1763

Consideratio formularum, quarum integratio per arcus sectionum conicarum absolvi potest

Leonhard Euler

CONSIDERATIO FORMVLARVM, QVARVM INTEGRATIO PER ARCVS SECTIONVM CONICARVM ABSOLVI POTEST.

Auctore

L. EVLERO.

Lemmata.

I. $\int dz\sqrt{\frac{f+gzz}{h+kzz}}=\frac{1}{k}\int dx\sqrt{\frac{fk-gh+gxx}{xx-h}}$

posito $x=\sqrt{(h+kzz)}$

II. $\int\frac{zzdz}{\sqrt{(f+gzz)(h+kzz)}}=\frac{1}{g}\int dx\sqrt{\frac{xx-f}{gh-fk+kxx}}$

$=\frac{1}{k}\int dy\sqrt{\frac{yy-h}{fk-gh+gyy}}$

posito $x=\sqrt{(f+gzz)}$ et $y=\sqrt{(h+kzz)}$

III. $\int\frac{dz\sqrt{(f+gzz)}}{(h+kzz)^{\frac{3}{2}}}=-\frac{1}{k}\int dx\sqrt{\frac{g+(fk-gh)xx}{1-hxx}}$

$=\frac{1}{h}\int dy\sqrt{\frac{f+(gh-fk)yy}{1-kyy}}$

posito $x=\frac{1}{\sqrt{(h+kzz)}}$ et $y=\frac{z}{\sqrt{(h+kzz)}}$

IV. $\int\frac{dz\sqrt{(h+kzz)}}{(f+gzz)^{\frac{3}{2}}}=-\frac{1}{g}\int dx\sqrt{\frac{k+(gh-fk)xx}{1-fxx}}$

$=\frac{1}{f}\int dy\sqrt{\frac{h+(fk-gh)yy}{1-gyy}}$

 posito

$$\text{posito } x = \frac{1}{\sqrt{(f+gzz)}} \text{ et } y = \frac{z}{\sqrt{(f+gzz)}}$$

$$\text{V. } \int \frac{dz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}} = \tfrac{1}{f}\int dx \sqrt{\frac{1-gxx}{h+(fk-gh)xx}}$$

$$= \frac{1}{fk-gh}\int dy \sqrt{\frac{k-gyy}{fyy-h}}$$

$$\text{posito } x = \frac{z}{\sqrt{(f+gzz)}} \text{ et } y = \sqrt{\frac{h+kzz}{f+gzz}}$$

$$\text{VI. } \int \frac{dz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}} = \tfrac{1}{h}\int dx \sqrt{\frac{1-kxx}{f+(gh-fk)xx}}$$

$$= \frac{1}{gh-fk}\int dy \sqrt{\frac{g-kyy}{hyy-f}}$$

$$\text{posito } x = \frac{z}{\sqrt{(h+kzz)}} \text{ et } y = \sqrt{\frac{f+gzz}{h+kzz}}$$

$$\text{VII. } \int \frac{zz\,dz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}} = -\tfrac{1}{g}\int dx \sqrt{\frac{1-fxx}{k+(gh-fk)xx}}$$

$$= \frac{1}{fk-gh}\int dy \sqrt{\frac{fyy-h}{k-gyy}}$$

$$\text{posito } x = \frac{1}{\sqrt{(f+gzz)}} \text{ et } y = \sqrt{\frac{h+kzz}{f+gzz}}$$

$$\text{VIII. } \int \frac{zz\,dz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}} = -\tfrac{1}{k}\int dx \sqrt{\frac{1-hxx}{g+(fk-gh)xx}}$$

$$= \frac{1}{gh-fk}\int dy \sqrt{\frac{hyy-f}{g-kyy}}$$

$$\text{posito } x = \frac{1}{\sqrt{(h+kzz)}} \text{ et } y = \sqrt{\frac{f+gzz}{h+kzz}}$$

Theore-

Theoremata.

I. $\int dz \surd \frac{f+gzz}{h+kzz} = \frac{1}{k}\int dx \surd \frac{fk-gh+gxx}{xx-h}$

posito $x = \surd (h+kzz)$

II. $\int dz \surd \frac{f+gzz}{h+kzz} = z\surd \frac{f+gzz}{h+kzz} - \int dx \surd \frac{hxx-f}{g-kxx}$

posito $x = \surd \frac{f+gzz}{h+kzz}$

III. $\int dz \surd \frac{f+gzz}{h+kzz} = z\surd \frac{f+gzz}{h+kzz} + \frac{gh-fk}{k}\int dx \surd \frac{1-hxx}{g+(fk-gh)xx}$

posito $x = \frac{1}{\surd (h+kzz)}$

IV. $\int dz \surd \frac{f+gzz}{h+kzz} = \frac{g}{k} z \surd \frac{h+kzz}{f+gzz} + \frac{fk-gh}{k}\int dx \surd \frac{1-gxx}{h+(fk-gh)xx}$

posito $x = \frac{z}{\surd (f+gzz)}$

V. $\int dz \surd \frac{f+gzz}{h+kzz} = \frac{g}{k} z \surd \frac{h+kzz}{f+gzz} + \frac{f}{k}\int dx \surd \frac{k-gxx}{fxx-h}$

posito $x = \surd \frac{h+kzz}{f+gzz}$

VI. $\int dz \surd \frac{f+gzz}{h+kzz} = \frac{f}{h}\int dz \surd \frac{h+kzz}{f+gzz} + \frac{gh-fk}{gh}\int dx \surd \frac{xx-f}{gh-fk+kxx}$

posito $x = \surd (f+gzz)$

VII. $\int dz \surd \frac{f+gzz}{h+kzz} = \frac{f}{h}\int dz \surd \frac{h+kzz}{f+gzz} + \frac{gh-fk}{hk}\int dx \surd \frac{xx-h}{fk-gh+gxx}$

posito $x = \surd (h+kzz)$

VIII. $\int dz \surd \frac{f+gzz}{h+kzz} = z \surd \frac{f+gzz}{h+kzz} + P + Q$

vbi $P = \frac{gh-fk}{gk}\int dx\sqrt{\frac{g+(fk-gh)xx}{1-hxx}} = \frac{fk-gh}{gh}\int dy\sqrt{\frac{f+(gh-fk)yy}{1-kyy}}$

posito $x = \frac{1}{\sqrt{(h+kzz)}}$ et $y = \frac{z}{\sqrt{(h+kzz)}}$

et $Q = \frac{-f(fk-gh)}{gh}\int dx\sqrt{\frac{1-kxx}{f+(gh-fk)xx}} = \frac{f}{g}\int dy\sqrt{\frac{g-kyy}{hyy-f}}$

posito $x = \frac{z}{\sqrt{(h+kzz)}}$ et $y = \sqrt{\frac{f+gzz}{h+kzz}}$

IX. $\int dz\sqrt{\frac{f+gzz}{h+kzz}} = \frac{fk}{gh}z\sqrt{\frac{f+gzz}{h+kzz}} + P + Q$

vbi $P = \frac{gh-fk}{gh}\int dx\sqrt{\frac{xx-f}{gh-fk+kxx}} = \frac{gh-fk}{hk}\int dy\sqrt{\frac{yy-h}{fk-gh+gyy}}$

posito $x = \sqrt{(f+gzz)}$ et $y = \sqrt{(h+kzz)}$.

