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Annotationes in locum quendam Cartesii ad circuli quadraturam spectantem

Leonhard Euler

ANNOTATIONES IN LOCVM QVENDAM CARTESII AD CIRCVLI QVADRATVRAM SPECTANTEM.

Auctore
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In excerptis ex Manuscriptis *Cartesii* paucis quidem verbis refertur constructio quaedam geometrica promtissime ad circuli veram dimensionem appropinquans, sed quae siue *Cartesius* ipse eam inuenerit, siue ab alio habuerit communicatam, acutissimum inuentoris ingenium, illo praesertim tempore, luculenter declarat. Qui deinceps hoc idem argumentum pertractarunt, quantum equidem memini, nullam huius eximiae constructionis mentionem faciunt, vt periculum sit, ne tandem penitus obliuione obruatur. Demonstratio quidem, quae non adiuncta repetitur, haud difficulter suppletur; verum non solum elegantia constructionis vberiorem explicationem meretur, sed etiam tam insignes conclusiones inde deduci possunt, quae per se omni attentione dignae videntur. Pulcherrima autem haec constructio ipsis *Cartesii* verbis ita est proposita:

„Ad quadrandum circulum nihil aptius inuenio, Tab. I.
„quam si dato quadrato bf adiungatur rectangulum cg Fig. 1.
„comprehensum sub lineis ac et bc, quod sit aequale
„quartae parti quadrati bf: item rectangulum db fa-
„ctum ex lineis da, dc, aequale quartae parti praece-

 „dentis;

„dentis; et eodem modo rectangulum *ei*, atque alia „infinita vsque ad *x*: et erit haec linea *ax* diameter „circuli, cuius circumferentia aequalis est circumferen„tiae quadrati *bf*.

Vis igitur huius constructionis in hoc consistit, vt continua appositione istiusmodi rectangulorum *cg*, *db*, *ei*, etc. quorum anguli superiores dextri in diagonalem quadrati productam cadunt, tandem ad punctum *x* perueniatur, quo terminatur diameter circuli *ax*, cuius peripheria aequalis est perimetro quadrati *bf*, seu quadruplo rectae *ab*.

Cum horum rectangulorum quodque aequetur parti quartae praecedentis, iam ipse *Cartesius* obseruat, summam omnium horum rectangulorum aequalem fore parti tertiae quadrati *bf*; quod quidem manifestum est, cum huius seriei $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} +$ etc. in infinitum continuatae summa sit $= \frac{1}{3}$.

Praeterea etiam *Cartesius* indicat rationem, cui haec constructio innititur; concipit scilicet polygona regularia 8, 16, 32, 64 etc. laterum, quorum perimetri sint inter se aequales simulque perimetro quadrati *bf*. Iam cum *ab* sit diameter circuli quadrato inscripti, ita affirmat fore *ac* diametrum circuli octogono inscripti, tum vero *ad* diametrum circuli 16gono, *ae* 32gono inscripti, et ita porro. Vnde liquet *ax* fore diametrum circuli polygono infinitorum laterum regulari inscripti, ideoque eius peripheriam aequari perimetro quadrati.

Quo facilius demonstrationem huius constructionis adornem, obseruo, quae hic de diametris circulorum dicuntur, etiam valere pro radiis, ita vt *ab*, *ac*, *ad*,

ae

ae etc. ſpectari poſſint tanquam radii circulorum, quibus ſi circumſcribantur polygona regularia 4, 8, 16, 32 etc. laterum, eorum perimetri futurae ſint inter ſe aequales.

Problema.

Dato circulo, cui polygonum [illegible] que ſit circumſcriptum, inuenire circulum alium, cui ſi polygonum regulare duplo plurium laterum circumſcribatur, perimeter huius polygoni aequalis ſit futura perimetro illius polygoni.

Solutio.

Sit ENM circulus datus et EP ſemilatus polygoni ipſi circumſcripti, centro exiſtente in C; CF autem ſit radius circuli quaeſiti, et FQ ſemilatus polygoni ipſi circumſcribendi. Neceſſe ergo eſt, vt ſit FQ ſemiſſis ipſius EP, et angulus FCQ ſemiſſis anguli ECP. Quare recta CQ angulum ECP, et recta QO ipſi CE parallela lineam EP biſecabit. Cum nunc Fig. 2.

