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Generalized twisted quantum doubles and the McKay correspondence

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Dedicated to Susan Montgomery

Abstract

We consider a class of quasiHopf algebras which we call *generalized twisted quantum doubles*. They are abelian extensions $H = \mathbb{C}[\bar{G}]^* \bowtie \mathbb{C}[G]$ (G is a finite group, $\bar{G}$ a homomorphic image, and * denotes the dual algebra), possibly twisted by a 3-cocycle, and are a natural generalization of the twisted quantum double construction of Dijkgraaf, Pasquier and Roche. We show that if G is a subgroup of $SU_2(\mathbb{C})$ then H exhibits an orbifold McKay Correspondence: certain fusion rules of H define a graph with connected components indexed by conjugacy classes of $\bar{G}$, each connected component being an extended affine Diagram of type ADE whose McKay correspondent is the subgroup of G stabilizing an element in the conjugacy class. This reduces to the original McKay Correspondence when $\bar{G} = 1$.
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1 Introduction

In an influential paper [DPR], Dijkgraaf, Pasquier and Roche introduced the twisted quantum double $D^\omega(G)$ of a finite group G. This is a quasiHopf algebra obtained by twisting the Drinfeld double $D(G)$ by a 3-cocycle of G. They also suggested that the irreducible representations of $D^\omega(G)$ naturally correspond to the irreducible representations of a certain kind of conformal field theory (i.e. vertex operator algebra), namely a holomorphic orbifold[1] V^G. Although this idea remains unproven at the level of mathematical rigor, it is almost certainly true. It is natural to ask if there are variants of the twisted double construction which might similarly correspond to other rational conformal field theories. Indeed, Dijkgraaf, Pasquier and Roche explicitly raised this question (*loc. cit.*) for the case of theories with central charge $c = 1$, where it is expected that most examples arise from G-orbifolds where G is a finite subgroup of $SU_2(\mathbb{C})$.

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[1]Here, V is a holomorphic vertex operator algebra

The purpose of the present paper is to consider a class of quasiHopf algebras which we call *generalized twisted quantum doubles*. They correspond to abelian extensions $H = \mathbb{C}[\bar{G}]^* \bowtie \mathbb{C}[G]$ where $\bar{G} = G/N$ with $N \trianglelefteq G$, and may be twisted by a 3-cocycle of G that is inflated from a 3-cocycle of $\bar{G}$. This produces a quasiHopf algebra that reduces to the DPR construction when $N = 1$. If G is a finite subgroup of $SU_2(\mathbb{C})$ and N has order at most 2, we will see that these quasiHopf algebras possess an *orbifold McKay Correspondence*. By this we mean that we can associate a graph $\mathcal{G}$ to H with connected components indexed by the conjugacy classes of $\bar{G}$. Each connected component of $\mathcal{G}$ is an extended affine diagram of type ADE. The edges represent fusion rules in H obtained by tensoring with the canonical 2-dimensional representation of G, which is naturally a representation of H as well. Thus the graph $\mathcal{G}$ is indeed an orbifold version of the original correspondence of McKay [Mc].

We discuss several aspects of the orbifold McKay correspondence.
1. It is well-known in the physics literature that there is an ADE classification of $c = 1$ rational conformal field theories (cf. [FMS] for more detail and further references). This arises in a manner that is rather different than our orbifold correspondence, but suggests nevertheless that generalized twisted quantum doubles may indeed be related to certain orbifold conformal field theories.
2. The calculation of fusion rules of vertex operator algebras is usually difficult (cf. [ADL] for an example of relevance to the present paper). The possibility of a McKay correspondence for fusion rules of vertex operator algebras suggests that, in some cases at least, there may be a more enlightening way to carry out the calculations. The case when $N = 1$, for example, shows (conjecturally) that *all* holomorphic orbifolds V^G ($G \subseteq SU_2(\mathbb{C})$) exhibit an orbifold McKay correspondence.
3. In current attempts to understand more general McKay Correspondences (e.g. [IR]), the duality that exists between conjugacy classes and representations plays a rôle as an analog of the duality between homology and cohomology. On the other hand, the graph $\mathcal{G}$ has a built-in duality: connected components are indexed by conjugacy classes, and nodes of a connected component are indexed by irreducible modules over the stabilizer.

The paper is organized as follows. In Section 2 we introduce our generalized twisted quantum doubles. We briefly develop some of their basic properties and the relationship between them and the usual twisted quantum doubles. In Section 3 we calculate fusion rules for generalized twisted quantum doubles. Some cases were already considered in [DPR]. In Section 4 we describe the orbifold McKay correspondence. We will not treat the application of these results to orbifold conformal field theory here, but hope to return to this topic in the future.