atque $Q = \frac{f(gh-fk)}{gh}\int dx\sqrt{\frac{1-kxx}{f+(gh-fk)xx}} = \frac{f}{g}\int dy\sqrt{\frac{g-kyy}{hyy-f}}$

posito $x = \frac{z}{\sqrt{(h+kzz)}}$ et $y = \sqrt{\frac{f+gzz}{h+kzz}}$

X. $\int dz\sqrt{\frac{f+gzz}{h+kzz}} = \frac{gh-fk}{gh}z\sqrt{\frac{f+gzz}{h+kzz}} + \frac{f}{h}\int dz\sqrt{\frac{h+kzz}{f+gzz}} + P$

vbi $P = \frac{gh-fk}{gk}\int dx\sqrt{\frac{g+(fk-gh)xx}{1-hxx}} = \frac{fk-gh}{gh}\int dy\sqrt{\frac{f+(gh-fk)yy}{1-kyy}}$

posito $x = \frac{1}{\sqrt{(h+kzz)}}$ et $y = \frac{z}{\sqrt{(h+kzz)}}$

XI. $\int dz\sqrt{\frac{f+gzz}{h+kzz}} = \frac{f}{h}z\sqrt{\frac{h+kzz}{f+gzz}} + P + Q$

vbi $P = \frac{gh-fk}{gh}\int dx\sqrt{\frac{xx-f}{gh-fk+kxx}} = \frac{gh-fk}{hk}\int dy\sqrt{\frac{yy-h}{fk-gh+gyy}}$

posito $x = \sqrt{(f+gzz)}$ et $y = \sqrt{(h+kzz)}$

atque

atque $Q=\frac{f(fk-gh)}{gh}\int dx\sqrt{\frac{1-fxx}{k+(gh-fk)xx}}=\frac{-f}{h}\int dy\sqrt{\frac{fyy-h}{k-gyy}}$

posito $x=\frac{1}{\sqrt{(f+gzz)}}$ et $y=\sqrt{\frac{h+kzz}{f+gzz}}$

XII. $\int dz\sqrt{\frac{f+gzz}{h+kzz}}=\frac{g}{k}z\sqrt{\frac{h+kzz}{f+gzz}}+P+Q$

vbi $P=\frac{f(gh-fk)}{ghk}\int dx\sqrt{\frac{k+(gh-fk)xx}{1-fxx}}=\frac{fk-gh}{hk}\int dy\sqrt{\frac{h+(fk-gh)yy}{1-gyy}}$

posito $x=\frac{1}{\sqrt{(f+gzz)}}$ et $y=\sqrt{\frac{z}{(f+gzz)}}$

atque $Q=\frac{f(fk-gh)}{gh}\int dx\sqrt{\frac{1-fxx}{k+(gh-fk)xx}}=\frac{-f}{h}\int dy\sqrt{\frac{fyy-h}{k-gyy}}$

posito $x=\sqrt{\frac{1}{(f+gzz)}}$ et $y=\sqrt{\frac{h+kzz}{f+gzz}}$

XIII. $\int dz\sqrt{\frac{f+gzz}{h+kzz}}=\frac{gh-fk}{hk}z\sqrt{\frac{h+kzz}{f+gzz}}+\frac{f}{h}\int dz\sqrt{\frac{h+kzz}{f+gzz}}+P$

vbi $P=\frac{f(gh-fk)}{gk}\int dx\sqrt{\frac{k+(gh-fk)xx}{1-fxx}}=\frac{fk-gh}{hk}\int dy\sqrt{\frac{h+(fk-gh)yy}{1-gyy}}$

posito $x=\frac{1}{\sqrt{(f+gzz)}}$, et $y=\frac{z}{\sqrt{(f+gzz)}}$.

Theorema Singulare.

$\int dz\sqrt{\frac{f+gzz}{h+kzz}}=\frac{-gxz}{\sqrt{p}}-\int dx\sqrt{\frac{f+gxx}{h+kxx}}$, vbi p denotat constantem arbitrariam, posita inter x et z hac relatione:

$gkxxzz - pxx - pzz - 2xz\sqrt{(p+fk)(p+gh)} + fh = 0$ siue

$$x = \frac{-z\sqrt{(p+fk)(p+gh)} + \sqrt{p(f+gzz)(h+kzz)}}{p-gkzz}$$

Hypothesis.

Haec scribendi formula $\Pi x\,[a]$ denotet sectionis conicae, cuius semiparameter $=1$, et semiaxis transversus $=a$, arcum a vertice sumtum, cui in axe transverso conueniat abscissa $=x$.

Corollarium.

Si a sit quantitas positiua, hoc modo designatur arcus ellipsis; sin negatiua, arcus hyperbolae. Si modo x fuerit quantitas positiua et minor quam $2a$.

Integrationes formulae $\int dz\sqrt{\frac{f+gzz}{h+kzz}}$ in 12 casus distributae.