ſit $EV : CE = FQ : CF$

et $EV : CE = EP : CE + CP$

erit $FQ : CF = EP : CE + CP$

ſed quia $FQ = \frac{1}{2}EP$, erit etiam $CF = \frac{1}{2}(CE + CP)$. Hinc auferatur CF, et habebitur $EF = \frac{1}{2}(CP - CE)$ ex quo erit rectangulum $CF.EF = \frac{1}{4}(CP^2 - CE^2) = \frac{1}{4}EP^2$ ideoque punctum F ita definiri debet, vt ſit rectangulum, ſub CF et EF comprehenſum, aequale parti quartae quadrati rectae EP, ſeu ipſi quadrato rectae FQ.

Coroll. 1.

Coroll. 1.

Cum sit $CF.EF = FQ^2$ erit $CF : FQ = FQ : EF$, vnde ducta recta QE, fiet triangulum FQE simile triangulo FCQ, vel ECV, ideoque angulus FQE aequalis angulo ECV.

Coroll. 2.

Cum sit $CE : EV = EO : EF$, punctum F etiam ita definiri poterit: ex O ducatur recta ad CV productam normalis, eaque basi CE in F occurret.

Coroll. 3.

Si polygonum circulo ENM circumscriptum sit n laterum, erit angulus $ECP = \frac{\pi}{n}$, denotante π mensuram duorum angulorum rectorum; et angulus $FCQ = \frac{\pi}{2n}$. Hinc si radius $CE = r$, erit $EP = r \,\text{tang.} \frac{\pi}{n}$ et $FQ = \frac{1}{2} r \,\text{tang.} \frac{\pi}{n}$.

Coroll. 4.

Iam quia angulus $FQE = \frac{\pi}{2n}$ erit $EF = FQ \,\text{tang.} \frac{\pi}{2n} = \frac{1}{2} r \,\text{tang.} \frac{\pi}{n} \,\text{tang.} \frac{\pi}{2n}$. Verum si vocemus $CF = s$, erit $FQ = s \,\text{tang.} \frac{\pi}{2n}$, vnde ob $FQ = \frac{1}{2} r \,\text{tang.} \frac{\pi}{n}$ fiet $s = \frac{1}{2} r \,\text{tang.} \frac{\pi}{n} \,\text{cot.} \frac{\pi}{2n}$.

Demonstratio Constructionis Cartesianae.

Fig. 3. Sit iam CE radius circuli quadrato inscripti, CF octogono inscripti, CG polygono regulari 16 laterum, CH

CH polygono 32 laterum et ita porro. Sit porro EP ſemilatus quadrati, FQ ſemilatus octogoni, GR polygoni 16, HS polygoni 32 laterum, etc. et quia haec polygona eiusdem perimetri aſſumuntur, erit $FQ = \frac{1}{2}EP$; $GR = \frac{1}{2}FQ = \frac{1}{4}EP$; $HS = \frac{1}{2}GR = \frac{1}{4}FQ = \frac{1}{8}EP$, etc. Iam ex problemate praemiſſo eſt $CF.\,EF = \frac{1}{4}EP^2 = FQ^2$; tum vero ex eodem ſimili modo

$$CG.\,FG = \tfrac{1}{4}FQ^2 = \tfrac{1}{4}CF.\,EF = GR^2$$
$$CH.\,GH = \tfrac{1}{4}GR^2 = \tfrac{1}{4}CG.\,FG = HS^2 \text{ etc.}$$

ſicque puncta F, G, H etc. eodem plane modo determinantur, vti habet conſtructio *Carteſiana*; et quia interualla EF, FG, GH etc. continuo fiunt minora, ſatis promte ad punctum vltimum x appropinquatur, eritque Cx radius circuli, cuius peripheria aequatur perimetro polygonorum praecedentium, ideoque rectae EP octies ſumtae. Q. E. D.

Coroll. 1.