Finally, it is a pleasure to record our debt to Susan Montgomery. She has had an enormous effect on our work as a friend, colleague, mentor, collaborator, and as the author of the most accessible book on Hopf algebras.

2 Generalized twisted quantum doubles

For background on Hopf algebras and related topics that we use here, the reader is referred to [Mo].

Fix the following data and notation: G is a finite group, $N \trianglelefteq G$ a normal subgroup, and $\bar{G} = G/N$. We use the 'bar convention' for elements in $\bar{G}$, i.e., if $g \in G$ then $\bar{g} = gN \in \bar{G}$. G acts by (right) conjugation on $\bar{G}$, so that $\bar{g}^x = \bar{g}^{\bar{x}} = \bar{x}^{-1}\bar{g}\bar{x}$. $\mathbb{C}[G]$ is the (complex) group algebra of G and $\mathbb{C}[G]^*$ the dual of the group algebra. Let $\omega \in Z^3(\bar{G}, \mathbb{C}^*)$ be a multiplicative, normalized 3-cocycle on $\bar{G}$ with $\omega' = \mathrm{Infl}_{\bar{G}}^{G}\, \omega$ the inflation of ω to G. Thus $\omega'(g, x, y) = \omega(\bar{g}, \bar{x}, \bar{y})$ for $g, x, y \in G$. The associated 2-cochains θ and γ are

$$\begin{aligned}
\theta_{\bar{g}}(\bar{x}, \bar{y}) &= \frac{\omega(\bar{g}, \bar{x}, \bar{y})\omega(\bar{x}, \bar{y}, \bar{g}^{xy})}{\omega(\bar{x}, \bar{g}, \bar{y}^{g})}, \\
\gamma_{\bar{g}}(\bar{x}, \bar{y}) &= \frac{\omega(\bar{x}, \bar{y}, \bar{g})\omega(\bar{g}, \bar{x}^{g}, \bar{y}^{g})}{\omega(\bar{x}, \bar{g}, \bar{y}^{g})}.
\end{aligned}$$

For clarity we sometimes use the notation θ' and γ' for the corresponding 2-cochains associated with ω', so that $\theta'_g(x, y) = \theta_{\bar{g}}(\bar{x}, \bar{y})$ and $\gamma'_g(x, y) = \gamma_{\bar{g}}(\bar{x}, \bar{y})$.

Define

$$D^{\omega}(G, N) = \mathbb{C}[\bar{G}]^* \bowtie \mathbb{C}[G],$$

where we use $\bowtie$ in place of $\otimes$ for notational convenience. The product, coproduct, associator, counit, antipode and α and β elements are defined as follows:

$$\begin{aligned}
e(\bar{g}) \bowtie x . e(\bar{h}) \bowtie y &= \delta_{\bar{g}^x, \bar{h}} \theta_{\bar{g}}(\bar{x}, \bar{y}) e(\bar{g}) \bowtie xy, \\
\Delta\ e(\bar{g}) \bowtie x &= \sum_{\bar{a}\bar{b} = \bar{g}} \gamma_{\bar{x}}(\bar{a}, \bar{b}) e(\bar{a}) \bowtie x \otimes e(\bar{b}) \bowtie x, \\
\Phi &= \sum_{\bar{g}, \bar{h}, \bar{k}} \omega(\bar{g}, \bar{h}, \bar{k})^{-1} e(\bar{g}) \bowtie 1 \otimes e(\bar{h}) \bowtie 1 \otimes e(\bar{k}) \bowtie 1, \\
\epsilon(e(\bar{g}) \bowtie x) &= \delta_{\bar{g}, 1}, \\
S(e(\bar{g}) \bowtie x) &= \theta_{\bar{g}^{-1}}(\bar{x}, \bar{x}^{-1}) \gamma_{\bar{x}}(\bar{g}, \bar{g}^{-1})^{-1} e(\bar{g}^{-\bar{x}}) \bowtie x^{-1} \\
\alpha &= \mathrm{Id} = \sum_{\bar{g}} e(\bar{g}) \bowtie 1. \\
\beta &= \sum_{\bar{g} \in \bar{G}} \omega(\bar{g}, \bar{g}^{-1}, \bar{g}) e(\bar{g}) \bowtie 1.
\end{aligned}$$

This definition is, of course, modeled after the original twisted quantum double of Dijkgraaf-Pasquier-Roche ([DPR], [Ka]), and the proof that it turns $D^{\omega}(G, N)$ into a quasiHopf algebra is the same. One simply has to make sure that the θ- and γ-coefficients behave properly, and this is taken care of because the cocycle ω' on G is inflated from ω. Note that $D^{\omega}(G, N)$ is also a cocentral abelian extension of Hopf algebras (cf. [KMM], Section 2). We call $D^{\omega}(G, N)$ a generalized (twisted) quantum double. If $N = G$ or $N = 1$, then $D^{\omega}(G, N)$ is the group algebra $\mathbb{C}[G]$ or the twisted quantum double $D^{\omega}(G)$ respectively.