Casus I. $\int dz\sqrt{\frac{f+gzz}{h+kzz}}$:

Integrale est immediate:

$$C - \frac{(fk+gh)}{k\sqrt{fk}}\Pi\frac{fk}{fk+gh}\left(1 - z\sqrt{\frac{k}{h}}\right)\left[\frac{fk}{fk+gh}\right]$$

vel etiam per Theor. I.

$$C + \frac{f}{\sqrt{(fk+gh)}}\Pi\frac{fk+gh}{fk}\left(1 - \frac{\sqrt{(h-kzz)}}{\sqrt{k}}\right)\left[\frac{fk+gh}{fk}\right]$$

Casus II. $\int dz\sqrt{\frac{f-gzz}{h-kzz}}$, existente $fk > gh$

Integrale est immediate:

$$C - \frac{(fk-gh)}{k\sqrt{fk}}\Pi\frac{fk}{fk-gh}\left(1 - z\sqrt{\frac{k}{h}}\right)\left[\frac{fk}{fk-gh}\right]$$

vel

vel etiam per Theor. I.

$$C+\frac{f}{\sqrt{(fk-gh)}}\Pi\frac{fk-gh}{fk}\left(1-\frac{\sqrt{(h-kzz)}}{\sqrt{h}}\right)\left[\frac{fk-gh}{fk}\right]$$

Casus III. $\int dz\sqrt{\frac{-f+gzz}{-h+kzz}}$, existente $fk<gh$

Integrale est immediate:

$$C+\frac{gh-fk}{k\sqrt{fk}}\,\Pi\,\frac{fk}{gh-fk}\left(z\sqrt{\frac{k}{h}}-1\right)\left[\frac{-fk}{gh-fk}\right]$$

Casus IV. $\int dz\sqrt{\frac{f+gzz}{h+kzz}}$, existente $fk<gh$

Integrale est per Theor. I.

$$C+\frac{f}{\sqrt{(gh-fk)}}\Pi\frac{gh-fk}{fk}\left(\frac{\sqrt{(h+kzz)}}{\sqrt{h}}-1\right)\left[\frac{-gh+fk}{fk}\right]$$

Casus V. $\int dz\sqrt{\frac{-f+gzz}{h+kzz}}$

Integrale est per Theor. III.

$$C+z\sqrt{\frac{-f+gzz}{h+kzz}}-\frac{f}{\sqrt{(fk+gh)}}\,\Pi\,\frac{fk+gh}{fk}\left(1-\frac{\sqrt{(fk+gh)}}{\sqrt{g(h+kzz)}}\right)\left[\frac{fk+gh}{fk}\right]$$

vel etiam per Theor. II.

$$C+z\sqrt{\frac{-f+gzz}{h+kzz}}+\frac{fk+gh}{k\sqrt{fk}}\Pi\frac{fk}{fk+gh}\left(1-\frac{\sqrt{k(-f+gzz)}}{\sqrt{g(h+kzz)}}\right)\left[\frac{fk}{fk+hg}\right]$$

Casus VI. $\int dz\sqrt{\frac{-f+gzz}{-h+kzz}}$, existente $fk>gh$

Integrale est per Theor. III.

$$C+z\sqrt{\frac{-f+gzz}{-k+kzz}}-\frac{f}{\sqrt{(fk-gh)}}\Pi\frac{fk-gh}{fk}\left(1-\frac{\sqrt{(fk-gh)}}{\sqrt{g(-h+kzz)}}\right)\left[\frac{fk-gh}{fk}\right]$$

vel

vel etiam per Theor. II.

$$C + z\sqrt{\frac{-f+gzz}{-h+kzz}} + \frac{fk-gh}{k\sqrt{fk}}\,\Pi\,\frac{fk}{fk-gh}\left(1-\frac{\sqrt{k(-f+gzz)}}{\sqrt{g(-h+kzz)}}\right)\left[\frac{fk}{fk-gh}\right]$$

Casus VII. $\int dz\sqrt{\frac{f-gzz}{h-kzz}}$, existente $fk<gh$

Integrale est per Theor. III.

$$C + \frac{gz}{k}\sqrt{\frac{h-kzz}{f-gzz}} - \frac{(gh-fk)}{k\sqrt{fk}}\,\Pi\,\frac{fk}{gh-fk}\left(\frac{\sqrt{f(h-kzz)}}{\sqrt{h(f-gzz)}}-1\right)\left[\frac{-fk}{gh-fk}\right]$$

Casus VIII. $\int dz\sqrt{\frac{-f+gzz}{h-kzz}}$, existente $fk<gh$

Integrale est per Theor. II.

$$C + z\sqrt{\frac{-f+gzz}{h-kzz}} - \frac{f}{\sqrt{(gh-fk)}}\,\Pi\,\frac{gh-fk}{fk}\left(\frac{\sqrt{(gh-fk)}}{\sqrt{g(h-kzz)}}-1\right)\left[\frac{-gh+fk}{fk}\right]$$

vel etiam per Theor. V.

$$C - \frac{gz}{k}\sqrt{\frac{h-kzz}{-f+gzz}} + \frac{f}{\sqrt{(gh-fk)}}\,\Pi\,\frac{gh-fk}{fk}\left(\frac{z\sqrt{(gh-fk)}}{\sqrt{h(-f+gzz)}}-1\right)\left[\frac{-gh+fk}{fk}\right]$$

Casus IX. $\int dz\sqrt{\frac{f+gzz}{h+kzz}}$, existente $fk>gh$

Integrale est per Theor. X.

$$C - \frac{(fk-gh)z}{gh}\sqrt{\frac{f+gzz}{h+kzz}} - \frac{(fk-gh)}{k\sqrt{fk}}\,\Pi\,\frac{fk}{gh}\left(1-\frac{z\sqrt{k}}{\sqrt{(h+kzz)}}\right)\left[\frac{fk}{gh}\right]$$

$$+ \frac{f}{\sqrt{(fk-gh)}}\,\Pi\,\frac{fk-gh}{gh}\left(\frac{\sqrt{(f+gzz)}}{\sqrt{f}}-1\right)\left[\frac{-fk+gh}{gh}\right]$$

vel etiam per Theor. XIII.

$$C - \frac{(fk-gh)z}{hk}\sqrt{\frac{h+kzz}{f+gzz}} + \frac{(fk-gh)}{k\sqrt{fk}}\,\Pi\,\frac{fk}{gk}\left(1-\frac{\sqrt{f}}{\sqrt{(f+gzz)}}\right)\left[\frac{fk}{gh}\right]$$

$$+ \frac{f}{\sqrt{(fh-gh)}}\,\Pi\,\frac{fk-gh}{gh}\left(\frac{\sqrt{(h+kzz)}}{\sqrt{h}}-1\right)\left[\frac{-fk+gh}{gh}\right]$$

Casus X.

