} = A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} \\
 + A \operatorname{tang.} \frac{1}{2} + \text{etc.}$$

$$A \operatorname{tang.} \frac{3}{2} = A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} \\
 + A \operatorname{tang.} \frac{1}{2} + \text{etc.}$$

$$A \operatorname{tang.} \frac{5}{2} = A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} + A \operatorname{tang.} \frac{1}{2} \\
 + A \operatorname{tang.} \frac{1}{2} + \text{etc.}$$

etc.

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6. Si statuatur $b=5$, sumi debet $u=5n+2$; hinc fit:

$$A \operatorname{tang.} \frac{1}{5} = A \operatorname{tang.} \frac{1}{5} + A \operatorname{tang.} \frac{1}{17} + A \operatorname{tang.} \frac{1}{41} + A \operatorname{tang.} \frac{1}{71} \\ + A \operatorname{tang.} \frac{1}{119} + A \operatorname{tang.} \frac{1}{174} + \text{etc.}$$

$$A \operatorname{tang.} \frac{1}{10} = A \operatorname{tang.} \frac{1}{10} + A \operatorname{tang.} \frac{1}{31} + A \operatorname{tang.} \frac{1}{47} + A \operatorname{tang.} \frac{1}{63} \\ + A \operatorname{tang.} \frac{1}{79} + A \operatorname{tang.} \frac{1}{95} + \text{etc.}$$

$$A \operatorname{tang.} \frac{1}{15} = A \operatorname{tang.} \frac{1}{15} + A \operatorname{tang.} \frac{1}{41} + A \operatorname{tang.} \frac{1}{71} + A \operatorname{tang.} \frac{1}{101} \\ + A \operatorname{tang.} \frac{1}{131} + \text{etc.}$$

$$A \operatorname{tang.} \frac{1}{20} = A \operatorname{tang.} \frac{1}{20} + A \operatorname{tang.} \frac{1}{47} + A \operatorname{tang.} \frac{1}{63} + A \operatorname{tang.} \frac{1}{79} \\ + A \operatorname{tang.} \frac{1}{95} + \text{etc.}$$

Primae terminus generalis est $A \operatorname{tang.} \frac{1}{5xx-x-1}$, secundae $A \operatorname{tang.} \frac{1}{5xx+x-1}$, sequentes autem ex prioribus facile deducuntur, quas ideo posthac omittemus.

7. Si $b=10$, sumi debet $a=10n+3$, unde oriuntur hae duae series:

$$A \operatorname{tang.} \frac{1}{10} = A \operatorname{tang.} \frac{1}{10} + A \operatorname{tang.} \frac{1}{20} + A \operatorname{tang.} \frac{1}{30} + A \operatorname{tang.} \frac{1}{40} \\ + A \operatorname{tang.} \frac{1}{50} + \text{etc.}$$

termino generali existente $A \operatorname{tang.} \frac{1}{10xx-x-2}$

$$A \operatorname{tang.} \frac{1}{20} = A \operatorname{tang.} \frac{1}{20} + A \operatorname{tang.} \frac{1}{40} + A \operatorname{tang.} \frac{1}{60} + A \operatorname{tang.} \frac{1}{80} \\ + A \operatorname{tang.} \frac{1}{100} + \text{etc.}$$

termino generali existente $A \operatorname{tang.} \frac{1}{10xx+x-2}$

8. Simili modo posito $b=13$, et $a=13n+5$, prodibunt

$$A \operatorname{tang.} \frac{1}{13} = A \operatorname{tang.} \frac{1}{13} + A \operatorname{tang.} \frac{1}{26} + A \operatorname{tang.} \frac{1}{39} + A \operatorname{tang.} \frac{1}{52} \\ + A \operatorname{tang.} \frac{1}{65} + \text{etc.}$$

termi-

termino generali existente: $A \operatorname{tang} \frac{r}{13xx - 3x - 5}$

$$A \operatorname{tang} \frac{1}{4} = A \operatorname{tang} \frac{1}{13} + A \operatorname{tang} \frac{1}{33} + A \operatorname{tang} \frac{1}{133} + A \operatorname{tang} \frac{1}{517} + \text{etc.}$$