There are maps

$$D^{\omega}(G, N) \xrightarrow{\varphi} D^{\omega'}(G) \xrightarrow{\psi} D^{\omega}(\bar{G}). \tag{1}$$

defined by

$$\varphi : e(\bar{g}) \bowtie x \mapsto \sum_{n\in N} e(gn) \bowtie x, \tag{2}$$
$$\psi : e(g) \bowtie x \mapsto e(\bar{g}) \bowtie \bar{x}. \tag{3}$$

Because ω' is inflated from ω, it is evident that ψ is a morphism of quasiHopf algebras. We assert that φ is also a morphism of quasiHopf algebras. We have

$$\begin{aligned}
\varphi(e(\bar{g}) \bowtie x).\varphi(e(\bar{h}) \bowtie y) &= \sum_{m,n\in N} e(gm) \bowtie x.e(hn) \bowtie y \\
&= \sum_{m,n\in N} \delta((gm)^x, hn)\theta'_{gm}(x,y)e(gm) \bowtie xy \\
&= \delta(\bar{g}^x, \bar{h})\theta_{\bar{g}}(\bar{x},\bar{y}) \sum_{m\in N} e(gm) \bowtie xy \\
&= \delta(\bar{g}^x, \bar{h})\theta_{\bar{g}}(\bar{x},\bar{y})\varphi(e(\bar{g}) \bowtie xy) \\
&= \varphi(e(\bar{g}) \bowtie x.e(\bar{h}) \bowtie y).
\end{aligned}$$

So φ preserves multiplication. Similarly,

$$\begin{aligned}
\Delta\varphi(e(\bar{g}) \bowtie x) &= \sum_{n\in N} \Delta e(gn) \bowtie x \\
&= \sum_{n\in N} \sum_{ab=gn} \gamma'_x(a,b)e(a) \bowtie x \otimes e(b) \bowtie x \\
&= \sum_{ab\in gN} \gamma_{\bar{x}}(\bar{a},\bar{b})e(a) \bowtie x \otimes e(b) \bowtie x,
\end{aligned}$$

and

$$\begin{aligned}
(\varphi\otimes\varphi)(\Delta e(\bar{g}) \bowtie x) &= \sum_{\bar{a}\bar{b}=\bar{g}} \gamma_{\bar{x}}(\bar{a},\bar{b})(\varphi\otimes\varphi)(e(\bar{a}) \bowtie x \otimes e(\bar{b}) \bowtie x) \\
&= \sum_{\bar{a}\bar{b}=\bar{g}} \gamma_{\bar{x}}(\bar{a},\bar{b}) \sum_{m,n\in N} e(am) \bowtie x \otimes e(bn) \bowtie x \\
&= \sum_{a,b\in G, \bar{a}\bar{b}=\bar{g}} \gamma_{\bar{x}}(\bar{a},\bar{b})e(a) \bowtie x \otimes e(b) \bowtie x,
\end{aligned}$$

so that φ preserves comultiplication. We also check that φ preserves counits, associator, antipode, α and β elements. Hence, φ is indeed a morphism of quasiHopf algebras. Let $H = im\ \varphi$.

Now consider the left adjoint action

$$\mathrm{ad}_l u(v) = \sum u_1 v(Su_2),$$

where we are using Sweedler notation $\Delta u = \sum u_1 \otimes u_2$. Taking $u = e(h) \bowtie y, v = \varphi(e(\bar{g}) \bowtie x)$,

$$\begin{aligned}
\mathrm{ad}_l u(v) &= \sum_{\substack{a,b\in G\\ ab=h}} \sum_{m\in N} \gamma'_y(a,b)\theta'_{b^{-1}}(y,y^{-1})\gamma'_y(b,b^{-1})^{-1}e(a) \bowtie y.e(gm) \bowtie x.e(b^{-y}) \bowtie y^{-1} \\
&= \sum_{\substack{a,b\in G\\ ab=h}} \sum_{m\in N} \gamma'_y(a,b)\theta'_{b^{-1}}(y,y^{-1})\gamma'_y(b,b^{-1})^{-1}\theta'_a(y,x)\theta'_a(yx,y^{-1})\delta_{a^y,gm}\delta_{a^{yx},b^{-y}}e(a) \bowtie x^{y^{-1}}. \quad (4)
\end{aligned}$$