Si ponatur $CE = a$, $CF = b$, $CG = c$, $CH = d$, etc. progreſſio harum quantitatum ita eſt comparata, vt ſit ob $EP = a$

$$b(b-a) = \tfrac{1}{4}aa;\quad c(c-b) = \tfrac{1}{4}b(b-a);\quad d(d-c) = \tfrac{1}{4}c(c-b) \text{ etc.}$$

ideoque

$$b = \frac{a + \sqrt{2aa}}{2};\quad c = \frac{b + \sqrt{(2bb - ab)}}{2};\quad d = \frac{c + \sqrt{(2cc - bc)}}{2} \text{ etc.}$$

et harum quantitatum infiniteſima eſt radius circuli cuius peripheria eſt $= 8a$.

Coroll. 2.

Cum sit angulus ECP semirectus, seu $ECP = \frac{\pi}{4}$, erunt anguli $FCQ = \frac{\pi}{8}$; $GCR = \frac{\pi}{16}$; $HCS = \frac{\pi}{32}$, etc. Quare ob $EP = a$; $FQ = \frac{1}{2}a$; $GR = \frac{1}{4}a$; $HS = \frac{1}{8}a$ etc. erit per cotangentes

$CE = a\cot.\frac{\pi}{4}$; $CF = \frac{1}{2}a\cot.\frac{\pi}{8}$; $CG = \frac{1}{4}a\cot.\frac{\pi}{16}$; $CH = \frac{1}{8}a\cot.\frac{\pi}{32}$ etc.

Vnde denotante n numerum infinitum, fit harum linearum vltima $= \frac{1}{n}a\cot.\frac{\pi}{4n}$.

Coroll. 3.

Sed $\cot.\frac{\pi}{4n} = 1 : \text{tang}.\frac{\pi}{4n}$; et quia angulus $\frac{\pi}{4n}$ est infinite paruus, erit $\text{tang}.\frac{\pi}{4n} = \frac{\pi}{4n}$, ideoque $\cot.\frac{\pi}{4n} = \frac{4n}{\pi}$. Quare linearum illarum vltima fit $= \frac{4a}{\pi}$, quo radio si circulus describatur, erit eius peripheria $= 2\pi.\frac{4a}{\pi} = 8a$.

Coroll. 4.

Deinde quia ex coroll. 4. praec. probl. est $EF = FQ\,\text{tang}.\,FCQ$, erit ob eandem rationem:

$FG = GR.\,\text{tang}.\,GCR$; $GH = HS.\,\text{tang}.\,HCS$ etc.

vnde haec interualla sequenti modo exprimentur:

$EF = \frac{1}{2}a\,\text{tang}.\frac{\pi}{8}$; $FG = \frac{1}{4}a\,\text{tang}.\frac{\pi}{16}$; $GH = \frac{1}{8}a\,\text{tang}.\frac{\pi}{32}$ etc.

antecedens vero ad analogiam $CE = a\,\text{tang}.\frac{\pi}{4} = a$.

Coroll. 5.

His cum praecedentibus collatis nanciscemur:

$$CF = a(\text{tang}.\tfrac{\pi}{4} + \tfrac{1}{2}\text{tang}.\tfrac{\pi}{8}) = \tfrac{1}{2}a\cot.\tfrac{\pi}{8}$$
$$CG = a(\text{tang}.\tfrac{\pi}{4} + \tfrac{1}{2}\text{tang}.\tfrac{\pi}{8} + \tfrac{1}{4}\text{tang}.\tfrac{\pi}{16}) = \tfrac{1}{4}a\cot.\tfrac{\pi}{16}$$
$$CH = a(\text{tang}.\tfrac{\pi}{4} + \tfrac{1}{2}\text{tang}.\tfrac{\pi}{8} + \tfrac{1}{4}\text{tang}.\tfrac{\pi}{16} + \tfrac{1}{8}\text{tang}.\tfrac{\pi}{32}) = \tfrac{1}{8}a\cot.\tfrac{\pi}{32}$$

etc.

sicque

ſicque omnium huiusmodi progreſſionum ſummae expedite aſſignari poſſunt.

Coroll. 6.