termino generali existente: $A \operatorname{tang} \frac{1}{13xx + 3x - 5}$

9. Sit $b = 17$, capiaturque $a = 17n + 4$, ac prodibunt

$$A \operatorname{tang} \frac{1}{4} = A \operatorname{tang} \frac{1}{5} + A \operatorname{tang} \frac{1}{17} + A \operatorname{tang} \frac{1}{133} + A \operatorname{tang} \frac{1}{229} + A \operatorname{tang} \frac{1}{377} + \text{etc.}$$

termino generali existente: $A \operatorname{tang} \frac{1}{17xx - 9x - 5}$

$$A \operatorname{tang} \frac{1}{13} = A \operatorname{tang} \frac{1}{53} + A \operatorname{tang} \frac{1}{133} + A \operatorname{tang} \frac{1}{177} + A \operatorname{tang} \frac{1}{353} + A \operatorname{tang} \frac{1}{487} + \text{etc.}$$

termino generali existente: $A \operatorname{tang} \frac{1}{17xx + 9x - 5}$

10. Sit $b = 25$, captoque $a = 25n + 7$, erit:

$$A \operatorname{tang} \frac{1}{7} = A \operatorname{tang} \frac{1}{9} + A \operatorname{tang} \frac{1}{25} + A \operatorname{tang} \frac{1}{127} + A \operatorname{tang} \frac{1}{319} + A \operatorname{tang} \frac{1}{575} + \text{etc.}$$

termino generali existente: $A \operatorname{tang} \frac{1}{25xx - 11x - 5}$

$$A \operatorname{tang} \frac{1}{13} = A \operatorname{tang} \frac{1}{53} + A \operatorname{tang} \frac{1}{117} + A \operatorname{tang} \frac{1}{353} + A \operatorname{tang} \frac{1}{479} + A \operatorname{tang} \frac{1}{675} + \text{etc.}$$

termino generali existente: $A \operatorname{tang} \frac{1}{25xx + 11x - 5}$

11. Hinc iam in genere colligere poterimus, si fuerit a numerus quicumque, fitque $aa + 1 = mn$, fore:

$$A \operatorname{tang} \frac{1}{a} = A \operatorname{tang} \frac{1}{a+m} + A \operatorname{tang} \frac{1}{5a+m+2n} + A \operatorname{tang} \frac{1}{9a+m+6n} + A \operatorname{tang} \frac{1}{13a+m+10n} + \text{etc.}$$

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termino generali existente $A \text{ tang. } \frac{x}{nxx + (2a-n)x - a + m}$

feu hoc modo: $A \text{ tang. } \frac{x}{n(x-1) + a(2x-1) + m}$

Si ergo summam huius seriei infinitae per $\int A \text{ tang. } \frac{x}{nxx + (2a-n)x - a + m}$ indicemus, consequemur hoc Theorema:

$$\int A \text{ tang. } \frac{x}{nxx + (2a-n)x - a + m} = A \text{ tang. } \frac{x}{a}, \text{ existente } aa + 1 = mm.$$

12. Videamus ergo, quibus casibus series, cuius terminus generalis est $A \text{ tang. } \frac{x}{Lxx + Mx + N}$, summari queat; et comparatione instituta deprehendemus, hoc fieri posse, quoties fuerit $MM + 4 = LL + 4LN$; ideoque

$$\text{vel } L = -2N + \sqrt{(MM + 4NN + 4)}, \text{ vel } M = \sqrt{(LL + 4LN - 4)}$$

$$\text{vel } N = \frac{MM - LL + 4}{4L}.$$

Atque si haec relatio inter coefficientes L ; M N locum habuerit, erit $\int A \text{ tang. } \frac{x}{Lxx + Mx + N} = A \text{ tang. } \frac{x}{L+M}$ siue erit

$$A \text{ tang. } \frac{x}{L+M} = A \text{ tang. } \frac{x}{L+M+N} + A \text{ tang. } \frac{x}{L + \frac{1}{2}M + \frac{1}{2}N} \\ + A \text{ tang. } \frac{x}{\frac{1}{2}L + \frac{1}{2}M + N} + A \text{ tang. } \frac{x}{\frac{1}{6}L + \frac{1}{4}M + N} + \text{etc.}$$