Suppose that H is a *normal* subquasiHopf algebra of $D^{\omega'}(G)$. Then (4) must lie in $im\ \varphi$ for all choices of g, h, x and y. A summand can only be nonzero in case $a = (gm)^{y^{-1}}, b = a^{-yxy^{-1}}$ and $h = ab = ((gm)(gm)^{-x})^{y^{-1}}$. If we therefore *choose* $h = (gg^{-x})^{y^{-1}}$ (corresponding to $m = 1$), then the coefficient of $e(g^{y^{-1}}) \bowtie x^{y^{-1}}$ is a product of theta- and gamma-values, and in particular is nonzero. Because (4) lies in $im\ \varphi$ then the coefficients of $e(g^{y^{-1}}m) \bowtie x^{y^{-1}}$ are nonzero for each $m \in N$. (Indeed, all such coefficients are equal to that of $e(g^{y^{-1}}) \bowtie x^{y^{-1}}$.) Therefore, the previous discussion shows that $gg^{-x} = (gm)(gm)^{-x}$ is *independent* of $m \in N$, and from this we readily find that $N \subseteq Z(G)$, the center of G.

Conversely, assume that $N \subseteq Z(G)$ with $h = (gg^{-x})^{y^{-1}}$. Setting $t = g^{y^{-1}}, u = x^{y^{-1}}$ and remembering that ω' is inflated from ω, (4) reads

$$\mathrm{ad}_l u(v) \quad = \quad \sum_{m \in N} \gamma_{\bar{y}}(\bar{t}, \bar{t}^{-u})\theta_{\bar{t}^u}(y, y^{-1})\gamma_{\bar{y}}(\bar{t}^{-u}, \bar{t}^u)^{-1}\theta_{\bar{t}}(y, x)\theta_{\bar{t}}(yx, y^{-1})e(tm) \bowtie u. \tag{5}$$

In particular, the coefficient of $e(tm) \bowtie u$ is independent of m, so that (5) indeed lies in H. One similarly checks that the right adjoint $\mathrm{ad}_r u(v)$ also lies in H. So we have proved (cf. [Mo], p.33) that H is a normal subquasiHopf algebra of $D^{\omega'}(G)$ if, and only if, $N \subseteq Z(G)$.

Let us continue to assume that $N \subseteq Z(G)$, and set $H^+ = H \cap \ker \epsilon, D = D^{\omega'}(G)$. Then DH^+ is a quasiHopf ideal in $D, D/DH^+$ a quasiHopf algebra, and the canonical projection $D \to D/DH^+$ is a morphism of quasiHopf algebras. As a basis of D/DH^+ we may take the (images of) the elements $e(m) \bowtie 1$ for $m \in N$. Because ω is normalized and ω' is inflated from ω, we have $e(m) \bowtie 1.e(n) \bowtie 1 = \delta_{m,n}e(m) \bowtie 1$. We thus obtain the following result.

Lemma 2.1 *Let φ be as in (1), (2) and set $D = D^{\omega'}(G), H = im\ \varphi$. Then φ is a morphism of quasiHopf algebras. $H \subseteq D$ is a normal subquasiHopf algebra if, and only if, $N \subseteq Z(G)$. In this case, there is an isomorphism of quasiHopf algebras $D/DH^+ \cong \mathbb{C}[N]^*$.* $\square$

As is well-known (cf.[Mo], Chapter 7), the group algebra $\mathbb{C}[G]$ can be expressed as a *crossed product* $\mathbb{C}[N]\#_\sigma\mathbb{C}[\bar{G}]$. Assuming for simplicity that $N \subseteq Z(G)$ (the case of most interest to us), multiplication in the crossed product is

$$(m\#\bar{g})(n\#\bar{h}) = mn\sigma(\bar{g}, \bar{h})\#\overline{gh}$$

where $\sigma \in Z^2(\bar{G}, N)$ is a 2-cocycle determined by the central extension $1 \to N \to G \to \bar{G} \to 1$. Much as in the earlier calculations leading to Lemma 2.1, we can check that there is an analogous description of $D^\omega(G, N)$ as a crossed product using the *same* 2-cocycle. Precisely, we have

Lemma 2.2 *There is an isomorphism of quasiHopf algebras*

$$D^\omega(G, N) \cong \mathbb{C}[N]\#_\sigma D^\omega(\bar{G}).$$

$\square$

We will not need this result so we skip the proof, which is similar to the group case (*loc. cit.*, p.103).