In infinitum ergo progrediendo obtinebimus ſummationem huius ſeriei:

$$\text{tang.}\,\tfrac{\pi}{4} + \tfrac{1}{2}\,\text{tang.}\,\tfrac{\pi}{8} + \tfrac{1}{4}\,\text{tang.}\,\tfrac{\pi}{16} + \tfrac{1}{8}\,\text{tang.}\,\tfrac{\pi}{32} + \text{etc.} = \tfrac{4}{\pi}$$

quae ergo per quadraturam circuli determinatur. Hinc occaſionem arripio ſequens problema ſoluendi.

Problema.

Denotante Φ arcum quemcunque circuli cuius radius $= 1$, inuenire ſummam huius ſeriei infinitae:

$$\text{tang.}\,\Phi + \tfrac{1}{2}\,\text{tang.}\,\tfrac{1}{2}\Phi + \tfrac{1}{4}\,\text{tang.}\,\tfrac{1}{4}\Phi + \tfrac{1}{8}\,\text{tang}\,\tfrac{1}{8}\Phi + \tfrac{1}{16}\,\text{tang.}\,\tfrac{1}{16}\Phi \text{ etc.}$$

Solutio.

Si in fig. 2. vti ſupra eſt conſtructa, ponatur angulus $ECP = \Phi$, erit $FCQ = \tfrac{1}{2}\Phi$: iam poſito $FQ = 1$ erit $EP = 2$, hincque $CE = 2\,\text{cot.}\,\Phi$; $CF = \text{cot.}\,\tfrac{1}{2}\Phi$ et $EF = \text{tang.}\,\tfrac{1}{2}\Phi$, ex quo habetur: Tab. I. Fig. 2.

$$2\,\text{cot.}\,\Phi = \text{cot.}\,\tfrac{1}{2}\Phi - \text{tang.}\,\tfrac{1}{2}\Phi \text{ et } \text{tang.}\,\tfrac{1}{2}\Phi = \text{cot.}\,\tfrac{1}{2}\Phi - 2\,\text{cot.}\,\Phi$$

eodemque modo $\text{tang.}\,\Phi = \text{cot.}\,\Phi - 2\,\text{cot.}\,2\Phi$. Collocentur hi valores tangentium per cotangentes expreſſi in ſerie propoſita

$$\begin{aligned}
\text{tang.}\,\Phi &= \text{cot.}\,\Phi - 2\,\text{cot.}\,2\Phi \\
\tfrac{1}{2}\,\text{tang.}\,\tfrac{1}{2}\Phi &= \tfrac{1}{2}\text{cot}\,\tfrac{1}{2}\Phi - \text{cot.}\,\Phi \\
\tfrac{1}{4}\,\text{tang.}\,\tfrac{1}{4}\Phi &= \tfrac{1}{4}\text{cot}\,\tfrac{1}{4}\Phi - \tfrac{1}{2}\text{cot.}\,\tfrac{1}{2}\Phi \\
\tfrac{1}{8}\,\text{tang.}\,\tfrac{1}{8}\Phi &= \tfrac{1}{8}\text{cot.}\,\tfrac{1}{8}\Phi - \tfrac{1}{4}\text{cot.}\,\tfrac{1}{4}\Phi \\
&\text{etc.}
\end{aligned}$$

et colligendo consequemur:

$$\mathrm{tang.}\,\Phi = \mathrm{cot.}\,\Phi - 2\,\mathrm{cot.}\,2\Phi$$
$$\mathrm{tang.}\,\Phi + \tfrac{1}{2}\mathrm{tang}\,\tfrac{1}{2}\Phi = \tfrac{1}{2}\mathrm{cot.}\,\tfrac{1}{2}\Phi - 2\,\mathrm{cot.}\,2\Phi$$
$$\mathrm{tang}\,\Phi + \tfrac{1}{2}\mathrm{tang}\,\tfrac{1}{2}\Phi + \tfrac{1}{4}\mathrm{tang}\,\tfrac{1}{4}\Phi = \tfrac{1}{4}\mathrm{cot.}\,\tfrac{1}{4}\Phi - 2\,\mathrm{cot.}\,2\Phi$$
$$\mathrm{tang.}\,\Phi + \tfrac{1}{2}\mathrm{tang}\,\tfrac{1}{2}\Phi + \tfrac{1}{4}\mathrm{tang.}\,\tfrac{1}{4}\Phi + \tfrac{1}{8}\mathrm{tang.}\,\tfrac{1}{8}\Phi = \tfrac{1}{8}\mathrm{cot.}\,\tfrac{1}{8}\Phi - 2\,\mathrm{cot.}\,2\Phi$$
etc.