Hypothesis 13. Cum haec ex progressionem harmonica sequantur, pro α , β , γ , δ etc. sumamus hanc seriem:

$$\frac{c}{a}; \frac{c+d}{a+b}; \frac{c+2d}{a+2b}; \frac{c+3d}{a+3b}; \frac{c+4d}{a+4b}; \text{ etc.}$$

vnde cum sit $\alpha = \frac{c}{a}$ et $\omega = \frac{d}{b}$, hanc adipiscemur summationem:

$$A \text{ tang. } \frac{bc-ad}{ab+ca} = A \text{ tang. } \frac{bc-ad}{a(a+b)+d(c+d)} + A \text{ tang. } \frac{bc-ad}{(a+b)(a+2b)+(c+d)(c+2d)} + \text{etc.}$$

cuius

cuius terminus generalis est $A \text{ tang. } \frac{bc-ad}{(a+b(x-1))(a+bx)+(c+d(x-1))(c+dx)}$

14. Vt iam numerator huius tangentis unitati fiat aequalis, vel esse debet $bc-ad=1$, vel denominator per numeratorem $bc-ad$ diuisibilis, quod posterius euenit sumendo:

$$a=pr+qs; c=ps-qr; b=pt+qu; d=pu-qt$$

dum sit $st-ru=1$.

tum enim fiet:

$A \text{ tang. } \frac{1}{rt+su} = f A \text{ tang. } \frac{1}{rr+ss+(rt+su)(xx-1)+(t+uu)x(x-1)}$
 Verum haec formula cum praecedente ita conuenit, ut hinc nullae nouae series eliciantur.

15. Fractiones autem continuae admodum idoneos praebent valores pro numeris $\alpha, \beta, \gamma, \delta$, etc. assumendos. Si enim fuerit:

Hypothesis
III.

$$x = a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \frac{1}{e + \frac{1}{f + \frac{1}{g + \text{etc.}}}}}}}$$

hinc sequens series fractionum constituitur, pro $\alpha, \beta, \gamma, \delta$ etc. capiendarum:

$$\frac{\alpha}{\beta}; \frac{\alpha\beta+\gamma}{\beta}; \frac{\alpha\beta\gamma+\delta}{\beta\gamma+\delta}; \frac{\alpha\beta\gamma\delta+\epsilon}{\beta\gamma\delta+\delta+\epsilon}; \text{ etc.}$$

qua-

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quarum vltima ipsi valori ipsius z est aequalis.

16. Quodsi hic eam notandi formam, quam in Algorithmi singularis specimine tradidi, introducamus, hae fractiones ita exprimentur:

$$\frac{a}{1}; \frac{\beta}{1}; \frac{\gamma}{(b)}; \frac{\delta}{(b, c)}; \frac{\epsilon}{(b, c, d)}; \frac{\zeta}{(b, c, d, e)} \text{ etc.}$$

vti notari oportet, esse

$$(a, b) = a(b) + 1 = b(a) + 1$$

$$(a, b, c) = a(b, c) + (c) = c(a, b) + (a)$$

$$(a, b, c, d) = a(b, c, d) + (c, d) = d(a, b, c) + (a, b)$$

$$(a, b, c, d, e) = a(b, c, d, e) + (c, d, e) = e(a, b, c, d) + (a, b, c) \text{ etc.}$$

17. Cum igitur sit $\alpha = \frac{1}{z}$, et $\omega = z$, erit

$$A \text{ tang. } \frac{1}{z} = A \text{ tang. } \frac{1}{\beta} + A \text{ tang. } \frac{\beta - \gamma}{\beta \gamma + 1} + A \text{ tang. } \frac{\gamma - \delta}{\gamma \delta + 1} + A \text{ tang. } \frac{\delta - \epsilon}{\delta \epsilon + 1} + \text{ etc.}$$

vbi est $\beta = a$; et $\frac{\beta - \gamma}{\beta \gamma + 1} = \frac{-1}{(a)(a, b) + (b)}$; tum vero

$$\frac{\gamma - \delta}{\gamma \delta + 1} = \frac{+1}{(a, b)(a, b, c) + (b)(b, c)}$$

$$\frac{\delta - \epsilon}{\delta \epsilon + 1} = \frac{-1}{(a, b, c)(a, b, c, d) + (b, c)(b, c, d)}$$

$$\frac{\epsilon - \zeta}{\epsilon \zeta + 1} = \frac{-1}{(a, b, c, d)(a, b, c, d, e) + (b, c, d)(b, c, d, e)}$$

etc.

ita vt omnes numeratores iam sint, vel $+1$, vel -1 .