By Lemma 2.2, $D^\omega(G, N)$ is a central extension of the twisted quantum double $D^\omega(\bar{G})$, so that the representations of $D^\omega(G, N)$ are *projective* representations of $D^\omega(\bar{G})$. From Lemma 2.1, we can also use Clifford theory and obtain representations of $D^\omega(G, N)$ by restriction of representations of $D^{\omega'}(G)$.

3 Fusion rules

The simple modules over $D^\omega(G,N)$ can be described as follows [KMM]. Choose $\{\bar g\}$ to be a set of representatives for the conjugacy classes of $\bar G$, and let $\{y_{\bar g,i}\}$ be a set of coset representatives of $C_G(\bar g)$. (In what follows, we often drop the subscripts in situations where the unique possible y can be determined from context, and we use z and w as coset representatives of other centralizers.) Then the simple modules are

$$\{\mathrm{Ind}_{C_G(\bar g)}^G V \mid V \text{ is a simple } \mathbb{C}^{\theta'_g}[C_G(\bar g)]\text{-module}\}.$$

If V is a module over $\mathbb{C}^{\theta'_g}[C_G(\bar g)]$, we let $\widehat V = \mathrm{Ind}_{C_G(\bar g)}^G(V)$ denote the corresponding $D^\omega(G,N)$-module. Fix $\bar g$ and let χ_V be the character afforded by V. The corresponding character $\widehat\chi$ of $\widehat V$ is then given by

$$\begin{aligned}\widehat\chi_{\widehat V}(e(\bar h)\bowtie x) &= \delta_{\bar h^y,\bar g}\ \delta_{x^y\in C_G(\bar g)}\ \theta'_{g^{y^{-1}}}(x,y)\theta'_g(y,x^y)^{-1}\chi_V(x^y)\\ &= \delta_{\bar h^y,\bar g}\ \delta_{x^y\in C_G(\bar g)}\ \chi_V(x^y)\Theta(g,x,y),\end{aligned}$$

where the first δ-function arises because the character value is zero unless $\bar h$ and $\bar g$ are conjugate, y is the chosen coset representative that conjugates $\bar h$ to $\bar g$ in $\bar G$, and

$$\Theta(g,x,y) := \theta'_{g^{y^{-1}}}(x,y)\theta'_g(y,x^y)^{-1}.$$

We choose the values of Θ to be roots of unity.

In what follows, we let U be a $C_G^{\theta'_f}(\bar f)$-module, V a $C_G^{\theta'_g}(\bar g)$-module, and W be a $C_G^{\theta'_h}(\bar h)$-module where $\bar f,\bar g,\bar h\in\bar G$ are three of the conjugacy class representatives. Let $\bar J$, $\bar K$, and $\bar L$ be the conjugacy classes of $\bar G$ that contain $\bar f$, $\bar g$, and $\bar h$, respectively. Define an inner product on the characters α,β of $D^\omega(G,N)$ as follows:

$$\langle\alpha,\beta\rangle := \frac{1}{|G|}\sum_{\bar k\in\bar G}\sum_{x\in G}\alpha(e(\bar k)\bowtie x)\overline{\beta(e(\bar k)\bowtie x)}.$$

In particular,

$$\left\langle\widehat\chi_{\widehat V},\widehat\chi_{\widehat U}\right\rangle = \frac{1}{|G|}\sum_{\bar k\in\bar G}\sum_{x\in G}\delta_{\bar k^y,\bar g}\ \delta_{\bar k^w,\bar f}\ \delta_{x\in Q}\ \chi_V(x^y)\overline{\chi_U(x^w)}\Theta(g,x,y)\overline{\Theta(f,x,w)}, \tag{6}$$

where $Q = yC_G(\bar g)y^{-1}\cap wC_G(\bar f)w^{-1}$. Notice that the inner product is zero unless $\bar f=\bar g$ (and thus $w=y$). In this case,

$$\begin{aligned}\left\langle\widehat\chi_{\widehat V},\widehat\chi_{\widehat U}\right\rangle &= \frac{1}{|G|}\sum_{\bar k\in\bar G}\sum_{x\in G}\delta_{\bar k^y,\bar g}\ \delta_{x\in yC_G(\bar g)y^{-1}}\ \chi_V(x^y)\overline{\chi_U(x^y)}\Theta(g,x,y)\overline{\Theta(g,x,y)}\\ &= \frac{|\bar K|}{|G|}\sum_{x^y\in C_G(\bar g)}\chi_V(x^y)\overline{\chi_U(x^y)}\\ &= \frac{1}{|C_G(\bar g)|}\sum_{z\in C_G(\bar g)}\chi_V(z)\overline{\chi_U(z)}\\ &= \langle\chi_V,\chi_U\rangle\,.\end{aligned}$$