Caſus X. $\int dz \surd \frac{f-gzz}{-h+kzz}$, exiſtente $fk > gh$

Integrale eſt per Theor. IX.

$$C + \frac{fkz}{gh} \surd \frac{f-gzz}{-h+kzz} + \frac{(fk-gh)}{k\surd fk} \Pi \frac{fk}{gh} \left(1 - \frac{\surd k(f-gzz)}{\surd(fk-gh)}\right) \left[\frac{fk}{gh}\right]$$
$$- \frac{f}{\surd(fk-gh)} \Pi \frac{fk-gh}{gh} \left(\frac{z\surd(fk-gh)}{\surd f(-h+kzz)} - 1\right) \left[\frac{-fk+gh}{gh}\right]$$

vel etiam per Theor. XI.

$$C - \frac{fz}{h} \surd \frac{-h+kzz}{f-gzz} + \frac{(fk-gh)}{k\surd fk} \Pi \frac{fk}{gh} \left(1 - \frac{\surd k(f+gzz)}{\surd(fk-gh)}\right) \left[\frac{fk}{gh}\right]$$
$$+ \frac{f}{\surd(fk-gh)} \Pi \frac{fk-gh}{gh} \left(\frac{\surd(fk-gh}{\surd k(f-gzz)} - 1\right) \left[\frac{-fk+gh}{gh}\right]$$

Caſus XI. $\int dz \surd \frac{f+gzz}{-h+kzz}$

Integrale eſt per Theor. XI.

$$C - \frac{fz}{h} \surd \frac{-h+kzz}{f+gzz} + \frac{f}{\surd(fk+gh)} \Pi \frac{fk+gh}{gh} \left(1 - \frac{\surd(fk+gh)}{\surd k(f+gzz)}\right) \left[\frac{fk+gh}{gh}\right]$$
$$+ \frac{(fk+gh)}{k\surd fk} \Pi \frac{fk}{gh} \left(\frac{\surd k(f+gzz)}{\surd(fk+gh)} - 1\right) \left[\frac{-fk}{gh}\right]$$

vel etiam per Theor. XII.

$$C + \frac{gz}{k} \surd \frac{-h+kzz}{f+gzz} + \frac{f}{\surd(fk+gh)} \Pi \frac{fk+gh}{gh} \left(1 - \frac{\surd(fk+gh)}{\surd k(f+gzz)}\right) \left[\frac{fk+gh}{gh}\right]$$
$$+ \frac{(fk+gh)}{k\surd fk} \Pi \frac{fk}{gh} \left(\frac{\surd f}{\surd(f+gzz)} - 1\right) \left[\frac{-fk}{gh}\right]$$

Caſus XII. $\int dz \surd \frac{f-gzz}{h+kzz}$

Integrale eſt per Theor. XIII.

$$C - \frac{(fk+gh)}{hk} \surd \frac{h+kzz}{f-gzz} + \frac{f}{\surd(fk+gh)} \Pi \frac{fk+gh}{gh} \left(1 - \frac{\surd(f-gzz)}{\surd f}\right) \left[\frac{fk+gh}{gh}\right]$$
$$+ \frac{(fk+gh)}{k\surd fk} \Pi \frac{fk}{gh} \left(\frac{\surd f}{\surd(f-gzz)} - 1\right) \left[\frac{-fk}{gh}\right]$$

Omnes ergo caſus formulae $\int dz \surd \frac{\alpha+\beta zz}{\gamma+\delta zz}$, quomodocunque litterae α, β, γ, δ fuerint comparatae, per arcus ſectionum conicarum integrari poſſunt.

Non ſolum igitur formulae initio commemoratae integrationem per arcus ſectionum conicarum admittunt, ſed etiam innumerabiles aliae, quae per ſubſtitutionem ad formam $\int dx \sqrt{\frac{\alpha + \beta xx}{\gamma + \delta xx}}$ ſe reduci patiuntur, cuiusmodi ſunt

$$1^\circ.\ \int \frac{dz}{zz}\sqrt{\frac{f+gzz}{b+kzz}} = -\int dx \sqrt{\frac{fxx+g}{bxx+k}} = -\frac{1}{b}\int dy \sqrt{\frac{fyy-fk+gb}{yy-k}}$$

poſito $x = \frac{1}{z}$ et $y = \frac{\sqrt{(b+kzz)}}{z}$

$$2^\circ.\ \int \frac{dz}{zz\sqrt{(f+gzz)(b+kzz)}} = -\frac{1}{f}\int dx \sqrt{\frac{xx-g}{bxx+fk-gb}} = \frac{1}{b}\int dy \sqrt{\frac{yy-k}{fyy-fk+gb}}$$

poſito $x = \frac{\sqrt{(f+gzz)}}{z}$ et $y = \frac{\sqrt{(b+kzz)}}{z}$

$$3^\circ.\ \int \frac{dz}{\sqrt{(f+gzz)(b+kzz)}} = \frac{k}{fk+gb}\int dz \sqrt{\frac{f+gzz}{b+kzz}} - \frac{g}{fk-gb}\int dz \sqrt{\frac{b+kzz}{f+gzz}}$$

cuius formulae reductio etiam ita inſtituitur:

$$\int \frac{dz}{\sqrt{(f+gzz)(b+kzz)}} = \frac{f}{fk-gb}\int dx \sqrt{\frac{k-gxx}{fxx-b}} + \frac{g}{fk-gb}\int dx \sqrt{\frac{fxx-b}{k-gxx}}$$

poſito $x = \sqrt{\frac{b+kzz}{f+gzz}}$

vel etiam ſic:

$$\int \frac{dz}{\sqrt{(f+gzz)(b+kzz)}} = \int dx \sqrt{\frac{1-gxx}{b+(fk-gb)xx}} - \int dy \sqrt{\frac{1-fyy}{k+(gb-fk)yy}}$$

poſito $x = \frac{z}{\sqrt{(f+gzz)}}$ et $y = \frac{1}{\sqrt{(f+gzz)}}$.

Ponamus $zz = v$ atque obtinebimus ſequentes formulas, quae pariter per arcus ſectionum conicarum conſtrui poterunt:

$$1^\circ.\ \int \frac{dv\sqrt{(f+gv)}}{\sqrt{v(b+kv)}}\,; \quad 2^\circ.\ \int \frac{dv\sqrt{(f+gv)}}{v\sqrt{v(b+kv)}}$$

$$3^\circ.\ \int \frac{dv\sqrt{v}}{\sqrt{(f+gv)(b+kv)}}\,; \quad 4^\circ.\ \int \frac{dv}{\sqrt{v(f+gv)(b+kv)}}$$

5°

$5^\circ. \int \frac{dv\sqrt{(f+gv)}}{(h+kv)^{\frac{3}{2}}\sqrt{v}}$; $6^\circ. \int \frac{dv}{v\sqrt{v(f+gv)(h+kv)}}$

$7^\circ. \int \frac{dv}{(f+gv)^{\frac{3}{2}}\sqrt{v(h+kv)}}$; $8^\circ. \int \frac{dv\sqrt{v}}{(f+gv)^{\frac{3}{2}}\sqrt{(h+kv)}}$

hae enim viciſſim, poſito $v=zz$, ad formas praecedentes reducuntur.

