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Using Composition Techniques to Improve Classroom Instruction and Students' Understanding of Proof

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Abstract

This paper describes an effort to incorporate standard composition exercises into a sophomore-level discrete mathematics class. It provides an example of how peer review can be integrated with a mathematical curriculum through the writing of proofs.

Key Words: peer review, critical thinking, mathematical writing, composition techniques

Introduction

The ability to write mathematically is an important part of becoming a good mathematician. Yet mathematical writing is rarely taught by itself at the college level. Students are expected to absorb good writing techniques while simultaneously trying to understand course material, such as in Discrete Mathematics, Linear Algebra, or Introductory Analysis. Recently, classes have sprung up which deal specifically with writing proofs mathematically; often, such a class is required before students can enroll in upper-level courses. One disadvantage with these courses is that they are often disconnected from fundamental aspects of composition. However, professional mathematicians "are expected to produce coherent, sometimes lengthy documents that incorporate both symbolic and 'natural' language [3, p.1]." Therefore, budding mathematicians are left to their own devices to learn how to write mathematically, since they receive little, if any, formal training in composition at the college level.

In Spring 2000, the author taught a discrete mathematics course that familiarized students with logic and different methods of proof while at the same time covering material on graph theory, sequences, matrices, relations, and the Fundamental Theorem of Arithmetic. Homework assignments regularly required the students to compose short proofs, explanations, or other writing samples. Students expressed difficulty in answering these questions.

In an effort to improve the students' ability to write mathematically, the author utilized some standard composition techniques, such as writing for peers, critiquing peer writing samples, and small group discussions. Specifically, students were asked to evaluate a proof which a peer had created, with the eventual goal of learning how to critique their own writing. The author created two different exercises to supplement and facilitate the mathematical writings required in the homework. These exercises were also designed to generate discussion about general aspects of good mathematical writing and how to apply these aspects specifically to proofs. Indeed, "[h]aving students write their proofs for their classmates...can improve their presentation [2, p.49]." While these exercises were developed to help the students understand proofs, an unexpected benefit was that classroom instruction improved as well, becoming more directly suited to the needs of the class.

A Brief Analysis of Peer-Oriented Learning

The benefits of a peer-oriented approach to learning extend far beyond course content. For example, students have shown "higher levels of academic achievement, improved interpersonal relationships among students, and greater personal and social development" using cooperative learning techniques as opposed to either competitive learning or individualized instruction [5, p.47]. Indeed, "[r]esearch and anecdotal evidence suggest that working cooperatively to learn mathematics affects students' attitudes and beliefs about ... how mathematics is created [1, p.21]."

There are however potential disadvantages to using a more collaborative approach. Time constraints can make it difficult to assess each student's progress adequately. Also, there is concern that not as much material can be covered because students move at a slower pace than would a teacher-driven lecture. Moreover, the instruction a student gets from a peer is not of the same quality as instruction obtained from the teacher [5, p.54]. We address these concerns in the concluding section.

The First Exercise

Students were asked to prove the following statement about X_A, the characteristic function of the set A.

$$X_{(\bigcap_{i=1}^{\infty} A_i)} = \prod_{i=1}^{\infty} X_{A_i}$$

They were reminded of a recent homework problem in which they showed that $X_{A \cap B} = X_A X_B$. Their assignment was to bring their proofs to class, at which point each student would read a few proofs, critique one of them, and have her or his proof critiqued by someone else. This relatively simple example was chosen so that, as *writers*, the students would focus on the writing and not on the mathematical content. All of them worked on the same problem so that, as *critics*, they would not be confused by the mathematical content, but could also focus on the writing. Obviously, "real" proofs in mathematics are not so divorced from their content nor are they necessarily easy, but since the goal was to improve the writing, an attempt was made to minimize interference from the mathematical content of the problem.

On the day the proof was due, the class was divided into groups of four and given the following instructions.