The last expression here is the usual inner product of characters of the group $C_G(\bar{g})$, although the characters may be projective. If the characters are *ordinary*, the orthogonality relations for group characters imply that $\langle \chi_V, \chi_U \rangle = \delta_{V,U}$ in case V, U are irreducible. In general, we view the characters as ordinary characters of a *covering group* of $C_G(\bar{g})$, and the orthogonality relations (applied to characters of the covering group) imply the same result. We conclude that the irreducible characters of $D^\omega(G,N)$ form an orthonormal basis with respect to (6).

Using the coproduct Δ, we derive the character of the tensor product module $\widehat{V} \otimes \widehat{W}$.

$$\begin{aligned}
\widehat{\chi}_{\widehat{V}\otimes\widehat{W}}(e(\bar{k}) \bowtie x) &= \sum_{\bar{a}\bar{b}=\bar{k}} \gamma_{\bar{x}}(\bar{a},\bar{b}) \widehat{\chi}_{\widehat{V}}(e(\bar{a}) \bowtie x) \widehat{\chi}_{\widehat{W}}(e(\bar{b}) \bowtie x) \\
&= \sum_{\bar{a}\bar{b}=\bar{k}} \delta_{\bar{a}^y,\bar{g}}\ \delta_{\bar{b}^z,\bar{h}}\ \delta_{x\in C_G(\bar{a})\cap C_G(\bar{b})} \chi_V(x^y)\chi_W(x^z)\, \cdot \\
&\quad \cdot\ \gamma_{\bar{x}}(\bar{a},\bar{b})\Theta(g,x,y)\Theta(h,x,z).
\end{aligned}$$

From this, we obtain

$$\begin{aligned}
\left\langle \widehat{\chi}_{\widehat{V}\otimes\widehat{W}}, \widehat{\chi}_{\widehat{U}} \right\rangle &= \frac{1}{|G|} \sum_{\bar{k}\in\bar{G}} \sum_{x\in G} \sum_{\substack{\bar{a},\bar{b} \\ (\bar{a}\bar{b}=\bar{k})}} \delta_{\bar{k}^w,\bar{f}}\ \delta_{\bar{a}^y,\bar{g}}\ \delta_{\bar{b}^z,\bar{h}}\ \delta_{x\in C_G(\bar{a})\cap C_G(\bar{b})} \chi_V(x^y)\chi_W(x^z)\overline{\chi_U(x^w)}\, \cdot \\
&\quad \cdot\ \gamma'_x(a,b)\Theta(g,x,y)\Theta(h,x,z)\overline{\Theta(f,x,w)}, \qquad (7)
\end{aligned}$$

that is

$$\begin{aligned}
\left\langle \widehat{\chi}_{\widehat{V}\otimes\widehat{W}}, \widehat{\chi}_{\widehat{U}} \right\rangle &= \frac{\delta_{\bar{J}\subseteq\bar{K}\bar{L}}}{|G|} \sum_{\substack{\bar{j}\in\bar{J} \\ (\bar{j}^w=\bar{f})}} \sum_{\substack{(\bar{a},\bar{b})\in\bar{K}\times\bar{L} \\ (\bar{a}\bar{b}=\bar{j}) \\ (\bar{a}^y=\bar{g}) \\ (\bar{b}^z=\bar{h})}} \sum_{x\in C_G(\bar{a})\cap C_G(\bar{b})} \chi_V(x^y)\chi_W(x^z)\overline{\chi_U(x^w)}\, \cdot \\
&\quad \cdot\ \gamma'_x(a,b)\Theta(g,x,y)\Theta(h,x,z)\overline{\Theta(f,x,w)}.
\end{aligned}$$

We specialize (7) to the case where $\bar{h} = \bar{1}$. Here, the previous displayed expression vanishes unless perhaps $\bar{f} = \bar{g}$ (and thus $w = y$). In this case,

$$\left\langle \widehat{\chi}_{\widehat{V}\otimes\widehat{W}}, \widehat{\chi}_{\widehat{U}} \right\rangle = \frac{1}{|G|} \sum_{\substack{\bar{j}\in\bar{J} \\ (\bar{j}^w=\bar{f})}} \sum_{\substack{x\in C_G(\bar{k}) \\ (x^w\in C_G(\bar{f}))}} \chi_V(x^w)\chi_W(x)\overline{\chi_U(x^w)}\Theta(f,x,w)\overline{\Theta(f,x,w)}.$$