Hinc patet, iſtam formulam ſatis late patentem ad arcus ſectionum conicarum reduci poſſe

$$\int \frac{(A+Bu)du}{\sqrt{(\alpha+\beta u)(\gamma+\delta u)(\varepsilon+\zeta u)}},$$

quae imprimis notari meretur. Ponatur enim $\alpha+\beta u=v$, vt ſit $u=\frac{v-\alpha}{\beta}$, haecque formula transmutabitur in hanc:

$$\int \frac{dv(A\beta-B\alpha+Bv)}{\beta\sqrt{v(\beta\gamma-\alpha\delta+\delta v)(\beta\varepsilon-\alpha\zeta+\zeta v)}},$$

quae ad binas formulas, ſub n°. 3 et 4 allatas, reuocatur. Quare, ſi $\alpha+\beta x+\gamma xx+\delta x^3$ habeat tres factores reales, haec formula

$$\int \frac{dx(A+Bx)}{\sqrt{(\alpha+\beta x+\gamma xx+\delta x^3)}}$$

modo expoſito integrari poterit: ſemper autem vnum factorem certe habet realem. Sin autem bini ſint imaginarii, formula $\alpha+\beta x+\gamma xx+\delta x^3$ ita referri poteſt $y(pp+2npqy+qqyy)$, exiſtente $nn<1$, vt definiendum ſit integrale harum formularum:

$$\int \frac{Cdy}{\sqrt{y(pp+2npqy+qqyy)}}+\int \frac{Ddy\sqrt{y}}{\sqrt{(pp+2npqy+qqyy)}}$$

Ponatur $\sqrt{(pp+2npqy+qqyy)}=p+qyz$, fietque $y=\frac{2p(z-n)}{q\sqrt{(1-zz)}}$, qua ſubſtitutione prior formula abit in $\frac{C\sqrt{2q}}{\sqrt{p}}\int \frac{dz}{\sqrt{(z-n)(1-z)(1+z)}}$ conſtruibilem: poſterior vero

 in

in hanc $\frac{2D\sqrt{2p}}{\sqrt{q}}\int\frac{dz\sqrt{(z-n)}}{(1-zz)^{\frac{3}{2}}}$; cum vero fit $\int\frac{dz\sqrt{(z-n)}}{(1-zz)^{\frac{3}{2}}}$ $= \frac{z\sqrt{(z-n)}}{\sqrt{(1-zz)}} - \frac{1}{2}\int\frac{z\,dz}{\sqrt{(z-n)(1-z)(1+z)}}$ etiam haec per superiora construi potest. Sicque in genere habetur constructio huius formulae $\int\frac{dx(A+Bx)}{\sqrt{(\alpha+\beta x+\gamma xx+\delta x^3)}}$.

Problema 1.

Integrationem huius formulae $\int\frac{dx}{\sqrt{(a+bx+cxx+dx^3+ex^4)}}$ per arcus sectionum conicarum perficere.

Solutio.

Quantitatem $a+bx+cxx+dx^3+ex^4$ semper in duos factores trinomiales reales resoluere licet, qui sint $(\alpha+2\beta x+\gamma xx)$ et $(\delta+2\varepsilon x+\zeta xx)$, ita vt habeatur haec formula integranda $\int\frac{dx}{\sqrt{(\alpha+2\beta x+\gamma xx)(\delta+2\varepsilon x+\zeta xx)}}$. Ponatur $\delta+2\varepsilon x+\zeta xx=(\alpha+2\beta x+\gamma xx)y$, vt formula proposita fiat $\int\frac{dx}{(\alpha+2\beta x+\gamma xx)\sqrt{y}}$. At aequatio assumta per radicis extractionem praebet

$$\varepsilon+\zeta x-\beta y-\gamma xy\sqrt{(pyy+qy+r)},$$

posito $p=\beta\beta-\alpha\gamma$; $q=\alpha\zeta-2\beta\varepsilon+\gamma\delta$; et $r=\varepsilon\varepsilon-\beta\zeta$. Tum vero eadem differentiata dat:

$$dx(\varepsilon+\zeta x-\beta y-\gamma xy)=\tfrac{1}{2}dy(\alpha+2\beta x+\gamma xx)$$

seu $\frac{dx}{\alpha+2\beta x+\gamma xx}=\frac{\frac{1}{2}dy}{\varepsilon+\zeta x-\beta y-\gamma xy}$

Quare.

Quare ſi pro hoc poſtremo denominatore valorem irrationalem modo inuentum ſubſtituamus, formula propoſita abit in hanc:

$$\int \frac{\frac{1}{2}dy}{\sqrt{y(pyy+qy+r)}},$$

cuius integratio per arcus ſectionum conicarum ſupra eſt oſtenſa.

Hic igitur naſcitur quaeſtio, quid tenendum ſit de hac formula:

$$\int \frac{dx(A+Bx+Cxx)}{\sqrt{(a+bx+cyy+dx^3+ex^4)}}.$$

Euidens enim eſt, non neceſſe eſſe, vt numeratori altiores poteſtates ipſius x tribuantur; quam etiam Cel. *d'Alembert* fatetur, ſe in genere ad rectificationem ſectionum conicarum perducere non poſſe. Conſiderat quidem in Vol. IV. Mem. Acad. R. Berol. pag. 254 caſum, quo $A=o$, $C=o$ et $a=o$, ita vt formula ſit $\int \frac{dx\sqrt{x}}{\sqrt{(b+cx+dx^2+ex^3)}}$, conaturque oſtendere (pag. 257.) eius integrationem caſu $dd=4ce$ per arcus ſectionum conicarum abſolui poſſe: verum methodus, qua vtitur, negotium minime conficere videtur, vti rem accuratius perpendenti mox patebit. Transformationes autem, quas deinceps tradit, caſus nonnunquam hoc modo tractabiles ſuppeditant. Quocirca haec inueſtigatio, vti eſt difficillima, merito omni attentione digna eſt cenſenda: vnde etiam mea tentamina ſuper hac quaeſtione propoſuiſſe iuuabit.