1. Read the other three proofs in your group.
2. Critique the last one you read. Include at least two sentences: "The strongest aspect of this proof is . . . ." and "This proof could be improved by . . . ." Include other comments if you wish.
3. Discuss as a group: What are some aspects of a good proof?
4. Create a group proof to be turned in.

The importance of being a constructive critic was emphasized to the students, as was the fact that their proofs were also being critiqued. As a result, the students worked earnestly and communicated well with each other. The exercise took about half of one 50-minute class session. The group proofs were reviewed and corrected by the instructor, and were shown to the entire class at a later date with comments provided. The class then discussed the results. Parts of the proofs from two different groups follow.

Group 1

Case 2: When $x \notin A_i$. Then $X_{A_i} = 0$; therefore, $X_{(\cap_{i=1}^{\infty} A_i)} = 0$, because $X_{A_1 \cap A_2 \cap A_3 \ldots}$ which would be equivalent to $\prod X_{A_i} = 0 \cdot 0 \cdot 0 \cdot 0 \ldots = 0$.

Group 2

Case 2: Suppose $x \notin \bigcap_{i=1}^{\infty} A_i$. This means that $x \notin A_i$ for all $i \in \mathbb{P}$. [...] So $\prod_{i=1}^{\infty} X_{A_i}(x) = \prod_{i=1}^{\infty}(0) = 0$.

These two examples highlight a frequent mistake made by various students: the confusion of "there exists some index i such that $x \notin A_i$" and "for every index i, $x \notin A_i$." The notation $x \notin A_i$ without further explanation is incomplete. Other common mistakes included writing "$X_{A_i} = 0$" instead of "$X_{A_i}(x) = 0$" and calling X a set instead of a function. Errors like these arise when students attempt to mix mathematical symbols with English. Finding an appropriate balance between symbolic language and written language can be a difficult task not only for students to learn but also for teachers to explain. The matter is complicated even further because different teachers have different writing styles in this regard.

Some might see these only as mathematical errors. However, they are also rich sources of information for the instructor. They highlight exactly which concepts the students find most challenging to understand, such as the difference between $x \notin \cap_i A_i$ and $x \notin A_i$ for all i. Through discussion, the class confronted these misunderstandings head-on and was able to generate a correct version of the proof from the different groups' proofs. This exercise provided the class with a concrete example of how interactive learning can be a constructive way to gain knowledge.

The Second Exercise

Although successful, the first exercise had its drawbacks. First, it did not directly study the persuasiveness of the proof. The students knew the answer was true before they read the proofs of their peers; they did not need to be persuaded by the proof of a fact they already believed. This caused them to supply details, filling in gaps as they read. Another drawback was that only the final group-generated proof was evaluated. There was no way to assess exactly how much improvement had occurred for each individual student.

The second exercise was developed to address these concerns. This time there were four different questions about the same topic: (binary) relations. Each student was assigned one proof and told to bring two copies of the proof to class. One copy would be turned in and one would be critiqued by another student. This time, each group of four was comprised of students who had been assigned different proofs. Thus the students would be familiar with the concept of a relation, but not with the specific problem they were critiquing. As writers, then, the students would need to persuade an audience who possessed general knowledge, but not specific information. Students are often asked to "pretend" like they are writing for their peers; one advantage of this exercise is that they are actually doing so. There is even some evidence that specifying an audience improves the level of student involvement with an assignment [4, p.97]. As critics, however, the students no longer knew why the specific problem was true. Hence they would not be so quick to insert corrections as they read.

The second assignment was as follows.

> Each question involves a relation $\mathcal{R}$ on a set S. Also, $\mathcal{R}^{\leftarrow}$ denotes the converse relation.
>
> 1. Prove that if $\mathcal{R}$ is reflexive, antireflexive, symmetric, antisymmetric, or transitive, then so is $\mathcal{R}^{\leftarrow}$. In other words, show that if a relation possess any one of these properties, then its converse does as well.
> 2. Prove that $(\mathcal{R}^{\leftarrow})^{\leftarrow} = \mathcal{R}$.
> 3. Find a relation which is reflexive and symmetric, but not transitive. Prove it.
> 4. Let $S \neq \emptyset$ and let $\mathcal{R}$ be a relation on S which is antireflexive, symmetric, and transitive. Prove that $\mathcal{R} = \emptyset$.