vnde in infinitum progrediendo, si n denotet numerum infinitum, quia $\mathrm{tang.}\,\frac{1}{n}\Phi = \frac{\Phi}{n}$, hincque $\mathrm{cot.}\,\frac{1}{n}\Phi = \frac{n}{\Phi}$, erit: $\frac{1}{n}\mathrm{cot}\,\frac{1}{n}\Phi = \frac{1}{\Phi}$, ideoque summa seriei propositae:

$$\mathrm{tang}\,\Phi + \tfrac{1}{2}\mathrm{tang}\,\tfrac{1}{2}\Phi + \tfrac{1}{4}\mathrm{tang}\,\tfrac{1}{4}\Phi + \tfrac{1}{8}\mathrm{tang.}\,\tfrac{1}{8}\Phi + \mathrm{etc.} = \tfrac{1}{\Phi} - 2\,\mathrm{cot.}\,2\Phi$$

Vnde si 2Φ est angulus rectus, seu $\Phi = \frac{\pi}{4}$, ob $\mathrm{cot}\,\frac{\pi}{2} = 0$ fit seriei summa $= \frac{1}{\Phi} = \frac{4}{\pi}$, qui est casus supra tractatus.

Ex hac serie plures aliae deriuari possunt non minus notitu dignae.

I. Ex eius differentiatione adipiscimur:

$$\frac{1}{\mathrm{cos.}\,\Phi^2} + \frac{1}{4\,\mathrm{cos.}\,\frac{1}{2}\Phi^2} + \frac{1}{4^2\,\mathrm{cos.}\,\frac{1}{4}\Phi^2} + \frac{1}{4^3\,\mathrm{cos}\,\frac{1}{8}\Phi^2} + \frac{1}{4^4\,\mathrm{cos}\,\frac{1}{16}\Phi}$$
$$+ \mathrm{et} = -\frac{1}{\Phi\Phi} + \frac{4}{\mathrm{sin.}\,2\Phi^2}$$

vel cum sit $\frac{1}{\mathrm{cos.}\,\Phi} = \mathrm{sec.}\,\Phi$ erit quoque

$$(\mathrm{sec.}\,\Phi)^2 + \tfrac{1}{4}(\mathrm{sec.}\,\tfrac{1}{2}\Phi)^2 + \tfrac{1}{4^2}(\mathrm{sec}\,\tfrac{1}{4}\Phi)^2 + \tfrac{1}{4^3}(\mathrm{sec.}\,\tfrac{1}{8}\Phi)^2 + \mathrm{etc.}$$
$$= \frac{1}{\mathrm{sin.}\,\Phi^2\,\mathrm{cos.}\,\Phi^2} - \frac{1}{\Phi\Phi}$$

II. Deinde ob $\mathrm{cos.}\,\Phi^2 = \frac{1+\mathrm{cos.}\,2\Phi}{2}$, et $\mathrm{sin.}\,2\Phi^2 = \frac{1-\mathrm{cos.}\,4\Phi}{2}$ erit vbique per 2 diuidendo:

$$\frac{1}{1+\mathrm{cos.}\,2\Phi} + \frac{1}{4(1+\mathrm{cos.}\,\Phi)} + \frac{1}{4^2(1+\mathrm{cos.}\,\frac{1}{2}\Phi)}$$
$$+ \frac{1}{4^3(1+\mathrm{cos.}\,\frac{1}{8}\Phi)} + \mathrm{etc.} = \frac{2}{1-\mathrm{cos.}\,4\Phi} - \frac{1}{2\Phi\Phi}.$$

seu

feu pro Φ fcribendo $\frac{1}{2}\Phi$

$$\frac{1}{1+\text{cof.}\,\Phi}+\frac{1}{4(1+\text{cof.}\,\frac{1}{2}\Phi)}+\frac{1}{4^2(1+\text{cof.}\,\frac{1}{4}\Phi)}$$

$$+\frac{1}{4^3(1+\text{cof.}\,\frac{1}{8}\Phi)}+\text{etc.}=\frac{2}{1-\text{cof.}\,2\Phi}-\frac{2}{\Phi\Phi}$$