 Proble-

Problema 2.

Inueſtigare conditiones, ſub quibus integrationem huius formulae $\int \frac{dy(\mathfrak{P}+\mathfrak{Q}y+\mathfrak{R}yy)}{\sqrt{(\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E})}}$ ad hanc ſimpliciorem $\int \frac{dx(P+Qx+Rxx)}{\sqrt{(Ax^4+Cxx+D)}}$ reducere liceat.

Solutio.

Statuatur inter variabiles x et y talis relatio:
$\alpha xxyy+2xy(\beta x+\gamma y)+\delta xx+\varepsilon yy+2\zeta xy+2\eta x+2\theta y+\varkappa=0$, cuius coefficientes ita determinentur, vt ſit

$$\beta\zeta-\alpha\eta-\gamma\delta=0; \qquad \zeta\theta-\gamma\varkappa-\varepsilon\eta=0$$
$$\gamma\gamma-\alpha\varepsilon=\mathfrak{A}; \qquad \gamma\zeta-\alpha\theta-\beta\varepsilon=\mathfrak{B}$$
$$\eta\eta-\delta\varkappa=\mathfrak{C}; \qquad \zeta\eta-\beta\varkappa-\delta\theta=\mathfrak{D}$$

et $\zeta\zeta+2\gamma\eta-\alpha\varkappa-\delta\varepsilon-4\beta\theta=\mathfrak{C}$

hincque erit pro denominatore transformatae:

$$\begin{matrix} A=\beta\beta-\alpha\delta; \\ E=\theta\theta-\varepsilon\varkappa; \end{matrix} \text{ et } C=\zeta\zeta+2\beta\theta-\alpha\varkappa-\delta\varepsilon-4\gamma\eta$$

Cum autem nouem habeantur litterae $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \varkappa$, his ſeptem conditionibus praeſcriptis vtique ſatisfieri poterit, relinqueturque adhuc vna arbitrio noſtro determinanda. Si iam breuitatis gratia ponamus:

$\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}y^2+2\mathfrak{D}y+\mathfrak{E}=Y$ et $Ax^4+Cxx+E=X$,

reſolutio aequationis aſſumtae praebet:

$$\alpha xyy+2\beta xy+\delta x+\gamma yy+\zeta y+\eta=\sqrt{Y}$$
$$\alpha xxy+2\gamma xy+\varepsilon y+\beta xx+\zeta x+\theta=\sqrt{X}$$

eius-

eiusque differentiatio ducit ad hanc aequationem:

$\frac{dy}{\sqrt{Y}} + \frac{dx}{\sqrt{X}} = 0.$ Ponamus ergo:

$$\int \frac{dy(\mathfrak{P} + \mathfrak{Q}y + \mathfrak{R}yy)}{\sqrt{(\mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}y^2 + 2\mathfrak{D}y + \mathfrak{E})}} = V - \int \frac{dx(P + Qx + Rxx)}{\sqrt{(Ax^4 + Cx^2 + E)}},$$

ac sit V talis functio algebraica:

$$V = mx + ny + pxy + \tfrac{1}{2}qxx + \tfrac{1}{2}ryy + txyy.$$

Hinc sumtis differentialibus terminisque homogeneis seorsim aequatis, reperientur sequentes determinationes:

$$m = \frac{\beta\mathfrak{R}}{\mathfrak{A}};\ n = \frac{\gamma\mathfrak{R}}{\mathfrak{A}};\ p = \frac{\alpha\mathfrak{R}}{\mathfrak{A}};\ q = 0,\ r = 0 \text{ et } t = 0,$$

praeterea vero haec determinatio accedit, vt sit $\mathfrak{A}\mathfrak{Q} = \mathfrak{B}\mathfrak{R}$. Deinde vero fit:

$$P = \mathfrak{P} + \frac{(\beta\theta - \gamma\eta)\mathfrak{R}}{\mathfrak{A}};\ Q = 0;\ \text{et } R = \frac{A\mathfrak{R}}{\mathfrak{A}}$$

Definitis ergo coefficientibus $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \varkappa$, quibus constat relatio inter x et y, ex iis innotescunt quantitates A, C, E, quibus inuentis, si fuerit $\mathfrak{A}\mathfrak{Q} = \mathfrak{B}\mathfrak{R}$, erit:

$$\int \frac{dy(\mathfrak{P} + \mathfrak{Q}y + \mathfrak{R}yy)}{\sqrt{(\mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}y^2 + 2\mathfrak{D}y + \mathfrak{E})}} = \text{Const.} + \frac{\mathfrak{R}}{\mathfrak{A}}(\beta x + \gamma y + \alpha xy)$$
$$- \int \frac{dx\left(\mathfrak{P} + \frac{(\beta\theta + \gamma\eta)\mathfrak{R}}{\mathfrak{A}} + \frac{A\mathfrak{R}}{\mathfrak{A}}xx\right)}{\sqrt{(Ax^4 + Cx^2 + E)}}$$

Dummodo ergo fuerit $\mathfrak{Q} = \frac{\mathfrak{B}\mathfrak{R}}{\mathfrak{A}}$, formulae propositae integratio reducta est ad hanc simpliciorem: $\int \frac{dx(P - Rxx)}{\sqrt{(Ax^4 + Cx^2 + E)}}$.

Corollarium I.

Determinatio coefficientium α, β, γ, etc. commodissime hoc modo instituetur: Primo quaeratur valor

lor ipſius s ex hac aequatione:

$$\mathfrak{C} = \frac{\mathfrak{B}\mathfrak{B} - \mathfrak{D}\mathfrak{D}ss}{\mathfrak{A} - \mathfrak{E}ss} + \frac{2\mathfrak{A}\mathfrak{D} - 2\mathfrak{B}\mathfrak{E}s}{\mathfrak{B} - \mathfrak{D}s},$$

quae, cum ſit cubica, certe valorem realem pro s ſuggerit: quo inuento, ſumtaque ad arbitrium quantitate t, ſit breuitatis gratia $\frac{\mathfrak{A} - \mathfrak{E}ss}{\mathfrak{B} - \mathfrak{D}ss} = u$, tum autem valores omnium 9 coefficientium ita ſe habebunt:

$$\zeta = u\sqrt{\frac{\mathfrak{A}\mathfrak{B} - 2\mathfrak{A}\mathfrak{D}s + 3\mathfrak{B}\mathfrak{E}ss - \mathfrak{D}\mathfrak{E}s^3}{s - uu}}$$

$$\gamma = \frac{\zeta s}{2u}; \qquad \alpha = \frac{u}{2t(s - uu)}$$

$$\eta = \frac{\zeta}{2u}; \qquad \delta = \frac{u}{2st(s - uu)}$$

$$\beta = \frac{1}{2t(s - uu)}; \quad \theta = \tfrac{1}{2}t\left(2(\mathfrak{A} + \mathfrak{E}ss) - \tfrac{s}{u}(\mathfrak{B} + \mathfrak{D}s)\right)$$

$$\varepsilon = \tfrac{1}{2}t(4\mathfrak{A}u - 3\mathfrak{B}s + \mathfrak{D}ss); \quad \kappa = \tfrac{1}{2}t(4\mathfrak{E}su + \mathfrak{B} - 3\mathfrak{D}s).$$

Coroll. 2.