Being familiar with the methodology, the students again worked earnestly. This exercise took them longer because they were having to read proofs of facts which they had not thought about beforehand. The most common type of mistake involved misconceptions about the precise definitions of reflexive, symmetric, etc. The quantifiers used in these definitions gave some students difficulty, such as not understanding the difference between "antireflexive" and "not reflexive". Also, many students fudged their way through the first proof by stating all the relevant facts, but in the wrong order. For instance, part of one proof read:

> Since $\mathcal{R}$ is symmetric, if $x\mathcal{R}y$, then $y\mathcal{R}x$.
> If $y\mathcal{R}^{\leftarrow}x$, then $x\mathcal{R}^{\leftarrow}y$. So $\mathcal{R}^{\leftarrow}$ is symmetric.

While the facts are present, no mention is made of how the definition of the converse relation plays a role. This example shows that the student does not truly understand why the converse relation of a symmetric relation is itself symmetric. But neither does the critic; otherwise the problem would have been corrected.

In principle, the instructor could more easily evaluate changes in student understanding with such an exercise, but in practice, the changes were not very great. Most students were unwilling to completely rewrite proofs, opting instead to make only cosmetic changes. It would be better to give the students time to rewrite the proof at home, rather than expecting them to do so in class.

Despite these difficulties in evaluation, here again was a wealth of information for the instructor. Examining the proofs makes it abundantly clear which concepts were most difficult for the students, which in turn can lead to more effective instruction. For example, to target the difference between "anti-" and "not" reflexive, the students could be shown more examples of relations which are neither reflexive nor antireflexive. The instructor could also point to the specific "pseudoproof" that the converse of a symmetric relation is symmetric and explain how to correct it. In short, instruction can become more direct – more appropriate to the particular needs of the students.

Student Responses

The students reflected on these writing exercises and were asked on the final course evaluation to comment on whether they thought the exercises had helped them to develop proof-writing skills. Some (unedited) comments follow.

- Writing assignments in class and out of class were helpful.
- writing proofs and having them revised in class was useful.
- Writing assignments were good. Group work helped to better understand concepts.
- Liked the in-class activities like the sessions we had on correcting proofs. Helped to see how other students did the proofs.
- The writing assignments were hard but helped greatly.
- the writing assignments were tedious but helped me to learn proofs.

Conclusions

First and foremost, these writing exercises assisted greatly in the identification of common misunderstandings among the students. Consequently, instruction time became more efficient: the instructor could clear up a common misconception on a major theme rather than answer specialized questions for one student at a time. So, even though writing exercises like these take classroom time that might be spent covering more material, they can help the instructor to be a more responsive teacher.

One concern is that it is difficult to evaluate each student's improvement. Indeed, it is always difficult to quantify an increase in the qualitative ability to write a good proof. However, the second exercise did highlight some of the "before" and "after" understandings of students and comments from the students themselves clearly indicate that they found the exercises helpful, particularly in understanding proofs.

Another concern is that the peer critic may offer poor or incorrect advice. One benefit of the framework of the second assignment is that the critic's performance can also be evaluated, at least indirectly. For instance, if a student changes a cogent argument to one that is incomprehensible or even incorrect, then they may have received bad advice. While such a situation is obviously not ideal, it still presents a valuable learning opportunity. Students should learn not to accept what other people say as incontrovertible. As independent thinkers, they should instead listen to criticism, understand it, and then evaluate it for themselves.

Introducing students to the notion of interactive learning will not only provide the instructor with useful feedback, it will also ultimately strengthen the students' understanding of how to read, write, and evaluate proofs. In some cases, such interaction in class might even lead to the formation of study groups outside of class. And finally, by both writing a proof and critiquing one, students begin to develop a critical disposition with which they can improve their own mathematical writing.

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