III. Si feries inuenta per $d\Phi$ multiplicetur et integretur, ob $\int d\Phi\,\text{tang.}\,\Phi=\int\frac{d\Phi\,\text{fin.}\,\Phi}{\text{cof.}\,\Phi}=-l\,\text{cof.}\,\Phi$, et $\int 2\,d\Phi\,\text{cot.}\,2\Phi=l\,\text{fin.}\,2\Phi$, habebitur

$$-l\,\text{cof.}\,\Phi-l\,\text{cof.}\,\tfrac{1}{2}\Phi-l\,\text{cof.}\,\tfrac{1}{4}\Phi-l\,\text{cof.}\,\tfrac{1}{8}\Phi-l\,\text{cof.}\,\tfrac{1}{16}\Phi-\text{etc.}$$
$$=l\Phi-l\,\text{fin.}\,2\Phi+\text{Conft.}$$

ad quam conftantem definiendam ponamus $\Phi=0$, et quia $l\,\text{cof.}\,0=l\,1=0$, ex priori parte habemus 0, ex pofteriori vero ob fin. $2\Phi=2\Phi$, habemus $l\Phi-l\,2\Phi+\text{Conft.}=-l\,2+\text{Conft.}$ vnde Conft. $=l\,2$. Hinc ad numeros progrediendo erit:

$$\frac{1}{\text{cof.}\,\Phi\;\text{cof.}\,\frac{1}{2}\Phi\;\text{cof.}\,\frac{1}{4}\Phi\;\text{cof.}\,\frac{1}{8}\Phi\;\text{cof.}\,\frac{1}{16}\Phi\;\text{etc.}}=\frac{2\Phi}{\text{fin.}\,2\Phi}$$

IV. Cum fit $\frac{1}{\text{cof.}\,\Phi}=\text{fec.}\,\Phi$, habebitur etiam hoc Theorema pro fecantibus:

$$\text{fec.}\,\Phi\;\text{fec.}\,\tfrac{1}{2}\Phi\;\text{fec.}\,\tfrac{1}{4}\Phi\;\text{fec.}\,\tfrac{1}{8}\Phi\;\text{fec.}\,\tfrac{1}{16}\Phi\;\text{etc.}=\frac{2\Phi}{\text{fin.}\,2\Phi}$$

vnde fi ratio diametri ad peripheriam ponatur $=1:\pi$ et q denotet angulum rectum, fi ftatuamus $2\Phi=q$, $=\frac{\pi}{2}$ erit:

$$\text{fec.}\,\tfrac{1}{2}q\;\text{fec.}\,\tfrac{1}{4}q\;\text{fec.}\,\tfrac{1}{8}q\;\text{fec.}\,\tfrac{1}{16}q\;\text{fec.}\,\tfrac{1}{32}q\;\text{etc.}=\tfrac{\pi}{2}.$$

Problema.

Innenire feriem quantitatum: a, b, c, d, e, f; etc. cuius haec fit proprietas, vt fit:

$c(c-b)=\frac{1}{4}b(b-a)$; $d(d-c)=\frac{1}{4}c(-b)$; $e(e-d)=\frac{1}{4}d(d-c)$ etc.

feu vt quantitates inde deriuatae

$$b(b-a);\ c(c-b);\ d(d-c);\ e(e-d);\ f(f-e),\ \text{etc.}$$

decrefcant fecundum rationem quadruplam.

Solutio.