Alio adhuc modo idem praeſtari poteſt. Extracto ſcilicet, vt ante, valore s ex hac aequatione:

$$\mathfrak{C} = \frac{\mathfrak{B}\mathfrak{B} - \mathfrak{D}\mathfrak{D}ss}{\mathfrak{A} - \mathfrak{E}ss} + \frac{2\mathfrak{A}\mathfrak{D} - 2\mathfrak{B}\mathfrak{E}s}{\mathfrak{B} - \mathfrak{D}s},$$

poſitoque breuitatis gratia $\frac{\mathfrak{A} - \mathfrak{E}ss}{\mathfrak{B} - \mathfrak{D}s} = u$, et, ſumto t pro arbitrio, erit:

$$\alpha = -\frac{1}{4tu}; \ \beta = 0; \ \gamma = \tfrac{1}{2}\sqrt{\frac{s(\mathfrak{B} + \mathfrak{D}s)}{u}}; \ \delta = \frac{s}{4tsu}$$

$$\varepsilon = t(4\mathfrak{A}u - \mathfrak{B}s - \mathfrak{D}ss); \ \zeta = \sqrt{\frac{u(\mathfrak{B} + \mathfrak{D}s)}{s}}; \ \eta = \tfrac{1}{2}\sqrt{\frac{\mathfrak{B} + \mathfrak{D}s}{us}}$$

$$\theta = 2tu; \ \kappa = t(\mathfrak{B} + \mathfrak{D}s - 4\mathfrak{E}su).$$

Coroll.

Coroll. 3.

Si fuerit $\mathfrak{A}:\mathfrak{E}=\mathfrak{B}\mathfrak{B}:\mathfrak{D}\mathfrak{D}$, aequatio cubica valori s definiendo fit inepta. Hoc autem incommodum facile tollitur, transformanda formula differentiali per positionem $y=y\pm a$; qua etiam forma numerationis non turbatur.

Scholion.

Posito $\mathfrak{N}=n\mathfrak{A}$, et $\mathfrak{Q}=n\mathfrak{B}$, integratio huius formulae:

$$\int \frac{dy(\mathfrak{P}+n\mathfrak{B}y+n\mathfrak{A}yy)}{\sqrt{(\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}y^2+2\mathfrak{D}y+\mathfrak{E})}},$$

semper reduci potest ad integrationem talis:

$$\int \frac{dx(P+Rxx)}{\sqrt{(Ax^4+Cxx+E)}},$$

quae, si dénominator $Ax^4+Cxx+E$ in huiusmodi duos factores reales $(f+gxx)(h+kxx)$ se resolui patitur, per rectificationem sectionum conicarum confici tur; at, si talis resolutio non succedit, sequenti artificio negotium absolui poterit.

Problema 3.

Si in formula $\int \frac{dx(P+Rxx)}{\sqrt{(Ax^4+Cx^2+E)}}$ quantitas Ax^4+Cx^2+E in factores reales huiusmodi $(f+gxx)(h+kxx)$ resolui nequeat, eam in aliam transformare, quae per arcus sectionum conicarum certo integrari queat.

Solutio.

Inducatur alia variabilis z, cuius relatio ad x hac aequatione exprimatur:

$$4Exxz^4 - 4xxzz\sqrt{AE} - 4Ezz + 2\sqrt{AE} - C = 0$$

vbi $\sqrt{A.E.}$ erit vtique quantitas realis, si quidem $Ax^4 + Cxx + E$ non habeat factores binomios reales. Hinc autem fiet:

$$\int \frac{dx(P + Rxx)}{\sqrt{(Ax^4 + Cx^2 + E)}} = \text{Const.} + \frac{Rx}{\sqrt{A}} - \frac{2R\sqrt{E}}{A} xzz$$
$$- 2\int \frac{dz\left(P - \frac{R\sqrt{E}}{\sqrt{A}} + \frac{2ER}{A} zz\right)}{\sqrt{\left(4Ez^4 + (C - 6\sqrt{AE})zz + 2A - \frac{C\sqrt{A}}{\sqrt{E}}\right)}}$$

in qua noua formula quantitas, in denominatore contenta, certe in duos factores binomios reales est resolubilis, cum sit $(C - 6\sqrt{AE})^2 > 16E\left(2A - \frac{C\sqrt{A}}{\sqrt{E}}\right)$; propterea quod hinc sequitur $CC + 4C\sqrt{AE} + 4AE = (C + 2\sqrt{AE})^2 > 0$.

Aliter.

Habeat noua variabilis z ad x talem relationem:

$$2Exxz^4 - Cxxzz + \frac{CC - 4AE}{8E} xx - 2Ezz = 0$$

eritque:

$$\int \frac{dx(P + Rxx)}{\sqrt{(Ax^4 + Cxx + E)}} = \frac{CR}{2A\sqrt{E}} x - \frac{2R\sqrt{E}}{A} xzz$$
$$- 2\int \frac{dz\left(P - \frac{CR}{2A} + \frac{2ER}{A} zz\right)}{\sqrt{\left(4Ez^4 - 2Czz + \frac{CC - 4AE}{4E}\right.}}$$

cuius

cuius denominator pariter certe in factores reales binomios est resolubilis.

Conclusio.