Since W is a G-module, we can reindex the second summation over $x^w(=t)$ to get

$$\begin{aligned}
\left\langle \widehat{\chi}_{\widehat{V}\otimes\widehat{W}}, \widehat{\chi}_{\widehat{U}} \right\rangle &= \frac{1}{|G|} \sum_{\substack{\bar{j}\in\bar{J} \\ (\bar{j}^w=\bar{f})}} \sum_{t\in C_G(\bar{f})} \chi_V(t)\chi_W(t)\overline{\chi_U(t)} \\
&= \frac{|\bar{J}|}{|G|} \sum_{t\in C_G(\bar{f})} \chi_V(t)\chi_W(t)\overline{\chi_U(t)} \\
&= \langle \chi_{V\otimes W}, \chi_U \rangle. \qquad (8)
\end{aligned}$$

The last expression is the usual fusion rule for the twisted group algebra $\mathbb{C}^{\theta_f}[C_G(\bar{f})]$, and we regard W as a module for $C_G(\bar{f})$ by restriction. We have therefore shown that fusion rules for $D^\omega(G,N)$ that involve a representation W coming from $G = C_G(\bar{1})$ can be computed locally in the stabilizer $C_G(\bar{f})$.

4 Orbifold McKay Correspondence

From now on we specialize to the case that $G \subseteq SU_2(\mathbb{C})$. We further assume that G contains the (unique) involution $t = -I \in SU_2(\mathbb{C})$. So we have the following commuting diagram with short exact rows.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & SU_2(\mathbb{C}) & \longrightarrow & SO_3(\mathbb{R}) & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & G & \longrightarrow & \bar{G} & \longrightarrow & 1
\end{array}$$

G is a so-called *binary polyhedral group*, defined abstractly via

$$\langle x, y, z \mid x^a = y^b = z^c = xyz \rangle,$$

often denoted simply by $\langle a, b, c \rangle$. The maximal cyclic subgroups of G have orders $2a, 2b$ and $2c$. Apart from the cyclic case, $\langle a, b, c \rangle$ is finite only in the following cases:

$$\langle a,b,c\rangle = \begin{cases} \langle 2,2,n\rangle \ (BD_{2n} = \text{binary dihedral, order } 4n), \\ \langle 2,3,3\rangle \ (SL_2(3) = \text{binary tetrahedral, order } 24), \\ \langle 2,3,4\rangle \ (SL_2(3).2 = \text{binary octahedral, order } 48), \\ \langle 2,3,5\rangle \ (SL_2(5) = \text{binary icosahedral, order } 120). \end{cases}$$

Each subgroup of G is also isomorphic to one of these groups. The quotient groups $\bar{G}$ are cyclic of order n (Z_n), dihedral of order $2n$ (D_{2n}), A_4, S_4 and A_5 respectively. The G-orbits on $\bar{G}$ are just the conjugacy classes of $\bar{G}$.

It is well-known ([B]) that the second cohomology group (Schur multiplier) $H^2(G, \mathbb{C}^*)$ is trivial for each of these groups and subgroups. Thus, each of the 2-cocycles θ'_g is a coboundary, so that the twisted group algebra $\mathbb{C}^{\theta'_g}[C_G(\bar{g})]$ is isomorphic to the corresponding untwisted group algebra. It follows that the number of irreducible modules for $D^\omega(G, N)$ is independent of ω and is equal to the sum of the number of irreducibles for the stabilizers $C_G(\bar{g})$, one from each conjugacy class of $\bar{G}$. If $\bar{g}$ has order greater than 2 then $C_G(\bar{g})$ is a cyclic group of order $2b$ or $2c$. The individual cases are readily computed. Taking N of order 2, for example, there are the following possibilities:

1. G cyclic of order $2n$.

$\bar{g}$	$C_G(\bar{g})$	# irreps of $C_G(\bar{g})$
any	G	$2n$

Total # irreps $= 2n^2$.

2. $G = \langle 2, 2, n \rangle$ (n odd).

$\bar{g}$	$C_G(\bar{g})$	# irreps of $C_G(\bar{g})$
$\bar{1}$	G	$n+3$
$(n-1)/2$ classes	Z_{2n}	$2n$
involution	Z_4	4

Total # irreps $= n^2 + 7$.

3. $G = \langle 2, 2, n \rangle$ (n even).

$\bar{g}$	$C_G(\bar{g})$	# irreps of $C_G(\bar{g})$
$\bar{1}$	G	$n+3$
central involution	G	$n+3$
$(n-2)/2$ classes	Z_{2n}	$2n$
involution (2 classes)	BD_2	5

Total # irreps $= n^2 + 16$.