Cum fit $\text{tang.}\,\frac{1}{2}\Phi = \cot.\frac{1}{2}\Phi - 2\cot.\Phi$, fi multiplicemus vtrinque per $\cot.\frac{1}{2}\Phi$, ob $\text{tang}\,\frac{1}{2}\Phi\,\cot.\frac{1}{2}\Phi = 1$ erit $\cot.\frac{1}{2}\Phi(\cot.\frac{1}{2}\Phi - 2\cot.\Phi) = 1$. Statuatur ergo $a = r\cot.\Phi$; $b = \frac{1}{2}r\cot.\frac{1}{2}\Phi$; $c = \frac{1}{4}r\cot.\frac{1}{4}\Phi$; $d = \frac{1}{8}r\cot.\frac{1}{8}\Phi$, etc. eritque

$$\frac{2b}{r}\left(\frac{2b}{r} - \frac{2a}{r}\right) = 1 \quad \text{hinc } b(b-a) = \frac{rr}{4}$$

$$\frac{4c}{r}\left(\frac{ac}{r} - \frac{4b}{r}\right) = 1 \quad \text{hinc } c(c-b) = \frac{rr}{4^2}$$

$$\frac{8d}{r}\left(\frac{8d}{r} - \frac{8c}{r}\right) = 1 \quad \text{hinc } d(d-c) = \frac{rr}{4^3}$$

etc.

Quare haec feries

$a = r\cot\,\Phi$; $b = \frac{1}{2}r\cot.\frac{1}{2}\Phi$; $c = \frac{1}{4}r\cot.\frac{1}{4}\Phi$; $d = \frac{1}{8}r\cot.\frac{1}{8}\Phi$; etc.

hanc habet proprietatem, vt quantitates inde formatae

$$b(b-a);\ c(c-b);\ d(d-c);\ e(e-d);\ \text{etc.}$$

in ratione quadrupla decrefcant.

Coroll. 1.

Coroll. 1.

Datis duobus terminis primis a et b reliqui c, d, e, f inde successiue ita determinantur, vt sit:
$c = \frac{b + \sqrt{(2bb + ab)}}{2}$; $d = \frac{c + \sqrt{(2cc - bc)}}{2}$; $e = \frac{d + \sqrt{(2dd - cd)}}{2}$ etc.
ideoque binis terminis initialibus pro lubitu assumtis, tota series ope harum formularum exhiberi potest.

Coroll. 2.

Datis autem terminis a et b, inde angulus Φ cum quantitate r ita definitur, vt fit:

$$\text{tang.}\,\Phi = \frac{2\sqrt{(bb - ab)}}{a} \text{ et } r = 2\sqrt{(bb - ab)}$$

vnde inuento angulo Φ reliqui termini etiam ita exprimuntur, vt sit:

$$c = \tfrac{1}{4} r \cot.\, \tfrac{1}{4}\Phi;\ d = \tfrac{1}{8} r \cot.\, \tfrac{1}{8}\Phi;\ e = \tfrac{1}{16} r \cot.\, \tfrac{1}{16}\Phi \text{ etc.}$$

Coroll. 3.

Hinc istius seriei termini infinitesimi fient $= \frac{r}{\Phi}$, ad quem valorem termini seriei satis cito conuergunt. Quaeratur scilicet in circulo radii $= 1$, arcus cuius tangens $= \frac{2\sqrt{(bb - ab)}}{a}$, qui arcus sit $= \Phi$, et seriei nostrae termini infinitesimi erunt $= \frac{2\sqrt{(bb - ab)}}{\Phi}$.

Scholion.

Caeterum hic monuisse iuuabit puncta P, Q, R, S, x sita esse in curua quadratrice veterum, propterea quod applicatae EP, FQ, GR, HS eandem inter se rationem tenent, quam anguli ECP, Tab. I. Fig. 3.
FCQ,

FCQ, GCR, HCS etc. Et quoniam x, vbi haec curua in basin incidit, iam olim circuli quadraturam indicare est inuentum, vnde ei istud nomen est inditum, constructio Cartesii cum hac veterum quadratura egregie quidem conuenit; sed multo commodius et accuratius puncta E, F, G, H etc. successiue praebet, quam a continua bisectione angulorum expectari queat.

DEMON-