18. Quodsi breuitatis gratia loco illius serici scribamus

$$\frac{a}{1}; \frac{b}{1}; \frac{c}{\beta}; \frac{d}{\gamma}; \frac{e}{\delta}; \frac{f}{\epsilon}; \frac{g}{\zeta}; \text{ etc}$$

vt

fin autem tantum fractiones alternae sumantur

$$\frac{1}{6}, \frac{2}{3}, \frac{7}{4}, \frac{26}{15}, \frac{97}{30}, \frac{362}{60} \text{ etc.}$$

obtinemus:

$A \operatorname{tang} \frac{1}{\sqrt{3}} = A \operatorname{tang} \frac{1}{2} + A \operatorname{tang} \frac{1}{4} + A \operatorname{tang} \frac{1}{8} + A \operatorname{tang} \frac{1}{16} + A \operatorname{tang} \frac{1}{32} + \text{etc.}$
 cuius denominatores omnes sunt duplicata quadrata; scilicet

$$A \operatorname{tang} \frac{1}{\sqrt{3}} = A \operatorname{tang} \frac{1}{2 \cdot 1^2} + A \operatorname{tang} \frac{1}{2 \cdot 2^2} + A \operatorname{tang} \frac{1}{2 \cdot 3^2} + A \operatorname{tang} \frac{1}{2 \cdot 4^2} + \text{etc.}$$

Sumtis autem alteris alternis, prodit ob $A \operatorname{tang} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$
 et $A \operatorname{tang} 1 = \frac{\pi}{4}$, $\frac{\pi}{12} = A \operatorname{tang} \frac{1}{2} + A \operatorname{tang} \frac{1}{3} + A \operatorname{tang} \frac{1}{6} + A \operatorname{tang} \frac{1}{12} + \text{etc.}$

qui denominatores sunt quadrata, quorum radices hanc progressionem constituunt:

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \dots \quad x$$

$$2, 8, 30, 112, 418, \dots \dots \frac{(2+\sqrt{3})^x - (2-\sqrt{3})^x}{\sqrt{3}}$$

21. Cum autem fractiones continuas ad huiusmodi series arcuum deduxerint, vicissim summa talis seriei ope fractionis continuas exhiberi poterit, quod commodissime sequenti modo praestabitur.

Sit $A \operatorname{tang} \frac{1}{2} = A \operatorname{tang} \frac{1}{a} - A \operatorname{tang} \frac{1}{b} + A \operatorname{tang} \frac{1}{c} - A \operatorname{tang} \frac{1}{d} + A \operatorname{tang} \frac{1}{e} - A \operatorname{tang} \frac{1}{f} \text{ etc.}$

ac ponatur

$$A \operatorname{tang} \frac{1}{2} = A \operatorname{tang} \frac{1}{a} - A \operatorname{tang} \frac{1}{b} \text{ erit } z = \frac{aB + 1}{B - a} = a + \frac{aB + 1}{-a + B}$$

$$A \operatorname{tang} \frac{1}{2} = A \operatorname{tang} \frac{1}{b} - A \operatorname{tang} \frac{1}{c} \text{ erit } B = \frac{bC + 1}{C - b} = b + \frac{bC + 1}{-b + C}$$

$$A \operatorname{tang} \frac{1}{2} = A \operatorname{tang} \frac{1}{c} - A \operatorname{tang} \frac{1}{d} \text{ erit } C = \frac{cD + 1}{D - c} = c + \frac{cD + 1}{-c + D}$$

$$A \operatorname{tang} \frac{1}{2} = A \operatorname{tang} \frac{1}{d} - A \operatorname{tang} \frac{1}{e} \text{ erit } D = \frac{dE + 1}{E - d} = d + \frac{dE + 1}{-d + E}$$

etc. etc.

G 2

hinc

hinc ergo colligendo habebitur per fractionem continuam :

$$z = a + \frac{aa + 1}{-a + b + \frac{bb + 1}{-b + c + \frac{cc + 1}{-c + d + \frac{dd + 1}{-d + e + \frac{ee + 1}{-e + f + \text{etc.}}}}}}$$

vnde valor ipsius z definitur.