4. $G = \langle 2, 3, 3 \rangle$.

$\bar{g}$	$C_G(\bar{g})$	# irreps of $C_G(\bar{g})$
$\bar{1}$	G	7
(12)(34)	BD_2	5
(123), (132)	Z_6	6

Total # irreps = 24.

5. $G = \langle 2, 3, 4 \rangle$.

$\bar{g}$	$C_G(\bar{g})$	# irreps of $C_G(\bar{g})$
$\bar{1}$	G	8
(12)	BD_2	5
(12)(34)	BD_4	7
(1234)	Z_8	8
(123)	Z_6	6

Total # irreps = 34.

6. $G = \langle 2, 3, 5 \rangle$.

$\bar{g}$	$C_G(\bar{g})$	# irreps of $C_G(\bar{g})$
$\bar{1}$	G	9
(12)(34)	BD_2	5
(123)	Z_6	6
(12345), (12354)	Z_{10}	10

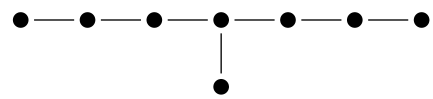
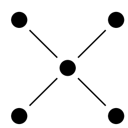
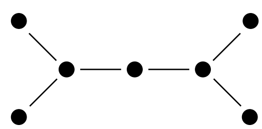
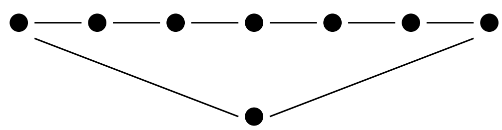
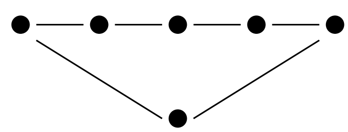
Total # irreps = 40.

Let W be the canonical 2-dimensional irreducible G-module affording the embedding $G \to SU_2(\mathbb{C})$ if G is *not* cyclic. If G is cyclic, we take W to be the direct sum of a 1-dimensional faithful G-module and its dual. As a $C_G(\bar{1})$-module, we may, and shall, consider W as a module over $D^\omega(G, N)$. In the notation of Section 3, $W = \widehat{W}$.

Following the original construction of McKay ([Mc]), introduce a graph $\mathcal{G}$ whose vertices are the irreducible modules over $D^\omega(G, N)$. If $\widehat{V}, \widehat{U}$ are two vertices, we connect them by $\langle \widehat{\chi}_{\widehat{V} \otimes \widehat{W}}, \widehat{\chi}_{\widehat{U}} \rangle$ edges. We saw in the last Section that $\langle \widehat{\chi}_{\widehat{V} \otimes \widehat{W}}, \widehat{\chi}_{\widehat{U}} \rangle = 0$ unless perhaps V and U are both irreducible modules for some $\mathbb{C}^{\theta'_f}[C_G(\bar{f})]$, in which case $\langle \widehat{\chi}_{\widehat{V} \otimes \widehat{W}}, \widehat{\chi}_{\widehat{U}} \rangle = \langle \chi_{V \otimes W}, \chi_U \rangle$. In particular, vertices indexed by modules in *distinct* stabilizers are *not* connected. On the other hand, for fixed $\bar{f}$, the multiplicities $\langle \chi_{V \otimes W}, \chi_U \rangle$ are *precisely* those which result from the McKay procedure (*loc. cit.*) applied to the stabilizer $C_G(\bar{f})$. Applying the McKay Correspondence in these cases, we arrive at our main result, which we state as follows.

Theorem 4.1 *Let the notation be as above. The connected components of the graph $\mathcal{G}$ are indexed by the conjugacy classes of $\bar{G}$. The connected component associated to the class of $\bar{g} \in \bar{G}$ is the affine Dynkin diagram $\Phi_{\bar{g}}$ which is the McKay correspondent of the stabilizer $C_G(\bar{g})$. If $\bar{g}$ has order greater than 2 then $\Phi_{\bar{g}}$ is of type $\widehat{A}_{2j-1}$ for some j.* □

The graph $\mathcal{G}$ is described in each case by the data given earlier in this Section. Here is a picture of the case when $G = \langle 2, 3, 4 \rangle$ and N has order 2.

Conjugacy Class	Diagram	Affine Lie Algebra
1		$\widehat{E}_7^{(1)}$
(12)		$\widehat{D}_4^{(1)}$
(12)(34)		$\widehat{D}_6^{(1)}$
(1234)		$\widehat{A}_7^{(1)}$
(123)		$\widehat{A}_5^{(1)}$

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