His demonstratis manifestum est, hanc formulam:

$$\int \frac{dy(\mathfrak{P} + n\mathfrak{B}y + n\mathfrak{A}yy)}{\sqrt{(\mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}y^2 + 2\mathfrak{D}y + \mathfrak{E})}}$$

semper per arcus sectionum conicarum construi posse. Cum igitur denominator semper in duos factores trinomiales reales resolui possit, hac formula ita exhiberi potest:

$$\int \frac{dy(\mathrm{A} + n(\alpha\varepsilon + \beta\delta)y + n\alpha\delta yy)}{\sqrt{(\alpha yy + 2\beta y + \gamma)(\delta yy + 2\varepsilon y + \zeta)}},$$

cuius ergo eadem datur constructio. Porro augendo vel diminuendo y quantitate constante, formula nostra etiam ita repraesentari potest:

$$\int \frac{dy(\mathrm{M} + \mathrm{N}yy)}{\sqrt{(\mathrm{A}y^4 + \mathrm{C}yy + 2\mathrm{D}y + \mathrm{E})}}.$$

In his autem fere omnes casus, quos quidem per rectificationem sectionum conicarum integrale licet, contineri videntur. Sed in medium afferamus adhuc aliam reductionem.

Problema IV.

Inuestigare conditiones, sub quibus integrationem huius formulae:

$\int \frac{dy(\mathfrak{P} + \mathfrak{Q}y + \mathfrak{R}yy)}{\sqrt{(\mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}yy + 2\mathfrak{D}y + \mathfrak{E})}}$ ad hanc simpliciorem $\int \frac{dx(\mathrm{P} + \mathrm{Q}x + \mathrm{R}xx)}{\sqrt{(2\mathrm{B}x^3 + \mathrm{C}x^2 + 2\mathrm{D}x)}}$ perducere liceat:

Solutio.

Statuatur inter variabiles x et y talis relatio:

$$\alpha xxyy + 2xy(\beta x + \gamma y) + \delta xx + \varepsilon yy + 2\zeta xy + 2\eta x + 2\theta y + \varkappa = 0,$$

cuius coefficientes ita determinentur, vt fit:

$$\beta\beta - \alpha\delta = 0;\ \gamma\gamma - \alpha\varepsilon = \mathfrak{A};\ \gamma\zeta - \alpha\theta - \beta\varepsilon = \mathfrak{B}$$

$$\theta\theta - \varepsilon\varkappa = 0;\ \eta\eta - \delta\varkappa = \mathfrak{E};\ \zeta\eta - \beta\varkappa - \delta\theta = \mathfrak{D}$$

atque $\zeta\zeta + 2\gamma\eta - \alpha\varkappa - \delta\varepsilon - 4\beta\theta = \mathfrak{C}$,

quem in finem definiatur primo p ex hac aequatione cubica:

$$p^3 - \tfrac{1}{2}\mathfrak{C}pp - (\mathfrak{A}\mathfrak{E} - \mathfrak{B}\mathfrak{D})p + \tfrac{1}{2}(\mathfrak{C}\mathfrak{A}\mathfrak{E} - \mathfrak{A}\mathfrak{D}\mathfrak{D} - \mathfrak{B}\mathfrak{B}\mathfrak{E}) = 0$$

Deinde, pro lubitu ſumto numero m, definiatur q ex hac aequatione quadratica: $qq - q(\mathfrak{D}m - \mathfrak{B}) + (m\mathfrak{E} - p)(mp - \mathfrak{A}) = 0$, quo facto, ſi denuo numerus arbitrarius accipiatur n, erit:

$$\beta = \frac{n(m\mathfrak{E} - p)}{\sqrt{(2mp - \mathfrak{A} - mm\mathfrak{E})}};\quad \theta = \frac{mp - \mathfrak{A}}{n\sqrt{(2mp - \mathfrak{A} - mm\mathfrak{E})}}$$

$$\alpha = \frac{nq}{\sqrt{(2mp - \mathfrak{A} - mm\mathfrak{E})}};\quad \varkappa = \frac{q}{n\sqrt{(2mp - \mathfrak{A} - mm\mathfrak{E})}}$$

$$\delta = \frac{n(m\mathfrak{E} + p)^2}{q\sqrt{(2mp - \mathfrak{A} - mm\mathfrak{E})}};\quad \varepsilon = \frac{(mp - \mathfrak{A})^2}{nq\sqrt{(2mp - \mathfrak{A} - mm\mathfrak{E})}}$$

$$\gamma = \frac{m\sqrt{(pp - \mathfrak{A}\mathfrak{E})}}{\sqrt{(2mp - \mathfrak{A} - \mathfrak{E}mm)}};\quad \eta = \frac{\sqrt{(pp - \mathfrak{A}\mathfrak{E})}}{\sqrt{(2mp - \mathfrak{A} - mm\mathfrak{E})}}$$

et $\zeta = \dfrac{\mathfrak{D}(mp - \mathfrak{A}) - \mathfrak{B}(m\mathfrak{E} - p)}{\sqrt{(pp - \mathfrak{A}\mathfrak{E})(2mp - \mathfrak{A} - mm\mathfrak{E})}}$.

Quibus inuentis erit:

$$B = \beta\zeta - \alpha\eta - \gamma\delta;\quad D = \zeta\theta - \gamma\varkappa - \varepsilon\eta$$

et $C = \zeta\zeta + 2\beta\theta - \alpha\varkappa - \delta\varepsilon - 4\gamma\eta$.

Pona-

Ponatur iam:

$$\int \frac{dy(\mathfrak{P} + \mathfrak{Q}y + \mathfrak{R}yy)}{\sqrt{(\mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}y^2 + 2\mathfrak{D}y + \mathfrak{E})}} = \text{Const.} + mx + ny + pxy - \int \frac{dx(P + Qx + Rxx)}{\sqrt{(2Bx^3 + Cx^2 + 2Dx)}}$$

atque reperitur, vt ante,

$$m = \frac{\beta\mathfrak{R}}{\mathfrak{A}};\quad n = \frac{\gamma\mathfrak{R}}{\mathfrak{A}};\ \text{et}\ p = \frac{\alpha\mathfrak{R}}{\mathfrak{A}}$$

deinde $P = \mathfrak{P} + \frac{(\beta\theta - \gamma\eta)\mathfrak{R}}{\mathfrak{A}}$; $\mathfrak{Q} = \frac{B\mathfrak{R}}{\mathfrak{A}}$ et $R = 0$.
Necesse autem est, vt in formula proposita sit $\mathfrak{A}\mathfrak{Q} = \mathfrak{B}\mathfrak{R}$, neque ergo haec reductio nouos casus suppeditat. At posito $x = zz$, formula transformata abit in hanc:

$$-2\int \frac{dz(P + Qzz)}{\sqrt{(2Bz^4 + Cz^2 + 2D)}}$$

quae reductio saepe facilius succedit, quam praecedens.
